Change in Hamiltonian General Relativity from the Lack of a Time-like Killing Vector Field

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Abstract

In General Relativity in Hamiltonian form, change has seemed to be missing, defined only asymptotically, or otherwise obscured at best, because the Hamiltonian is a sum of first-class constraints and a boundary term and thus supposedly generates gauge transformations. Attention to the gauge generator G of Rosenfeld, Anderson, Bergmann, Castellani *et al.*, a specially *tuned sum* of first-class constraints, facilitates seeing that a solitary first-class constraint in fact generates not a gauge transformation, but a bad physical change in electromagnetism (changing the electric field) or General Relativity. The change spoils the Lagrangian constraints, Gauss's law or the Gauss-Codazzi relations describing embedding of space into space-time, in terms of the physically relevant velocities rather than auxiliary canonical momenta. But the resemblance between the gauge generator G and the Hamiltonian H leaves still unclear where objective change is in GR.

Insistence on Hamiltonian-Lagrangian equivalence, a theme emphasized by Castellani, Sugano, Pons, Salisbury, Shepley and Sundermeyer among others, holds the key. Taking objective change to be ineliminable time dependence, one recalls that there is change in vacuum GR just in case there is no time-like vector field ξ^α satisfying Killing's equation $\pounds_\xi g_{\mu\nu} = 0$, because then there exists no coordinate system such that everything is independent of time. Throwing away the spatial dependence of GR for convenience, one finds explicitly that the time evolution from Hamilton's equations is real change just when there is no time-like Killing vector. The inclusion of a massive scalar field is simple. No obstruction is expected in including spatial dependence and coupling more general matter fields. Hence change is real and local even in the Hamiltonian formalism.

The considerations here resolve the Earman-Maudlin standoff over change in Hamiltonian General Relativity: the Hamiltonian formalism is helpful, and, suitably reformed, it does not have absurd consequences for change and observables. Hence the classical problem of time is resolved. The Lagrangian-equivalent Hamiltonian analysis of change in General Relativity is compared to Belot and Earman's treatment. The more serious quantum problem of time, however, is not automatically resolved due to issues of quantum constraint imposition.

Keywords: constrained Hamiltonian dynamics, General Relativity, problem of time, quantum gravity, variational principles

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1 Introduction

1.1 Hamiltonian Change Seems Missing but Lagrangian Change Is Not

It has been argued that General Relativity, at least in Hamiltonian form, lacks change, has change only asymptotically and hence only for certain topologies, or appears to lack change with no clear answer in sight (*e.g.*, (Anderson, 1962a; Isham, 1993; Belot and Earman, 2001; Earman, 2002; Rickles, 2006; Huggett et al., 2013)). Such a conclusion calls to mind earlier philosophical puzzles, whether ancient (the paradoxes of Zeno, whom James Anderson mentions (Anderson, 1962b; Anderson, 1962a), and the views of Parmenides, whom Kuchař mentions (Kuchař, 1993)) or modern (the argument conclusion that real time requires something contradictory and hence is impossible by McTaggart (McTaggart, 1908), mentioned in a memorable philosophical exchange (Earman, 2002; Maudlin, 2002)). The new conclusion, following apparently with mathematical rigor from our standard theory of gravity, is not as readily ignored as Zeno, Parmenides and McTaggart. On the other hand, if one breathes the fresh, clean air of numerical relativity from time to time, it is difficult not to notice that there really is change in GR. Thus one might suspect that any formal Hamiltonian results to the contrary are mistaken. Such a conclusion is all the clearer if one recalls that the Lagrangian GR formalism and 4-dimensional differential geometry are not thought to have any analogous problem. Either the canonical standards are inappropriately strict, or the 4-dimensional Lagrangian standards are too loose. But no one thinks the latter.

The Earman-Maudlin philosophical exchange provides a good starting point (Earman, 2002; Maudlin, 2002). Maudlin displays liberal amounts of common sense about point individuation, observables (in the non-technical sense of what can be observed), *etc.*, whereas Earman displays standard glosses on standard mathematical physics. Neither common sense rooted in scientific practice nor a common interpretation of mathematical physics is to be taken lightly, but can one have both? Is there a point in which mathematical physics becomes so bizarre as to undermine itself by excluding grounds for any possible empirical confirmation (Healey, 2002)? Earman has elsewhere composed an "Ode" commending the Dirac-Bergmann constrained dynamics formalism to philosophers (Earman, 2003). The reader of the Earman-Maudlin exchange gets the impression that each side declares victory. The progress of physics has been so great, and often enough counterintuitive, that beating back Poisson brackets with appeals to common sense does not yield full conviction, and rightly so. If Earman unwittingly exhibits "How to Abuse Gauge Freedom to Generate Metaphysical Monstrosities," as Maudlin's subtitle claims, then what is the right way to handle gauge freedom? On this question Maudlin is less full than one would prefer. Ultimately I will side with Maudlin's common-sense conclusions, though not his dismissive view of the Hamiltonian formalism. Change will be defended not in defiance of or indifference to mathematical physics, but through careful engagement in it and reform motivated by more solid mathematical physics.

1.2 Lagrangian Interpretive Strategy Brings Clarity

Attending to the Lagrangian formalism of General Relativity and to 4-dimensional differential geometry holds the key to clarity in all these matters. It seems to be widely agreed on diverse grounds that the Lagrangian formulation of mechanics (broadly construed) is more fundamental than the Hamiltonian one (Gotay and Nester, 1979; Curiel, 2013). It is also widely believed that the two are equivalent (apart perhaps from topological restrictions), or at least that they should be. Yet there are controversies in the literature on constrained dynamics about whether such equivalence actually holds, and various proofs presented have too narrow a scope (such as addressing the equivalence of equations of motion but neglecting to address the equivalence of the gauge transformations). One possible view is described by Pons, Salisbury and Sundermeyer (Pons et al., 2010):

> [t]he position on one side is that there ought to be no debate at all [about the physical interpretation of General Relativity or any generally covariant theory] because the phase space formalism is equivalent to the formalism in configuration-velocity space, and no one has claimed that any interpretational problem exists in the latter framework. Entire books have been devoted to the experimental tests of GR, and this very language implies that observables exist - alive and kicking. Thus the entire debate must be a consequence of misunderstandings. (Pons et al., 2010, p. 3).

Such a view suggests a Lagrangian-first interpretive strategy.

This view does not seem to be the view of Kuchař, though he, very unusually, is willing to tinker with the Dirac-Bergmann formalism to uncover real change in General Relativity (Kuchař, 1993). Kuchař's reinterpretation of the Hamiltonian constraint is not systematic—the common-sense arguments about observing temporal change work equally well for the momentum constraint and spatial change. Neither is Kuchař's view clearly inspired by the need for equivalence with the Lagrangian formalism. He allows that observables should commute with the momentum constraint $\mathcal{H}_i$, because we cannot directly observe spatial points. But he denies that observables should commute with the Hamiltonian constraint $\mathcal{H}_0$. He notes that one cannot directly observe which hypersurface one is on, either—which provides pressure to think that observables should commute with the Hamiltonian constraint. But he does not treat the constraints even-handedly despite their fitting his argument-form equally well:

> However, the collection of the canonical data $g_{(1)}$, $p_{(1)}$ on the first hypersurface is clearly distinguishable from the collection $g_{(2)}$, $p_{(2)}$ of the evolved data on the second hypersurface. If we could not distinguish those two sets of the data, we would never be able to observe dynamical evolution. (Kuchař, 1993, p. 137)

Indeed so. But who makes observations over an entire space-like hypersurface? We observe that the world is spatially varying, so, by parity of reasoning, why should observables commute with the momentum constraint $\mathcal{H}_i$, either? Hence making his argument more consistent provides a reason to doubt that observables should commute with the Hamiltonian or momentum constraints. One could also note that neither H_0 by itself nor H_i actually generates a gauge transformation due to neglect of the lapse and shift vector, basically the part of the space-time metric that is not contained in the spatial metric.

Kuchař's common-sense remarks on what experimental physicists routinely observe and Maudlin's common-sense remarks about observables point in the same direction. But the view described isn't quite that view of Maudlin, either (Maudlin, 2002). While Maudlin is satisfied with the Lagrangian formalism and its implicit concept of observables in terms of tensor calculus, he takes the problems in Hamiltonian General Relativity as a reason to reject it, not to reform it. Such a response is both unnecessarily drastic and incompatible with regarding the phase space formalism as equivalent to the configuration-velocity formalism.

The view described above by Pons, Salisbury and Sundermeyer (Pons et al., 2010) is attractive, if suitably developed to reform the Dirac-Bergmann formalism to achieve Hamiltonian-Lagrangian equivalence, and is adopted here. A principled way to implement the common sense that Kuchař and Maudlin forcefully deploy is to apply consistently the standards of observability/physical reality in Lagrangian/4-dimensional GR, where tensor fields (if not hobbled by other gauge freedoms like electromagnetism) are observable. Something like this view is implicit in the Pons-Salisbury-Shepley-Sundermeyer principles of reforming the Dirac-Bergmann formalism to ensure Hamiltonian-Lagrangian equivalence and paying due attention to the gauge generator G (Anderson and Bergmann, 1951; Castellani, 1982). One can take as a motto a remark of Pons and Shepley that has characterized that series of works (*e.g.*, (Pons et al., 1997; Shepley et al., 2000; Pons and Salisbury, 2005; Pons and Shepley, 1998)):

> We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent... in coming to the conclusion that they in fact are. (Pons and Shepley, 1998, p. 17)

Indeed one can hardly do otherwise in constrained dynamics without falling into inconsistency or error. But such an innocuous principle, they have shown, has unappreciated consequences that yield greater conceptual clarity. In this paper I aim to push further, primarily on conceptual rather than technical matters, in the same direction. The descriptive claim that the q-p formalism is equivalent to the q-$\dot{q}$ formalism is too quick; that equivalence *should* hold, but work has been needed to make it hold.

1.3 Taproot of Confusion: First-Class Constraints and Gauge Transformations

Indeed still more work is needed to make the q-p formalism equivalent to the q-$\dot{q}$ formalism. Recently I showed that if one accepts the longstanding claim that a first-class constraint generates a gauge transformation, then one spoils Gauss's law, the Lagrangian q-$\dot{q}$ constraint $\nabla \cdot \vec{E} = 0$ in (vacuum) electromagnetism (Pitts, 2013b). The electric field, a familiar function of the derivatives of the potentials A_μ (Jackson, 1975), changes by the gradient of the arbitrary smearing function and hence ceases to be divergenceless even in the absence of charge. It is easy not to notice this problem because the conjugate momentum that one expected to be the electric field, remains divergenceless. If one isn't careful to retain Hamiltonian-Lagrangian equivalence, the conjugate momenta can cease to mean what one expected, as Anderson and Bergmann pointed out long ago (Anderson and Bergmann, 1951).

To see the problem, one can add the two independently smeared constraints' actions together:

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y[p^0(y)\xi(t,y) + p^i{}_{,i}(y)\epsilon(t,y)]\} = \delta^0_\mu \xi - \delta^i_\mu \partial_i \epsilon. \tag{1}$$

Then one calculates the 4-dimensional curl of the transformed 4-vector potential less the curl of the original potential, getting the curl of the gauge transformation due to linearity. (The fact that the electric field itself is not defined on phase space is not relevant. Once one knows A_μ, one can take its curl to find the electromagnetic field.) The magnetic field is unchanged (Sundermeyer, 1982, p. 134). But the electric field, which curiously has been neglected, *does* change (Pitts, 2013b)—a result that dashes expectations that a first-class constraint alone generate a gauge transformation. The combined change in $\vec{E}$ is given by

$$\delta F_{0n} = -\delta \vec{E} = -\partial_n \xi - \partial_n \partial_0 \epsilon. \tag{2}$$

If one puts the constraints to work together as a *team* by setting $\xi = -\dot{\epsilon}$ to make the $\delta F_{0n} = 0$, then

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y[-p^0(y)\dot{\epsilon}(t,y) + p^i{}_{,i}(y)\epsilon(t,y)]\} = -\delta^0_\mu \dot{\epsilon} - \delta^i_\mu \partial_i \epsilon = -\partial_\mu \epsilon, \tag{3}$$

which is good. Not surprisingly in light of the form of the gauge generator (Anderson and Bergmann, 1951; Castellani, 1982; Pons et al., 2000)

$$G = \int d^3x(p^i{}_{,i}\, \epsilon - p^0 \dot{\epsilon}), \tag{4}$$

p^0 and $p^i{}_{,i}$ generate *compensating* changes in $\vec{E}$ when suitably combined. Indeed one can piece together G by demanding that the changes in $\vec{E}$ cancel out. Two wrongs, with opposite signs and time differentiation, make a right. This tuning, not surprisingly, is a special case of what Sundermeyer found necessary to get first-class transformations to combine suitably to get the familiar gauge transformation for the potentials for Yang-Mills (Sundermeyer, 1982, p. 168). The benefit of my taking the curl *before* fixing the coefficients of the constraints to make a team is to show that getting the usual Lagrangian gauge transformations is compulsory on pain of spoiling elementary facts of electromagnetism, not merely optional and comfortingly familiar as it has seemed to be in the constrained dynamics literature. The commutative diagram makes the point.

$$\begin{tikzcd}
A_\mu \arrow[r, "L-equiv."] \arrow[d, "\int d^3x(p^0\xi+\epsilon p^i{}_{,i})"'] & G = \int d^3x(-p^0\dot{\epsilon} + \epsilon p^i{}_{,i}) \arrow[rr] & & \delta A_\mu = -\partial_\mu \epsilon \arrow[d, "curl"] \\
\delta A_\mu = \delta^0_\mu \xi - \delta^i_\mu \epsilon_{,i} \arrow[r, "curl"] & \delta F_{\mu\nu} = (\delta^0_\nu \xi_{,\mu} - \delta^i_\nu \epsilon_{,i\mu}) - \mu \leftrightarrow \nu \arrow[rr, "L-equiv."', "\xi=-\dot{\epsilon}"] & & \delta F_{\mu\nu} = 0
\end{tikzcd}$$

While the top line is not unknown in works advocating Hamiltonian-Lagrangian equivalence, the bottom line appears to be novel. It is of course unacceptable to have $\delta F_{\mu\nu} \neq 0$, so requiring Lagrangian equivalence from the Hamiltonian resolves the trouble. Hamiltonian-Lagrangian equivalence is the law, not just a good idea.

The equality (up to a sign) of the electric field and the momentum p^i conjugate to A_i *using Hamilton's equations* $\dot{A}_i = \frac{\delta H}{\delta p^i}$ has tempted many authors to treat that equality as if it were something stronger. These are among the boring Hamilton's equations, ones that recover what one already knew in the Lagrangian context and then forget in performing the Legendre transformation. Boring or not, that relationship must be forgotten in calculating Poisson brackets. Some authors, far from forgetting that equality, even use the letter E for the canonical momentum, reflecting in notation an overly hasty identification that is often less transparently (Faddeev, 1982; Belot and Earman, 2001). But that canonical momentum is no longer equivalent to the electric field after either first-class constraint has acted on A_μ. Likewise, one spoils the Hamiltonian and momentum constraints in GR—at least the physically relevant versions, which are in terms of q and $\dot{q}$ (the 3-metric, lapse, shift vector, and extrinsic curvature tensor, roughly) and pertain to intrinsic curvature and the embedding of space into space-time, not q and off-shell physically meaningless canonical momenta π^{ij} by acting with a primary first-class constraint or a secondary first-class constraint. Hence a first-class constraint in these representative theories, whether primary or secondary, generates a bad physical change, not a gauge transformation. Conjugate momenta don't ultimately matter; they are just auxiliary fields for making $\dot{q}$ (and hence q) do the right thing over time. Being auxiliary fields, they appear essentially algebraically in the canonical action $\int dt d^3x(p\dot{q} - \mathcal{H})$ and can be 'eliminated' (reduced to functions of derivatives of A_μ in physically possible worlds) using their own equations of motion, leading back to the original Lagrangian action $\int dt d^3x \mathcal{L}$. Such an ontologically superfluous entity clear is not the primordial observable electric field. In electromagnetism, coupling to charge happens through an interaction term $A_\mu J^\mu$; p^i is nowhere in sight. If the q's—in this case A_μ—do the right thing over time, all is well; if they don't (no matter how well the conjugate momenta behave), all is lost. In the context of GR, if the momentum π^{ij} conjugate to the 3-metric h_{ij} behaves properly, but the extrinsic curvature tensor K_{ij} behaves improperly, then space doesn't fit rightly into space-time and Einstein's equations are false. Usually one doesn't bother with this distinction because the boring Hamilton's equations $\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}}$ return the correct relationship that one forgets in setting free π^{ij} in the Legendre transformation. But that relationship is not an identity. It is just a field equation ('on-shell'), whereas gauge transformations are (typically, including electromagnetism and the momentum constraint $\mathcal{H}_i$ in GR) defined off-shell. Hence an inept identification of gauge transformations can spoil $\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}}$ or its electromagnetic analog, making the momenta lose their usual relationship to the velocities. $\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}}$ arrives on the scene logically too late to prevent trouble; instead it just makes a mess. Hence good behavior of the momenta does not guarantee good behavior of the velocities. The latter can, and should, be checked directly. Naturally there will be agreement in a correct Hamiltonian formalism. With such issues in mind Anderson and Bergmann urged attending to the Lagrangian constraint surface, not only the Hamiltonian constraint surface (Anderson and Bergmann, 1951).

