Topos-Theoretic Approaches to Quantum Theory

Part III Essay

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2 May, 2012.

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Signed ....

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Topos-Theoretic Approaches
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To the Chairman of Examiners for Part III Mathematics.

Dear Sir, I enclose the Part III essay of ....

Signed .... (Director of Studies)

Abstract

Starting from a naive investigation into the nature of experiments on a physical system one can argue that states of the system should pair non-degenerately with physical observables. This duality is closely related to that between space and quantity, or, geometry and algebra. In particular, it is grounded in the mathematical framework of both classical and quantum mechanics in the form of a pair of duality theorems by Gelfand and Naimark. In particular, they allow us to construct a classical phase space, a compact Hausdorff space, for an algebra of classical observables. In the case of quantum mechanics, this construction breaks down due to non-commutative nature of the algebra of quantum observables. However, we can construct a Hilbert space as the geometry underlying quantum mechanics.

Although this Hilbert space approach to quantum mechanics has proven to be very effective, it does have its drawbacks. In particular, these arise when one associates propositions to observables and investigates what kind of logical structure they form. One way of doing this is by realising the propositions as certain subsets of the phase space. In classical mechanics this procedure indeed gives one the structure one would expect: a Boolean algebra. However, although the case of quantum mechanics yields a nice mathematical structure, an orthocomplemented lattice, the physical interpretation of this logic is rather subtle, due to its crude notion of truth.

Recent work by Isham, Butterfield, Doering, Landsman, Spitters, Heunen et al., attempting to address these problems, has led to an alternative method for dealing with non-commutative algebras of observables and with that an alternative framework for quantum kinematics. Moreover, it stays much closer to our intuition from classical physics, in some sense, the motto being: Quantum kinematics is exactly like classical kinematics, that is, not in Set, but internal to some other topos!

This review paper gives a first introduction to the subject. It attempts to provide a stepping stone towards more serious papers.

Contents

1 Introduction

A theory of physics should be grounded in experiments: we formulate a *hypothesis*, we do measurements on a prepared *system* by performing a certain *experiment* on it and we compare the two to see if our experiment falsifies our hypothesis. Somehow, these three concepts should correspond (not necessarily bijectively) to the mathematical concepts of respectively *proposition*, *state* and *observable*. A theory of physics should specify which systems, experiments and hypotheses we deal with.

Starting from a naive investigation into the nature of experiments on a physical system one can argue that, regardless of what the theory is, observables should pair non-degenerately with states. One can argue that the observables naturally embed in a C*-algebra on which the states act as linear functionals. The crucial difference between classical and quantum mechanics is found in the Heisenberg uncertainty relation, which entails that the quantum algebra should be a non-commutative one.

Theorems by Gelfand and Naimark allow one to construct a geometry, a so called phase space, underlying the states: a compact Hausdorff space in the case of quantum mechanics and a Hilbert space for quantum mechanics, given by the GNS-construction. For this reason, it might be useful to think of classical mechanics taking place internally in some category of topological spaces, while some category of inner product spaces is rather the domain of quantum mechanics. I mean that in the sense that the categorical structures at hand in both categories (e.g. objects, morphisms, (co)limits, monoidally closed structure) have similar physical interpretations.

This is a very reasonable but traditional approach for arguing the similarities between classical and quantum mechanics. However, it has its drawbacks. In particular, one can associate *propositions* to observables and investigate what kind of logical structure they form. One way of doing this is by realising the propositions as certain subsets of the phase space. In classical mechanics this procedure indeed gives one the structure one would expect: a Boolean algebra. However, although the case of quantum mechanics, that was first dealt with by Birkhoff and Von Neumann [5], yields a nice mathematical structure, an orthocomplemented lattice, that has been intensively studied since, the physical interpretation of this logic is rather subtle, due to its crude notion of truth.

