

Does cosmological expansion affect local physics?

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Abstract

In this contribution I wish to address the question whether, and how, the global cosmological expansion influences local physics, like particle orbits and black hole geometries. Regarding the former I argue that a pseudo-Newtonian picture can be quite accurate if “expansion” is taken to be an attribute of the inertial structure rather than of “space” in some substantialist sense. This contradicts the often-heard suggestion to imagine cosmological expansion as that of “space itself”. Regarding isolated objects in full General Relativity, like black holes, I emphasise the need for proper geometric characterisations in order to meaningfully compare them in different spacetimes, like static and expanding ones. Examples are discussed in some detail to clearly map out the problems.

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1 Introduction

One of the most stunning statements in modern cosmology—i.e. cosmology after Einstein’s seminal paper of 1917 ([49]), Vol. 6, Doc. 43, p. 541-552)—is that “the Universe is expanding”, at least on average. This provokes the question of what it is that expands, i.e., what object or structure is “expansion” really an attribute of? This is the question I wish to address in this contribution. But before outlining its structure, let me briefly recall some historical background.

As a theoretical possibility within the framework of General Relativity (henceforth abbreviated GR), global cosmological expansion was first conceived by Alexander Friedmann (1888-1925) in his 1922 paper [20] and slightly later also in his more popular book “The World as Space and Time” of 1923, of which a German translation is available [21]. However, the “discovery” of the Expanding Universe is nowadays attributed to Georges Lemaître (1894-1966) who was the first to use it as a possible explanation for the observed redshifts (mostly immediately interpreted as due to recession velocities) in the optical spectra of “nebulae” by Vesto Slipher (1875-1969).¹

Slipher’s results were brought to the attention of others mainly by Arthur S. Eddington (1882-1944), who included a list of 41 radial velocities of spiral nebulae in the 2nd edition of his book *The Mathematical Theory of Relativity* of 1924. The list, provided by Slipher, is shown in Figure 1 and contains 41 “nebulae” (galaxies) in a distance range (modern values) roughly between 0.6 and 25 Mpc.² Only five nebulae in Slipher’s list show blueshifts, three of which are members of the local group and hence less than a Mpc away. Roughly speaking, according to this table and modern (independent!) distance estimates, the dominance of recession sets in at about 10 Mpc. However, this table gives redshifts only up to $z \approx 10^{-2}$ and neglects “southern nebulae”, as Eddington regretfully remarks.

It was Hubble who in his famous paper [27] of 1929 explicitly suggested a *linear* relation (to leading order) between between distances and redshifts/velocities, as shown in the well known plot from that paper, which we here reproduce in Figure 2. Note that Hubble underestimated distances by a factor of about 8. For example, Hubble states the distance to the Virgo cluster as 2 Mpc, the modern value being 16.5 Mpc for our distance to its centre.

Modern Hubble plots include type Ia supernovae as standard candles. Figure 3 shows a plot from the final 2001 publication [19] of the *Hubble Key Project*, which includes closer ($z \lesssim 0.1$) Ia supernovae calibrated against Cepheids. Note that this plot already extends in distance scale Hubble’s original one (Figure 2) by a factor of about 25, and that the supernovae investigated by *Supernova Cosmology Project* reach up to redshifts of about $z = 1$, hence extending this plot by another factor of 10. Deviations from a linear Hubble law in the sense of an accelerating expansion

¹ The English translation of the originally french title of Lemaître’s 1927 paper is: “A homogeneous universe of constant mass and increasing radius, accounting for the radial velocity of extragalactic nebulae”.

² Mpc:=Megaparsec= 10^6 parsec. 1 parsec \equiv 1 pc \approx 3.26 ly \approx 3.1×10^{16} m; note ly \equiv lightyear.

RADIAL VELOCITIES OF SPIRAL NEBULAE

+ indicates receding, - approaching

N. G. C.	R. A.		Dec.		Rad. Vel.	N. G. C.	R. A.		Dec.		Rad. Vel.
	<i>h</i>	<i>m</i>	°	'	km. per sec.		<i>h</i>	<i>m</i>	°	'	km. per sec.
221	0	38	+40	26	- 300	4151*	12	6	+39	51	+ 980
224*	0	38	+40	50	- 300	4214	12	12	+36	46	+ 300
278†	0	47	+47	7	+ 650	4258	12	15	+47	45	+ 500
404	1	5	+35	17	- 25	4382†	12	21	+18	38	+ 500
584†	1	27	- 7	17	+1800	4449	12	24	+44	32	+ 200
598*	1	29	+30	15	- 260	4472	12	25	+ 8	27	+ 850
936	2	24	- 1	31	+1300	4486†	12	27	+12	50	+ 800
1023	2	35	+38	43	+ 300	4526	12	30	+ 8	9	+ 580
1068*	2	39	- 0	21	+1120	4565†	12	32	+26	26	+1100
2683	8	48	+33	43	+ 400	4594*	12	36	- 11	11	+1100
2841†	9	16	+51	19	+ 600	4649	12	40	+12	0	+1090
3031	9	49	+69	27	- 30	4736	12	47	+41	33	+ 290
3034	9	49	+70	5	+ 290	4826	12	53	+22	7	+ 150
3115†	10	1	- 7	20	+ 600	5005	13	7	+37	29	+ 900
3368	10	42	+12	14	+ 940	5055	13	12	+42	37	+ 450
3379*	10	43	+13	0	+ 780	5194	13	26	+47	36	+ 270
3489†	10	56	+14	20	+ 600	5195†	13	27	+47	41	+ 240
3521	11	2	+ 0	24	+ 730	5236†	13	32	-29	27	+ 500
3623	11	15	+13	32	+ 800	5866	15	4	+56	4	+ 650
3627	11	16	+13	26	+ 650	7331	22	33	+33	23	+ 500
4111†	12	3	+43	31	+ 800						

Figure 1: Table taken from Eddington's *The Mathematical Theory of Relativity*. N.G.C. refers to the "New General Catalogue" as published by John Louis Emil Dreyer in the *Memoirs of the Royal Astronomical Society*, Vol. 49, in 1888. (The "New" refers to the fact that it succeeded John Herschel's "General Catalogue of Nebulae" of 1864.) This table was prepared for the second edition of Eddington's book by V. Slipher, containing, as of February 1922, a complete list of measured red shifts (velocities). Entries marked with * are said to be confirmed by others, those marked by † are said not to be as accurate as others. Of the five entries with negative radial velocities those approaching fastest are the Andromeda Galaxy NGC 224 (M 31), its elliptical dwarf satellite galaxy NGC 221 (M 32), and the Triangulum Galaxy NGC 598 (M 33), all of which are well within the Local Group. Of the other two, NGC 3031 (M 81) is Bode's Galaxy, whose modern (1993) distance determination gives 3,7 Mpc. The other one, NGC 404 (not listed in Messier's catalogue) is also known as "Mirach's Ghost" for it is hard to observe being close to the second magnitude star Beta Andromedae. Up to ten years ago its distance was very uncertain. Since 2001, and with independent confirmations in the subsequent years, it has been determined to be at 3.1-3.3 Mpc. The largest positive redshift in the list is shown by NGC 584, which is $z \approx 6 \times 10^{-3}$ corresponding to a recession velocity of 1800 km/s. This value is confirmed by modern measurements (NASA/IPAC Extragalactic Database). This is an elliptical galaxy in the constellation Cetus, the modern distance estimate of which is 23.4 according to the OBEY survey (Observations of Bright Ellipticals at Yale).

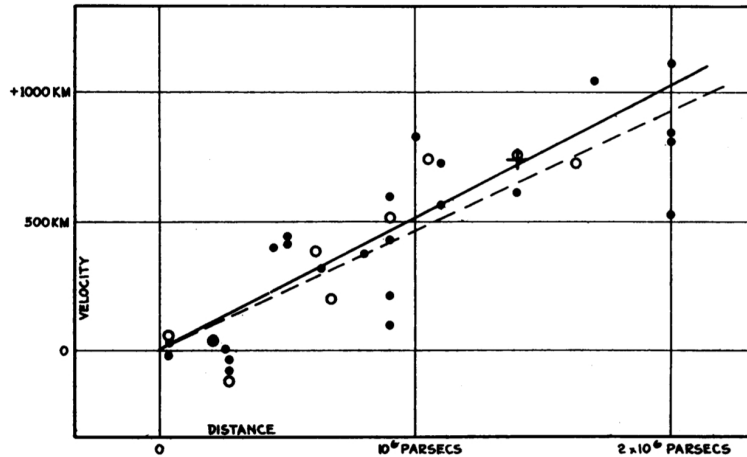


Figure 2: Hubble's original plot (shown on p. 172 of [27]) of radial recession velocity, inferred from the actually measured redshifts by assuming Doppler's formula, versus distance. The latter is underestimated by a factor up to about 8.

are seen roughly above $z = 0.4$. See, e.g., [17, 42] for more on the intriguing history of the "expanding universe".

At first sight, the above statement concerning the Universe being in a state of expansion is ambiguous and hardly understandable. It cannot even be taken face value unless we have a good idea of what "Universe" refers to. But this can be said precisely within the limits of relativistic *cosmological models* about which I will make some general remarks in the next section.

In this contribution I will rather focus on the follow-up question of how to characterise structures that do and structures that do not participate in the expansion. For this I will first consider orbits of "test" masses (structureless masses of arbitrarily small spatial extent whose own gravitational field is negligible) in the gravitational field of a central mass, the whole system being embedded in an expanding universe. In section 2 I will employ a simple pseudo-Newtonian model for the dynamics of point particles in expanding universes. In this model the absolute simultaneity structure as well as the geometry for space and time measurements will be retained from Newtonian physics, but the inertial structure is changed so as to let the cosmological expansion correspond to force-free (inertial) motion. This will be achieved by adding an additional term to Newton's second law, in full analogy to the procedure one applies in other cases when rewriting Newton's second law relative to non-inertial reference frames. This model, the intuitive form of which will be later justified in the context of GR, will lead us to reasonable first estimates for those systems that are themselves reasonably well described by Newtonian physics (i.e. excluding things like black holes). In doing this I will stress that the thing that undergoes expansion, either accelerated or decelerated, is the inertial structure and not some kind of "space" in the sense of substantivalism. The

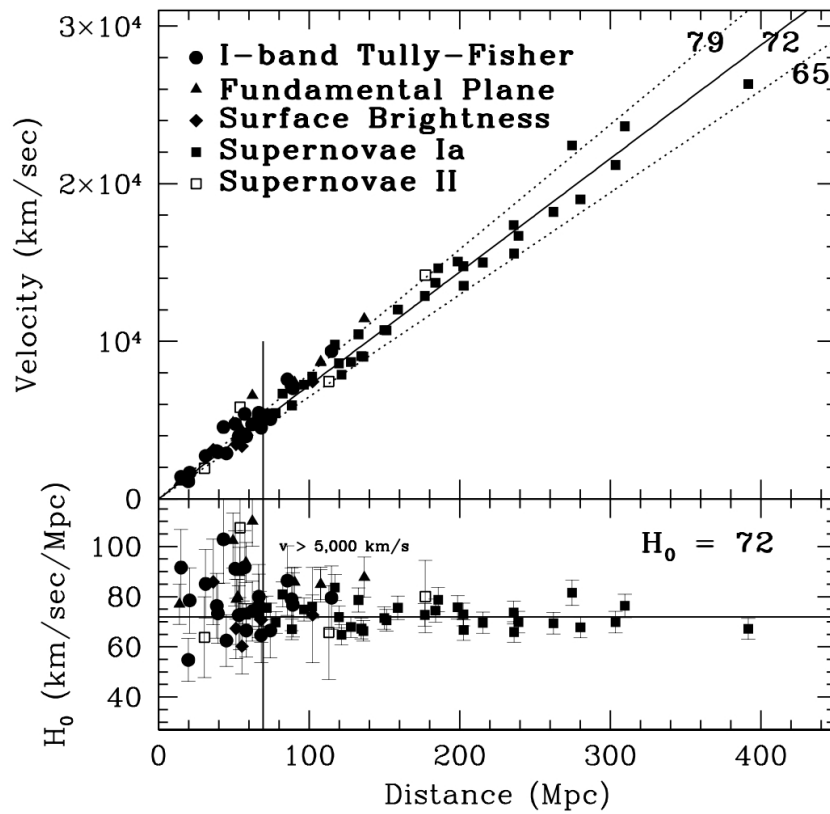


Figure 3: Modern Hubble plot including the closer type Ia supernovae up to distances of 400 Mpc and redshifts of 0.1 (recession velocities of almost 10% of the velocity of light). This plot is taken from the final analysis [19] of the *Hubble Space Telescope Key Project*. A slope of $H_0 = 72 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$ is shown flanked by $\pm 10\%$ lines. The plot below shows the apparent variation of H_0 with distance.