After 30 years the force of Castellani's recovered insight about the team rather than individual character of gauge transformation generation by first-class constraints still has not been generally recognized. He said that

> Dirac's conjecture that all secondary first-class constraints generate symmetries is revisited and replaced by a theorem.... The old question whether secondary first-class constraints generate gauge symmetries or not ... is then solved: they are *part* of a gauge generator G ... (Castellani, 1982, pp. 357, 358). (emphasis in the original)

The force of "replaced" requires the *elimination of the old erroneous claim*, not just the introduction of a new true claim. Nowadays one sees a curious coexistence of beliefs in first class constraints as generating gauge transformations (or lingering consequences of that old belief pertaining to change or observables) and belief in the gauge generator G, a special *combination* of first class constraints, as generating gauge transformations. Indeed the gauge generator is already a very old part of the constrained dynamics literature (Anderson and Bergmann, 1951; Rosenfeld, 1930; Salisbury, 2010), so in one sense there wasn't much of a new claim to introduce even in 1982 (apart from the use of a $3+1$ split, the 1958 trivialization of the primaries by a well chosen boundary term (Dirac, 1958; Anderson, 1958), and elimination of $\dot{q}$ from the gauge generator in favor of Legendre projectability)—though the gauge generator had been almost entirely forgotten apart from recent work by Mukunda (Mukunda, 1980). In a sense the Dirac conjecture is two-sided: it claims that every first-class constraint is involved in generating gauge transformations and that each first-class constraint that generates a gauge transformation does so by itself. (Actually the conjecture pertains only to secondary and higher generations of constraints, because Dirac thought that he had proved the result for primary constraints (Dirac, 1964, p. 21).) The former claim, which is generally received, seems to be true apart from ineffective constraints (such as squared constraints, which have vanishing gradient and hence vanishing Poisson brackets on the constraint surface). This former claim seems to account for some recent literature that purports to prove the Dirac conjecture while in fact arriving at the gauge generator G instead (Gitman and Tyutin, 2006). The latter claim is false.

While Castellani's attention (following Dirac) was initially directed to secondary constraints, it is easy to show by direct calculation that a *primary* constraint in Maxwell's electromagnetism (or in General Relativity) does not generate a gauge transformation either. In electromagnetism it changes the electric field $\vec{E}$ by a gradient, just as the secondary constraint does (Pitts, 2013b), again spoiling Gauss's law. Not surprisingly, a special combination of the two leaves the electric field (and the magnetic field, which the constraints do not touch) unchanged. That combination is just the gauge generator (Anderson and Bergmann, 1951; Castellani, 1982; Pons et al., 2000) $G = \int d^3x (p^i,_i \xi - p^0 \dot{\xi})$. Dropping a boundary term gives the more suggestive form $G = -\int d^3x p^\nu \partial_\nu \xi$. G acts as $\{G, A_\mu\} = \partial_\mu \xi$ and $\{G, p^\mu\} = 0$. Hence one can reinvent the gauge generator in electromagnetism simply by requiring that the changes in the electric field due to the first-class constraints cancel out. One can revisit Dirac's argument that a first-class constraint generates a gauge transformation (Dirac, 1964, p. 21) with electromagnetism in mind. Part of the problem is that Dirac, by comparing two solutions with identical initial data, neglects the effect that a gauge transformation can have instantaneously on the initial data surface, thus underestimating the influence of the primary constraints. Relatedly, he neglects the fact that the secondary first-class constraint (the one like Gauss's law) appears in the Hamiltonian with a gauge-dependent coefficient $-A_0$, so two evolutions will (eventually) differ not simply by the coefficients of the primary constraint (the scalar potential's velocity), but also by what the Gauss-like constraint generates (Pitts, 2013b; Pons, 2005). To see the difference, notice the effects on the vector potential A_i, which is certainly altered (by a gradient) by a gauge transformation, but is not altered by the primary constraint. Unfortunately Dirac's error is widely followed (Govaerts, 1991; Henneaux and Teitelboim, 1992; Wipf, 1994; Rothe and Rothe, 2010). (By contrast Sundermeyer computes relevant Poisson brackets providing the raw material to diagnose the problem (Sundermeyer, 1982, p. 134).) This oversight of Dirac's motivates the extended Hamiltonian formalism, which is intended to recover the full gauge freedom which is actually already there in the primary Hamiltonian with the primary and secondary first-class constraints.

For General Relativity, there is an additional problem in Dirac's argument regarding the primary constraint conjugate to the lapse N (Barbour and Foster, 2008; Thébault, 2012a). The reason is that his argument assumes that physically equivalent points should have equal values of the coordinate time—in effect, that the time is observable, as was noticed by Henneaux and Teitelboim (Henneaux and Teitelboim, 1992, pp. 17-19). They attempt to fill the gap another way (Henneaux and Teitelboim, 1992, p. 107), but give up Hamiltonian-Lagrangian equivalence in the process.

This paper continues the project of resolving problems in canonical General Relativity by enforcing the equivalence of the (obscure and troubled) Hamiltonian formalism to the (perspicuous and correct) Lagrangian formalism. The clear formalism interprets the unclear one. Maudlin's common sense about change and observables is inspirational, but his disregard for the Hamiltonian formalism is not satisfactory. What is needed is conceptual-technical work that identifies distinctively Hamiltonian conjectures about gauge freedom, change, and observables and eliminates them in favor of Lagrangian-equivalent notions. Paradoxical features of the Hamiltonian formalism are a good place to look for such problems. Sometimes one can show that seemingly paradoxical features are equivalent to the plainly correct Lagrangian or differential geometric concepts and hence, on further reflection, no longer paradoxical. In other cases paradoxical Hamiltonian claims are erroneous. The predecessor paper (Pitts, 2013b) carried out that task for gauge freedom and first class constraints for electromagnetism. As a consequence one wants observables to be defined in terms of suitable Poisson brackets not with first-class constraints separately, but with the gauge generator G (Pons et al., 2010). Another paper (in preparation) carries out the same task for General Relativity (Pitts, 2013a). This current paper carries out the task regarding change in GR. A following paper will carry out the task for observables in GR, where it turns out that there is a key distinction between internal and external symmetries; as a result, Hamiltonian-Lagrangian equivalence for observables involves not *vanishing* Poisson bracket (with G), but a Poisson bracket with G that gives the Lie derivative of a geometric object. While the ideas about change and observables can be understood in isolation, the overall coherence of the package is best appreciated collectively. Clearing away confusion about first-class constraints clears the field of entrenched errors about change and observables to make the Lagrangian-equivalent definitions more plausible.

1.4 Illustration *via* Homogeneous Truncation of General Relativity

This paper aims to deploy just enough technical apparatus to make the point about change clear—that is, warranted and also not buried in irrelevancies—using a reparametrization-invariant 'mechanical' theory derived by simply dropping the spatial dependence from GR. Such work sometimes goes by the name "minisuperspace"; the particularly simple case addressed here corresponds to Bianchi I cosmological models. My aim is not primarily to illuminate a very narrow sector of GR (though in many respects the toy theory does so), but to have a 'mechanical' system that is analogous to GR in most ways relevant to the task at hand, yet allows easy calculations. It has a physically undetermined function relating physical time to coordinate label time. This function, the "lapse function" N, appears undifferentiated in the Lagrangian and the Hamiltonian. It relates to the space-time metric as $g^{00} = -N^{-2}$ (Misner et al., 1973). The constrained Hamiltonian formalism due to Dirac, Bergmann, *et al.* associates with N a conjugate momentum p, which is appended linearly to the Hamiltonian with an arbitrary function $v(t)$, giving the "primary" (or sometimes "total," but some authors use the term differently (Govaerts, 1991)) Hamiltonian $H_p = N\mathcal{H}_0 + vp$, which generates time evolution (Sundermeyer, 1982). Whether this time evolution constitutes objective change will be explored. The distinction between this Hamiltonian (generator of time evolution from initial data) and the gauge generator G will be made, used and discussed in detail. What will be achieved is a positive and intelligible Hamiltonian story about change that replaces the mysterious one built on the assumption that a first-class constraint generates a gauge transformation.

My treatment differs from Pons and Shepley's treatment of homogeneous cosmologies (Pons and Shepley, 1998) primarily due to a different aim. My aim is technically much simpler, namely, to find a theory that has enough GR-like behavior to threaten to lose time evolution just as GR does—time reparametrization invariance with a Hamiltonian constraint quadratic in the momenta—but simple enough that explicitly finding time evolution is easy when one looks in the right place. Confusingly, the counting of degrees of freedom in Bianchi cosmologies depends on the intended interpretation and also the global topology (Pons and Shepley, 1998; Ashtekar and Samuel, 1991; Salisbury et al., 2008). If one views such cosmologies as a sector of General Relativity (without preferred simultaneity), then solutions related by a 4-dimensional coordinate transformation will count as equivalent. Thus one achieves the same number of degrees of freedom in the Hamiltonian and Lagrangian formalisms, as one should, if one is careful and insistent on doing so (Pons and Shepley, 1998). While the Lagrangian and Hamiltonian treatments of the 'mechanical' system ought to agree with each other, for my conceptual purposes, neither many-fingered time nor the momentum constraints and their first-class character, are deliberately retained, sought, or valued. There will be no occasion to worry about homogeneity-preserving diffeomorphisms or the number of degrees of freedom. Leaving redundant components in the spatial metric is useful in order to preserve the visual-mathematical resemblance between a tensor (matrix) field in GR and a matrix, as opposed to whittling it down to its diagonal elements (*c.f.*, (Goldberg and Klotz, 1973; Ryan and Shepley, 1975; Salisbury et al., 2008)). My treatment also assumes the simplest version of homogeneity, namely, with the three spatial Killing vector fields all commutating (vanishing structure constants, Bianchi type I), so one can simply keep $x^0 = t$ and drop x^i $(i = 1, 2, 3)$. Simultaneity is absolute, but the labels of the slices are not quantitatively meaningful. I expect that the treatment of change in terms of the lack of a time-like Killing vector field would hold for GR just as for the mechanical theory, but with tedious and irrelevant calculations tending to obscure the main point. It will be necessary to do a bit of logic, however, paying attention to or's ($\vee$), and's ($\wedge$), not's ($\neg$), and existential and universal quantifiers ($\exists$) and ($\forall$), in order to show the equivalence of the Hamiltonian and differential geometric conceptions of change. Some classical differential geometry, especially involving scalar densities and their Lie derivatives (Schouten, 1954; Anderson, 1967), will be needed in making the $3+1$ split of the space-time Killing vector equation into space (largely trivial) and time (the focus here).

2 Change Unproblematic in 4-Dimensional Geometry

2.1 Hamiltonian Impersonators Obscure Change Generator

If one attempts to define real change in terms of what some Hamiltonian-like entity generates, one has to try to figure out which Hamiltonian-like entity to use. Even then one might not be sure that even one's favorite candidate generates real change. Is it the Hamiltonian constraint $\mathcal{H}_0$ (also called $\mathcal{H}_\perp$)? One might find this line from Sundermeyer's book tempting: "[s]o we really infer that $\mathcal{H}_\perp$ is responsible for the dynamics." (Sundermeyer, 1982, p. 241) Is it perhaps instead the weight 2 Hamiltonian constraint $\sqrt{h}\mathcal{H}_0$, which avoids awkward powers of the square root of the determinant of the metric and relates more closely to hyperbolic formulations of the field equations (Anderson and York, 1998)? (Indeed the former property can be achieved also for weight 4, weight 6, weight 8, *etc.* by throwing more powers of h onto $\sqrt{h}\mathcal{H}_0$ and compensating with an oppositely weighted lapse, now taken as primitive.) Is it the canonical Hamiltonian $H_c = N\mathcal{H}_0$? It has the virtue of being what results from $p\dot{q} - L$ once the primary constraint is used to annihilate the coefficient of $\dot{N}$. Is it the primary Hamiltonian $H_p = N\mathcal{H}_0 + vp$, which adds in the primary constraint(s)?[1] Is it the gauge generator G, which is built out of the same secondary and primary constraints as the primary Hamiltonian, and which appears to include the primary Hamiltonian as a special case? Is it the extended Hamiltonian, which adds the secondary constraints by hand, as Dirac invented to try to find the most general motion possible (Dirac, 1964)? Dirac's attitude about whether

[1] If one is willing to keep $\dot{N}$ in the Hamiltonian formalism, then one can use the Sudarshan-Mukunda Hamiltonian instead (Sudarshan and Mukunda, 1974, p. 93) (Castellani, 1982).

any particular Hamiltonian was the right one was quite relaxed.[2] But one might expect that at most one Hamiltonian would yield real change equivalent to the reliable Lagrangian/differential geometric definition. With at least six moderately plausible candidates, it is best to look elsewhere besides the Hamiltonian formalism for some more reliable criterion to decide which, if any, generates real change.

Whether there really is change in GR cannot depend on whether the theory is described in terms of the Lagrangian or the Hamiltonian formalism. But since the answer is a clear "yes" for the Lagrangian case, while the Hamiltonian formalism has long been obscure on the matter, it follows that the Hamiltonian answer needs to be "yes" as well. If some Hamiltonian formalism does not give a positive answer, then one needs to rethink the formalism until a positive answer and equivalence to the Lagrangian formalism are achieved. Thus we will find that there is an answer that generates real change, and which of the six (or more) candidates it is.

2.2 Change Unambiguous in 4-dimensional Differential Geometry

It is best to get back to basics—something even more basic than the Lagrangian formalism, in fact. Change, one might say, is being different at different times. At least that definition would be adequate if General Relativity didn't pose the risk of fake change due to funny labeling. A revised, more GR-aware definition would be that change is being different at different times in a way that is not an artifact of funny labeling. A solution of Einstein's equations displays objective change iff (and where) it depends on time for all possible time coordinates. At this point differential geometry comes in.

General Relativity expressed in the language of Lagrangian field theory and four-dimensional differential geometry is generally not believed to suffer from a problem or time or a lack of observable quantities (Pons et al., 2010). One of course needs to account for the coordinate (gauge) freedom, but that is not difficult to do—tensors and all that. (All transformations are interpreted *passively*, thereby averting one gratuitous source of confusion, namely, primitive point identities moving around relative to the physical properties.) There is change if (or where) the metric is not "stationary," that is, if (or where) there exists no time-like Killing vector field (Wald, 1984; Kramer et al., 1980). (A local section suffices for having no change locally—in a neighborhood where there is a time-like Killing vector, there change does not happen. The Schwarzschild solution outside the horizon is a familiar example.) A time-like Killing vector field ξ^{μ} is a vector field such that is Killing relative to the metric,

$$\pounds_{\xi} g_{\mu\nu} = \xi^{\alpha} g_{\mu\nu},_{\alpha} + g_{\mu\alpha} \xi^{\alpha},_{\nu} + g_{\alpha\nu} \xi^{\alpha},_{\mu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0, \tag{5}$$

making the metric independent of a coordinate adapted to ξ^{α} (Ohanian and Ruffini, 1994, p. 352), and that is time-like relative to the metric ($g_{\mu\nu}\xi^{\mu}\xi^{\nu} < 0$ using $-+++$ signature). Lie differentiation $\pounds_{\xi}$ is a more or less tensorial directional derivative; when acting on tensors or connections, it gives a tensor (Yano, 1957). (Some of the less widely known features of Lie derivatives (Yano, 1957; Tashiro, 1950; Tashiro, 1952; Szybiak, 1966) will be relevant in the successor paper on observables (Pitts, 2014).) The nonexistence of a time-like Killing vector field is the coordinate-invariant statement of change: change is not being stationary (neighborhood by neighborhood) (Kramer et al., 1980). Quantities are observable if they are tensors, tensor densities, or more general geometric objects (Nijenhuis, 1952; Schouten, 1954; Anderson, 1967) and are not compromised by some other convention-dependence such as electromagnetic gauge freedom. All of this is uncontroversial in the 4-dimensional Lagrangian context (Wald, 1984). I mention it in some detail only because of the general failure to attend adequately to the Hamiltonian analogs in the appropriate contexts.

Because change is locally defined in terms of the lack of a time-like Killing vector field (or more generally, the lack of a time-like vector such that the Lie derivative of everything, metrical and material, vanishes, if matter is present), one need not attend to boundary terms to define change. Thus non-trivial topologies also admit change, even if they have no boundaries. Even if the world is like a doughnut, there is change in GR. Just watch for change in a room with no windows, such as a basement. The difficulty in knowing what happens at infinity, or knowing whether there is any such place, will be no hindrance. This Lagrangian platitude amounts to a revisionist Hamiltonian research project.

[2] I thank Edward Anderson for a useful comment on this point.

3 Bergmann *vs.* Change as Only from Global Observables

Bergmann wrote rather more on observables than is usually recognized in the literature on the subject. More importantly, the aspects of his work that are generally recalled, though the most technically precise, are neither the most central in his understanding nor the most plausible parts of his work. Hence it is far from obvious that the condition that usually passes for a definition of "Bergmann observables" is something that Bergmann would endorse on reflection. Instead that condition involves a specialization to the Hamiltonian formalism, which Bergmann clearly did not intend to be decisive, as well as an unsuccessful argument from his actual definitions. That condition is thus only a pseudo-lemma.