This, more than anything else[1], raises the question if the category of vector spaces really is the right framework for a theory of quantum mechanics, at least if one wants to do logic. Recent work by Isham, Butterfield, Doering, Landsman, Heunen, Spitters et al. (e.g. [11, 20, 21, 22, 29]) suggests that a fundamentally different approach might lead to new insights. In their work, the quantum phase space is not a Hilbert space, but a compact Hausdorff space[2] internal to a different topos then Set, namely the topos of (co)presheaves over the poset of commutative subalgebras of the algebra of observables[3]. One then constructs a quantum phase space as a topological space internal to this topos by a version of the commutative Gelfand-Naimark correspondence. In this way the GNS-construction and with it the framework of vector spaces is avoided. This leads to an approach of quantum logic that is, as I will argue, more reasonable from an operational point of view. Moreover, apart from stressing a very strong parallel with classical mechanics, it

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[1]Of course, there are also other issues with the treatment of quantum mechanics in Vect. For instance, the GNS-construction, unlike the commutative Gelfand-Naimark correspondence, does not extend to a functor. Even worse, it is non-canonical: the construction depends on the choice of designated states. Of course, one can take all states, to obtain the universal GNS-representation. However, due to size issues, this is often not a practical Hilbert space to deal with.

[2]Rather, a completely regular locale.

[3]As we will explain, this is strongly inspired by Bohr's doctrine of classical contexts.

also gives insight into the origin of fundamental differences between the classical and quantum worlds.

However, there are currently two competing approaches that realise this idea in a slightly different way. The first approach, which I shall call 'the contravariant approach' was developed by Isham and Butterfield and later by Doering and Isham, uses the topos of presheafs on the poset of subalgebras. The second 'covariant approach', due to Heunen, Landsman and Spitters, uses the topos of copresheaves on this poset. The two approaches were recently extensively compared by Wolters [43] and appear to be compatible in a number of ways.

This review paper attempts to give a brief outline of the subject, culminating in a comparison of the contravariant and covariant approach. Rather than to explore new territory, the aim of this text is to give an overview, leaving out technicalities if possible; focussing on ideas rather than calculations. All omitted details can be found in the excellent papers of the bibliography.

> *"Well,* I *never heard it before,"* said the Mock Turtle; *"but it sounds uncommon nonsense."*
>
> *(Lewis Caroll's 'Alice's Adventures in Wonderland', Chapter X.)*

2   Naive Physics (Syntax)

Let us forget about the standard formalism for theories of physics for a moment. In this informal paragraph we shall take a naive operational point of view (in the sense of Bridgman), to motivate a minimal core of syntax[4] that should get an interpretation in each theory of physics. We hope that this philosophical background will help the reader to appreciate the topos approaches to quantum physics.

Physical theories originate from the experimental practice. We perform experiments on prepared systems to yield outcomes[5]: we have sorts[6] {systems}, {experiments} and $\mathcal{P}$\{outcomes\} and a function symbol

$$\{\text{systems}\}, \{\text{experiments}\} \xrightarrow{\text{Measurement}} \{\text{outcomes}\}.$$

The same information as the outcome of all measurements should be contained in the truth status of all hypotheses[7]: we can also formulate physics using a function

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[4]We will not make a strict destinction between syntax and semantics. In particular, we shall not work out the details of the syntactical side of the story. Instead we hope that our handwavy comments in this paragraph will be enough to transfer some intuitions about the formalism used in theories physics.

[5]The sort {outcomes} will be interpreted as some object of real numbers in a topos.

[6]In the sense of many-sorted logics.

[7]In an interpretation in a topos this is obtained from the last line by the counit of the exponential adjunction.

symbol

$$\{\text{systems}\}, \{\text{experiments}\}, \mathcal{P}\{\text{outcomes}\} \overset{\text{Hypothesis}}{\longrightarrow} \Omega.$$

Here, Ω is some abstract sort of truth values[8], the sort $\mathcal{P}\{\text{outcomes}\}$ represents the idea of an object of subobjects of $\{\text{outcomes}\}$ and the term $\text{Hypothesis}(X, Y, \Delta)$ should be read as the statement "If we perform experiment X on system Y, we will obtain a measurement value in $\Delta \in \mathcal{P}\{\text{outcomes}\}$.".