latter notion would suggest the emergence of frictional or viscous forces on any body moving relative to that kind of “space”. But, as we will also see in section 2, this is not the right picture even though it is often stated (in particular in popular accounts) that “it is space itself that expands”. Section 3 discusses cosmological models in GR proper and justifies the pseudo-Newtonian approach regarding test-mass orbits of section 2 from first principles in GR. It also comments on another effect that cosmological expansion has on the mapping of trajectories. This effect is more of a kinematical nature and arises from the fact that the notions of *simultaneity* and *instantaneous distance*, as defined by the geometry in standard cosmological models, are not identical with the corresponding notions using the exchange of light signals (Einstein simultaneity). Section 4 discusses black holes in expanding universes, which are outside the realm of applicability of the pseudo-Newtonian picture. Rather, here we have to employ proper geometric techniques from GR in order to be sure to characterise the physical situations independently of the coordinates used. Known exact solutions representing spherically symmetric black holes in expanding universes are discussed with an attempt to meaningfully characterise the impact of expansion. Finally, generalisations of a specific class of solutions are discussed along the lines of [9].

A standard picture for global expansion is that of a rubber balloon being gradually filled with air; see, e.g., Figure 27.2 in [38]. In such a picture the “Universe” is identified with the rubber sheet of a balloon. The two dimensional sheet is meant to represent three dimensional space. Points in real space off that sheet are simply not part of the model and do not represent anything real. On that sheet we paint little circular discs and also glue some coins of about the same size. The painted elements of the rubber material continue to expand unhindered from each other, but underneath the coins the glue holds them rigidly in positions of unchanging mutual distances. We ask: which physical structures in the real world are meant to correspond to the painted and which to the glued coins? What sort of physical interactions can act like the glue in this picture?

Note that in the pseudo-Newtonian discussion we pretend a clear split between space and time and that the “Universe” at an “instant” corresponds to three-dimensional space filled with all there is. It is clear that in relativistic cosmology this corresponds to more structure than just a spacetime (four-dimensional differentiable manifold with Lorentzian metric) that satisfies the coupled field equations for the gravitational (metric) and matter fields. What structure one needs in order to be allowed to talk in a Newtonian fashion will we recalled below. Until then let us proceed unworried guided by Newtonian intuition.

If for the moment we assume that cosmological expansion were active within our solar system, we might be tempted to suspect it to cause dynamical anomalies, like extra radial accelerations. Such an anomalous acceleration had indeed been found for the Pioneer 10 and 11 spacecrafts [2, 35]. Its possible cosmological origin is suggested by its observed magnitude

$$\Delta a = (8.6 \pm 1.34) \times 10^{-10} \text{ m} \cdot \text{s}^{-2}, \quad (1)$$

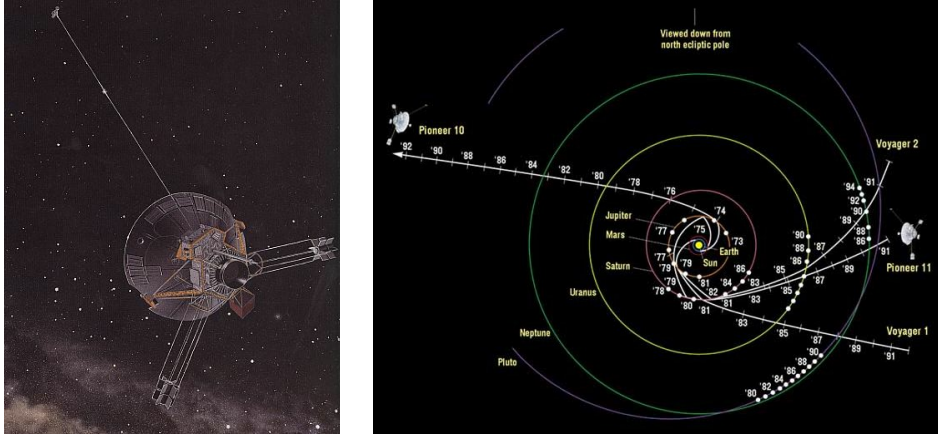


Figure 4: Left: Impression of the Pioneer satellite. Right: View from the north ecliptic pole onto the ecliptic showing the escape orbits of Pioneer 10 and Pioneer 11. The dots and numbers along the trajectories denote the positions and dates for each satellite.

which is close to the product of the currently measured Hubble constant H_0 with the velocity of light

$$H_0 \cdot c \approx (74 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}) \cdot (3 \times 10^5 \text{ km} \cdot \text{s}^{-1}) \approx 7 \times 10^{-10} \text{ m} \cdot \text{s}^{-2}. \quad (2)$$

Now, even though the *Pioneer anomaly*, as it was called, has most likely received a far more mundane explanations recently [4, 45, 48], and even though it always seemed hard to believe that such a connection should exist at all, it was not entirely easy to show such an impossibility within a scheme of controlled approximations. Note also that the sign was contrary to intuition: Whereas an accelerated expansion (like the one we presently seem to undergo) would give rise to extra outward-pointing accelerations (see below), the measured anomalous accelerations of the Pioneer spacecrafts pointed inwards, more or less towards the Sun-Earth system.

2 A Pseudo-Newtonian Picture

The rubber-sheet picture of cosmological expansion is useful to capture some *kinematical* aspect. But, if naïvely interpreted, it is definitely misleading as far as the *dynamical* aspects are concerned. The naïve and misleading interpretation is to think of expanding space as moving substance, composed of individuated points that can be assigned a local state of motion, like for a fluid. This picture would suggest that a body in relative motion to the “fluid” will eventually be dragged with it due to frictional or viscous forces. If this were true, the equations of motions should contain such force terms whose effect is to universally diminish velocities relative to the cosmic flow. But this is not the case, as has been discussed at length in [3, 43, 55] and as we will see below. Rather than causing *frictional forces* proportional to the *first* time-derivative of expansion parameter, the correct picture of

the expansion’s dynamical effect is to cause *apparent forces* proportional to the *second* time-derivative of the expansion parameter. In other words, what cosmic expansion does is to change the inertial structure of space, so that coordinates based on metrically equidistant marks (as usually employed in the Newtonian equations of motion) become non inertial.

2.1 Changing the inertial structure

A natural way to fit the general-relativistic concept of cosmological expansion into a pseudo-Newtonian framework is to recall that in GR the expanding structures move inertially (if represented by dust-like matter without pressure). This is transcribed into Newtonian language by keeping the absolute simultaneity structure and the geometries for space and time measurements (instantaneous space being still modelled by \mathbb{R}^3 with its euclidean metric), but replacing the usual inertial structure such that inertial motion is now represented by worldlines of particular time-dependent instantaneous mutual separation (rather than being straight and parallel). This can be done by adding a suitable term to Newton’s second law, just as it is done by rewriting this law so as to be valid in non inertial reference frames. We now give a simple derivation of this extra term.

We require the local inertial frames to radially move apart according to Hubble’s law:

$$\dot{R}(t) = H(t)R(t). \quad (3)$$

Here $R(t)$ is the instantaneous (with respect to some cosmological time) distance between two inertial frames and H is the Hubble constant (the “constant” refers to it not depending on space). $H(t)$ is usually related to some cosmological scale parameter $a(t)$ via

$$H(t) := \dot{a}(t)/a(t). \quad (4)$$

This gives (suppressing arguments from now on)

$$\dot{H} = (\ddot{a}/a) - H^2 = -H^2(1 + q), \quad (5)$$

where

$$q := -\ddot{a}a/\dot{a}^2 \quad (6)$$

is usually called the *deceleration parameter*. Note that $H = \dot{a}/a$ and \ddot{a}/a are decreasing with time $\propto t^{-1}$ and $\propto t^{-2}$, respectively, if $a(t)$ is a power-law and are constant in time if $a(t)$ grows exponentially. In any case, we assume that the typical timescales on which both quantities vary are much larger than the timescale over which the motions of the objects that we consider take place, so that we can replace H and q by their current values, indicated by a subscript 0 (for $t = t_0 =$ ‘now’). A recent evaluation [30], based on the 7-year WMAP data, quotes as “seven-year mean” the following value for the Hubble constant³

$$H_0 = (70.4 \pm 2.5) \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}, \quad (7)$$

³ This result includes as prior the independent determination of the Hubble constant from a differential distance ladder [44].

and as values for the relative contributions of matter and (dark/vacuum-) energy to the overall gravitating energy

$$\Omega_m = 0.274 \pm 0.03, \quad \Omega_\Lambda = 0.727 \pm 0.03. \quad (8)$$

The deceleration parameter follows from this via the relation $q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$, which is an immediate consequence of the Friedmann equations. Hence we may take as currently best value

$$q_0 = -0.59. \quad (9)$$

Proceeding in a pseudo-Newtonian fashion, we recall that the Newtonian force is proportional to the acceleration relative to the local inertial frames. The local inertial frames move radially according to (3). Hence, using (3) and (5), their outward acceleration is

$$\ddot{R} = \dot{H}R + H\dot{R} = -HqR = \frac{\ddot{a}}{a}R. \quad (10)$$

With respect to a given inertial frame, which we may choose as origin of \mathbb{R}^3 , the other inertial systems move with instantaneous velocity and acceleration

$$\begin{aligned} \dot{\vec{x}} &= H\vec{x}, \\ \ddot{\vec{x}} &= -qH^2\vec{x} = \frac{\ddot{a}}{a}\vec{x}, \end{aligned} \quad (11)$$

which is independent of the chosen origin.⁴

This means that the Newtonian equations of motion of a test particle in an expanding universe are obtained from the usual one by replacing

$$\ddot{\vec{x}} \mapsto \ddot{\vec{x}} - (\ddot{a}/a)\vec{x}. \quad (12)$$

We will later discuss how this replacement finds a simple explanation in GR. With it, the modified Newtonian equation then reads

$$m(\ddot{\vec{x}} - (\ddot{a}/a)\vec{x}) = \vec{F}. \quad (13)$$

We end this subsection by making the simple general observation that (13) involves only the *second* time derivative of $a(t)$. Hence the sign of \dot{a} does not matter, as would be the case if the impact of cosmological expansion would be analogous to some sort of viscous or frictional force due to the relative motion against a substantialist's "space". A given positive \dot{a}/a can either be caused by a universe in a state of accelerated expansion or decelerated contraction. Likewise, a given negative \dot{a}/a can either be caused by a universe of decelerated expansion or accelerated contraction.