3.1 Why Bergmann Defined Observables

The presence or absence of a time-like Killing vector field has been contemplated briefly as relevant to the problem of finding change in Hamiltonian GR (Earman, 2002; Healey, 2004), but it has not been taken seriously. The glare of supposed Hamiltonian insights until now has distracted attention away from the time-like Killing vector question, which is in fact decisive. For Earman, such matters as time-like Killing vectors pertain only to the "surface structure" of GR. The "deep structure" for Earman evidently involves distinctively Hamiltonian notions such as first-class constraints as generating gauge transformations and observables as Poisson-commuting with the first-class constraints. If physical reality is confined to "observables," then real change requires changing observables. But observables that change—even observables that are observable (in the ordinary non-technical sense related to experiment)—are hard to come by (Torre, 1993; Tambornino, 2012).

But Bergmann, who has at least as good a claim as anyone to define observables in General Relativity because he invented them, clearly denied that physical reality was confined to observables. He also insisted that observables in General Relativity were local (Bergmann, 1962, p. 250). Rather than simply recalling theorems from moderately recent literature about observables, it is important to figure out what problem Bergmann was trying to solve in putting forward the concept.

The main problem that motivated defining observables is how to predict the future in a way that is not infected by future choices of coordinate conventions, given that space-time coordination of the present and past leaves unspecified the coordinatization in the future in GR (Bergmann, 1961). One doesn't know today what the future's coordinates mean in terms of distances and times, independent of calculating the field(s) from now till next year. In Special Relativity it makes sense at (t, x, y, z) to ask what the field values are at some finite coordinate distance into the future—at $(t + 1, x, y, z)$, for example. One knows where and when that space-time point will be, though one does not know what will happen there. The question makes sense, but the answer is unavailable without filling in the field values in the past null cone of the point in question. In General Relativity the situation is more subtle. The question at (t, x, y, z) of the field values at $(t + 1, x, y, z)$ isn't even a precise question, because one has little idea where $(t + 1, x, y, z)$ is. Part of the task of the space-time metric is to define the coordinates implicitly by 'inverting' $g_{\mu\nu}(x)$. This lack of a precise question is a problem that does not arise non-trivially in special relativistic theories, because there one already knows in advance what the coordinates mean metrically and/or in terms of Killing-type conditions and can simply extend the preferred coordinates into the future (Bergmann, 1957; Bergmann, 1961). (This point bears on the hole argument and whether it arises equally in Special Relativity and General Relativity, as argued by Earman and Norton (Earman and Norton, 1987). On Bergmann's view the hole argument is interesting only in General Relativity (Bergmann, 1957; Stachel, 1993).) But in some respects it doesn't matter, because in General Relativity the question becomes specific enough to be well defined exactly when the answer becomes available. The question becomes well defined through a combination of physical time evolution and arbitrary coordinate stipulation. (Some of that arbitrariness can be avoided by using more physically meaningful descriptions—from whence such notions as Weyl curvature scalar coordinates (Komar, 1955), Rovelli's partial observables (Rovelli, 1991), *etc.* However, such strategies are not very convenient for addressing partial differential equations.) In terms of the Cauchy problem in Special Relativity, there is always a shrinking but infinite supply of well defined but presently unanswered questions about the future, questions just sitting on the shelf until their time comes to be answerable.[3] In General Relativity,

[3] There is an amusing partial analogy to rival logistical philosophies in the automotive industry some time back. Japanese

the value of the field(s) at a given coordinate point in the future becomes a well-defined question just as the answer appears.

It is worth noting that Bergmann works passively, dealing in coordinate transformations rather than active diffeomorphisms. The passive coordinate language clearly is adequate. In my view it is in fact preferable in the mature form that also recognizes physical points with unique names like p (Trautman, 1965) (as opposed to a mere multiplicity of labels corresponding to the multiplicity of coordinate systems), for reasons to be discussed below, involving Einstein's point-coincidence argument. Admitting only passive coordinate transformations excludes one extra potential source of confusion.

3.2 Bergmann Defined Observables as Local

Bergmann defined observables several times. Here are some relevant passages. According to Bergmann, observables "can be predicted uniquely from initial data." (Bergmann, 1961) They are "invariant under a coordinate transformation that leaves the initial data unchanged." (Bergmann, 1961) "General relativity was conceived as a local theory, with locally well defined physical characteristics. We shall call such quantities *observables*." (Bergmann, 1962, p. 250) "We shall call *observables* physical quantities that are free from the ephemeral aspects of choice of coordinate system and contain information relating exclusively to the physical situation itself. Any observation that we can make by means of physical instruments results in the determination of observables" (Bergmann, 1962, p. 250). *Pace* Earman's and others' claim that only mysterious non-local "observables" are real and can sustain real change (Earman, 2002), Bergmann says that the metric and matter components everywhere (4-dimensionally) in some coordinates, satisfying the field equations, "include all the physical characteristics of the situation to be described. . . ." The problem is merely that the metric and matter fields in all coordinates are *redundant* in two respects. Cauchy data at one moment should suffice—3 dimensions instead of 4—and some of the $g_{\mu\nu}$ component information is just coordinate (gauge) choice, not part of the physical situation (Bergmann, 1962, p. 250). But a Killing vector field is a coordinate-invariant feature of the metric components, hence physically real, regardless of its relation to observables, which are intended by Bergmann to be both coordinate-invariant (hence real) *and economical*. Redundancy doesn't imply non-reality (*c.f.* (Earman, 2002)). It isn't clear that these two goals, finding a set of Cauchy data and removing the subjective taint of arbitrary descriptive choice, can be simultaneously realized, given that a Cauchy surface can be a level surface of one time coordinate but not of some other one, a problem with no analog in electromagnetism. (The electromagnetic gauge symmetry is internal and unrelated to simultaneity. One can use a different Cauchy surface by choosing a different inertial coordinate system or an arbitrary coordinate system, but such changes are not related to electromagnetic gauge transformations.) But that worry does not concern us here.

3.3 Bergmann Observables Are Not Inherently Hamiltonian

Bergmann's concept of observables was clearly intended to be independent of the Hamiltonian formalism (Bergmann and Komar, 1962, p. 314) (Bergmann, 1961; Bergmann, 1962). Consequently it must satisfy Hamiltonian-Lagrangian equivalence when sorted out properly. Thus one can see that no statement involving Poisson brackets with first-class constraints can be part of Bergmann's definition, though most treatments of Bergmann observables take such a claim as definitive (*e.g.*, (Torre, 1993)). One finally finds Hamiltonian notions introduced only rather late in the widely read review article on the subject:

> If we employ a Hamiltonian formalism to describe our theory, then there is a certain set of generators that produce infinitesimal coordinate transformations. An observable is then a dynamical variable that has vanishing Poisson brackets with all the generators of infinitesimal coordinate transformations. (Bergmann, 1961, pp. 511, 512)

manufacturers used a "just-in-time" strategy in which parts arrived when needed and not much before, whereas American manufacturers had more traditional stockpiles of parts waiting to be used (Alternburg et al., 1999). The Japanese style evidently was more efficient and spread, but it also required more reliable suppliers. Because the physical world neither faces storage costs nor risks labor strikes or natural disasters in relation to the supply of physically meaningful questions, both styles are fine for a physical theory.

This Hamiltonian pseudo-lemma serves as the starting point for many subsequent discussions of observables.

Unfortunately this passage contains multiple problems. One is that it suggests (as Bergmann indeed held at that time, in tacit contrast to ((Anderson and Bergmann, 1951))) that a first-class constraint by itself generates a gauge transformation, a claim that is much easier to believe if one has forgotten about the space-time metric in favor of the spatial metric, much as Dirac advised (Dirac, 1958). Another is that the requirement that whatever generates a gauge transformation (whether the gauge generator (Anderson and Bergmann, 1951) or an individual smeared first-class constraint) have *vanishing* Poisson bracket with the field in question seems to be motivated by nothing stronger than mere analogy with electromagnetism (Bergmann, 1961; Bergmann, 1956). But an analogy between a theory with an internal symmetry (electromagnetism) and one with an external symmetry (General Relativity) is hardly stronger than the requisite equivalence between the Lagrangian-differential geometric formulation of GR and Hamiltonian GR. The difference between internal symmetries and external symmetries (which differ in that only the latter have a transport term such as $\xi^{\alpha} g_{\mu\nu},_{\alpha}$ differentiating the field in the infinitesimal transformation law) implies that one should seek only *covariance* for external symmetries, not invariance as with internal symmetries.[4] These problems will be discussed in more detail elsewhere (Pitts, 2014).

Bergmann apparently neglected to consider the internal *vs.* external symmetry distinction because he neglected Hamiltonian-Lagrangian equivalence for GR, despite ((Anderson and Bergmann, 1951)). He simply imported his concept of observables from electromagnetism (with an internal gauge symmetry) to GR (with an external gauge symmetry) by analogy (Bergmann, 1956; Bergmann, 1961). Poisson-commuting with the gauge generator G is in fact a Killing-type condition in GR, which usually has no solutions, at any rate not among local fields. Having vanishing Poisson bracket with the gauge generator for any smearing vector field, obviously, implies constancy over time and space. A more appropriate infinitesimal condition for an observable in GR is for the gauge generator G (not the first-class constraints separately) to yield an expression that is *the Lie derivative of a geometric object*. For linear (including inhomogeneous) geometric objects, the Lie derivative is a tensor (Yano, 1957). Because Lie and coordinate differentiation commute (Yano, 1957), some expressions have a Lie derivative without being geometric objects. One example is Einstein's $\Gamma-\Gamma$ Lagrangian density, which differs from a scalar density $\sqrt{-g}R$ by a divergence. Another example is the coordinate gradient of a vector $V^{\mu},_{\nu}$. Hence there are Lie derivatives of entities that are not geometric objects. Thus the requirement that the gauge generator yield a Lie derivative of a geometric object is not empty; it excludes the two examples and many similar ones from the realm of observables, as one would hope; there are certain identities involving structure constants that must be satisfied to ensure the group property (Bergmann, 1949; Tashiro, 1950; Tashiro, 1952; Szybiak, 1966). This requirement admits tensor fields and more general geometric objects as observables—again as one would expect on Lagrangian/4-dimensional grounds. When Bergmann imported the definition of observables from electromagnetism to GR (Bergmann, 1956; Bergmann, 1961), he implicitly took Hamiltonian electromagnetism to be more definitive than Lagrangian GR in ascertaining the observables in Hamiltonian GR. But Hamiltonian electromagnetism has no idea what the gauge transformations in GR are, and in particular has no idea what disanalogies there might be between internal and external symmetries. Logical-mathematical equivalence must trump electromagnetic-gravitational analogy when a tension arises.

4 Vacuum GR without Spatial Dependence

In this section, which forms the heart of the paper, I will show how a proper Hamiltonian treatment of GR does indeed involve change. Change will be exhibited in a truncated form of GR obtained by simply dropping the spatial dependence. Such a simplified toy theory has most of the features that threaten to

[4] Invariance is appropriate for dealing with either the metric-in-itself $\mathbf{g}$ (which equals $g_{\mu\nu}\mathbf{d}x^{\mu}\otimes\mathbf{d}x^{\nu}$ locally in any coordinates) or the geometric object comprising the metric components in *all* coordinate systems (Trautman, 1965). The former, if not quite ineffable, is not something for which it is widely known how to take a Poisson bracket, though perhaps one could find a way by devising some prescription for Poisson brackets of the basis vectors $\mathbf{d}x^{\mu}$. The latter has a painfully high cardinality and also does not clearly lend itself to the taking of a Poisson bracket. The natural fall-back position is to retreat from invariance to covariance.

obscure time and change in GR, apart from "many-fingered time" (freedom to define various simultaneity hypersurfaces). The treatment of the toy theory will also show by a tractable and interesting example how Dirac-Bergmann constrained dynamics works. Then I will recall my recent result that a first-class constraint generates a bad physical change—not a gauge transformation as is often held. Because a first-class constraint does not generate a gauge transformation, one is no longer tempted to infer from the fact that the Hamiltonian of GR is a sum of first-class constraints that time evolution is only a gauge transformation and hence that there is no real evolution or change. With that erroneous Hamiltonian story undermined, one is well positioned to attend to the correct Hamiltonian story, which involves detailed attention to the equations of motion and to gauge transformations, which must be equivalent to the Lagrangian coordinate transformations (or at any rate to the ones involving time in this spatially truncated toy theory) at least for solutions of the equations of motion. (Inconveniently, it is provable that kinematically possible but dynamically impossible trajectories have a gauge freedom that is not simply related to changing the time coordinate.) One already knows from differential geometry what it is for there to be change in a solution of Einstein's equations, that is, for the solution not to be stationary: change is just the absence of a time-like Killing vector field. It will be found that the Hamiltonian equations of motion imply that, for all choices of (time) coordinate, something depends on time if and only if there is no (time-like) Killing vector field. It will then be shown that the gauge generator G implements using Hamiltonian resources exactly the infinitesimal changes of time coordinate *via* Lie differentiation, at least on-shell.

4.1 Homogeneous Truncation

One can capture just enough of the real physics of GR, while maintaining manifest relevance and avoiding most complication, by using a toy theory, vacuum GR with all spatial coordinate derivatives dropped. Such a restriction is both a substantial physical restriction compared to GR, and an adaptation of coordinates to fit the restricted physics. Because the coefficient of the shift vector N^i in the Lagrangian, namely the momentum constraint, vanishes when nothing varies with the spatial coordinates, N^i no longer appears in the formalism at all. Thus no corresponding primary constraints exist. What remains is reparametrization invariance, with the lapse function $N(t)$ an arbitrary function (assuming positivity, boundedness away from 0, differentiability to some order, *etc.* as usual). Calculations are easy, but the resemblance to GR is clear.

For a partial treatment of GR in Hamiltonian form, two standard texts treat the subject (Misner et al., 1973; Wald, 1984) (but watch for opposite conventions in defining the extrinsic curvature). These texts take a short-cut by dropping the primary constraints and treating the lapse function and shift vector as Lagrange multipliers with no conjugate momenta—a procedure that is adequate for getting the field equations of these theories, but does not work for general field theories and that leaves one quite unable to express 4-dimensional gauge transformations in Hamiltonian form. So one turns elsewhere (Sundermeyer, 1982), especially given the recent G-related appreciation of primary constraints. If one sets $\frac{\partial}{\partial x^i}$(anything) $= 0$ (where i=1,2,3), then the Lagrangian density $\mathcal{L}$—now rather a Lagrangian L, simplifies to

$$L = N\sqrt{h}(K^{ij}K_{ij} - K^2), \tag{6}$$

where the extrinsic curvature tensor (a sort of 3-dimensional tensorial 'velocity' in GR) simplifies due to disappearance of the shift vector and its derivatives:

$$K_{ij} = \frac{1}{2N}\dot{h}_{ij},$$

with the dot denoting the coordinate time derivative, h_{ij} the spatial 3-metric, and the lapse function N being related to the space-time metric by $N = \frac{1}{\sqrt{-g^{00}}}$. It is perhaps better to use the slicing density $\alpha = N/\sqrt{h}$ (or even the logarithm thereof) as a canonical coordinate (Anderson and York, 1998), partly to enforce positivity (Goldberg and Klotz, 1973), but N is quite entrenched. The spatial Ricci curvature scalar has disappeared. There being, so to speak, no potential energy, the actual dynamics will be rather dull. That does not matter, however, because all the formal properties that we need will appear. Note that the time derivative of N does not appear in the Lagrangian.

4.2 Generalized Legendre Transformation from L to H

Defining canonical momenta as usual, one has

$$\pi^{ij} =_{def} \frac{\partial L}{\partial h_{ij},_0} = \sqrt{h}(K^{ij} - K h^{ij})$$

and

$$p =_{def} \frac{\partial L}{\partial N,_0} = 0.$$

In going over to the Hamiltonian formalism, one forgets these definitions, recovering the π^{ij} equation as an equation of motion and requiring of p merely that it stay 0. That condition is called a "primary constraint" because it arises at the first stage of the Hamiltonian analysis.