From a physical point of view it is clear that these pairings will generally be degenerate in some sense. Indeed, there might be two different experimental procedures (described in some alphabet) that yield the same measurement results on all physical systems or equivalently assign the same truth value to all associated hypotheses. In that case, we like to think that we are actually measuring the same 'physical quantity'. The equivalence classes under this equivalence relation should correspond 1-1 with what physicists call *observables*[9]. Let us write $\mathcal{A}$ for the sort of observables. Similarly, we would like to identify certain (descriptions of) systems, when they agree on all hypotheses, involving all experiments (or observables). We say that they are in the same *state*. We write $\mathscr{S}$ for the sort of all states. These identifications should induce pairings: i.e. function symbols

$$\mathcal{A}, \mathscr{S} \longrightarrow \{\text{outcomes}\}$$

and

$$\mathscr{S}, \mathcal{A}, \mathcal{P}\{\text{outcomes}\} \overset{\text{Proposition}}{\longrightarrow} \Omega.$$

of which the interpretation should be non-degenerate in some sense. What is important is the result that from purely practical, syntactical considerations it follows that states and observables should be dual to each other.

There is also a reasonable way to construct a logical structure out of these data. If we fix terms a of sort $\mathcal{A}$ and Δ of sort $\mathcal{P}\{\text{outcomes}\}$ in the function symbol Proposition, we obtain a function symbol

$$\mathscr{S} \overset{[a \epsilon \Delta]}{\longrightarrow} \Omega.$$

This should be interpreted as a subobject of the object of states[10]. We note that the propositions of this form should give rise to a kind of Lindenbaum-Tarski algebra related to the theory of physics, generated by the operations induced from $\text{Sub}\mathscr{S}$.

3 Canonical Formalism (Semantics)

3.1 C*-algebras and their States in Physics

Of course, we will not be able to derive many consequences from the fuzzy fundaments of the previous section. We have to narrow down the concepts of observable and state. We therefore turn to the theory of C*-algebras, where these ideas take concrete form. The formalisms we will set up in this section for classical and quantum kinematics are among the canonical ones, being particularly popular among

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[8]In the case of classical mechanics, it is clear that it should just be the set $\{\bot, \top\}$. In our topos interpretation of quantum kinematics (i.e. all matters accept dynamics) it will be the subobject classifier of another topos.

[9]One can compare the distinction between experiments and observables with that between functions in intension and functions in extension.

[10]In the formalism we will set up, we will also interpret these propositions as certain specific observables. The idea is that in practice (in the interpretation) we have a mono $\Omega \hookrightarrow \{\text{outcomes}\}$ (for instance, $\{0, 1\} \hookrightarrow \mathbb{R}$) so we can identify subobjects of $\mathscr{S}$ with certain arrows $\mathscr{S} \longrightarrow \{\text{outcomes}\}$. If we choose our object of observables big enough, we may hope that these arrows are included. (Recall that under currying we have a map $\mathcal{A} \longrightarrow \{\text{outcomes}\}^{\mathscr{S}}$.)

mathematical physicists. The reader might want to think of them as an attempt at constructing a semantics representing the core syntax we presented in the previous paragraph.

There exist good physical (operational) arguments to support the idea that the set of observables can be embedded in (the set of self-adjoint elements of) a C*-algebra (over $\mathbb{C}$), which we shall also denote by $\mathcal{A}$. Here a polynomial $p(a)$ in $a \in \mathcal{A}$ gets the interpretation of first performing a measurement of a and then applying the polynomial to the measurement outcome. The operational interpretations of $a + b$ and $a \cdot b$ for arbitrary observables a and b are more subtle. The measurement of the first observable might alter the state of the system we are trying to measure and therefore disturb the measurement of the second observable. This certainly does not agree with the interpretation of the sum of observables we have in existing theories of physics. Identically prepared states do not really solve this problem either, according to the various no-cloning theorems we have in physics (e.g. [15]).