⁴ Let $\vec{v}(t, \vec{x})$ be an arbitrary velocity field in \mathbb{R}^3 , say of a fluid. An observer who at time t and position \vec{x} is co-moving with the fluid sees the velocity distribution $\vec{h} \mapsto \vec{w}_{\vec{x}}(t, \vec{h}) := \vec{v}(t, \vec{h} + \vec{x}) - \vec{v}(t, \vec{x})$. This is independent of the observers location \vec{x} if and only if $\vec{v}(t, \vec{x})$ is an affine function of \vec{x} , i.e. $\vec{v}(t, \vec{x}) = A(t) \cdot \vec{x} + \vec{a}(t)$ for some matrix-valued function A and vector-valued function \vec{a} of time. Sufficiency is obvious and necessity follows since $\vec{w}_{\vec{x}}(t, \vec{h}) = \vec{w}_{\vec{0}}(t, \vec{h})$ can be simply rewritten into $\vec{w}_{\vec{0}}(t, \vec{h} + \vec{x}) = \vec{w}_{\vec{0}}(t, \vec{h}) + \vec{w}_{\vec{0}}(t, \vec{x})$, showing that $\vec{w}_{\vec{0}}$ must be linear in its second argument. The result now follows from $\vec{v}(t, \vec{x}) = \vec{w}_{\vec{0}}(t, \vec{x}) + \vec{v}(t, \vec{0})$.

2.2 Inertial motion

In this subsection we briefly discuss solutions to (13) for $\vec{F} = 0$ and special expansion laws $a(t)$. This is meant to illustrate the last point of the previous subsection. We start with the so-called matter-dominated universe in a state of decelerated expansion, given by $a(t) \propto t^{2/3}$. This gives

$$\ddot{a}(t)/a(t) = -\frac{2}{9} \cdot t^{-2}. \quad (14)$$

Integrating (13) for the initial conditions $\vec{x}(1) = (R, 0, 0)$ and $\dot{\vec{x}}(1) = \vec{0}$, corresponding to a body that at time $t = 1$ is released at distance R on the x -axis with zero velocity, we obtain the following solution for the x coordinate (the y and z coordinates stay zero)

$$x(t) = R(2t^{1/3} - t^{2/3}). \quad (15)$$

So even though “space expands”, the particle first approaches the origin $x = 0$, hits it at $t = 8$ (i.e. after seven further unfoldings of the “world age”), and then recedes from it along the negative x -axis in an asymptotically co-moving fashion. That despite expansion the particle first starts to approach $x = 0$, rather than recede from it, as a frictional force would imply, is not too surprising in view of the fact that the initial condition $\dot{x}(1) = 0$ means that, *relative to the inertial frame at the initial position* $x(1) = R$, the particle is moving with velocity \dot{a}/a towards the origin. The fact that the particle asymptotically approaches a co-moving state is not universally implied by the equations of motion, as one can see from the remark that (14), and hence (15), are the same for $a(t) \propto t^{1/3}$, in which case the particle overshoots the cosmic expansion for large times.⁵

Similarly, for an exponential scale factor $a(t) \propto \exp(\lambda t)$, where t now ranges over the full real axis, we have

$$\ddot{a}(t)/a(t) = \lambda^2. \quad (16)$$

Picking the corresponding initial data $x(0) = R$ and $\dot{x}(0) = 0$ for (13) we now get

$$x(t) = R \cosh(\lambda t). \quad (17)$$

Note that this is insensitive of the sign of λ . For $\lambda > 0$ this might confirm the naïve expectation, since the particle immediately starts to recede from the origin and asymptotically approaches a co-moving state. But this is certainly not true for a contracting universe where $\lambda < 0$.

⁵ Quite generally, an FLRW universe with perfect-fluid matter and equation of state $p = w\rho c^2$, where $w > -1$, expands like $a(t) \propto t^n$ with $n = 2/3(w + 1)$, so that $n = 1/3$ corresponds to $w = 1$, the extreme positive-pressure case in view of energy dominance.

2.3 Coulomb-like potential

Let us now apply (13) to the case where \vec{F} is a time independent radial force $\propto 1/r^2$. To keep matters simple we will assume \ddot{a}/a to be constant. This is exactly true if $a(t) \propto \exp(\lambda t)$, or approximately for motions during timescales in which \ddot{a}/a changes very little. The second term in (13) then acts like a time-independent radial pseudo-force, which is outward pointing for either accelerated expansion or decelerated contraction, and inward pointing for decelerated expansion or accelerated contraction. The current value of this relative acceleration of inertial frames per unit of separation distance can be inferred from the data given above. Using $\ddot{a}_0/a_0 = -q_0 H_0^2$ and the values from (7) and (9) gives

$$A := \frac{\ddot{a}_0}{a_0} = -q_0 H_0^2 \approx 10^{-13} \cdot \text{m} \cdot \text{s}^{-2} \cdot \text{Mpc}^{-1}, \quad (18)$$

which looks very small indeed! Here we introduced the letter A as abbreviation for later notational convenience.

We now put a time-independent radial $1/r^2$ force on the right-hand side of (13),

$$\vec{F} = -m \vec{\nabla} \left(-\frac{C}{r} \right), \quad (19)$$

where C is a constant which we assume to be positive since we are only interested in attractive central forces.

As usual, time independence and rotational symmetry allow us to infer conservation laws for energy and angular momentum. The latter implies in particular that the motion stays in a plane, which we coordinatise by planar polar coordinates (r, φ) . Denoting by E the energy per unit mass and by L the modulus of angular momentum per unit mass, the conservation laws are

$$\frac{1}{2} \dot{r}^2 + U(r) = E, \quad r^2 \dot{\varphi} = L, \quad (20)$$

where

$$U(r) = \frac{L^2}{2r^2} - \frac{C}{r} - \frac{A}{2} r^2. \quad (21)$$

This differs from the usual expression by the last term. In order to gain a first estimate of its impact consider the case $A > 0$, e.g. accelerated expansion. In this case it leads to an additional local maximum to the right of the minimum describing stable circular orbits. For values of r greater than that of the additional maximum the system is doomed to expand forever by following the cosmic expansion. The critical radius at which cosmic expansion just balances the attraction for a particle at rest ($L = 0$) is

$$r_c = \left[\frac{C}{A} \right]^{1/3}. \quad (22)$$

As we will see in more detail below, this critical radius is an approximate upper bound for radii of stable circular orbits. Let us estimate its size. If attraction is due

to the gravitational force of a central mass M upon a test mass m , or the Coulomb force of a central charge Ze upon an elementary charge e (also of mass m), we have

$$C = \begin{cases} GM & \text{gravitational field} \\ \frac{Qe}{4\pi\epsilon_0 m} & \text{electric field.} \end{cases} \quad (23)$$

The corresponding critical radii (22) can then be expressed as weighted geometric mean of the Hubble radius

$$R_H := \frac{c}{H_0} = 4.23 \times 10^9 \text{ Mpc} = 13.7 \times 10^9 \text{ ly} = 1.3 \times 10^{26} \text{ m} \quad (24)$$

with the Schwarzschild radius for the central mass M (gravity case)

$$R_M := \frac{2GM}{c^2} \approx \left(\frac{M}{M_\odot} \right) \cdot 3 \times 10^3 \text{ m}, \quad (25)$$

or with the classical charge-radius⁶ for the charge Q with mass m (electric case)

$$R_Q := \frac{Q^2}{8\pi\epsilon_0 mc^2} \approx \left(\frac{Q}{e} \right)^2 \cdot 1.4 \times 10^{-15} \text{ m}. \quad (26)$$

Explicitly, the simple expression for the critical radius in the gravitational case is

$$r_c^{(gr)} = [-1/2q_0]^{1/3} \cdot [R_M R_H^2]^{1/3} \approx \left(\frac{M}{M_\odot} \right)^{1/3} 352 \text{ ly}, \quad (27)$$

and in the electric case⁷

$$r_c^{(el)} = [-2e/Qq_0]^{1/3} \cdot [R_Q R_H^2]^{1/3} \approx \left(\frac{Q}{e} \right)^{1/3} 30 \text{ AU}. \quad (28)$$

The critical radii are the characteristic scales above which systems, which were bound if no expansion existed, disintegrate as a result of accelerated cosmological expansion or decelerated contraction.

The last equations (28) shows that disintegration of a Hydrogen atom is inevitable if the electron-proton distance is of the order of 30 astronomical units, that is about the semi-major axis of the Neptune orbit! Atoms, humans, and all things around us are essentially unaffected by cosmic expansion. For gravitating systems, (27) sets the scale above which the solar system starts to disintegrate beyond 300 lightyears. Recall that the next star (Proxima Centauri) is about 4 lightyears away. If the central body is of about 10^{12} solar masses, like our Galaxy, expansion sets in above a scale above 3 million lightyears, that is 1.5 times farther than the distance to the Andromeda galaxy. Finally, if the consider a central mass of 10^{15} solar

⁶ The classical charge-radius of a charge Q with mass m is defined to be the radius outside which the energy stored in the Coulomb field of charge Q equals mc^2 . Sometimes twice of that is called the ‘‘classical charge radius’’.

⁷ AU \equiv astronomical unit $\approx 1.5 \times 10^{11}$ m.

masses, like for the Virgo cluster, expansion sets in above 30 million lightyears or 10 megaparsecs. This is now, finally, smaller than the distance to other (smaller) clusters, like Fornax (≈ 30 megaparsecs), and even larger than the distance to the local group (≈ 20 megaparsecs). This leads one to roughly estimate the scale in our Universe above which gravitationally interacting systems start to follow the Hubble flow to be that of large galaxy clusters. Structures below that size are expected to stay bounded. This fits well with the observational status described in the introduction. Note that the absence of any expansion below certain distances does not fit with the statement that “space itself expands”, as there is space everywhere.

2.4 Existence of stable circular orbits

Circular orbits are those of unchanging radius, i.e. $r(t) = r_*$. Hence $\dot{r}(t) = 0$ for all times t . It follows from (20) that that U must have a stationary point at $r = r_*$:

$$U'(r = r_*) = -\frac{L^2}{r_*^3} + \frac{C}{r_*^2} - Ar_* = 0. \quad (29)$$

If the circular orbit is to be stable, the stationary point must be a minimum:

$$U''(r = r_*) = 3\frac{L^2}{r_*^4} - 2\frac{C}{r_*^3} - A > 0. \quad (30)$$

Adding $(3/r_*)$ times (29) to (30) gives

$$\frac{C}{r_*^3} - 4A > 0. \quad (31)$$

Since we restrict to $C > 0$ this is automatically satisfied if $A < 0$, i.e. for decelerated expansion or accelerated contraction. However, for accelerated expansion or decelerated contraction we get the non-trivial constraint

$$r_* < 4^{-1/3}r_c \approx 0.63 \cdot r_c, \quad (32)$$

where r_c was defined in (22). This gives the precise upper bound on stable circular orbits.