Performing the Legendre transformation to get from L to H (and using $p = 0$ where needed), one gets the canonical Hamiltonian

$$H_c = N\mathcal{H}_0, \tag{7}$$

where $\mathcal{H}_0 = \pi^{ij}\pi^{ab}(h_{ia}h_{bj} - \frac{1}{2}h_{ij}h_{ab})/\sqrt{h}$. (Given that the Hamilton density has turned into the Hamiltonian, one could drop the script font for the Hamiltonian constraint $\mathcal{H}_0$. But to emphasize the link to GR, I will retain it.) Time evolution is generated by the primary Hamiltonian (which adds the primary constraints to the canonical Hamiltonian (Sundermeyer, 1982):

$$H_p = N\mathcal{H}_0 + vp, \tag{8}$$

where $v(t)$ is an arbitrary function, which one readily sees is equated to $\dot{N}$ using the basic Poisson bracket $\{N, p\} = 1$ and calculating $\{N, H_p\}$.[5] It is evident that both constraints are first-class. $\{\mathcal{H}_0, \mathcal{H}_0\} = 0$ because dropping the spatial dependence makes everything difficult about the 'Dirac algebra' disappear and Poisson brackets are anti-symmetric, in contrast to spatially varying GR's $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$ which involves derivatives of Dirac δ functions (Sundermeyer, 1982, p. 239). $\{p, \mathcal{H}_0\} = 0$ because $\mathcal{H}_0$ does not depend on N. Finally, $\{p, p\} = 0$ by antisymmetry and the fact that p is one of the canonical coordinates. Letting ψ be an arbitrary function of h_{ij}, π^{ij}, N, and p, one can calculate the time evolution of ψ:

$$\{\psi, H_p\} = \frac{\partial \psi}{\partial h_{ij}}\frac{\partial H_p}{\partial \pi^{ij}} - \frac{\partial \psi}{\partial \pi^{ij}}\frac{\partial H_p}{\partial h_{ij}} + \frac{\partial \psi}{\partial N}v - \frac{\partial \psi}{\partial p}\cdot 0 \tag{9}$$

using $\mathcal{H}_0 = 0$ after taking the Poisson bracket. (The symbol $\approx$ is often used for equality on a constraint surface, but I will not bother.) It follows that

$$\begin{aligned}
\{h_{ij}, H_p\} = \frac{\partial H_p}{\partial \pi^{ij}} &= \frac{2N}{\sqrt{h}}(h_{ai}h_{bj} - \frac{1}{2}h_{ij}h_{ab})\pi^{ab} = \dot{h}_{ij}, \\
\{\pi^{ij}, H_p\} = -\frac{\partial H_p}{\partial h_{ij}} = -\frac{N}{\sqrt{h}}(2\pi^{ia}\pi^{j}_{a} - \pi\pi^{ij} &- \frac{1}{2}\pi^{ab}\pi_{ab}h^{ij} + \frac{1}{4}\pi^2 h^{ij}) = \dot{\pi}^{ij}, \\
\{N, H_p\} &= \frac{\partial H_p}{\partial p} = v = \dot{N}, \\
\{p, H_p\} &= -\frac{\partial H_p}{\partial N} = -\mathcal{H}_0 = 0 = \dot{p}.
\end{aligned} \tag{10}$$

4.3 A First-Class Constraint Generates a Bad Physical Change

Much as in the electromagnetic case, in GR or its homogeneous reduction a first-class constraint generates not a gauge transformation, but a bad physical change. (One expects such a result to be typical for theories in which primary first-class constraints have descendents, secondary and maybe higher generations of constraints that follow from the first-class primaries.) In the case of electromagnetism (Pitts, 2013b) and full GR (in preparation (Pitts, 2013a)), one way to spot the bad physical change is in the spoilage of the Lagrangian constraint equations, Gauss's law in terms of the electric field (not canonical momenta) or the Gauss-Codazzi relations describing the embedding of space into space-time. Such spoilage of a Lagrangian constraint is a sufficient condition for a change to be bad, but presumably is not necessary.

[5] If one keeps $\dot{N}$ itself in the formalism (Sudarshan and Mukunda, 1974; Castellani, 1982), then one does not suppress $\dot{N}$ *via* the primary constraint $p = 0$, introduce a new function v unrelated to phase space, and then recover $v = \dot{N}$.

What does p do to the $q-\dot{q}$ Hamiltonian constraint, the normal-normal part of Einstein's equations? To answer that question in GR, it is convenient to define a lapse-less factor in the extrinsic curvature tensor: $L_{ij} =_{df} NK_{ij} = \frac{1}{2}(\dot{h}_{ij} - D_i N_j - D_j N_i)$. Thus

$$\{\int d^3y\epsilon(y)p(y), K^{ij}K_{ij} - K^i_i K^j_j - R(x)\} = \int d^3y\epsilon(y)\{p(y), N^{-2}(x)(L^{ij}L_{ij} - L^2(x))\}$$
$$= 2\epsilon(x)N^{-1}(K^{ij}K_{ij} - K^2), \qquad (11)$$

which in full GR is generally nonzero due to the spatial Ricci scalar term $-R$. In the homogeneous truncation, R and the shift vector N^i are absent. But the former contributes nothing anyway, while the latter's contribution (by virtue of being multiplied by a factor involving the lapse) is accounted for in the K_{ij} terms, so the absence of those contributions in the homogeneous truncation does not change the result:

$$\{\int d^3y\epsilon(y)p(y), K^{ij}K_{ij} - K^i_i K^j_j\} = 2\epsilon(x)N^{-1}(K^{ij}K_{ij} - K^2). \qquad (12)$$

In GR, that expression would be generically nonzero (and not a Lie derivative either) because $K^{ij}K_{ij} - K^i_i K^j_j = R$; hence that Poisson bracket would clearly manifest a bad physical change, spoiling the constraint equation that is the time-time part of Einstein's equations. For the homogeneous truncation, the vanishing of the 3-dimensional Ricci scalar R makes that Poisson bracket look not so bad. Sometimes very special cases produce potentially misleading simplifications, so it is best not to forget the guidance from full GR, given that it (not the homogeneous truncation) is ultimately what we seek to understand. For GR, one can show that each of the first-class constraints p, p_i, $\mathcal{H}_0$ and $\mathcal{H}_i$ spoils all of the Lagrangian constraints (Pitts, 2013c; Pitts, 2013a). One expects that the appropriate teaming arrangements (the gauge generators) will cause all the badness to cancel out, which has been show to occur in one case and presumably will occur when the remaining three are completed (Pitts, 2013c; Pitts, 2013a). The calculations are analogous to the electromagnetic case except, of course, much harder. Much as one notices for electromagnetism that the electric field is changed by the primary first-class constraint and *therefore* Gauss's law is violated, spoilage of the Lagrangian constraints is a generally logically weaker consequence of first-class constraint transformation behavior that differs from the known gauge transformations.

One can go on to show that varying the lapse function N in an arbitrary way changes what ought to be physically invariant quantities, namely, Komar's intrinsic Weyl curvature scalar coordinates. (Here one assumes a generic but not completely general situation, namely, the absence of Killing vector fields; see also (Komar, 1955).) The Weyl tensor, the totally traceless part of the Riemann curvature tensor for space-time, is the part of the Riemann tensor not specified by Einstein's equations. Typically it has 4 independent scalar concomitants, two quadratic, two cubic (Bergmann and Komar, 1960; Bergmann, 1962; Pons and Salisbury, 2005), that remain after Einstein's equations (source-free, coupled to electromagnetism, or perhaps more generally) have been imposed. One can of course devise other functions of these (Bergmann and Komar, 1960). (In that sense, any coordinate system is a Weyl scalar coordinate system, a fact that has significant implications.) The Weyl scalars can be written, using Einstein's equations if necessary, in terms of the spatial metric tensor, the extrinsic curvature tensor, and their derivatives. In terms of canonical coordinates, one can re-express these expressions in terms of the canonical momenta π^{ij} instead of the extrinsic curvature K_{ij}. In terms of the 3-metric and its canonical momenta, the Weyl scalars are independent of the lapse and the shift vector. But such replacement uses the Hamilton equations $\frac{\delta H_p}{\delta \pi^{ij}} = \dot{h}_{ij}$, which are not identities. Hence a more primordial form of the Weyl scalars leaves them in terms of h_{ij} and K_{ij}. But K_{ij} is not primitive; it is derived in terms of the definition $K_{ij} = \frac{1}{2N}\dot{h}_{ij} + \ldots$. Hence the Weyl scalars in terms of q and $\dot{q}$ depend on N inversely from how they depend on the lapse-independent quantity $L_{ij} = NK_{ij}$, which depends on $\dot{h}_{ij}$. Hence one could write the Weyl scalars in terms of h_{ij}, L_{ij}, and N in order to make the dependence on q and $\dot{q}$ clearer. But then it is evident that varying N arbitrarily (while leaving $\dot{h}_{ij}$ alone), as the smeared primary constraint p does, will *alter* the Weyl scalar(s)—something that a real gauge transformation could never do. Hence what the primary first-class constraint p generates is a bad physical change, not a gauge transformation.

With that reminder to check the full transformation behavior, not just what happens to the Lagrangian constraints, one can consider the Hamiltonian constraint $\mathcal{H}_0$. One can find the relation between $\mathcal{H}_0$ and space-time coordinate transformations by starting with the gauge generator G (Castellani, 1982) and throwing away some terms to isolate $\mathcal{H}_0$. As will appear below, G has a bunch of terms involving the primary constraints, the lapse and shift, and (in some cases) the spatial 3-metric (Castellani, 1982); these will not affect the 3-metric h_{ij}, which only sees it own conjugate momentum due to the structure of the Poisson bracket. Homogeneous truncation also eliminates $\mathcal{H}_i$. Thus $\{h_{ij}(x), G\} = \{h_{ij}(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\}$, where $\epsilon^{\perp}$ is the normal projection of the 4-vector ξ^{μ} describing an infinitesimal coordinate transformation, $\epsilon^{\perp} = N\xi^0$. One has

$$\{h_{ij}(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\} = \delta^{\mu}_i \delta^{\nu}_j \pounds_{(\epsilon^{\perp} n^{\alpha})} g_{\mu\nu}(x).$$

That looks good for getting a coordinate transformation out of $\mathcal{H}_0$, but then $\{N(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\} = 0$ shows that part of the remainder of the space-time Lie derivative formula is violated: the lapse N certainly changes under a change of time coordinate, but $\mathcal{H}_0$ fails to effect such a change. Thus $\mathcal{H}_0$ does not generate a coordinate transformation either. As with electromagnetism, a first-class constraint, primary or secondary, generates a bad physical change. Only the special combination G generates a gauge transformation.

4.4 Change from Hamilton's Equations

Change is (real) time dependence. There is a risk of fake change in GR by a funny time choice or a funny labeling of space over time. Hence one needs a savvy definition: there is real change if and only if, for *all* choices of coordinates, there is time dependence. In reparametrization-invariant theories, one can try to generate fake change by speeding up (bunching up) or slowing down (spreading out) the labeling of the time slices. In GR the possibilities for fake change are much more varied due to many-fingered time and to the possibility of letting spatial coordinates slide around over time *via* the shift vector. But I'll continue with the toy theory that is simple, but not too simple. Fake change is then apparent change that exists only for some choices of time coordinate. Note that the question is whether *something or other* depends on time; it isn't obviously required that some single quantity depend on time in all coordinate systems.

If there is real change, then there is time dependence for all choices of labeling of the time slices in that interval: $\forall$ labelings ($\dot{h}_{ij} \neq 0 \vee \dot{\pi}^{ij} \neq 0 \vee \dot{N} \neq 0$). (If there is no real change, then $\exists$ labeling such that $\dot{h}_{ij} = 0 \wedge \dot{\pi}^{ij} = 0 \wedge \dot{N} = 0$.) It would be nice if one didn't have to worry about $\dot{N}$, a rather slippery quantity that is determined by conventional gauge (coordinate) choice: it vanishes for some time coordinates but not others. Thus $\dot{N} \neq 0$ doesn't hold reliably; but can $\dot{N}$ team up with $\dot{h}_{ij}$ or $\dot{\pi}^{ij}$ so that at least one of them is nonzero for any time coordinate? Not usefully: $\dot{h}_{ij}$, π^{ij}, and $\dot{\pi}^{ij}$ are all scalar densities (of whatever weights) or scalars under change of time coordinate. Thus if they vanish, they do so invariantly; if they fail to vanish, they do so invariantly. So they can't use help from $\dot{N}$. If there is a time coordinate such that $\dot{h}_{ij} = 0$ and $\dot{\pi}^{ij} = 0$, there is a time coordinate that also yields $\dot{N} = 0$ because $\dot{N}$ has an affine rather than strictly linear time coordinate transformation law. It therefore cannot bear the load of being the locus of real change, and indeed cannot even help to bear that load. So the quantifier over time coordinates (labelings of the slices) can be dropped, as can the disjunct $\dot{N} \neq 0$. Thus there is a nice

Result: There is real change in the homogeneous vacuum theory if and only if $\dot{h}_{ij} \neq 0 \vee \dot{\pi}^{ij} \neq 0$ in some coordinate system (and hence in all coordinate systems).

One could simplify a bit further using Hamilton's equations for the toy theory, but there is no need to do so and some of the simplifications would not carry over to GR.

4.5 Change from Differential Geometry: No Time-like Killing Vector

Now let us ascertain the conditions for the non-existence or existence of a time-like Killing vector, or the relevant analog thereof, and see how it lines up with the time evolution generated by H_p. We need the $3+1$ split of the Lie derivative formula for an infinitesimal coordinate transformation $\pounds_{\xi} g_{\mu\nu} = \xi^{\alpha} g_{\mu\nu},_{\alpha} + g_{\mu\alpha}\xi^{\alpha},_{\nu} + g_{\alpha\mu}\xi^{\alpha},_{\mu}$ (Mukhanov and Wipf, 1995). Actually we need only the x^i-independent homogeneous truncation of the $3+1$ split, which is much cleaner. One has, besides rigid affine spatial coordinate transformations (which would be important if we cared about counting degrees of freedom (Ryan and Shepley, 1975; Ashtekar and Samuel, 1991; Pons and Shepley, 1998), but which I ignore),

$$\delta N = \xi^0 \dot{N} + N \dot{\xi^0},$$

$$\delta h_{ij} = \xi^0 \dot{h}_{ij}.$$

The formula for the variation of the momenta $\delta\pi^{ij}$ is derived using $K_{ij} = N\Gamma^0_{ij}$ and Hamilton's equations (Pons et al., 2000; Thiemann, 2007). The result simplifies enormously for the homogeneous toy theory: $\delta\pi^{ij} = \xi^0 \dot{\pi}^{ij}$. One also needs p to stay 0 somehow. One can show that p is a weight -1 scalar density under change of time coordinate, so $\delta p = \pounds_\xi p = \xi^0 \dot{p} - \dot{\xi^0} p$, which will stay 0 using both the constraints.

Hence the condition for the existence of a time-like Killing vector field is equivalent (using Hamilton's equations as needed) to

$$(\exists \xi^0)(\xi^0 \dot{N} + N\dot{\xi^0} = 0 \wedge \xi^0 \dot{h}_{ij} = 0 \wedge \xi^0 \dot{\pi}^{ij} = 0). \tag{13}$$

Thus the condition for change is just the negation of this triple conjunction. Below some logical manipulations will be performed from this point.

4.6 G Generates Lie Differentiation On-shell in Hamiltonian Formalism

The gauge generator G by design ought to give these same formulas just obtained from 4-dimensional differential geometry, at least for solutions of all the Hamiltonian equations and constraints. G is a bit complicated in GR (Castellani, 1982; Pons et al., 2000): it is a specific sum of the secondary constraints $\mathcal{H}_0$ and $\mathcal{H}_i$ *and* the primary constraints p and p_i with coefficients savoring of Lie differentiation. While it will not be necessary here to use the whole expression, it might be useful to have it anyway. Making sure that G and the changes that it generates live in phase space requires eliminating $\dot{N}$ by taking not the time component ξ^0, but a $3+1$ projected relative $\epsilon^\perp = N\xi^0$ as primitive and hence having vanishing Poisson brackets with everything (Castellani, 1982; Pons et al., 2000). (My notation, opposite to Pons, Salisbury and Shepley's but partly in line with Castellani's, uses ξ^μ for the coordinate basis components of the vector ξ and lets ϵ be the $3+1$ projected descriptors.)

It is most readily given in two parts, one for gauge transformations that do not preserve the simultaneity hypersurface, and one for spatial coordinate transformations. Note that I am not worrying about spatial boundary terms (on which the two sources above disagree), partly because there is no point in trying to run before walking. Castellani's notation also tends to hide dependence on the spatial metric, which is made explicit here. The normal gauge generator is

$$G[\epsilon, \dot{\epsilon}] = \int d^3x [\epsilon^\perp \mathcal{H}_0 + \epsilon^\perp p_j h^{ij} N,_i + \epsilon^\perp (N p_i h^{ij}),_j + \epsilon^\perp (p N^j),_j + \dot{\epsilon}^\perp p]. \tag{14}$$

It generates on phase space-time a transformation that, for solutions of the Hamiltonian field equations, changes the time coordinate in line with 4-dimensional tensor calculus. The spatial gauge generator is

$$G[\epsilon^i, \dot{\epsilon}^i] = \int d^3x [\epsilon^i \mathcal{H}_i + \epsilon^i N^j,_i p_j - \epsilon^j,_i N^i p_j + \epsilon^i N,_i p + \epsilon^i,_0 p_i]. \tag{15}$$

This spatially projected descriptor ϵ^i, on account of the shift vector that allows the spatial coordinates to slide around over time, is not simply the spatial components ξ^i of ξ. Instead it is given by $\epsilon^i = \xi^i + N^i \xi^0$. It generates spatial coordinate transformations that respect the simultaneity hypersurfaces.