In this picture states ω are interpreted to give the expectation value $\omega(a)$ for measurements of observables a on a corresponding system. One can argue that these should form linear functionals on $\mathcal{A}$. (Linearity over polynomials in one observable at least should be clear.) To agree with the physical intuition of an expectation value, we want states to be positive functionals, i.e. they should yield positive results on positive elements of $\mathcal{A}$ (i.e. elements of the form a^*a). Then the self-adjoint elements of $\mathcal{A}$ will be the candidates for observables. In this interpretation the zero-th power of any observable should have expectation value 1 for any state. Indeed, $\omega(a) = \omega(aa^0) = \omega(a)\omega(a^0) = \omega(a)\omega(1)$, for all $\omega \in \mathscr{S}$, $a \in \mathcal{A}$. This means that $\omega(1) = 1$, for all ω. This leads to the following definition.

Definition 1 (State on a C*-algebra). With a *state on a C*-algebra* we mean a positive normalised linear functional. It is easily checked that these form a convex set. The extreme points of this set are called *pure state*, while the term *mixed state* is used to refer to a state that might not be pure[11].

One can show that $\|a\| = \sup_{\omega \in \mathscr{S}} |\omega(a)|$. Moreover, it is easily verified that the states separate the elements of $\mathcal{A}$ and conversely, as desired. I hope that this brief digression gives the reader enough motivation to believe that a C*-algebra framework is not too restrictive for a theory of physics. An excellent motivation of the use of C*-algebras in physics (including all the details I omitted) can be found chapter 1 of [41].

3.2 Classical and Quantum C*-algebras

As it will turn out, the C*-algebra of observables of classical mechanics should be commutative, while we use a non-commutative C*-algebra to describe quantum mechanics. Strocchi [41] gives a very nice argument for this fact, starting from Heisenberg's uncertainty relation. It goes as follows. Recall we gave a state ω on a C*-algebra $\mathcal{A}$ the following physical interpretation: if a is a self-adjoint element of $\mathcal{A}$, then we interpret $\omega(a)$ as the expectation value of a measurement of A on a system in state ω. Note that therefore the variance $\Delta_\omega(a)^2$ would be $\omega(a^2) - \omega(a)^2$.

Now, Heisenberg's uncertainty relation states that

$$\Delta_\omega(q_j)\Delta_\omega(p_j) \geq \hbar/2,$$

where we write q_j and p_j for respectively the position and momentum observables in the j-th direction. Since the bound given by the Heisenberg uncertainty relation

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[11]This terminology reflects the physics. Pure states represent the physical idea of a state of which we have complete knowledge. That is, as far as our theory of physics allows this. (E.g. in quantum mechanics, even pure states exhibit statistical behaviour.) Mixed states represent generally statistical (ignorance) ensembles of states.

is independent of the state, it is natural to assume that its origin can be found in the algebra of observables.

Let $a, b \in \mathcal{A}$ such that $a^* = a$ and $b^* = b$. Since $(a - i\lambda b)(a + i\lambda b)$ is a positive element, positivity of ω gives us that

$$\omega(a^2) + |\lambda|^2 \omega(b^2) + i\lambda \omega([a, b]) \geq 0,$$

where $[a, b] := ab - ba$. This tells us that the last term is real and therefore, by positive definiteness of the quadratic form in λ $4\omega(a^2)\omega(b^2) \geq |\omega(i[a, b])|^2$ or, put differently,

$$\Delta_\omega(a)\Delta_\omega(b) \geq |\omega([a, b])|/2.$$

It is natural to demand that this algebraic bound that follows from the mathematical formalism coincides with the experimentally determined bound given by the Heisenberg uncertainty relation: $\hbar = |\omega([q_j, p_j])|$, for all states ω. This implies that

$$[q_j, p_k] = \pm i\hbar\delta_{jk}1,$$

where δ_{jk} denotes the Kronecker delta. Therefore, our conclusion is that the algebra of quantum observables should be a non-commutative one[12].

Our algebra of observables is a very abstract entity. As is often the case in algebra, we would like to think of the elements of our algebra as functions of some kind. The typical example of a commutative C*-algebra one might come up with is perhaps $\mathbb{C}$ and if one has a bit more fantasy some algebra of bounded functions from a fixed space to $\mathbb{C}$. In the non-commutative case one would perhaps think of an algebra of matrices with complex coefficients or more generally an algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ of bounded operators on some Hilbert space $\mathcal{H}$. As it turns out, these are, in fact, the only examples. This was proved by Gelfand, Naimark (and I. Segal).