The stability properties of the potential (21) are summarised in Figure 5, where we plotted the rescaled potential $u := \left(\frac{r_*}{C}\right) \cdot U$ as a function of $x := \frac{r}{r_*}$. Here r_* is the radius at which $U'(r_*) = 0$, i.e. it satisfies (29). We then have

$$u(x) = \frac{1 - \alpha}{2x^2} - \frac{1}{x} - \frac{\alpha}{2}x^2, \quad (33)$$

with

$$\alpha := r_*^3 \cdot \frac{A}{C}. \quad (34)$$

By construction $u(x)$ has an extremum at $x = 1$. In Figure 5 we plotted (33) for various values of $\alpha \in [-1, 1]$, with $-1 \leq \alpha < 0$ corresponding to the case of

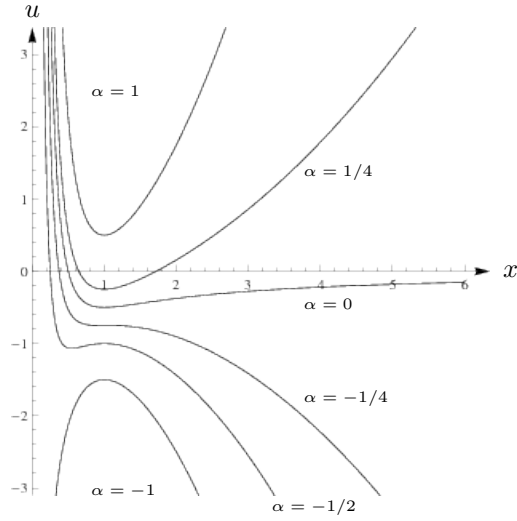


Figure 5: This figure, taken from [10], shows various plots of the potential (21) in the rescaled form (33). Circular orbits correspond to the extremum at $x = 1$, which are stable (minimum) for $\alpha < 1/4$ and unstable $\alpha > 1/4$.

decelerated expansion or accelerated contraction and $0 < \alpha \leq 10$ to accelerated expansion or decelerated contraction. For increasing values of α the right slope of the graph comes down so as to turn the extremum at $x = 1$, which is a minimum for $\alpha < 1/4$, to a maximum for $\alpha > 1/4$. This corresponds to the transition from stable to unstable circular orbits.

The angular frequency ω_* of a circular orbit follows from the expression (29) for $U'(r_*) = 0$ if we set $L = r_*^2 \omega_*$. We get

$$\omega_* = \omega_K \sqrt{1 - r_*^3 \frac{A}{C}} \quad (35)$$

where

$$\omega_K := \sqrt{\frac{C}{r_*^3}} \quad (36)$$

is the ‘Keplerian orbital frequency of the unperturbed problem. Switching on A changes Kepler’s 3rd law from the familiar form $\omega_* = \omega_K$ to its modified form (35)

If we compare circular orbits of a fixed radius $r = r_*$ of the unperturbed ($A = 0$) with the perturbed problem, (35) tells us that the angular frequency is diminished if $A > 0$ and enhanced if $A < 0$. This clearly fits well with intuition since the additional force due to $A \neq 0$ is directed parallel ($A > 0$) or opposite ($A < 0$) the centrifugal force, so that in order to maintain equilibrium of forces we have to diminish or enhance the latter. If we adiabatically switched on cosmological expansion, the system would just readjust itself according to the modified

Keplerian law (35). For small A changes in the periods and radii are of the order $(r_*/r_c)^3$, where $r_c = (C/|A|)^{1/3}$, which are small indeed (compare discussion above).

3 The General-Relativistic Picture

General relativistic models of spacetime differ from such models in Newtonian physics in several aspects. Very loosely speaking, less structure is assumed in GR than is in Newtonian contexts. The word *spacetime* in GR usually just refers to a tuple (M, g) , where M is a smooth 4-dimensional manifold and g is a (piecewise) smooth Lorentzian metric. Sometimes the word *spacetime* is reserved to those tuples (M, g) where g obeys Einstein’s equations with suitable matter sources, but here we need not be specific about that. The first thing we wish to stress – following Hermann Weyl – is that a *cosmological model* comprises more structure than just a spacetime.

3.1 On the notion of “cosmological model” in GR

As already remarked in the introduction, we have to first understand what “Universe” refers to if we want to understand the statement concerning its alleged expansion. In order to answer this question in the context and geometric spirit of GR (i.e. in terms of geometric structures rather than special coordinate system), we recall that according to Weyl [53] the definition of a *cosmological model* comprises not only a spacetime (M, g) , but also a normalised timelike vector field V on M . In GR a cosmological model thus comprises at least a triple (M, g, V) . The rôle of V is to represent the flow of cosmological dust matter (sometimes referred to as “privileged observers”). Already at this point we may speak of (local) expansion/contraction, namely if V (locally) has positive/negative divergence. Weyl stresses [54] that without the geometric structure supplied by, and the interpretation given to, V one could not even attempt to calculate the cosmological redshifts, which is always to be thought of as taking place between systems moving along different flow lines of V .⁸ Now, generally one would of course require that g and V be related by Einstein’s equation. For example, the energy momentum tensor of a perfect fluid (no friction, no heat conduction) is just

$$T = \rho V \otimes V + p(c^{-2}V \otimes V - g^{-1}), \quad (37)$$

where ρ is the fluid’s mass density and p is its pressure, both measured in the local rest frame. V is the fluid’s four-velocity vector field (normalised to $g(V, V) = c^2$)

⁸ Here we do not wish to enter into the discussion of *Weyl’s principle*, which further asserts that the flow lines of V should be contained in a region of common causal dependence, i.e. that no particle horizons should exist [13]. See, e.g., [24] and [47] for discussions of partly alternative viewpoints on this principle.

and g^{-1} is the “inverse metric” field (metric in the cotangent spaces). The integrability condition of Einstein’s equation, which says that T must have vanishing covariant divergence with respect to g , then implies the relativistic Euler equations for V , which in the pressureless case, $p = 0$, imply that V is geodesic with respect to g . If, in addition, V is hypersurface orthogonal, i.e. of vanishing vorticity, we may address the hypersurfaces orthogonal to V as “space” (at least locally). Such a hypersurface may then be called an “instant”. The “Universe at an instant” would consist of the instant and all metric and matter fields comprising Cauchy data. That “the Universe (locally) expands” usually just means that Cauchy surfaces (locally) increase their volume along the flow of V .

Sometimes one just considers geodesic timelike vector fields V on a given spacetime (M, g) that solves Einstein’s equation *without* (37) as source, that is, one neglects the back-reaction of the dust matter onto the geometry of spacetime. Then it should not come as a surprise that the choice of V is ambiguous. The most trivial example is perhaps flat Minkowski space, where, in standard coordinates, we could just take

$$V_1 = \partial/\partial t \quad (38)$$

which is even a geodesic Killing field. Another choice, defined in the upper wedge region $W := \{(t, x, y, z) \in \mathbb{R}^4 \mid ct > r := \sqrt{x^2 + y^2 + z^2}\}$, would be as follows: Instead of (t, r) use coordinates (T, ρ) in W , where

$$t = T\sqrt{1 + \rho^2}, \quad r = cT\rho \quad (39)$$

with inverse

$$T = \sqrt{t^2 - r^2/c^2}, \quad \rho = \frac{r}{\sqrt{c^2 t^2 - r^2}}. \quad (40)$$

The hypersurfaces of constant T are spacelike hyperboloids of constant timelike distance T from the origin. Their intrinsic metric is of constant negative curvature T^{-2} . They are orthogonal to the timelike geodesic vector field

$$V_2 = \frac{\partial}{\partial T} = \frac{1}{\sqrt{t^2 - r^2/c^2}} \left(t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \right). \quad (41)$$

The Minkowski metric, restricted to W , can be written as

$$\begin{aligned} g|_W &= -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \\ &= -c^2 dT^2 + c^2 T^2 \left(\frac{d\rho^2}{1 + \rho^2} + \rho^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right). \end{aligned} \quad (42)$$

This shows that the region W of Minkowski spacetime can be written as an “open” (i.e. spatially negatively curved) FLRW model with scale function $a(T) = T^2$, which is usually called the Milne model. Clearly V_2 is not Killing.

A more interesting variety exists for the de Sitter spacetime, which is a solution (M, g) to the vacuum Einstein equations with cosmological constant $\Lambda > 0$. It can

be represented as one-sheeted hyperboloid in five-dimensional Minkowski space:⁹

$$M = \{(x^0, x^1, x^2, x^3, x^4) \in \mathbb{R}^5 \mid -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 3/\Lambda\} \quad (43)$$

and where g is the metric induced from ambient Minkowski space. Various time-like and hypersurface orthogonal geodesic vector fields V exist, either globally or on sub-domains of M , which allow to represent these domains as FLRW universes with positive, zero, or negative spatial curvature, expanding with cosh, exp and sinh of time respectively. The corresponding homogeneous spacelike hypersurfaces are given by taking the intersection of M with the following three families of spacelike, lightlike, and timelike hyperplanes in \mathbb{R}^5 : $x^0 = \text{const.}$, $x^0 + x^1 = \text{const.} > 0$, and $x^1 = \text{const.} > 1$ with $x^0 > 0$.

It is not necessary to write down explicit expressions for the various vector fields V . This can be done, e.g., from the expressions one obtains for the FLRW forms of the metric by setting $V = \partial/\partial T$, where T is the FLRW time. Coordinate expressions of the FLRW forms corresponding to the slicings just mentioned are well known¹⁰

This shall suffice to characterise the ambiguity in representing a given space-time (M, g) as a sequence of Universes, i.e. in turning a given (M, g) into a “standard model”. This is almost always expressed in terms of different coordinate systems on (subsets of) M , which hides the fact that a cosmological model can, of course, be fully characterised by proper geometric structures, independent of any choices of coordinates.

3.2 Standard Models

The so-called standard models in relativistic cosmology are based on metrics of the FLRW (Friedman, Lemaître, Robertson, Walker) form

$$g = c^2 dt^2 - a^2(t)g_{cc}^{(3)}, \quad (44)$$

where $g^{(3)}$ is a 3-dimensional Riemannian metric of constant curvature (hence subscript cc). A set of minimal assumptions leading to (44) are given in [50]. The topologies of the constant time hypersurfaces $t = \text{const.}$ are restricted if one requires for them the condition of completeness (equivalent to geodesic completeness). This implies closure (compact without boundary) and hence finite volume in the positive curvature case, whereas in the zero or negative curvature cases open and closed universes may exist.¹¹

The geodesic vector field V is given in these coordinates simply by

$$V = \frac{\partial}{\partial t}. \quad (45)$$

⁹ The global structure of de Sitter’s solution was revealed by Lanczos, Weyl, and, last not least, Felix Klein. The contribution of Klein is often overlooked. See [46] for a fair account.

¹⁰ See, e.g., http://en.wikipedia.org/wiki/De_Sitter_space

¹¹ Sometimes constant negative curvature is taken synonymously for openness, which is unfortunate.

It is Killing if and only if $\dot{a} = 0$. In the cases we are interested in here a will have a non trivial time dependence, which means that neighbouring geodesics change separation. For infinitesimally nearby geodesics the perpendicular connecting vector, X , changes according to Jacobi's equation

$$\nabla_V \nabla_V X = -R(X, V)V, \quad (46)$$

where R is the Riemann tensor.¹² For given V the right-hand side of (46) should be read as a map $X \mapsto -R(X, V)V$ that at each point p in spacetime linearly maps the 3-dimensional orthogonal complement of V in tangent space to itself. It is called the Jacobi map J . For (44) one finds

$$J = \frac{\ddot{a}}{a} \text{id} \quad (47)$$

where id refers to the identity map (in the respective orthogonal complement of V).

This is the general-relativistic rationale for the heuristic step taken in (12): If we write down (46) in terms of Fermi normal coordinates (t, \vec{x}) centred around some integral curve of V (along which the eigentime equals cosmological time t), it follows immediately from (47) that the nearby geodesics obey

$$\ddot{\vec{x}} - (\ddot{a}/a)\vec{x} = 0, \quad (48)$$

where an overdot stands for differentiation with respect to t . This equation characterises inertial motions in a small neighbourhood of a reference observer. It can be generalised to the motion of electric charges in a FLRW background (i.e. without taking into account any back-reaction of the charges onto the spacetime geometry) in the following manner: Let a charge Q move on an integral line of V (i.e. geodesically). Determine its electromagnetic field by solving Maxwell's equations in that background.¹³ Consider another charge, e , that moves in the combined backgrounds of the FLRW universe and the electromagnetic field of the other charge. The law of motion for e is that where the Lorentz four-force divided by the rest mass m of e replaces the zero on the right hand side of the geodesic equation. In a slow-motion approximation, where terms quadratic and higher in v/c are neglected (here v refers to the velocity of e relative to the notion of rest defined by V), the result takes the form [10]

$$\ddot{\vec{x}} - \frac{\ddot{a}}{a} \vec{x} = \frac{eQ}{4\pi\epsilon_0 m} \cdot \frac{\vec{x}}{\|\vec{x}\|^3}, \quad (49)$$

where a dot denotes derivatives with respect to cosmic time t (proper time along integral curves of V) and $\vec{x} := (x^1, x^2, x^3)$ are the Fermi proper-length coordinates

¹² Its definition is $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$.