Roughly speaking, the coefficients of the primary constraints are those needed to transform the lapse and shift in such a way as to fill the holes left by the secondary constraints (*c.f.* the common error that the secondary constraints by themselves do generate coordinate transformations). For spatial coordinate transformations, which have the virtues of being defined even without using the equations of motion and manifestly relating to fragments of tensor calculus, one has the good news (Sundermeyer, 1982, p. 241)

$$\{h_{ij}(x), \int d^3y \xi^k(y) \mathcal{H}_k(y)\} = \pounds_\xi h_{ij}(x),$$

$$\{\pi^{ij}(x), \int d^3y \xi^k(y) \mathcal{H}_k(y)\} = \pounds_\xi \pi^{ij}(x) \tag{16}$$

that might tempt one to think that a coordinate transformation is being made, but also the bad news

$$\{\mathcal{H}_i(x), N^j(y)\} = 0,$$
$$\{\mathcal{H}_i(x), N(y)\} = 0 \tag{17}$$

that shows that no coordinate transformation is made unless quite specific assistance from the primary constraints p_i and even p is brought in. A similar story holds for H_0, which is loosely tied to both time evolution and changes of time coordinate, but by itself generates neither one.

In our x^i-independent homogeneous truncation, G simplifies nicely. One can ignore the spatial gauge generator. Throwing away spatial dependence, besides discarding the momentum constraint $\mathcal{H}_i = 0$ and the primary constraints $p_i = 0$ tying down the momenta conjugate to the shift vector, also annihilates the structure constant $C^0_{00} = 0$ for the Hamiltonian constraint with itself. One has for the spatially truncated normal gauge generator

$$G = \epsilon^{\perp}\mathcal{H}_0 + p\dot{\epsilon}^{\perp}. \tag{18}$$

One finds that G has the following Poisson brackets with the basic canonical variables:

$$\begin{aligned} \{h_{ij}, G\} &= \xi^0 N \frac{2}{\sqrt{h}}(h_{ai}h_{bj} - \frac{1}{2}h_{ij}h_{ab})\pi^{ab} = \xi^0\{h_{ij}, H_p\}, \\ \{\pi^{ij}, G\} &= -\xi^0 \frac{N}{\sqrt{h}}(2\pi^{ia}\pi^j_a - \pi\pi^{ij} - \frac{1}{2}\pi^{ab}\pi_{ab}h^{ij} + \frac{1}{4}\pi^2 h^{ij}) = \xi^0\{\pi^{ij}, H_p\}, \\ \{N, G\} &= \dot{\epsilon^{\perp}} = \xi^0\dot{N} + \dot{\xi^0}N, \\ \{p, G\} &= 0 \end{aligned} \tag{19}$$

identically. Using Hamilton's equations, one has "on-shell"

$$\begin{aligned} \{h_{ij}, G\} &= \xi^0 \dot{h}_{ij}, \\ \{\pi^{ij}, G\} &= \xi^0 \dot{\pi}^{ij}, \\ \{N, G\} &= \xi^0\dot{N} + \dot{\xi^0}N, \\ \{p, G\} &= 0. \end{aligned} \tag{20}$$

One sees that these equations match the Lie derivative formulas $\delta N = \xi^0\dot{N} + N\dot{\xi^0}$ (N being a weight 1 scalar density under change of time coordinate and thus having this Lie derivative (Anderson, 1967)), $\delta h_{ij} = \xi^0\dot{h}_{ij}$, and $\delta\pi^{ij} = \xi^0\dot{\pi}^{ij}$, while $\delta p = 0$ is also fine even without fitting the weight -1 density character of p, so the gauge generator G deserves its name. The Hamiltonian formalism implements 4-dimensional coordinate transformations at least on solutions of the Hamilton equations (Thiemann, 2007); here the one-dimensional temporal analog is explicit and convenient.

If and only if there exists a (time-like) Killing vector 'field' ξ^μ, the Poisson brackets of the canonical variables with G should all vanish (in all time coordinates):

$$\begin{aligned} \{h_{ij}, G\} &= \xi^0 \dot{h}_{ij} = 0, \\ \{\pi^{ij}, G\} &= \xi^0 \dot{\pi}^{ij} = 0, \\ \{N, G\} &= \xi^0\dot{N} + \dot{\xi^0}N = 0, \\ \{p, G\} &= 0. \end{aligned} \tag{21}$$

This is just the Killing vector condition

$$(\exists\xi^0)(\xi^0\dot{N} + N\dot{\xi^0} = 0 \wedge \xi^0\dot{h}_{ij} = 0 \wedge \xi^0\dot{\pi}^{ij} = 0) \tag{22}$$

from the previous subsection, along with a suitable claim about the primary constraint (the boring canonical momentum that is always 0). Change is related to a *lack* of a time-like Killing vector field, so let us negate. The primary constraint p has to remain 0 no matter what. The Killing vector condition, being tensorial, holds in all coordinate systems or fails in all of them, so there is no need to quantify over labelings (time coordinates). Thus the lack of a Killing vector is

$$(\forall\xi^0)(\xi^0\dot{h}_{ij} \neq 0 \quad \vee \quad \xi^0\dot{\pi}^{ij} \neq 0 \quad \vee \quad \xi^0\dot{N} + \dot{\xi^0}N \neq 0).$$

Vanishing ξ^0 would not count as Killing (or time-like), so the no-Killing condition is

$$(\forall \xi^0)(\dot{h}_{ij} \neq 0 \quad \vee \quad \dot{\pi}^{ij} \neq 0 \quad \vee \quad (\xi^0 N),_0 \neq 0).$$

All three disjuncts are scalar densities of some weight or other, so their vanishings or not are invariant—recall that no quantification over time labelings was needed. There is always some ξ^0 that can make $(\xi^0 N),_0 = 0$; $\xi^0 = N^{-1}$ or some constant multiple thereof will do. Because the last disjunct $(\xi^0 N),_0 \neq 0$ is unreliable and the other two don't depend on ξ^0, the quantification over ξ^0 can also be dropped. Hence one has another

Result: the non-existence of a time-like Killing vector field is equivalent to $\dot{h}_{ij} \neq 0 \vee \dot{\pi}^{ij} \neq 0$.

That condition holds in all (time) coordinates if it holds in any.

This condition is exactly the one found above using the time evolution generated by H_p and asking it to be nonzero in all coordinate systems. Thus the primary Hamiltonian's time evolution gives exactly the same result as 4-dimensional differential geometric Lie differentiation. The primary Hamiltonian is thus vindicated out of the right choice out of the six or more candidates—no surprise in light of the known equivalence of the primary Hamiltonian to the Lagrangian.

The Lie derivative formula, in turn, is implemented in the Hamiltonian formalism (at least on-shell) using the gauge generator G, as was already known (Castellani, 1982). There is real change just in case there is no time-like Killing vector field. Expecting the Hamiltonian formalism to match the unproblematic Lagrangian/differential geometric formalism has resolved the problem in terms of the role of a time-like Killing vector field, as promised in the title. This agreement makes change in classical canonical GR, or at any rate in the toy theory, luminously clear and satisfying. Working in full GR would add messy terms that tend to obscure the point. GR adds the further issue of many-fingered time. I expect that an analysis of this sort would work fine even for GR.

Defining change in terms of the lack of a time-like Killing vector field provides an attractive way to remain non-committal regarding a choice between, for example, "intrinsic time" involving h or "extrinsic time" involving $\pi^{ij} h_{ij}$, at least classically. The expressions involving quantifiers and conjunctions ($\wedge$, and) or disjunctions ($\vee$, or) give a tensorial statement that h_{ij} or π^{ij} (or both) depends on time t. It might well be the case that a time could be dug out of h_{ij} in some regions but not others, but that π^{ij} can fill the gaps; a recollapsing Big Bang model at the moment the expansion stops and reverses is a familiar example. Of course it would be ideal not to have to choose at all. But if one must choose, the equivalence of the expressions using quantifiers and conjunctions or disjunctions to the (negation of the) tensorial Killing equation implies that there is always something that could be chosen as time, and that the baton is smoothly passed back and forth as needed (classically).

5 Change with Matter: A Massive Scalar Field

5.1 Change in Hamiltonian Formalism from H_p

Thus far change has been sorted out for (a homogeneous truncation of) vacuum General Relativity, but not for (a homogeneous truncation of) General Relativity with sources. To address the latter question, one can introduce a scalar field ϕ, for generality a massive scalar field. Recalling that change is ineliminable time dependence, there is no change in a space-time region R just in case there exists everywhere in R a time-like vector field ξ^μ that is 'Killing' in a generalized sense for both the metric and the matter:

$$\begin{aligned}(\forall p \in R)(\exists \xi^\mu)(\pounds_\xi g_{\mu\nu} = 0 \wedge \\ \pounds_\xi \phi = 0 \wedge \\ g_{\mu\nu}\xi^\mu \xi^\nu < 0).\end{aligned} \tag{23}$$

What does this condition come to in terms of a $3+1$ Hamiltonian formalism? It will give a Hamiltonian definition of change, one potentially involving the matter as a change-bearer, not just gravity. As usual, throwing away spatial dependence will simplify matters.

The starting Lagrangian density is now (Sundermeyer, 1982)

$$\mathcal{L} = \mathcal{L}_{GR} - \frac{1}{2}\phi,_{\mu}\phi,_{\nu} g^{\mu\nu}\sqrt{-g} - \frac{m^2}{2}\phi^2\sqrt{-g}. \tag{24}$$

Making the 3+1 ADM split and discarding spatial dependence, one has

$$L = N\sqrt{h}(K^{ij}K_{ij} - K^2) + \frac{\sqrt{h}}{2N}\dot{\phi}^2 - \frac{m^2}{2}N\sqrt{h}m^2\phi^2. \tag{25}$$

The canonical momentum π^{ij} for gravity is as before, while the new canonical momentum for the scalar field ϕ is

$$\pi_\phi = \frac{\partial L}{\partial\dot{\phi}} = \frac{\sqrt{h}}{N}\dot{\phi}, \tag{26}$$

which is trivially inverted. The primary Hamiltonian is

$$H_p = N(\mathcal{H}_0 + \mathcal{H}_{0\phi}) + vp = N\mathcal{H}_0 + N\left(\frac{\pi_\phi^2}{2\sqrt{h}} + \frac{m^2\sqrt{h}\phi^2}{2}\right) + vp. \tag{27}$$

Note that a massless scalar field ($m = 0$) would behave as a free particle, making ϕ evolve monotonically and hence be a pretty good clock, but a massive scalar field behaves as a harmonic oscillator, with the ϕ and its momentum π_ϕ oscillating. Hence a massive scalar field avoids unrealistic simplicity and thus is more representative of other matter fields and even what happens in spatially inhomogeneous contexts than is a massless scalar field.

One can now find Hamilton's equations.

$$\{h_{ij}, H_p\} = \frac{\partial H_p}{\partial\pi^{ij}} = \frac{2N}{\sqrt{h}}(h_{ai}h_{bj} - \frac{1}{2}h_{ij}h_{ab})\pi^{ab} = \dot{h}_{ij}$$

as before.

$$\{\pi^{ij}, H_p\} = -\frac{\partial H_p}{\partial h_{ij}} = -\frac{N}{\sqrt{h}}(2\pi^{ia}\pi^j_a - \pi\pi^{ij} - \frac{1}{2}\pi^{ab}\pi_{ab}h^{ij} + \frac{1}{4}\pi^2 h^{ij}) + \frac{N}{4\sqrt{h}}\left(\pi_\phi^2 h^{ij} - m^2 h h^{ij}\phi^2\right) = \dot{\pi}^{ij}.$$

$\{N, H_p\} = \frac{\partial H_p}{\partial p} = v = \dot{N}$ as before. $\{p, H_p\} = -\frac{\partial H_p}{\partial N} = -\mathcal{H}_0 - \mathcal{H}_{0\phi} = 0 = \dot{p}$, sprouting a contribution from ϕ. The novel equations are $\{\phi, H_p\} = \frac{N\pi_\phi}{\sqrt{h}} = \dot{\phi}$, which inverts the Legendre transformation back from q, p to q, $\dot{q}$ for matter ϕ, and

$$\{\pi_\phi, H_p\} = -m^2 N\sqrt{h}\phi = \dot{\pi}_\phi, \tag{28}$$

which gives the interesting part of the dynamics of the massive scalar field.

There is real change if and only if something depends on time for every choice of time coordinate (labeling):

$$(\forall labeling)(\dot{h}_{ij} \neq 0 \vee \dot{\pi}^{ij} \neq 0 \vee \dot{N} \neq 0 \vee \dot{\phi} \neq 0 \vee \dot{\pi}_\phi \neq 0).$$

One can always set $\dot{N}$ to 0, because its transformation law is affine, but $\dot{h}_{ij}$, $\dot{\pi}^{ij}$, $\dot{\phi}$, and $\dot{\pi}_\phi$ are all scalar densities, vanishing or not invariantly. Hence one can drop both the $\dot{N}$ disjunct and the quantifier $\forall$ over time labelings. Thus there is change iff

$$\dot{h}_{ij} \neq 0 \vee \dot{\pi}^{ij} \neq 0 \vee \dot{\phi} \neq 0 \vee \dot{\pi}_\phi \neq 0. \tag{29}$$

Thus the burden of bearing change can be shared among these four quantities. One can, for example, pass the baton around as needed among h_{ij}, π^{ij}, ϕ and π_ϕ.

5.2 Differential Geometric Change: No Generalized Killing Vector

From the standpoint of differential geometry, a solution of Einstein's equations with a scalar field should be regarded as changeless (a generalization of stationarity) just in case there is no time-like vector field ξ^μ satisfying the generalized Killing condition

$$\pounds_\xi g_{\mu\nu} = 0 \wedge \pounds_\xi \phi = 0. \tag{30}$$

Neither gravity nor matter changes, and nothing else is present, so nothing changes. In principle this pair of equations might be redundant if the equations are not independent. Can matter change without making gravity change also? That is of no concern for present purposes, partly because such a result might be model-dependent: it might depend on what types and numbers of fields are used as matter. What is of interest is not special features of a massive scalar field, but features likely to be representative of a broad class of matter sources in GR.

As before, a $3+1$ split, followed by throwing away spatial dependence, is useful. The nontrivial new part is that we will need something like '$\pounds_\xi \pi_\phi$.' From the experience with the vacuum case, we expect to need the relation $\pi_\phi = \frac{\sqrt{h}}{N}\dot{\phi}$, which holds due to $\{\phi, H_p\} = \frac{N\pi_\phi}{\sqrt{h}} = \dot{\phi}$. Under time relabelings, $\sqrt{h}$ is unmoved, whereas $N' = N\left|\frac{\partial x^0}{\partial x^{0\prime}}\right|$ (weight 1 and not flipping signs under time reversal) and $\dot{\phi}' = \dot{\phi}\frac{\partial x^0}{\partial x^{0\prime}}$ (also weight 1 but flipping signs under time-reversal). Hence π_ϕ is a scalar under time coordinate transformations not involving reversal (good enough for Lie differentiation), yielding

$$\pounds_\xi \left(\frac{\sqrt{h}}{N}\dot{\phi}\right) = \xi^0 \frac{\partial}{\partial t}\left(\frac{\sqrt{h}}{N}\dot{\phi}\right). \tag{31}$$

Using Hamilton's equations that last expression equals $\xi^0 \frac{\partial}{\partial t}\pi_\phi$.

Now we can write the Hamiltonian version of the Killing-like condition for no change:

$$(\exists \xi^0)(\xi^0 \dot{h}_{ij} = 0 \quad \wedge \quad \xi^0 \dot{\pi}^{ij} = 0 \quad \wedge \quad \xi^0 \dot{N} + \dot{\xi^0} N = 0 \quad \wedge \quad \xi^0 \dot{\phi} = 0 \quad \wedge \quad \xi^0 \dot{\pi}_\phi = 0).$$

(Naturally p needs to stay 0 also.) Negating to find the condition for *change*, one has

$$\begin{aligned}
&\neg(\exists \xi^0)(\xi^0 \dot{h}_{ij} = 0 \quad \wedge \quad \xi^0 \dot{\pi}^{ij} = 0 \quad \wedge \quad \xi^0 \dot{N} + \dot{\xi^0} N = 0 \quad \wedge \quad \xi^0 \dot{\phi} = 0 \quad \wedge \quad \xi^0 \dot{\pi}_\phi = 0) \quad \leftrightarrow \\
&(\forall \xi^0) \quad \neg(\xi^0 \dot{h}_{ij} = 0 \quad \wedge \quad \xi^0 \dot{\pi}^{ij} = 0 \quad \wedge \quad \xi^0 \dot{N} + \dot{\xi^0} N = 0 \quad \wedge \quad \xi^0 \dot{\phi} = 0 \quad \wedge \quad \xi^0 \dot{\pi}_\phi = 0) \quad \leftrightarrow \\
&\quad (\forall \xi^0) \quad (\xi^0 \dot{h}_{ij} \neq 0 \quad \vee \quad \xi^0 \dot{\pi}^{ij} \neq 0 \quad \vee \quad \xi^0 \dot{N} + \dot{\xi^0} N \neq 0 \quad \vee \quad \xi^0 \dot{\phi} \neq 0 \quad \vee \quad \xi^0 \dot{\pi}_\phi \neq 0).
\end{aligned} \tag{32}$$

The condition $(\xi^0 N),_0 \neq 0$ is completely unreliable (strongly coordinate-dependent), whereas the other four disjuncts are all invariant. Hence the disjunct $(\xi^0 N),_0 \neq 0$ can be dropped. Vanishing ξ^0 would not count as Killing, so that factor can be dropped. Now nothing depends on ξ^0, so the quantification over it can be dropped. Thus the generalized no-Killing condition for change is

$$\dot{h}_{ij} \neq 0 \quad \vee \quad \dot{\pi}^{ij} \neq 0 \quad \vee \quad \dot{\phi} \neq 0 \quad \vee \quad \dot{\pi}_\phi \neq 0. \tag{33}$$

This is exactly what was derived above from the Hamiltonian time evolution from H_p. Hence the generalized (no) Killing definition of change agrees with the Hamiltonian definition, as advertised in the title.