¹³ This is easy since Maxwell's equations are conformally invariant and (44) is conformally static (pull out $a^2(t)$ and introduce "conformal time" η via $d\eta = dt/a(t)$).

in the surfaces of constant cosmic time. As far as this Hydrogen-atom-type situation is concerned, the upshot is that we may indeed just take the familiar flat-space non-relativistic equation and replace $\ddot{\vec{x}}$ with $\ddot{\vec{x}} - \frac{\ddot{a}}{a} \vec{x}$, as heuristically anticipated in the pseudo Newtonian picture of section 2.

Since the gravitational interaction of two bodies works via their back-reaction onto the geometry of space time, it is clear that for the gravitational analogue of (49) we cannot just work in the FLRW background. Even under the simplifying assumption that one mass can be treated as test particle (no back reaction) we still need to consider solutions to Einstein's equation representing a central mass in an expanding universe. This will be discussed in section 4. But before that we will have to say a few more words on the apparently simple equation (48).

3.3 Mapping out trajectories

Equation (48) characterises an inertial structure in a neighbourhood of a reference observer (who moves himself on a geodesic). This is achieved through the characterisation of inertial trajectories (i.e. characterising a path structure [14]) in particular coordinates, here Fermi normal coordinates. In this section we merely wish to emphasise the dependence of this procedure of “mapping out” nearby trajectories on the simultaneity structure employed, which should be made explicit in order to avoid confusion.

With reference to a single selected observer O (i.e. worldline; here even a geodesic), mapping out a nearby worldline W usually means to establish a coordinate system K in a tubular neighbourhood U of O so that W is contained in U . Such a system may, e.g., consist of a parametrisation of O in terms of eigentime and a foliation of U by spacelike hypersurfaces intersecting O orthogonally. A point q is then given the following coordinates of *time* and *distance* (we ignore the angular part here): Let p be the unique point on O that lies on the same leaf L of our foliation as q . Then the time coordinate of q is the eigentime of p . Further, the distance of q is the L -geodesic distances between p and q . Here “ L -geodesic” means that the curve is running entirely in L and is a geodesic with respect to the metric that L inherits from the ambient spacetime.

Fermi normal coordinates are of that type and exist in any spacetime. Here the 3-dimensional leafs of the spacelike foliation are spanned by the geodesics emanating from, and perpendicular to, O . This obviously implies that each leaf has zero extrinsic curvature at its intersection point with O .

On the other hand, in an FLRW spacetime, we do have the natural foliation by hypersurfaces of homogeneity, i.e. constant cosmological time t . Here t coincides with the eigentime for the (geodesic) reference observer. However, the cosmological foliation is certainly *not* the same as that resulting from Fermi coordinates. This can immediately be seen from the fact that the extrinsic curvature of a leaf of constant cosmological time is proportional to its intrinsic metric (hence the hypersurface is totally umbilical) and the Hubble constant as constant of proportionality. In particular, it never vanishes.

One should therefore not expect (48) to be of exactly the same analytic form as the equation one gets from the exact geodesic equation in FLRW spacetime in the standard FLRW coordinates. Let us be explicit here: A geodesic in a spatially flat (just for simplicity) FLRW universe is most easily obtained from the variational principle

$$\delta \int d\tau \frac{1}{2} \left[c^2 \left(\frac{dt}{d\tau} \right)^2 - a^2(t) \left(\frac{d\vec{y}}{d\tau} \right)^2 \right]. \quad (50)$$

Its Euler Lagrange equations are

$$\begin{aligned} c^2 \frac{d^2 t}{d\tau^2} + a\dot{a} \left(\frac{d\vec{y}}{d\tau} \right)^2 &= 0, \\ \frac{d^2 \vec{y}}{d\tau^2} + 2 \frac{\dot{a}}{a} \frac{dt}{d\tau} \frac{d\vec{y}}{d\tau} &= 0. \end{aligned} \quad (51)$$

In order to compare this with (48) we need to rewrite this in a twofold way. The first is to replace τ with t as a parameter. The equation for $\vec{y}(t)$ then becomes

$$\ddot{\vec{y}} + 2 \frac{\dot{a}}{a} \dot{\vec{y}} - \dot{a} a \frac{\|\dot{\vec{y}}\|^2}{c^2} \dot{\vec{y}} = 0. \quad (52)$$

The second step is to use instead of \vec{y} the spatial coordinate $\vec{x} := a\vec{y}$, which has direct metric relevance since the modulus of \vec{x} gives the geodesic distance within the surface of constant cosmological time, unlike the ‘‘co-moving’’ coordinate \vec{y} , the modulus of which corresponds to a distance proportional to $a(t)$. In terms of \vec{x} equation (52) reads

$$\ddot{\vec{x}} - (\ddot{a}/a)\vec{x} = \frac{\|\dot{\vec{x}} - (\dot{a}/a)\vec{x}\|^2}{c^2} \frac{\dot{a}}{a} \left(\dot{\vec{x}} - \frac{\dot{a}}{a}\vec{x} \right). \quad (53)$$

This equation is exact. It differs from (48) by the terms on the right-hand side, which are small of order v^2/c^2 . But one should still keep in mind that (52) refers to cosmological simultaneity whereas in (48) we used simultaneity of Fermi normal coordinates.

In many applications within our solar system so called *radar coordinates* are used, which define yet another relation of simultaneity. These coordinates are again based on an observer O and exist in a tubular neighbourhood U . We take O to be parametrised by eigentime τ and use it to assign a time and distance value to any event q in U as follows: Let $p_+(q)$ and $p_-(q)$ be the intersections of the future and past light-cone at q with O respectively and $\tau_{\pm}(q)$ the corresponding parameter values. Then

$$T(q) = \frac{1}{2}(\tau_+(q) + \tau_-(q)), \quad R(q) = \frac{c}{2}(\tau_+(q) - \tau_-(q)) \quad (54)$$

are the time and distance assigned to q . Event q is then simultaneous with event p on O half way between p_+ and p_- , i.e. the eigentime at p is the arithmetic

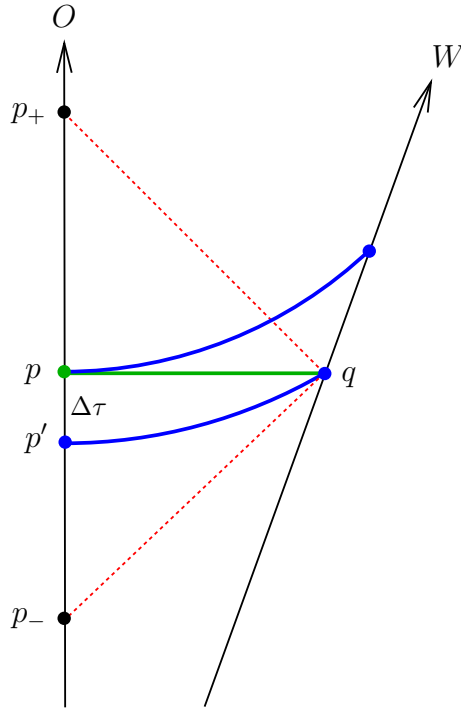


Figure 6: Radar versus cosmological simultaneity. See main text for explanation.

mean $\frac{1}{2}(\tau_+ + \tau_-)$. The hypersurface $T = T(q)$ intersects O at p perpendicularly. The “radius” $R(q)$ is generally different from the hypersurface geodesic distance between p and q (unlike the other cases discussed above). This follows already from the simple observation that the coordinate functions T and R are invariant under all those conformal transformations $g \mapsto \phi^2 g$ of the metric for which $\phi|_O = 1$, whereas this is certainly not true for Fermi normal coordinates.

Figure 6 shows two worldlines: the observer O and another one, W , corresponding to a spacecraft, say. The future and past light cones (dashed lines) of the event q on W intersect O at p_+ and p_- respectively. This can be interpreted as a light signal being emitted by the observer at p_- , reflected by a spacecraft moving at q , and received back by the observer at event p_+ . In radar coordinates the event p on the observer’s worldline O that is simultaneous with q is that half way (in terms of eigentime along O) between p_- and p_+ . In contrast, the hypersurfaces $t = \text{const.}$ of cosmological simultaneity correspond to the horizontal upward-bent curves. The lower one is that intersecting the reflection event q . Its intersection with the observer’s worldline is at p an eigentime $\Delta\tau$ prior to p .

Assigning a distance to event q on W with respect to O can either mean to assign the geodesic distance along the curved line $p'q$ to the event p' on O , or to assign $R(q)$ as in (54) to the event p on O . Depending on which one is used the distance of the spacecraft as function of proper time along O is given by different

functions. Consequently, this is also true for the relative velocity and acceleration. The following relations have been shown to hold in leading order [10]:

$$\tilde{r} = r - (H_0 c) \frac{1}{2} (v/c) (r/c)^2 \quad (55a)$$

$$\tilde{v} = v - (H_0 c) (v/c)^2 (r/c) \quad (55b)$$

$$\tilde{a} = a - (H_0 c) \left\{ (v/c)^3 + (r/c)(v/c)(a/c) \right\}. \quad (55c)$$

Here (r, v, a) are the distance, velocity, and acceleration with respect to cosmological simultaneity and proper geodesic distance, whereas the quantities with tildes refer to radar coordinates. H_0 is the Hubble constant. Note in particular the first term on the right-hand side of (55c) which shows an apparent additional inward pointing acceleration proportional to $H_0 c$ in radar coordinates as compared to cosmological coordinates. Recall also that the Pioneer spacecrafts were tracked by Doppler methods so that indeed \tilde{a} rather than a was measured. However, as seen from (55c), the additional term $\propto H_0 c$ is multiplied with $(v/c)^3$, which for the Pioneer spacecraft is of the order of 10^{-12} . Hence this additional acceleration, albeit proportional to $H_0 c$, is strongly suppressed by the third power of v/c . Finally we mention that cosmic expansion also affects the method of Doppler tracking *per se*. A detailed study of this has been performed in [8].

4 Black holes and expansion

Intuitively we expect the local geometric properties of black holes to be affected if the black hole is placed into a cosmological environment. Anticipated changes could concern the mass, the horizon structure, and certainly the orbits of bound systems. More generally, one might fear that the very notion of a “black hole” does not generalise in an obvious way. But before going into some of these aspects, we must mention that the very meaning of “being placed” is unclear in view of the fact that solutions to Einsteins equations cannot be simply superposed. Hence it is not obvious at first how to meaningfully compare an ordinary black hole solution corresponding to an asymptotically Minkowskian spacetime to an inhomogeneous cosmological solution that asymptotically approaches a FLRW universe and contains an inner event (or apparent) horizon. There is no natural notion of “sameness” by which we could identify black-hole solutions with different asymptotics. A somewhat pragmatic way, that wish to follow here, is to use (quasi-)local geometric properties as characterisations. One could then ask for the effect of cosmic expansion on the relation of such local geometric features, like, e.g., the change of horizon size for given mass. But for this to make sense “horizon size” and “mass” must first be defined geometrically. This can be done at least in the spherically symmetric case. To see this, we need to recall a few mathematical facts.

4.1 Geometric background

We are used to the fact that standard cosmological spacetimes have a preferred foliation by spacelike hypersurfaces of constant cosmic time. In fact, the hypersurfaces are defined geometrically as the orbits of the symmetry group of spatial homogeneity and isotropy, and “cosmic time” is essentially the parameter that labels these hypersurfaces. In this subsection we wish to point out that the weaker condition of spherical symmetry (without homogeneity) in fact suffices to geometrically characterise a foliation by spacelike hypersurfaces. Roughly speaking, the hypersurfaces are the spacelike hypersurfaces within which the areas of the 2-spheres traced out by the action of the rotation group (its orbits) increase or decrease fastest, as compared to all other spatial directions perpendicular to the orbits themselves.

In order to say this more precisely, we first recall that a spacetime (M, g) is called spherically symmetric if and only if there is an action of the rotation group $SO(3)$ on (M, g) by isometries, such that the orbits of this action are spacelike 2-spheres. M is then foliated by these 2-spheres, which means that each point p in M lies on precisely one such sphere. Let $a(p)$ be the (2-dimensional) volume of that sphere (i.e. its “surface area”), as measured by the spacetime metric g . Then we can define the following function

$$R : M \mapsto \mathbb{R}_+, \quad R(p) := \sqrt{\frac{a(p)}{4\pi}}. \quad (56)$$

This function is called the *areal radius*. It assigns a positively-valued “radius” to each $SO(3)$ orbit, such that the 2-dimensional volume (surface area) of this orbit is $4\pi R^2$, just as in ordinary flat space. Note that it is not proper to think of this radius as some kind of (geodesic-)distance to an “origin”, since such an origin would correspond to a fixed point of the $SO(3)$ action which we excluded here explicitly. It may not exist at all, even though the manifold is inextendible (i.e. no point has been artificially removed), like e.g. in the maximally extended Schwarzschild-Kruskal spacetime.

The function R is $SO(3)$ invariant by construction. Hence the exterior differential, dR , which is a co-vector field on M , is also $SO(3)$ invariant. We assume, or restrict attention to that part of spacetime where this is true, that dR is spacelike, so that R is a good spatial radial coordinate and that the hypersurfaces of constant R are timelike. This gives us a first and rather obvious foliation of spacetime by hypersurfaces. But there is another one: At each point in spacetime there is a unique line (unoriented direction) perpendicular to the $SO(3)$ orbit which is also annihilated by dR . Up to overall sign it can be represented by a normalised vector field k orthogonal to the $SO(3)$ orbits and satisfying $dR(k) = 0$, meaning that R does not change along the flow of k . Now, k is always hypersurface orthogonal (this statement only depends only on the line field represented by k). Hence we have a foliation of M by spacelike hypersurfaces Σ orthogonal to k , so that a leaf $R = \text{const.}$ of the former foliation intersects a leaf Σ of the latter in precisely one

$SO(3)$ orbit. In this way, the geometric structure imposed by spherical symmetry leads to a *specific* foliation of spacetime into spacelike hypersurfaces, each leaf of which is itself foliated by $SO(3)$ orbits.¹⁴ If the latter are parametrised by spherical polar coordinates (θ, φ) , then (R, θ, φ) parametrise each spacelike leaf Σ . The most general spherically symmetric spacetime metric is then of the form

$$g = A^2(T, R)c^2 dT^2 - B^2(T, R) dR^2 - R^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (57)$$

where A, B are some non-vanishing dimensionless functions.

4.2 Reissner-Nordström-de Sitter

For example, a spherically symmetric solution to the coupled Einstein-Maxwell system with cosmological constant Λ (and no sources for the Maxwell field) is the Reissner-Nordström-de Sitter solution, where

$$A^2 = B^{-2} = 1 - \frac{2m}{R} + \frac{q}{R^2} - \frac{\Lambda}{3}R^2. \quad (58)$$

The parameters m and q have physical dimension of length (geometric units). They can be converted to parameters M and Q with physical dimensions of mass and electric charge in MKSA-units via

$$m = \frac{GM}{c^2}, \quad q^2 = \frac{1}{4\pi\epsilon_0} \frac{GQ^2}{c^4}. \quad (59)$$

Here we continue to work in geometric units. The identification of q as electric charge is unproblematic because q can be shown to be the flux of the electric field through any of the 2-spheres of constant R (and hence any 2-surface homologous to it). The identification of m as mass is less obvious due to the lack of an unanimously accepted definition of (quasi-)local mass in GR. It is true that (59) becomes the Reissner-Nordström solution for $\Lambda = 0$, which is asymptotically flat and has M as its well defined overall mass (i.e. at spacelike infinity) But for general Λ (58) is not asymptotically flat and we cannot just continue to call the parameter m its mass without further justification.

4.3 Misner-Sharp mass and its refinement

Fortunately, for spherically symmetric situations there is a reasonable notion of mass associated with each 2-sphere $R = \text{const.}$ that is also easy to work with. Based on [37] it is called the Misner-Sharp mass (or energy). It has been shown to have a number of physically appealing properties [26] and agrees with the more

¹⁴ That spherically symmetric spacetimes admit spacelike foliations each leaf of which is foliated by $SO(3)$ orbits is not at all surprising: There are zillions of it. The point made here is, that there is, in a sense, a natural one.

generally defined Hawking mass [25] whenever the Misner-Sharp mass can be defined. A geometric definition can be given as follows [10]:

$$m_{\text{MS}}(S) = -\frac{R^3}{2} \text{Sec}(S), \quad (60)$$

where $\text{Sec}(S)$ denotes the sectional curvature of spacetime (M, g) tangent to the sphere S of constant R .¹⁵ of constant R . The overall minus-sign on the right hand side of (60) is due to our “mostly-minus” signature convention. Note that m_{MS} can be considered as function on the spacetime manifold M . Its value $m_{\text{MS}}(p)$ at the point p is simply $m_{\text{MS}}(S)$, where S is the unique 2-sphere ($SO(3)$ orbit) through p . It may then be shown that [10]

$$m_{\text{MS}} = \frac{R}{2} \left(1 + g^{-1}(dR, dR) \right). \quad (61)$$

The result for metrics of the form (57) is immediate:

$$m_{\text{MS}}(T, R) = \frac{R}{2} \left(1 - B^{-2}(T, R) \right). \quad (62)$$

Further specialised to (58) we get

$$m_{\text{MS}}(R) = m - \frac{q^2}{2R} + \frac{\Lambda}{6} R^3. \quad (63)$$

The interpretation of the second and third term on the right-hand side is rather obvious. First, $q^2/2R$ is, in geometric units, the electrostatic field energy stored in that part of space where the areal radius is larger than R . This can be easily verified by direct computation from the explicit form of the electromagnetic field and its energy-momentum tensor. Without Λ , m would be the total mass of the hole, *including* its electrostatic field. $m_{\text{MS}}(R)$ is then its total mass minus the electromagnetic energy located outside S . Second, the Λ term in Einstein’s equations corresponds to a mass density, which in geometric units is

$$\rho_{\Lambda} = \frac{G}{c^2} \cdot \frac{1}{c^2} \cdot \frac{c^4}{8\pi G} \Lambda = \frac{\Lambda}{8\pi}. \quad (64)$$

Hence the last term on the right-hand side of (63) equals $\rho_{\Lambda}(4\pi/3)R^3$. This is a familiar formula in GR: a mass density multiplied by $(4\pi/3)R^3$ gives the total mass within radius R *diminished by the gravitational binding energy*. The last fact is hidden by this deceptively simple formula, but note that $(4\pi/3)R^3$ is generally *not* the geometric volume enclosed by the sphere of areal radius R .

Given (63) we may ask how to separate the black hole mass from the other components due to the electric field and the cosmological constant *in a geometric way*. It has been argued in [10] quite generally that this can be achieved by splitting the sectional curvature in (60) into its Weyl and its Ricci part. Indeed, in the special case at hand the first term on the right-hand side of (63) (the m) is the Weyl part, the other two terms comprise the Ricci part. This also works in other spherically-symmetric situations, as we shall discuss below.

¹⁵ Note that $\text{Sec}(S)$ is not the intrinsic curvature of S .

4.4 Applications

Reissner–Nordström–de Sitter spacetimes

We now have the tools at hand to discuss relations between geometrically defined quantities of spherically symmetric black holes in different environments. If we identify the mass of the black hole with the Weyl part of the Misner-Sharp mass, we may meaningfully compare the areal radii of its horizon for fixed mass with or without Λ . In the present example (58) this just boils down to discussing the dependence on Λ of the root of $A^2(R)$. The root corresponds to the hole’s horizon (event or apparent, as we are in a static situation).

For $\Lambda = 0$ a horizon exists if $m \geq |q|$ and lies at an areal radius

$$R_{\text{Hor}} = m + \sqrt{m^2 - q^2}. \quad (65)$$

(The “inner” root at $m - \sqrt{m^2 - q^2}$ corresponds to a Cauchy horizon and does not interest us here.) Switching on Λ shifts this root to

$$R_{\text{Hor}} \rightarrow \tilde{R}_{\text{Hor}} := R_{\text{Hor}}(1 + \epsilon), \quad (66a)$$

where, to for small ΛR_H^2 , we get in leading order

$$\epsilon = \frac{\Lambda}{6} R_{\text{Hor}}^3 \frac{1}{\sqrt{m^2 - q^2}}. \quad (66b)$$

Hence \tilde{R}_{Hor} is larger/smaller than R_{Hor} if Λ is larger/smaller than zero. This might be taken to conform with one’s intuition that an accelerated/decelerated expansion somehow “pulls/pushes” the horizon to larger/smaller radii. But this is again deceptive since the horizon is not a material substratum acted upon by forces. Since $u := A^{-1}\partial/\partial T$ is the four velocity of the static observer, his acceleration measured in his instantaneous rest frame is

$$a = \nabla_u u = \frac{c^2}{2} B \frac{dA^2}{dR} e_1 = c^2 \frac{\frac{m}{R^2} - \frac{q^2}{R^3} - \frac{\Lambda R}{3}}{\sqrt{1 - \frac{2m}{R} + \frac{q^2}{R^2} - \frac{\Lambda}{3} R^2}} e_1, \quad (67)$$

where $e_1 := B^{-1}\partial/\partial R$ is the normal vector in radial direction. Hence for Λ switched on the acceleration already diverges at a radius larger than R_{Hor} because of its effect on the relevant zero of the expression under the root in the denominator (which is A^2), not because of its effect in the numerator. Recall also that the extreme case occurs for such (m, q, Λ) where the two hole horizons coincide. For $\Lambda = 0$ this happens for $|q| = m$, but for $\Lambda > 0$ at values $|q| > m$. This last fact generalises to the rotation parameter in the Kerr–de Sitter family of rotating holes, as was recently discussed in some detail in [1].

Einstein-Straus vacuoles

Another elegant application of the Misner-Sharp mass is to derive the radius of the vacuole in the Einstein-Straus model [15]. This model, also called the ‘‘Swiss-Cheese model’’ consists of matching a Schwarzschild-de Sitter (or Kottler [32]) solution to a FLRW universe along some radius, along which the outward pull from the cosmological masses are just balanced by the inward pull from the central black hole. The matching surface in spacetime is a timelike hypersurface foliated by the $SO(3)$ orbits. Each such orbit represents the matching surface at a time. We wish to determine its radius.