5.3 Lie Derivative from Gauge Generator with Matter Field

One also wants to be able to implement coordinate transformations in the Hamiltonian formalism, at least using the Hamiltonian equations of motion and constraints as needed. With a matter field ϕ present, one needs to think about what the new gauge generator G is. Fortunately one needn't think for long (*c.f.* (Castellani, 1982)). There are no new constraints for this very simple matter theory—a feature that wouldn't hold for electromagnetism as the source for gravity, for example. The gauge generator depends on the primary constraint and the secondary constraint. The primary is just as before. The secondary constraint gets a new term, but, crucially, the Poisson bracket 'algebra' is unchanged (Sundermeyer, 1982). Hence one only needs to introduce the modified secondary constraint $\mathcal{H}_0 + \mathcal{H}_{0\phi}$ in place of $\mathcal{H}_0$. The result is

$$G = \epsilon^\perp(\mathcal{H}_0 + \mathcal{H}_{0\phi}) + p\dot{\epsilon}^\perp. \tag{34}$$

One finds that the matter-inclusive gauge generator G has the following Poisson brackets with the canonical variables (re-expressed using ξ^0 once outside the Poisson bracket):

$$\begin{aligned}
\{h_{ij}, G\} &= \xi^0\{h_{ij}, H_p\}, \\
\{\pi^{ij}, G\} &= \xi^0\{\pi^{ij}, H_p\}, \\
\{N, G\} &= \xi^0 \dot{N} + \dot{\xi^0} N, \\
\{p, G\} &= 0
\end{aligned} \tag{35}$$

as before, and now also

$$\begin{aligned}\{\phi, G\} &= \xi^0 \{\phi, H_p\},\\ \{\pi_\phi, G\} &= \xi^0 \{\pi_\phi, H_p\}.\end{aligned} \tag{36}$$

Using Hamilton's equations, one has

$$\begin{aligned}\{h_{ij}, G\} &= \xi^0 \dot{h}_{ij},\\ \{\pi^{ij}, G\} &= \xi^0 \dot{\pi}^{ij},\\ \{N, G\} &= \xi^0 \dot{N} + \dot{\xi}^0 N,\\ \{p, G\} &= 0,\\ \{\phi, G\} &= \xi^0 \dot{\phi},\\ \{\pi_\phi, G\} &= \xi^0 \dot{\pi}_\phi.\end{aligned} \tag{37}$$

Excluding p, which at least stays 0, all equations match the Lie derivative formulas appropriate given the transformation properties of the quantities under small change of time coordinate (respectively, scalar, scalar, weight 1 density, weight -1 density for p, scalar, and scalar). Hence with a massive scalar field present, the gauge generator G does what is expected in this homogeneously truncated stub of General Relativity. The Hamiltonian definition of change, in addition to being equivalent to the differential geometric definition of change, can be expressed in the differential geometric way using distinctively Hamiltonian resources, which is quite satisfying.

5.4 Restoring Space and Including More General Matter

To complete the analysis for GR, one would need, of course, to restore spatial dependence. One would also want to include more general matter fields, including electromagnetism (which introduces first-class constraints unrelated to GR's), Yang-Mills fields (which add to electromagnetism the complexity of a gauge-covariant rather than gauge-invariant field strength and so more obviously require reckoning with the gauge freedom of the potentials), massive Proca electromagnetism (which has second-class rather than first-class constraints, with nontrivial implications for the gauge generator), and spinor fields, concerning which the ideas of gauge symmetry and Lie differentiation have a large and conflicted literature (Pitts, 2012). To pursue all such matters now would start to submerge the conceptual points about change in GR amidst a longer and more technical discussion. I hope to address such matters on another occasion.

6 Comparison of Hamiltonian and Gauge Generator

By now it is clear that Hamiltonian GR has real change and that it is just where one should have expected in on Lagrangian/differential geometric grounds. Thus time evolution is not just a gauge transformation. The Hamiltonian H_p takes a moment of time and predicts the future (and retrodicts the past) from it (3-d to 4-d), whereas the gauge generator G takes a temporally extended trajectory (satisfying Hamilton's equations) into a qualitatively identical (physically the same) trajectory with different coordinate labels (4-d to 4-d). The Hamiltonian is like a large machine that builds a road. The gauge generator is like a golf cart-sized machine that restripes a road.

Still one might feel a bit puzzled by the similarity of the primary Hamiltonian $H_p = N(\mathcal{H}_0 + \mathcal{H}_{0\phi}) + vp$ generating time evolution and the gauge generator $G = \epsilon^\perp(\mathcal{H}_0 + \mathcal{H}_{0\phi}) + \dot{\epsilon}^\perp p$ that generates something like time evolution (to be cancelled off using H_p) with a few extra bits savoring of a coordinate transformation. Why do H_p and G (and their Poisson bracket actions) look so similar, especially for rigid time translation $\xi^0 = 1$? Part of the problem is that one can only implement coordinate transformations for solutions, due to the quadratic form of the Hamiltonian constraint (Mukhanov and Wipf, 1995). Thus one gets a lot of terms that catch the eye and yet vanish using Hamilton's equations, along with terms that look insignificant and yet are tensorial (Lie derivative terms) and typically nonzero. Recall the equations (35,36) from above for the Poisson brackets of the gauge generator G with the fundamental canonical variables (re-expressed using ξ^0 outside the brackets to make clearer the tie to differential geometry).

Besides the widely shared factor of ξ^0, the two odd-balls in comparison to Hamilton's equations are $\dot{\xi}^0 N$ in $\{N, G\}$ and the *identical* vanishing of $\{p, G\}$ (*c.f.* $\{p, H_p\} = -(\mathcal{H}_0 + \mathcal{H}_{0\phi})$, which vanishes on the constraint surface). The odd-ball term $\dot{\xi}^0 N$ and the factor of ξ^0 elsewhere disappear if one chooses $\xi^0 = 1$. But one can always choose a time-like vector field to take that form by a suitable coordinate choice (Bergmann, 1958; Pitts, 2006). So does a 'rigid' time coordinate translation—that is, a change of coordinates described by an arbitrary time-like vector field (in some neighborhood), specialized to its own adapted coordinate system—swallow up time evolution? The tensorial argument above disproves that conclusion, but one still might feel puzzled by the close resemblance.

Viewed from the right angle, this similarity is in fact just what one should expect intuitively. (This discussion resembles Pons, Salisbury and Sundermeyer's (Pons et al., 2010) on some points.) Evolving from the start of 2013 to the start of 2014 leaves one in the same state of the world's history, the same space-time slice as it were, as setting one's calendar back a year (changing coordinates by $t' = t - 1\ yr$) and picking the moment with $t' =$ start of 2013 (which is $t =$ start of 2014). This is just what a passive infinitesimal coordinate transformation does—except for that for expository clarity I used a year rather than an infinitesimal interval. (Doing a finite translation honestly requires exponentiation.) At least, that is what the more famous of the two kinds of infinitesimal coordinate transformations does (Bergmann, 1962). The difference depends on whether one uses the new or the old coordinate system to express the result. The more physically natural coordinate transformation, as in the tensor transformation law, stays at the same space-time point. But the more mathematically convenient coordinate transformation, at least infinitesimally, involves comparing two different space-time world points, but with the same coordinate values in different coordinate systems. That's an odd comparison to make, but it tends to give tensorial answers. It is easy to see how forgetting what this transformation means could lead to odd conclusions, as in the literature on observables.

The main physical difference between evolving from the start of 2013 to the start of 2014 (what H_p does) on the one hand, and setting one's calendar back a year and then looking at a moment labeled as the start of 2013 in the primed coordinate system (what $G_{\xi=1yr}$ does), is one of viewpoint. Evolving from now assumes that the future isn't already there (3-dimensional), whereas setting the calendar back presupposes it is already there (4-dimensional). That metaphysical difference is not visible in physics.

A second difference is that time evolution leads to calling that future state the start of 2014, whereas the gauge generator (using Hamilton's equations) calls that same state of the world the start of 2013—perhaps one should say the start of 2013′ to emphasize that the reappearance of the label "the start of 2013" applies in a different coordinate system, that is, on a different calendar. In this sense the Hamiltonian demonstrably is *not* a special case of the gauge generator.

6.1 Geometric Objects and Space-time Point Individuation

The issue is most clearly seen by recalling the classical definition of a geometric object as given by Trautman, which labels field components in terms of *both* physical points with names like p *and* coordinate labels like x^μ or $x^{\mu\prime}$:

> Let X be an n-dimensional differentiable manifold.... [I]t is convenient to define... the geometric object, which includes nearly all the entities needed in geometry and physics, so that definitions and theorems can be given in terms of geometric objects so as to hold for all the more specialized cases that we may require.
>
> Let $p \in X$ be an arbitrary point of X and let $\{x^a\}, \{x^{a'}\}$ be two systems of local coordinates around p. A geometric object field y is a correspondence
>
> $$y : (p, \{x^a\}) \to (y_1, y_2, \cdots y_N) \in R^N$$
>
> which associates with every point $p \in X$ and every system of local coordinates $\{x^a\}$ around p, a set of N real numbers, together with a rule which determines $(y_{1'}, \cdots y_{N'})$, given by
>
> $$y : (p, \{x^{a'}\}) \to (y_{1'}, \cdots y_{N'}) \in R^N$$
>
> in terms of the $(y_1, y_2, \cdots y_N)$ and the values of [*sic*] p of the functions and their partial derivatives which relate the coordinate systems $\{x^a\}$ and $\{x^{a'}\}$.... The N numbers $(y_1, \cdots y_N)$

> are called the <u>components</u> of y at p with respect to the coordinates $\{x^a\}$. (Trautman, 1965, pp. 84, 85)

In contrast to very old-fashioned tensor calculus (*e.g.*, (Landau and Lifshitz, 1975)) that uses only coordinate labels as arguments, Trautman's definition gives invariant names like p to individuals, space-time points, which are individuated by virtue of their physical properties. For example, a firecracker explodes at p. To pick out the value of a scalar field ϕ at p, one might use five scalar fields—the other four besides ϕ being 'used up' in picking out p.

Modern 'coordinate-free' geometry, by contrast, individuates points by mathematical fiat with names like p; but most of this individuation is physically superfluous and gets effectively stripped away by taking equivalence classes for physical individuation. On the modern view, the firecracker might explode at various mathematically distinct points, but these mathematical points are declared physically identical. Thus mathematical identity differs from physical identity. Trautman's definition, reflecting the maturity of classical differential geometry, is neither tardy in awarding names like p as the older classical style was, nor hasty in doing so like the modern style. Instead it reflects the lessons that Einstein learned from his point-coincidence argument (Einstein, 1923, pp. 117, 118), which he used to escape his hole argument (Howard and Norton, 1993; Norton, 1987). For a time Einstein took his hole argument to threaten determinism in generally covariant theories, which he consequently rejected before devising his point-coincidence argument. While of course one should follow the modern understanding that the hole argument was intended to be read actively (which was denied prior to the 1980s), I take the point-coincidence argument to make plausible that if one tries in GR to make an active diffeomorphism, the dependence of physical point identities on what happens there implies that instead the physical points go with the fields and the attempted active diffeomorphism instead does nothing at all. This view bears some resemblance to Stachel's (Stachel, 1993), but my remedy is simpler because it is applied earlier. What remains are passive coordinate transformations, which are perspicuously handled in classical differential geometry by tensor calculus, Lie differentiation, *etc.* (Nijenhuis, 1952; Schouten, 1954; Yano, 1957). Such a reading of the point-coincidence argument by no means trivializes it. Rather, it *forestalls* formulating mathematical physics in terms of active diffeomorphisms and then invoking a sophisticated doctrine, Leibniz equivalence, to recover from them. One can no longer even formulate the active diffeomorphisms needed to set up the hole argument and then recover from it by Leibniz equivalence. If one foresees not needing a building, it is prudent and economical to avoid erecting one, rather than erecting and promptly demolishing it.

Such a view seems congenial to Einstein in the following late passage:

> We are now in a position to see how far the transition to the general theory of relativity modifies the concept of space. In accordance with classical mechanics and according to the special theory of relativity, space (space-time) has an existence independent of matter or field. In order to be able to describe at all that which fills up space and is dependent on the co-ordinates, space-time or the inertial system with its metrical properties must be thought of at once as existing, for otherwise the description of " that which fills up space " would have no meaning.[footnote suppressed] On the basis of the general theory of relativity, on the other hand, space as opposed to " what fills space ", which is dependent on the co-ordinates, has no separate existence. Thus a pure gravitational field might have been described in terms of the g_{ik} (as functions of the co-ordinates), by solution of the gravitational equations. If we imagine the gravitational field, *i.e.* the functions g_{ik}, to be removed, there does not remain a space of the type (I), [with $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$] but absolutely *nothing*, and also no " topological space ". For the functions g_{ik} describe not only the field, but at the same time also the topological and metrical structural properties of the manifold. A space of the type (I), judged from the standpoint of the general theory of relativity, is not a space without field, but a special case of the g_{ik} field, for which--for the co-ordinate system used, which in itself has no objective significance–the functions g_{ik} have values that do not depend on the co-ordinates. There is no such thing as an empty space, *i.e.* a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field. (Einstein, 1961, pp. 154, 155)

Einstein's claim, if perhaps too strong, at least represents an effort to take very seriously the dependence of space-time points' identities on what happens at them, from which effort one might draw inspiration. While modern coordinate-free geometry as applied to General Relativity hasn't wholly forgotten such lessons—it recognizes that *physical* points in GR are individuated by what happens there—it partially forgets them by awarding mathematical identities (unique names like p), not just coordinate labels (one for each chart covering a point), to points independently of their physical properties. For the mature classical style with x^μ and p, some contexts in which one might want to talk about fields at p require speaking proleptically of the point p in advance (logically speaking) of the recognition of the properties that make the physical space-time point p be what it is. If one is solving the Cauchy problem to build space-time out of space, for example, then one can only paint on the physically meaningful point identities after reaching them *via* time evolution. (This point is much of the motivation for Bergmann's work on observables (Bergmann, 1961).) Such prolepsis, arguably, is a lower price to pay than the alternatives require. One can then try to flesh it out with one or another metaphysical story among those offered in answer to the hole argument (*e.g.*, (Butterfield, 1989; Maudlin, 1990; Hoefer, 1996))—but without the presumption in favor of primitive mathematical point individuation that often appeared in that literature, not only in presentations of the hole argument, but even in responses to it. The natural affinity between some answers to the hole argument and the mature classical theory of geometric objects has not been widely emphasized. It might be that the 1980s reappearance of Einstein's hole argument as an important issue for philosophy was made possible by the weakening of the influence of Einstein's point-coincidence argument that was embedded in classical differential geometry in favor of the modern-style mathematical haecceities tacitly embraced by philosophers in the preceding decade or two.

It might seem superfluous to discourage the p-less ("point"-less?) older component style at this late date, because no one explicitly advocates it or would consciously draw foundational conclusions from such expressions. It is not superfluous, however, because the older p-less notation tends to reappear tacitly as working physicists in the midst of long difficult calculations quite reasonably strip away notational bits that seem irrelevant (*e.g.*, (Pons et al., 2000) for one example out of infinitely many—or see the not-so-difficult calculations above); indeed one tends not to bother writing even the coordinate argument(s). An abbreviation made once or twice saves only a little time, but an abbreviation made (say) 50 times (5 terms in each of 10 lines for a mildly nontrivial calculation) saves quite a bit of time and tedium. This abbreviation can happen even when an author's official view is the modern one with active diffeomorphisms, hence providing further opportunity for confusion. By contrast, admitting only passive coordinate transformations (as in the classical views) and having physical point individuation with names like p, as in Trautman's mature classical definition, brings clarity. The Hamiltonian H_p takes a certain space-like hypersurface (a level surface of x^0) with spatial coordinates to a different space-like hypersurface, a *later* one in the most obvious case, with spatial coordinates and a different value of x^0. The gauge generator G, specialized to rigid time translation, takes the same original space-like hypersurface with spatial coordinates and value of x^0, to physically the same 'later' space-like hypersurface, with the same spatial coordinates as H_p yielded, but not the same value of x^0 as H_p yielded. In this sense it is best not to say that H_p appears as a special case of G. The Hamiltonian's job is like building and painting (labeling) a road; the gauge generator's job is to repaint a road that already exists. This distinction can be seen clearly with both coordinates x and physical point individuation p as in Trautman's definition; otherwise it becomes obscure.