The matching conditions are traditionally given by the continuity of the induced metrics (first fundamental forms) and extrinsic curvatures (second fundamental forms) defined on either sides of the hypersurface, the so-called Lanczos-Darmois-Israel conditions [33][12][28, 29]. Another but equivalent set of conditions involves the areal radii and Misner-Sharp energies, stating their equality for each pair of $SO(3)$ orbits to be matched [7, 10].¹⁶

The Schwarzschild–de Sitter metric is given by (57)(58), where $q = 0$. The FLRW cosmological model is given by the metric

$$g = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right), \quad (68)$$

and the four-velocity field $V = \partial/\partial t$. For an energy-momentum tensor of the form (37) (perfect fluid) Einstein’s equations are then equivalent to Friedmann’s equations

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda c^2}{3} + \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}, \quad (69a)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right). \quad (69b)$$

From (68) we immediately read of the areal radius $R(t, r) = a(t)r$, so that $dR = \dot{a}r dt + a dr$. Inserting this into the expression (61) for the Misner-Sharp mass gives

$$m_{\text{MS}}(R) = \frac{1}{2} R r^2 (k + c^{-2} \dot{a}^2) = \frac{4\pi}{3} R^3 [\rho_{\text{matter}} + \rho_{\Lambda}], \quad (70)$$

where ρ_{matter} and ρ_{Λ} are the mass densities of the dust and cosmological constant, respectively, in geometric units, that is $\rho_{\text{matter}} = \rho G/c^2$ and ρ_{Λ} as in (64). In the second step in (70) we used the first Friedmann equation (69a) to eliminate \dot{a}^2 . Note that neither the pressure p of the matter nor the curvature k of space enters the final expression in (70).

In (63) we already calculated the Misner-Sharp mass for the Reissner-Nordström–de Sitter case. Setting $q = 0$ we obtain the Misner-Sharp mass for

¹⁶ The partial statement, that the (in our terminology) Misner-Sharp masses of the excised ball and the inserted inhomogeneity have to be equal in order for the resulting spacetime to satisfy Einstein’s equation, is also known as *Eisenstaedt’s Theorem* [16].

the Schwarzschild–de Sitter case. It contains two terms, the second being equal to the second in (70); compare (64) and line below. Here we already used that the matching conditions require the areal radii of the matching spheres to be equal. Hence, in order for the Misner–Sharp masses of the matching spheres in the FLRW universe and the Schwarzschild–de Sitter universe to be equal, the first terms must also coincide. This immediately gives the simple condition

$$m = \frac{4\pi}{3} R^3 \rho_{\text{matter}} \quad (71a)$$

or, equivalently,

$$R = \left(\frac{3m}{4\pi \rho_{\text{matter}}} \right)^{1/3}. \quad (71b)$$

Again note that this expression makes no explicit reference to the pressure (and hence no reference to an equation of state for the matter) and the curvature.

If space were flat (71a) just said that the mass concentrated in the black hole just equals that fraction that formerly had been homogeneously distributed inside the excised ball (the vacuole), and (71b) said that in order to include a black hole of mass m into a FLRW universe one has to remove the cosmological mass surrounding it up to that radius inside which the cosmological dust has an integrated mass of just m . In the general (curved) cases (71) is almost deceptively simple, since for spaces of constant positive/negative curvature the expression $(4\pi/3)R^3$ grows slower/faster with the areal radius R than the actual geometric volume.¹⁷

Mc Vittie–type spacetimes

Finally I wish to mention attempts to interpolate between spacetime metrics representing black holes in a near zone and FLRW cosmologies in a far zone. Most of what follows will be based on [10] and [9].

First attempts in this direction go back to McVittie in the early 1930s [36], who attempted a particle-like interpretation. The basic idea is to literally interpolate analytically between two metrics. Since the result of a naïve interpolation will generally depend on the coordinates used, it is necessary to do this in a geometrically meaningful way. We recall that any spherically symmetric spatial metric is conformally flat. So the first thing to do is to write the black-hole metric as well as the FLRW metric in a manifest spatially conformally flat form, like

$$g_{\text{BH}} = \left[\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right]^2 c^2 dt^2 - \left[1 + \frac{m}{2r} \right]^4 (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)) \quad (72)$$

and

$$g_{\text{Cosm}} = c^2 dt^2 - a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)). \quad (73)$$

¹⁷ The first correction for the volume inside a 2-sphere of areal radius R in a 3-space of constant curvature $k = \pm 1$ is $\text{Vol}_k(R) = (4\pi/3)R^3(1 + 3kR^2/10 + \dots$

Expression (72) corresponds to (57) with $q = \Lambda = 0$ and after redefinition of the radial coordinate. Expression (73) is just (68) with $k = 0$ set for simplicity, so that the metric is already in manifest conformally flat form. The interpolation now consists in writing the McVittie form (hence subscript MV)

$$g_{\text{MV}} = \left[\frac{1 - \frac{m(t)}{2r}}{1 + \frac{m(t)}{2r}} \right]^2 c^2 dt^2 - a^2(t) \left[1 + \frac{m(t)}{2r} \right]^4 (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)). \quad (74)$$

Note that formally this just changes (72) by 1) writing a $a^2(t)$ in front of the spatial part and 2) allowing m to become time dependent. The general strategy is now to evaluate the left-hand side of Einstein's equations using (74), and then "see" - basically by trial and error - what can reasonably be put on the right hand side.¹⁸

The relaxation of allowing m to be time dependent seems clearly necessary if we wish to discuss processes like the accretion of cosmological matter by the black hole through radial infall.¹⁹ However, the change from (72) to (74) will also bring about a change in the very notion of mass. Here it becomes important that we can distinguish between the mass of the central object and that of cosmological matter (forming overdensities, say). This can be done using the refinement of the Misner-Sharp mass discussed above [10].

An obvious way to proceed is to just read off the areal radius from (74):

$$R(t, r) = \left[1 + \frac{m(t)}{2r} \right]^2 a(t)r \quad (75)$$

and then calculate the Misner-Sharp according to (61). If we denote by e_0 the timelike unit vector normal to the hypersurfaces $t = \text{const.}$, i.e. parallel to $\partial/\partial t$, and by e_1 the spacelike unit vector in radial direction, i.e. parallel to $\partial/\partial r$, we can rewrite (61) as²⁰

$$m_{\text{MS}} = \frac{R}{2} \left(1 - [e_1(R)]^2 \right) + \frac{R}{2} [e_0(R)]^2. \quad (76)$$

It can be shown [10] that the first term is the Weyl part of the Misner-Sharp mass, whereas the second part is its Ricci part. Straightforward evaluation of the first part using (75) gives ma , whereas the second term can be shown to be related to the (e_0, e_0) -component of the Einstein Tensor. In total we get

$$m_{\text{MS}} = am + \frac{1}{6} R^3 \mathbf{Ein}(e_0, e_0) = m_{\text{MS}}^{\text{Weyl}} + m_{\text{MS}}^{\text{Ricci}}. \quad (77)$$

We can use Einstein's equation to replace the Einstein tensor in the second term on the right-hand side with the energy momentum tensor of matter and the cosmological constant. A possible cosmological constant adds a term $R^3 \Lambda/6$, just like

¹⁸ This may be called the "poor man's way to solve Einstein's equations".

¹⁹ Clearly it has to be radial since we restricted to spherical symmetric situations.

²⁰ We recall that, because of spherical symmetry, the hypersurfaces $t = \text{const.}$ and the spatial radial directions (being tangential to these hypersurfaces and normal to the $SO(3)$ orbits in them) can be characterised purely geometrically, as explained in section 4.1.

in (63), which again may be interpreted as additional homogeneous mass density given by (64). If we assume the energy-momentum tensor to be that of a perfect fluid (37) moving along trajectories orthogonal to the hypersurfaces $t = \text{const.}$, that is

$$V = ce_0, \quad (78)$$

the (e_0, e_1) -component of Einstein's equation implies

$$m(t)a(t) =: m_0 = \text{const.} \quad (79)$$

Equation (77) then takes the form

$$m_{\text{MS}} = m_0 + \frac{4\pi}{3} R^3 [\rho_{\text{matter}} + \rho_{\Lambda}], \quad (80)$$

in which the second term is just that which we already obtained in (70) for the pure FLRW case and which here, as before, appears as the Ricci part of the Misner-Sharp mass. Its Weyl part turns out to be a constant m_0 given by ma (not just $m!$). Again we stress that in general $(4\pi/3)R^3$ is not the space volume of a ball or radius R so that the right-hand side of (80) is *not* the space integral over the mass density; the difference being due to the gravitating binding energy.

Clearly, the constancy of the Weyl part, which we defined with the mass of the inhomogeneity, is a consequence of our assumption (78), which means that there is no accretion of cosmological matter by the central object. This “object” need not be a black hole. An interior solution modelled on a homogeneous perfect-fluid “star” of positive mass density is known [41]. The four-velocity of the star's surface is V , i.e. it is co-moving, so that (78) indeed is a condition for no-accretion. Note that “co-moving” means constant r , which according to (75) implies an expanding areal radius R , essentially proportional to $a(t)$ when $m_0 \ll ar$.

It is clear that the no-accretion condition can only be achieved through a special pressure function that just balances gravitational attraction. It turns out that this can be relaxed but, somewhat surprisingly, only at the price of introducing heat flows. This means that (37) needs to be generalised to

$$T = \rho V \otimes V + p(c^{-2}V \otimes V - g^{-1}) + c^{-2}(Q \otimes V + V \otimes Q), \quad (81)$$

where Q is a spacelike vector perpendicular to V that represents the residual current-density of energy (heat) in the rest system of the fluid. The matter is now not moving along e_0 , but rather has a non-vanishing radial velocity relative to the frame (e_0, e_1) . Expressed in terms of the rapidity χ one has

$$V = c(\cosh \chi e_0 + \sinh \chi e_1), \quad (82a)$$

$$Q = q(\sinh \chi e_0 + \cosh \chi e_1). \quad (82b)$$

Now, equations (81) and (82) allow for solutions of Einstein's equation in the form (74) only if [9]

$$(c^2 \rho + p) \tanh \chi + 2q/c = 0. \quad (83)$$

This constraint shows that radial infall of matter $\chi < 0$ must be accompanied by a radial outflow of heat $q > 0$, and vice versa. For pressures p small compared to ρc^2 the modulus of the infalling matter-energy exceeds that of the outflowing heat by a factor of two.²¹ This means that infalling matter always increases the mass $m_{\text{MS}}^{\text{Weyl}}$ at a rate roughly half of the net infall of matter energy divided by c^2 . (This is just what the (0, 1)-component of Einstein's equation expresses.) But the increase of inertial mass implies an increase in gravitational mass. A slow-motion and weak-field approximation of the geodesic equation in (74), which gives it a pseudo-Newtonian form, shows that the Newtonian potential is proportional to $m_{\text{MS}}^{\text{Weyl}}/R$, so that the gravitational pull increases with increasing mass. As a result, orbits of test particles will spiral "inwards", i.e. to smaller R -values [9, 10].

Note that at the level of our discussion (83) is of purely geometric origin and does not lie in any deeper physical insight into heat production by infalling matter due to friction, as we had not specified any such model. The geometric origin is inherent in the Ansatz (74) and has been identified in [9] as the condition of *Ricci isotropy*, which means that the four-dimensional Ricci tensor and the four-dimensional spacetime metric are pointwise proportional after being pulled back to the hypersurfaces $t = \text{const}$. This condition would have to be relaxed in order to specify radial flows of matter and heat independently, as seems physically desirable.

Finally we comment on the singularity structure of (74), following [9]. As expected, it has a singularity at $r = 0$ in the Weyl part of the curvature and, somewhat unexpected, a singularity in the Ricci part of the curvature at $r = m/2$. The latter is absent only in the following limiting cases: 1) $m = 0$ and a arbitrary (pure FLRW), 2) m and a constant (pure Schwarzschild), and 3) $am = \text{const}$. and $\dot{a}/a = \text{const}$. (Schwarzschild de Sitter case). The singularity at $r = m/2$ will lie within a trapped region, i.e. behind an apparent horizon. Since here the asymptotic structure of spacetime is considerably more difficult to analyse than in the previous cases, we restrict attention to the local concept of apparent horizons. Recall that a spacelike 2-sphere S is said to be trapped, marginally trapped, or untrapped if the product $\theta^+\theta^-$ of the expansions for the outgoing and ingoing future-pointing null vector fields normal to S is positive, zero, or negative, respectively. An apparent horizon is the boundary of a trapped region.

Now, in our case, the product $\theta^+\theta^-$ can be shown to equal

$$\theta^+\theta^- = \frac{g^{-1}(dR, dR)}{R^2} = \frac{2m_{\text{MS}} - R}{R^3}, \quad (84)$$

where the last equality following from (61). This shows that a trapped, marginally trapped, or untrapped region corresponds to dR being timelike, lightlike, and spacelike respectively, or, equivalently, $2m_{\text{MS}} - R$ being positive, zero, and negative respectively.

According to the last equality in (84) the location of the apparent horizons is given by the zeros of the function $2m_{\text{MS}} - R$. In order to write this function in

²¹ Note that the rapidity χ is related to the ordinary velocity by $\tanh \chi = v/c$.

a convenient form, we restrict to expansion ($\dot{a} > 0$) and introduce the following non-negative quantities:

$$R_M := 2m_{\text{MS}}^{\text{Weyl}} = 2ma, \quad R_H := \frac{c}{H_0} = \frac{ca}{\dot{a}}, \quad x := \frac{R}{R_M}. \quad (85)$$

R_M is the mass-radius just as in (25), but now “mass” refers specifically to the mass of the central object that we identified with the Weyl part of the Misner-Sharp mass. R_H is the Hubble radius defined in terms of the Hubble constant as before (24). Finally it will be convenient to use the dimensionless variable x . The last expression in (84), divided by R_M , is then readily seen from (77) to be

$$F(x) := \frac{2m_{\text{MS}} - R}{R_M} = 1 - x + \frac{1}{3}x^3 R_M^2 \mathbf{Ein}(e_0, e_0). \quad (86)$$

The (e_0, e_0) -component of the Einstein tensor for the metric (74) can be computed as function of (t, r) and then re-expressed as function of (t, x) . The result is

$$F(x) := \frac{2m_{\text{MS}} - R}{R_M} = 1 - x + x^3 \left[\frac{R_M}{R_H} + \dot{R}_M \Theta(x) \right]^2, \quad (87)$$

where Θ is some positive function which we need not specify here. Note the proportionality of the second term with \dot{R}_M , i.e. twice the time rate of change of the mass of the central black hole. It vanishes in the case of no mass accretion, in which case $F(x)$ just becomes a simple polynomial of third order which is already in reduced form (no x^2 terms). Its zeros can be written down explicitly using Cardano’s formula, but the essential features can be seen directly. As $F(x)$ is positive for $x = 0$ and $x \rightarrow \infty$, it has two zeros if and only if $F(x)$ assumes a negative value at its minimum, which is at $x = R_H/R_M\sqrt{3}$. This is the case if and only if $R_M < 2R_H/3\sqrt{3}$, which is the case of interest here (the hole’s horizon radius being much smaller than the Hubble radius). A leading order expansion for the location of the zeros of $F(x)$ for small values of R_M/R_H is now simple. For $R_M/R_H = 0$ we have $x = 1$, i.e. $R_{\text{Hor}} = R_M = 2ma$. Switching on cosmological expansion shifts this root to

$$R_{\text{Hor}} \rightarrow \tilde{R}_{\text{Hor}} := R_M(1 + \epsilon), \quad (88a)$$

where for small R_M/R_H we now get to leading order

$$\epsilon = \left(\frac{R_M}{R_H} \right)^2. \quad (88b)$$

This is precisely the generalisation of (66) for $q = 0$ and $\Lambda > 0$, where $R_H = \sqrt{3/\Lambda}$ for de Sitter spacetime. In the Mc Vittie case, too, the areal radius of the apparent horizon is enlarged by expansion.

This discussion can be generalised to the case of non-zero mass accretion [9]. The result is that mass accretion further enlarges the (instantaneous) value of the apparent horizon’s areal radius. This concludes our discussion of black holes in expanding universes. This is a difficult subject and not much is known in terms of exact solutions.

Black-Hole cosmologies

We have seen in some detail how a single black hole may inhibit cosmological expansion locally. This was most pronounced in the Einstein-Strauss construction where the metric becomes static throughout the vacuoles of areal radius (71b). As we have discussed at the end of section 2.3, a realistic size for such a vacuoles would be that of galaxy clusters. So it seems reasonable to approximate the dynamics of a closed universe above galaxy-cluster size by a finite number of vacuoles each with a galaxy-cluster mass at the centre. Could this, in turn, be approximated by the same number of black holes, each with galaxy-cluster mass, and without any other forms of matter? In this case Einstein's vacuum equations, possibly with cosmological constant, would suffice to discuss the dynamics of the universe, at least in this approximation. This idea has indeed been entertained by Lindquist & Wheeler in a seminal paper in 1957 [34]. They consider a "lattice universe" made out of a finite number of black holes distributed on the surface of a 3-sphere according to the vertices of a convex regular polytope in four dimensional euclidean space (in which we think of the 3-sphere as being embedded such that the vertices of the polytope lie on it). Once the black holes are introduced the geometry changes of course and it is assumed, as noted by Lindquist & Wheeler, that "This approximation demands that the distribution of gravitational influences just external to each sphere should depart relatively little from spherical symmetry". This is modelled by the fact that *regular* polytopes are chosen, which meets as close as possible the usual requirement of isotropy around each point (here vertex). In two space dimensions regularity means that the black holes are situated at the vertices of one of the five platonic solids inscribed into a 2-sphere. Such a spatially two-dimensional universe would consist of 4, 6, 8, 12, or 20 black holes, corresponding to the vertices of the tetrahedron, octahedron, cube, icosahedron, or dodecahedron. In three spatial dimensions there exist five analogs of the platonic solids carrying 5, 8, 16, 120 and 600 vertices, respectively, and one more with 24 vertices that has no direct analog. The even simpler case of just two black holes sitting at antipodal points of the 3-sphere has not been considered by Lindquist & Wheeler and will be discussed below. Initial data for the vacuum Einstein equations corresponding to a given number of equal-mass black holes at given locations on a 3-sphere can be constructed by standard methods. In the particular case of time symmetry (zero extrinsic curvature) and spatial conformal flatness such data can be written down explicitly; see, e.g., [6], where Fig. 3 shows the geometry of three black holes in a closed universe, and [23] for a general discussion. The time evolution according to Einstein's equations is a more complicated matter that cannot be dealt with without numerical integration. The quantity of interest is the distance between black-hole neighbours (suitably defined) as a function of time. This one may attempt for the simpler cases of the 5- and 8-hole universe [34] and the qualitative behaviour is quite easily understood and does indeed resemble the standard FLRW dynamics; see, e.g., [11]. The full field-theoretic problem remains of course a formidable task.

Regarding the last point, we may try to get rid of the complicated dynami-

cal issues by considering static situations, in the most simple case involving only two black-holes. To keep the holes at a constant distance we either have to introduce singularities on (segments of) the symmetry axis, or, more interestingly, a positive cosmological constant (positive energy density) whose negative pressure does the job. Such a solution has indeed already been envisaged by Erich Trefftz in 1922 [51], who obtained just the Schwarzschild-de Sitter solution (first found by Kottler [32]) but endowed it with a different global interpretation, namely as representing *two* Schwarzschild black holes located at antipodal points of a 3-sphere. Note that this solution still has the rotational $SO(3)$ -symmetry. Einstein, in a critical reply to Trefftz paper ([31], Doc. 387, pp. 595-596), observed from direct inspection of the metric written down by Trefftz that stationary points of the areal-radius function correspond to zeros of the metric coefficient in front of dt^2 . As in Trefftz' solution the areal radius is not constant in the region between the black holes, Einstein concluded that it must assume a stationary point and hence that the time-time component of the metric must vanish somewhere. This Einstein (erroneously) interpreted as the indubitable sign of an additional singularity between the black holes, which would indeed render the interpretation given to it by Trefftz impossible.

However, the proper geometric meaning of the vanishing of this particular metric coefficient is that the static Killing vector field becomes lightlike. Physically it means that the acceleration of the static observers approaching this critical set diverge. This indicates a horizon rather than a singularity. (See [22] and [52] for a discussion of the global properties of the Schwarzschild-de Sitter solution in terms of Penrose diagrams.) Hence we arrive at the following question: Are there solutions to Einstein's equations with cosmological constant representing two spherically symmetric stars at constant distance without horizons in the region between the stars? This question has recently been addressed and answered in the negative for black holes in [52] and for perfect fluid stars in [5]. However, the non-existence result for black holes presented in [52] depends on an assumption (non-constancy of the areal radius) which may be dropped so that the actual set of solutions is larger than anticipated.

Einstein's mathematical observation regarding the connection between stationary points of the areal-radius function and the location of horizons was made on the basis of Trefftz' formulae for the Schwarzschild-De Sitter solution. But it is not hard to see that it is really of a more general kind. This is already implicit in equation (84), which informs us that zeros of dR correspond to marginally trapped regions. Let us therefore see what can be said on the basis of Einstein's equations alone. We are interested in static and spherically symmetric solutions to Einstein's vacuum equations with cosmological constant in which the areal radius might assume stationary points or be even piecewise constant. Hence we write the metric in the form

$$g = f^2(r) c^2 dt^2 - dr^2 - R^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (89)$$

We have chosen the radial coordinate such that $g_{rr} = -1$. The areal radius R may

then be a general function of r , i.e. not restricted to $dR \neq 0$, as it would be if R were taken to be a coordinate. Einstein's vacuum equations with cosmological constant are then equivalent to the following set of three equations, corresponding to the tt , rr , and $\theta\theta$ (equivalently $\varphi\varphi$) components respectively:

$$\Lambda + f''/f + 2f'R'/fR = 0, \quad (90a)$$

$$\Lambda + f''/f + 2R''/R = 0, \quad (90b)$$

$$\Lambda + R''/R + f'R'/fR - R^{-2}(1 - R'^2) = 0. \quad (90c)$$

Taking the difference between (90a) and (90b) gives

$$fR'' = f'R'. \quad (91)$$

Now suppose $r = r_*$ is a stationary point for R , i.e. $R'(r_*) = 0$. Then (91) shows that $f(r_*) = 0$ if $R''(r_*) \neq 0$, i.e. if R assumes a proper extremal value at r_* . But zeros of f correspond to horizons, which is Einstein's observation (in modern terminology and interpretation). But we can also see that there is precisely one way to avoid this argument, namely if R is constant; $R = R_0$. Equation (90c) then gives

$$R_0 = 1/\sqrt{\Lambda}, \quad (92)$$

whereas equations (90a) and (90b) both give $f'' = -\Lambda f$ (harmonic-oscillator equation). The two integration constants (amplitude and phase) can be absorbed by redefining the scale of the t and the origin of the r coordinate. This leads to the *Nariai* metric (in static form), known since 1950 [40][39]:

$$g = \cos^2(r/R_0) c^2 dt^2 - dr^2 - R_0^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (93)$$

The possibility to connect two black holes via a piece of the *Nariai* metric, which is not considered in [52], is presently investigated [18]. Some solutions where the *Nariai* metric connects two perfect-fluid stars were discussed in [5].

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