The instantaneous changes of the canonical variables $h_{ij}(x)$, *etc.* are exactly the same either way—whether from H_p or from G—as they should be for starting at the same state in the same description and arriving at the same state in slightly different descriptions related by rigid translation and hence invisible in expressions like $\frac{\partial x^{\mu\prime}}{\partial x^\nu}$, which becomes δ^μ_ν. Adding only a constant to the time coordinate makes the $\dot{\xi}^0$ terms vanish and suppresses the distinction between a scalar, a scalar density of weight 1, and a scalar density of weight -1. If one follows the common practice of suppressing the time coordinate label and writing $h_{ij}(x)$ or h_{ij} rather than $h_{ij}(x,t)$ or $h_{ij}(t)$, *etc.*, then this second difference is invisible also. Thus one realizes that there are differences, but also sees why they have tended not to be noticed. A bit more formally, H_p takes fields $\langle h_{ij}, \pi^{ij}, N, p\rangle$ at all points p in *space* with coordinates $x^\mu = (t, x^i)$ and yields the fields at *different* (typically, later) space-time points p' with coordinates $x^{\mu\prime} = (t', x^i)$.[6] By contrast (in part), G takes fields $\langle h_{ij}, \pi^{ij}, N, p \rangle$ at every point p in space-time (or a piece thereof, the intersection of two charts) with coordinates x^μ and yields the fields at the same point p with new coordinates $x^{\mu\prime}$, components that differ from the original by a Lie derivative (infinitesimally). In trivial cases (where the descriptive vector field and the coordinates relate suitably), the Lie derivative term can vanish, making the relabeling look almost like time evolution; but the time coordinate labeling is different due to G nonetheless. So gauge relabeling by G and time evolution from H_p are always different in some respect.

What ultimately remains of the supposed problem that H_p looks like a special case of G is just the fact that one gets the same spatial description (state and coordinate system) of the same physically identified moment (slice) with almost the same time description (time coordinates the same up to 'rigid' translation) if one either evolves an initial data surface for a given coordinate interval (from the start of 2013 to the start of 2014, for example), or if one relabels the coordinates of space-time by the same amount and redirects one's attention from that old surface (the erstwhile initial data surface, which is the start of 2013 and the start of 2012′) to a surface a coordinate year later in both coordinate systems. But that is equally true in such theories as a massive scalar field in flat space-time (paradigmatically special relativistic) or a harmonic oscillator in Newtonian mechanics, which are wholly unproblematic as far as time evolution is concerned. One interesting difference that does not get in the way is the varying multiplicity of rigid coordinate time translations in the theories: one in classical mechanics, a 3-parameter family for the massive scalar field in flat space-time, and as many as there are time-like vector fields (or local pieces thereof) in GR.

An issue that arises novelly for General Relativity (with no electromagnetic analog) is that while Hamiltonian techniques are typically applied to phase space, for General Relativity and other theories with velocity-dependent gauge transformations one should use phase space extended by time (Marmo et al., 1983; Sugano et al., 1986; Sugano et al., 1985; Lusanna, 1990)—one might call it phase space-time. In General Relativity, gauge transformations take the form $\pounds_\xi g_{\mu\nu} \sim \xi^0 \dot{g}_{\mu\nu} + \ldots$, which expression is velocity-dependent. Hence it is no surprise that something goes wrong in treating GR on phase space. Some authors even use a histories formalism for General Relativity, thereby giving space-time rather than space a still more prominent role (Savvidou, 2004; Kouletsis, 2000)—though one might hope for a milder revision of the standard phase space formalism with basically the usual Poisson brackets.

Clarity about time evolution in Hamiltonian GR is achieved in eight easy steps that yield Hamiltonian-Lagrangian equivalence:

1. Remove the unclarity induced by active diffeomorphisms (which one would have discarded eventually anyway *via* equivalence classes) in favor of passive coordinate transformations.

2. Define geometric objects in terms of both coordinates x^μ, $x^{\nu\prime}$, and physically individuated points p instead of just x^μ, $x^{\nu\prime}$.

3. Extend the phase space by x^0 in view of the velocity-dependent character of the gauge transformations, obtaining phase space-time.

4. Restore the lapse and shift vector to recover the space-time metric, not just the spatial metric.

5. Require proper coordinate transformation behavior for the lapse and shift vector somehow or other.

6. Recognize that a first-class constraint does not generate a gauge transformation in GR because it violates physical equivalence and physical law (*c.f.* 4-dimensional differential geometry).

7. Restore the canonical momenta p, p_i conjugate to the lapse and shift vector to the phase space(-time).

8. Implement 4-dimensional coordinate transformations (at least on-shell) by Poisson bracket with G, obtaining 4-dimensional Lie differentiation for solutions of field equations.

Then no confusion remains. Rather than striving to acquiesce in the mysteries supposedly disclosed by Hamiltonian GR, one can clear them up. Given that such mysteries do not appear in Lagrangian GR, they must be defects in the typical Hamiltonian formalism rather than features of GR. Time evolution

[6] I trust that the use of p for both a space-time point and the canonical momentum conjugate to the lapse N will not be confusing.

in GR is intricate and in some respects novel, but not bizarre or mysterious—much as with gravitational energy localization (Pitts, 2010).

One can now see why discussions of reduced phase space for GR have been problematic. The velocity-dependent character of foliation-changing coordinate transformations implies that phase space-time, not phase space, is the proper arena in which to work. There just isn't room in phase space to wander off into the future or the past. It is thus unclear what it would be to construct a reduced phase *space* for GR (Thébault, 2012b) (*c.f.* the description of the process in ((Belot and Earman, 2001))). Both the original phase space-time and the reduction process need, so to speak, to reach out into the future (and past) in a way that is disanalogous to simpler theories. Additional difficulty in constructing a (fully) reduced phase space(-time) for GR arises from the fact, encountered in detail above, that the Hamiltonian formalism implements the equivalent of time-involving coordinate transformations only for solutions of the equations of motion. Hence the usual expectation of constructing a space where antecedently recognizably physically equivalent gauge-related configurations that might not be solutions of the equations of motion have been identified as one point, and then formulating dynamics on it, apparently cannot be realized in the usual way.

7 Resolution of Earman-Maudlin Standoff

Successfully finding change in Hamiltonian GR resolves the standoff between Earman and Maudlin (Earman, 2002; Maudlin, 2002). Earman invoked Hamiltonian GR and inferred no real change (more or less—one is left contending with a neologistic "D series" in the spirit of McTaggart and delving into the philosophy of mind), whereas Maudlin invoked common sense's real change and rejected Hamiltonian GR. Earman's enthusiasm for Hamiltonian GR is not misplaced, but various errors commonly found in the physics literature needed to be corrected. Earman's mere "surface structure" of tensor calculus and the presence or absence of Killing vector fields in fact provided the key: it gives same answer as the corrected Hamiltonian treatment. His "deep structure" is erroneous in ways that I discuss here, the prequel (Pitts, 2013b) and its GR-oriented companion (in preparation (Pitts, 2013a)), and the sequel on observables (Pitts, 2014). Maudlin's common-sense detection of change is vindicated, but his dismissal of the Hamiltonian formalism isn't. Instead a Lagrangian-guided reform of the Hamiltonian formalism has been achieved, the continuing need for which (extending conceptually what Mukunda, Castellani, Sugano, Kimura, Pons, Salisbury, Shepley, Sundermeyer and a few others have done technically) might not yet have been recognized without Maudlin's critique.

Of course requiring that the Hamiltonian formalism agree with the Lagrangian formalism simply had to work. Change in Hamiltonian GR is neither distinctively canonical nor problematic, at any rate not classically. Presumably there are quantum consequences, but I do not attempt here to say what they are. It might be that resolving problems in classical canonical GR simultaneously loosens the presumed connections between classical and quantum canonical GR that Bergmann expected. Enforcing the equivalence of the unclear constrained Hamiltonian formalism with the clear Lagrangian formalism is evident in early works by Bergmann and collaborators (*e.g.*, (Anderson and Bergmann, 1951)). Eventually Bergmann took shortcuts about observables, as Dirac did about gauge transformations and even the size of the phase space. Recovering Hamiltonian-Lagrangian equivalence removes confusion. Real change has been found without delving much into the "observables" thicket, though addressing such issues in a Lagrangian-equivalent way can bring additional clarity (Pitts, 2014).

8 Philosophical Accounts of Time and Change in GR

There isn't any conceptual difficulty in locating change in General Relativity, either in the Lagrangian formalism or in the Hamiltonian formalism, once the latter is set up properly. The former claim is fairly widely accepted. The latter differs from widely shared views among philosophers (Belot, 2007; Belot and Earman, 2001; Belot and Earman, 1999; Huggett et al., 2013; Rickles, 2006), where the idea that a first-class constraint generates a gauge transformation has been widely influential.

Demanding and successfully achieving a Hamiltonian formalism equivalent to the Lagrangian one will ensure that change is equally discernible both places. But much of the conceptual reflection on

GR and much of the effort to quantize it canonically have been carried out with Hamiltonian formalism(s) inequivalent to the Lagrangian. A main root of the difficulty is the doctrine that a first-class constraint generates a gauge transformation. That doctrine, widely assumed by high authorities in theoretical/mathematical physics (Henneaux and Teitelboim, 1992; Gotay et al., 1978) and already apparent in the work of Dirac and Bergmann in the 1950s (but not in ((Anderson and Bergmann, 1951)), which employs G), is nonetheless readily falsifiable by direct calculation using the example of electromagnetism (Pitts, 2013b). This doctrine is often associated with the extended Hamiltonian, Dirac's idea that one may add all the first-class constraints (not just first-class primary constraints) to the Hamiltonian with arbitrary coefficients, and needs to do so to exhibit the full gauge freedom (Dirac, 1964). Instead, a gauge transformation is actually generated by a *special combination* of first-class constraints, the gauge generator G of Anderson and Bergmann, which disappeared for about 30 years and started resurfacing slowly in the 1980s (Castellani, 1982; Sugano et al., 1986; Gràcia and Pons, 1988; Sugano et al., 1992; Pons et al., 1997; Shepley et al., 2000; Pons and Salisbury, 2005). The gauge generator G is associated with the primary Hamiltonian, which adds to the canonical Hamiltonian only the primary first-class constraints. The primary Hamiltonian is equivalent to the Lagrangian, whereas the extended Hamiltonian is not. Proponents of the extended Hamiltonian claim that it is equivalent to the Lagrangian for observable quantities, the difference being additional extra gauge freedom (Henneaux and Teitelboim, 1992). That turns out not to be the case once one is clear about primordial observables *vs.* auxiliary fields like canonical momenta (Pitts, 2013b). The canonical momentum conjugate to A_i, far from being the primordial observable electric field as is sometimes claimed, is just an auxiliary field that one can integrate out using its algebraic equation of motion due to its nonlinear algebraic appearance in the Hamiltonian action $S = \int dtd^3x(p\dot{q} - \mathcal{H})$. By such moves one thereby recovers the Lagrangian action $\int dtd^3x\mathcal{L}$, which gets by just fine without the conjugate momenta. A_i and A_0 are what couple to current and charge densities, so they (through a gauge-invariant combination of their derivatives (Jackson, 1975)) constitute the primordial observable electric field.

8.1 Comparison to Belot and Earman

Belot and Earman's treatments, being especially thorough, early and influential, will repay detailed attention. One should note that, at least on technical grounds, Earman is perhaps of two minds, assuming that a first class constraint generates a gauge transformation, but also briefly exhibiting the gauge generator (Earman, 2003). But the former view is clearly dominant in his work.

It appears that the difference between my conclusions about the ease of finding change in GR and Belot's conclusions about the difficulty of finding change in GR are caused largely by the divide between my adopting the Lagrangian-equivalent primary Hamiltonian and the specially tuned sum of first-class constraints G as generating gauge transformations on the one hand, and Belot's and Earman's adopting the Lagrangian-inequivalent extended Hamiltonian and an arbitrary sum of first-class constraints as generating gauge transformations on the other (Belot and Earman, 2001).[7]

Before delving into the challenges of canonical GR, it is advisable to look at the standard test bed theory, Maxwell's electromagnetism, as presented in Belot's version of a constrained Hamiltonian formalism (Belot, 1998). One first notices the omission not merely of the canonical momentum p^0 conjugate to the scalar potential A_0 (which is common enough and not automatically disastrous), but also the omission of A_0 itself, from the formulation. Is this elimination achieved by gauge fixing $A_0 = 0$ *via* the Dirac bracket, the usual gauge-fixing technology of Dirac-Bergmann constrained dynamics (Sundermeyer, 1982)? Such a formalism would be possible, though not trivial (*e.g.*, what is the canonical generator of whatever gauge transformations might remain (Mukunda, 1980)?)—but Belot's approach seems to be less systematic. It would be awkward to couple Belot's Hamiltonian electromagnetism to charge without A_0, because $A_0\rho$ is the standard interaction term with charge density. But difficulties exist even for the vacuum case. One dilemma pertains to the origin of the Gauss-like constraint $\nabla \cdot \vec{E} = 0$. (I have here employed Belot's notation of using E for the canonical momentum. Such a notation in fact is a temptation to conflate entities that are quite distinct in their gauge transformation properties, F_{0i} (a familiar function of derivatives of A_μ) and the canonical momentum. These quantities are in fact not

[7] I thank Oliver Pooley for very helpful discussion on this matter.

even related until one uses the equations of motion: p is independent of $\dot{q}$. That is why I called the constraint Gauss-*like*.) The Hamiltonian is given as (simplifying to flat space and Cartesian coordinates for simplicity) $\int d^3x(\vec{E} \cdot \vec{E} + |\nabla \times \vec{A}|^2)$.

Now one faces a dilemma. If the condition $\nabla \cdot \vec{E} = 0$ is not employed in the variational principle, then how does it arise later? This is not a systematic treatment of constraints. Note that gauge-fixing $A_0 = 0$ in the usual treatment would require that any lingering gauge transformations be time-independent in view of the gauge freedom $\delta A_\mu = -\partial_\mu \epsilon$. Thus Belot's notation $\Lambda(t)$ is misleading in implying time dependence of the descriptor Λ (basically my $-\epsilon$). Note also that, in contrast to the more usual formulation of electromagnetism that includes A_0, Belot's alleged gauge transformation descriptor Λ doesn't cancel out of the Maxwell equation $\dot{A} = -\vec{E}$—not unless one discards the advertised time dependence of $\Lambda(t)$. Thus there actually is no gauge freedom in Belot's formulation, and no threat of indeterminism.

On the other hand, suppose that $\nabla \cdot \vec{E} = 0$ is imposed in the variational principle (though Belot does not seem to mention it). That explains why the constraint exists later. But there is a cost in the variational principle, which must make use of a transverse-longitudinal decomposition (*e.g.*, (Deser, 1972; Marzban et al., 1989)). Then one gets not $\dot{\vec{A}} = -\vec{E}$, but only its transverse part $\frac{\partial}{\partial t}(\vec{A} - \frac{\nabla(\nabla \cdot \vec{A})}{\nabla^2}) = -\vec{E}$: the gradient part is projected away, leaving only what is spatially divergenceless. This expression is perhaps most readily motivated using spatial Fourier transformations, and can also be understood in terms of Green's functions. This transverse projection ensures that $\nabla\Lambda$, the supposed gauge transformation, is immediately eliminated by the projection operation. Resorting to Fourier space or inverting a differential operator using a Green's function is of course a spatially *nonlocal* operation. Such a blatant threat to locality will pose grave difficulties for the project of understanding senses of locality, the Aharonov-Bohm effect, *etc.* later in the paper.

Turning to Belot's article on time and change in mechanics generally and in GR in particular, one finds that section 3.3 (Belot, 2007) on presymplectic manifolds presents a collection of stipulative definitions, some of them about familiar words like "gauge," some of them less familiar like "presymplectic": "We call," "We define," *etc.* While clearly the definitions of long-familiar words are intended to be roughly equivalent to more traditional ones like $A_\mu \rightarrow A_\mu - \partial_\mu \epsilon(t, x)$ (Belot, 2007, p. 189), that equivalence is not shown—not there, and not successfully elsewhere (*e.g.* (Gotay et al., 1978; Henneaux and Teitelboim, 1992)) either. Beneath the surface, evidently, is the doctrine that a first-class constraint generates a gauge transformation (Belot and Earman, 2001)—which, alas, isn't true (Pitts, 2013b). Gauge transformations for electromagnetism make sense "off-shell" (without using any of Hamilton's equations), and there just isn't any relationship at all between the canonical momentum and the electric field (a function of derivatives of A_μ—the field that couples to charge density in the term $A_\mu \mathcal{J}^\mu$) in that context. Hence preserving the magnetic field (the curl of the 3-vector A_m) and the canonical momentum, Belot's necessary and sufficient conditions for physical equivalence (Belot, 2007, p. 189), is necessary but *not* sufficient. One also needs to preserve the electric field, which equals the canonical momenta (up to a sign) only *on*-shell *via* $\dot{q} = \frac{\delta H}{\delta p}$. (Belot and Earman also present the electric field as though it were itself the canonical momenta (Belot and Earman, 2001).) That equality is spoiled by arbitrary combinations of first-class constraints; it is preserved only by the specially tuned combination G. Indeed one can derive the form of G by requiring that the change in the electric field from the primary constraint and the change in the electric field from the secondary straight cancel out (Pitts, 2013b). The relation between the canonical momentum and the electric field comes from calculating a Poisson bracket and so cannot be used inside another Poisson bracket (such as in calculating a gauge transformation).

Turning to General Relativity itself, one finds both analogs of the issues for electromagnetism and new ones. One issue that becomes important is whether one uses active diffeomorphisms (as Belot does) or passive coordinate transformations. Active diffeomorphisms presuppose that space-time points are individuated mathematically independently of what happens there. But the lesson of Einstein's point-coincidence argument seems to be that space-time points are individuated physically by virtue of what happens there. It seems to start off on the wrong foot to presuppose a mathematical formalism ill-adapted to the lessons of Einstein's point-coincidence argument by individuating space-time points primitively for mathematical purposes, and then to repudiate the physical meaning of that individuation. That curious piece of mathematical metaphysics lacks two merits of coordinate systems, namely, being descriptively rich enough to do tensor calculus, and being manifestly just one of many equally good options and hence less tempting to take with undue seriousness. Geographers and classical differential geometers have rules for changing coordinate systems, but it seems awkward to change names of individuals, or to introduce individuals and then annihilate or conflate them. Clarity about passive coordinate transformations will play a role below in identifying gauge transformations and hence gauge-invariant quantities.

A second issue that arises novelly for General Relativity, as noted above, is the velocity-dependent gauge transformations and consequent need to use phase space extended by time. While one can attempt to carry over to GR a reduced phase space construction that works for theories with internal velocity-independent transformations (Belot and Earman, 2001), the results are very unlikely to mean what one hoped. This fact relates to the admission elsewhere in the paper that it isn't obvious how changes of time coordinate are implemented in their formalism (Belot and Earman, 2001). Indeed when one does implement changes of time coordinate (as discussed above), one doesn't know which points in phase space-time are physically equivalent under the gauge transformations that (on-shell!) change the time coordinate until *after* the equations of motion are used. Hence the usual idea of reduced phase space as implementing dynamics on a space where gauge-related descriptions have been *identified in advance* is impossible. One would therefore need to rethink reduced phase space-time from the ground up for GR, as far as changes of time coordinate are concerned, before philosophizing about it.

There is a methodological lesson here about the risks of abstraction and the role of examples. If one uses serious examples, interesting examples that are antecedently well understood using other formalisms, then one can use them to test the formalism at hand, not merely to illustrate it. The difference is that tests of a formalism involve considerable knowledge of appropriate conclusions and some doubt about appropriate premises, whereas illustrations use little knowledge of appropriate conclusions and great confidence in appropriate premises. In short, a test allows the example to push back against the formalism, to show its failings or restrictions, whereas a mere example does not.

A third issue involves what to do with the less interesting field components and their momenta. Although Belot now keeps the electric scalar potential A_0 (in contrast to (Belot, 1998)), for GR he gives short shrift to the lapse and the shift vector, the 40% of the space-time metric by which it transcends the spatial metric. He apparently discards the lapse and shift in choosing a Gaussian slicing (Belot, 2007, p. 201). Belot and Earman discard the electric scalar potential and claim to follow Beig (Beig, 1994) in eliminating the lapse and shift (Belot and Earman, 2001). For Beig that means only keeping them as freely prescribed functions of space and time without the Hamiltonian apparatus of canonical momenta and Hamilton's equations; it might mean something stronger for Belot and Earman. Losing some of the q's, besides leaving one unable to infer the space-time metric, makes it well-nigh impossible to express 4-dimensional coordinate transformations.

To discuss further points at issue particularly in GR, it is best to quote the end of p. 201 and much of p. 202 for contrast.

> The gauge orbits of [the presymplectic form] ω have the following structure: initial data sets (q, π) and (q', π') belong to the same gauge orbit if and only if they arise as initial data for the same solution g.[Footnote suppressed]
>
> 2. *Construct a Hamiltonian.* Application of the usual rule for constructing a Hamiltonian given a Lagrangian leads to the Hamiltonian $h \equiv 0$.
>
> 3. *Construct dynamics.* Imposing the usual dynamical equation, according to which the dynamical trajectories are generated by the vector field(s) X_h solving $\omega(X_h, \cdot) = dh$, leads to the conclusion that dynamical trajectories are those curves generated by null vector fields. So a curve in $\mathcal{I}$ [the space of initial data] is a dynamical trajectory if and only if it stays always in the same gauge orbit. This is, of course, physically useless – since normally we expect dynamical trajectories for a theory with gauge symmetries to encode physical information by passing from gauge orbit to gauge orbit. But in the present case, nothing else could have been hoped for. A non-zero Hamiltonian would have led to dynamical trajectories which passed from gauge orbit to gauge orbit – but this would have been physical nonsense (and worse than useless). For such dynamics would have carried us from an initial state that could be thought of as an instantaneous state for solution g to a later instantaneous state that could not be thought of as an instantaneous state for solution g. In doing so, it would have turned out to encode dynamical information very different from that encoded in Einsteins field equations.

(Belot, 2007, pp. 201, 202)

The claim that the Hamiltonian vanishes *identically* (also in (Belot and Earman, 1999)) is not correct. The method for constructing a Hamiltonian in constrained dynamics gives a Hamiltonian that is only weakly equal to 0 (plus boundary terms, which do not matter) (Sundermeyer, 1982); the gradient is not 0, so Poisson brackets with the Hamiltonian need not be 0. Weak equality is by construction compatible with non-zero Poisson brackets and hence a non-zero Hamiltonian vector field. Belot and Earman point to chapter 4 of ((Henneaux and Teitelboim, 1992)). But there one finds the following: "If the q's and p's transform as scalars under reparametrizations, the $p\dot{q}$-term in the action transforms as a scalar density, and its time integral is therefore invariant by itself.... *Thus, if q and p transform as scalars under time reparametrizations, the Hamiltonian is* **(weakly)** *zero for a generally covariant system.* (Henneaux and Teitelboim, 1992, pp. 105, 106, italics in the original, but boldface is my addition) As appeared above in the flurry of Lie derivatives, the relevant q's and p's are scalars under time reparametrization. (The lapse fits in with Henneaux and Teitelboim's u's, which are densities, as is the lapse.) In GR the Hamiltonian is weakly 0 (apart from possible boundary terms), but it nonetheless has nonzero Poisson brackets and so is not prohibited from generating real time evolution.

The primary Hamiltonian leads to equations equivalent to Einstein's equations (Sundermeyer, 1982). Such dynamics takes one set of initial data to another set of (what one could regard as initial) data with (typically) different properties—the universe has expanded, for example, or gravitational waves have propagated, or some such. Change has occurred. Both moments are parts of the same space-time. Gauge transformations can be divided into purely spatial ones and those changing the time as well. Purely spatial ones, generated by $G[\epsilon^i, \dot{\epsilon}^i]$ depending on a spatially projected 3-vector and its velocity, take each moment under one coordinate description into that same moment under another coordinate description with the same time coordinate but different spatial coordinates. Coordinate transformations involving time cannot be implemented on phase space, but live rather on phase space extended by time, because they are velocity-dependent. Such a gauge transformation, which essentially involves the normally projected gauge generator $G[\epsilon, \dot{\epsilon}]$, acts on an entire space-time (trajectory, history) and repaints coordinate labels onto it (or at any rate acts in that way in the overlap of the two relevant coordinate charts); the relabeling happens on a 4-dimensional blob, not a 3-dimensional one. Relabeling a space-time with coordinates leaves one on the same gauge orbit, but that in no way implies the absence of change. Change is indicated by nonzero Lie derivatives with respect to all time-like vector fields.

The representation of changeable quantities proves to be rather harder for Belot than it is on my view. He surveys various spaces on which one might try to represent changeable quantities—the space of solutions, the reduced space of solutions, and the reduced space of initial data (Belot, 2007, pp. 203, 204) and comes up empty every time. "And on the space of initial data we face an unattractive dilemma: if we seek to represent changeable quantities by non-gauge invariant functions, then we face indeterminism; if we employ gauge-invariant functions, then we are faced with essentially the same situation we met in the reduced space of initial data." (Belot, 2007, p. 204) At this point it becomes important to employ exclusively passive coordinate transformations in order to have a clear idea of what gauge-invariance is and hence what the gauge-invariant functions are. Thus gauge transformations are coordinate transformations. Famously, scalars are coordinate-invariant. Hence scalars that depend on time exhibit change; an independent set of Weyl curvature scalars to form a coordinate system is one example. Quantities that aren't scalars (gauge-invariant) might still be geometric objects (gauge-*covariant*)—contravariant vectors, covariant vectors, various kinds of tensors, tensor densities, *etc.*, the meat and drink of classical differential geometry (Schouten, 1954; Anderson, 1967). Lie differentiation is the tool to ascertain time dependence. In cases where (there exists a coordinate system such that) the metric is independent of time, one has a time-like Killing vector field: there exists a vector field ξ^α such that $\pounds_\xi g_{\mu\nu} = 0$ and ξ^α is time-like. The components $g_{\mu\nu}$ are not gauge-invariant, but they are gauge-covariant, which is good enough. (The metric-in-itself $\mathbf{g}(p) = g_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$ is gauge-invariant, but the usual modern definition of Lie derivatives involves active diffeomorphisms; the geometric object $\{g_{\mu\nu}\}$ (in *all* coordinates (Nijenhuis, 1952; Anderson, 1967; Trautman, 1965; Schouten, 1954)) is also gauge-invariant, but dealing with every coordinate system at once introduces needless complication. Hence the classical approach of using any arbitrary coordinate system as representative is attractive.) It makes no difference whether the space-time is spatially closed, asymptotically flat, or neither, because change is defined locally. Just go to a basement with no windows and watch for change.

The nexus of the problem of time, as identified by Belot, is related to the space on which one formulates the Hamiltonian dynamics. "This is the nexus of the problem of time: time is not represented in general relativity by a flow on a symplectic space and change is not represented by functions on a space of instantaneous or global states." (Belot, 2007, p. 209). But one ought not to try to represent time and change in GR in phase space; one needs phase space extended by a time coordinate because the gauge transformations are velocity-dependent (Marmo et al., 1983; Sugano et al., 1986; Sugano et al., 1985; Lusanna, 1990). To express 4-dimensional coordinate transformations, one also needs the lapse and the shift vector. If one wishes to generate 4-dimensional coordinate transformations *via* a Poisson bracket, then one needs the canonical momenta conjugate to the lapse and shift—the momenta that vanish according to the primary constraints. As noted above, it isn't terribly clear what has become of the lapse and the shift vector in Belot's treatment; the quantities that he (like many authors) mostly discusses are merely the 3-metric and its canonical momentum, giving 12 functions at each point in space, satisfying 4 constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ at each point in space. When one restores the lapse and shift vector to include the whole space-time metric and hence a better shot at expressing coordinate transformations involving time, and restores their conjugate momenta (vanishing as primary constraints) to improve one's chances at expressing 4-dimensional coordinate transformations using Hamiltonian resources, namely the gauge generator G, one gets 20 functions of time at each point in space, satisfying 8 constraints at each point in space: $p = 0$, $p_i = 0$, $\mathcal{H}_0 = 0$, and $\mathcal{H}_i = 0$ at each point in space. But to include velocity-dependent gauge transformations (such as change the time slice in GR), one should also include time in an extended phase space. Hence instead of $12\infty^3$ functions satisfying $4\infty^3$ constraints and (maybe) changing over a time that isn't part of the space in question, one needs $20\infty^3$ quantities satisfying $8\infty^3$ constraints on a slice of a space of $20\infty^3 + 1$ dimensions. (Admittedly the primary constraints p, p_i are $4\infty^3$ quantities with the boring task of being 0 according to $4\infty^3$ of the constraints.) Such a space admits a Hamiltonian formalism equivalent to the Lagrangian formalism and hence makes change (or its absence) unproblematic in terms of the absence (or presence, respectively) of a time-like Killing vector field.

8.2 Thébault on Time, Change and Gauge in GR and Elsewhere

Unusually among philosophers, Thébault's work has expressed at least selective skepticism about whether a first-class constraint generates a gauge transformation, especially in relation to time and the Hamiltonian constraint in theories or formulations that have one (Thébault, 2012a; Thébault, 2012b). Such skepticism is partly informed by, among other things, views of Kuchař, Barbour and Foster's work, and of the Lagrangian equivalence-oriented reforms of Pons, Salisbury and Shepley. Clearly such selective skepticism can only be bolstered by the recognition (Pitts, 2013b) that a first-class constraint *typically* does not generate a gauge transformation. Then the quick argument from the fact that the Hamiltonian of GR is a sum of first-class constraints (and a boundary term) to the conclusion that it generates just a pile of gauge transformations is no longer tempting. When familiar general presumptions about gauge freedom and first-class constraints disappear, less work is required to motivate taking GR as partly violating those presumptive conclusions. For example, Kuchař's and Thébault's exceptional treatment of the Hamiltonian constraint *vis-a-vis* the other constraints in GR becomes partly unnecessary, because the other constraints are no longer viewed as having some of the features that Kuchař *et al.* deny of the Hamiltonian constraint. (Of course the considerations about reduced phase space-time and the merely on-shell nature of Hamiltonian coordinate transformations due to the quadratic-in-momenta character of the Hamiltonian constraint imply that there are still some exceptional features of $\mathcal{H}_0$, rightly highlighted by Thébault's doubts about reduced phase space.) The dual role of the Hamiltonian constraint in relation to both evolution and gauge transformations (Thébault, 2012b) is clarified when one notices that $\mathcal{H}_0$ does *neither* of these jobs by itself; both are accomplished by teaming up with other constraints, whether in H_p or in G. These teaming arrangements are easy enough to see when one retains the lapse, shift vector, and associated canonical momenta p, p_i and associated primary constraints, but impossible to see when one truncates the phase space in the way common since Dirac (Dirac, 1958; Salisbury, 2010; Salisbury, 2006). The idea of extending phase space by t in order to accommodate velocity-dependent gauge transformations also fits well with Thébault's project. In short, making the constrained Hamiltonian formalism equivalent to the Lagrangian formalism as far as possible will facilitate drawing various conclusions for which Thébault has argued on partly different grounds.

9 Problem of Time in Quantum Gravity Not Resolved

While I think that change in Hamiltonian General Relativity is unproblematic at the classical level, the same does not hold at the quantum level. In other words, what remains of the problem of time, what actually exists of the problem, begins at quantization. It is many-faceted (Butterfield and Isham, 1999; Butterfield and Isham, 2001; Anderson, 2012). The usual Dirac method of imposition of constraints by requiring that a physical state be annihilated by them seems to close the door to change at the quantum level in a way with no classical analog.[8]

One might need to rethink aspects of how the constraints are imposed quantum mechanically in order to parallel the reformed classical treatment (Sugano et al., 1985; Sugano and Kimura, 1985; Sugano et al., 1992), especially in light of the association (Henneaux and Teitelboim, 1992, p. 18) between standard quantization methods and the doctrine that a first-class constraint generates a gauge transformation. If the classical idea of vanishing weakly, with a distinction between being 0 on the constraint surface and having nonzero gradient (in the Poisson bracket), survived at the quantum level (such as by a distinction between vanishing expectation value and nonzero commutator), then there might be room for change at the quantum level. The usual Dirac condition of regarding physical states as those annihilated by the constraint operator leaves no room for a quantum analog of vanishing weakly. Hence understanding the extent to which the usual Dirac imposition of constraints is mandatory will be very important for ascertaining the degree to which a problem of time exists in canonical quantum gravity.

10 Conclusion

One could confidently affirm real change without attending at all to arguments about the Hamiltonian formalism, because nothing about the Hamiltonian formalism's treatment of change could be more decisive than the meaning of the presence or absence of a time-like Killing vector. This claim bears a faint resemblance to the response to skepticism by G. E. Moore (Moore, 1939), as well as the spirit of Maudlin's and Healey's responses to Earman (Maudlin, 2002; Healey, 2002). But the Moorean-like fact, in my view, is not (or not only) some deliverance of common sense, accessible by simple bodily gestures (Moore's displaying his hands, Samuel Johnson's kicking a stone), but rather, a deliverance of Lagrangian field theory. (Of course Lagrangian field theory is itself responsible not to do violence to common sense about change, *etc.* to the point of being self-undermining.) Yet this is no justification for dismissing the Hamiltonian formalism (*c.f.* (Maudlin, 2002)). It is, rather, a recipe for reform.

Having delved into the Hamiltonian formalism, one finds that, when set up properly, its verdict on change agrees with that of the condition of having no time-like Killing vector. This is of course no accident: making the Hamiltonian match the Lagrangian has been the basic policy of reform employed by Pons, Salisbury and Shepley's series of works, which I quote again:

> We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent... in coming to the conclusion that they in fact are. (Pons and Shepley, 1998, p. 17)

Enforcing the equivalence of the unclear Hamiltonian formalism with the clear Lagrangian formalism is very evident in the early work of Bergmann's school—*e.g.*, (Anderson and Bergmann, 1951). But eventually, perhaps by accident, certain shortcuts were taken, by Bergmann about observables, by Dirac about gauge transformations, by both about whether the lapse, shift vector, and their canonical momenta should be retained in the phase space (*e.g.*, (Dirac, 1958))—shortcuts yielding 'insights' that have sustained confusion for decades. Recovering Hamiltonian-Lagrangian equivalence has been underway for some time (Castellani, 1982; Sugano et al., 1986; Gràcia and Pons, 1988). While the gauge generator by now is moderately famous again, it has by no means swept the field. Furthermore, and more importantly

[8] I thank Claus Kiefer for discussing this point.

for present purposes, it remains to clear away the conceptual errors generated during the gauge generator's period of eclipse. This paper has aimed to do that regarding time evolution. Earman's healthy respect for a constrained Hamiltonian formalism and Maudlin's healthy respect for common sense are reconciled.

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