Theorem 3.1 ((Commutative) Gelfand-Naimark theorem,[32]). *Write* cCStar *for the category of commutative complex C*-algebras and* CHaus *for that of compact Hausdorff topological spaces. Then the functors*

$$\text{cCStar}^{op} \underset{\text{Max}}{\overset{C(-,\mathbb{C})}{\leftrightarrows}} \text{CHaus}$$

define an equivalence of categories, where $C(-, \mathbb{C}) \subset \text{Set}(-, \mathbb{C})$ *sends* Σ *to the algebra of continuous functions from* Σ *to* $\mathbb{C}$ *and* Max *sends an algebra A to its maximal spectrum, equipped with the Zariski topology and an algebra morphism* $A \xrightarrow{f} A'$ *to a continuous map* $\text{Max}(A') \xrightarrow{\text{Max}(f)} \text{Max}(A)$ *that sends a maximal ideal to its inverse image under* f.

Maximal ideals in a C*-algebra correspond precisely with *-homomorphisms to $\mathbb{C}$, or pure states. (See theorem 4.7.) When viewed this way, the maximal spectrum is commonly referred to as the Gelfand spectrum of the C*-algebra.

We like to think of this space Σ as the phase space from classical mechanics[13] and of $C(\Sigma, \mathbb{C})$ as the algebra of observables. Respecting the duality between states

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[12]This heuristic argument also shows one of the weaknesses of the C*-algebra formalism. Indeed, if one has two self adjoint elements $q, p \in \mathcal{A}$ such that $[q, p] = \lambda \cdot 1$, where λ is some constant, one easily derives that $\|q\| \|p\| \geq n|\lambda|/2$, for all $n \in \mathbb{N}$, i.e. this cannot happen. In practice this means that either q or p will be an unbounded operator, thereby falling outside our C*-algebra formalism. However, one can argue that by the practical restrictions on our measurements and because of relativistic considerations, the observables we measure in practice are in fact bounded approximations of q and p. Cf. the next footnote.

[13]A physicist might object that we the phase space in clssical mechanics is usually a cotangent bundle of some manifold Q and therefore not compact. The compactness of our space reflects the fact that we started out with a normed algebra of observables. This means that we restricted our

and observables, we should also explain what the geometric analogue of a mixed state is. The answer is given by a celebrated theorem by Riesz and Markov. This brings us back to the situation of states as probability distributions on the phase space that is common in (statistical) classical physics.

Theorem 3.2 (Riesz-Markov,[41]). *Let Σ be a compact Hausdorff space. Then, there is a 1-1 correspondence between states ω on $C(\Sigma, \mathbb{C})$ and radon measures*[14] *μ_ω on Σ, the correspondence being*

$$\omega(f) = \int_\Sigma f d\mu_\omega.$$

The construction for non-commutative C*-algebras is not nearly as nice, as it is non-functorial.

Theorem 3.3 (GNS-construction,[41]). *Any C*-algebra $\mathcal{A}$ is *-isomorphic to an algebra of bounded operators $\mathcal{A}' \subset B(\mathcal{H})$ on some*[15] *Hilbert space $\mathcal{H}$. There is a canonical*[16] *construction to realise this, called the (universal) GNS-representation.*

This construction is an argument commonly given in favour of the Hilbert space framework in which quantum mechanics is usually dealt with. States, as physicists know them, fit in this framework as follows.

Corollary 3.4. *Let $\mathcal{A}$ be a C*-algebra with any representation $\mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Then every trace 1 operator ρ on $\mathcal{H}$ defines a state ω_ρ of $\mathcal{A}$ by*

$$\omega_\rho(a) := \operatorname{tr}(\rho a).$$

Conversely, if we take π to be the universal GNS-representation every state in fact arises (non-uniquely) in such a way, the pure states precisely corresponding to traces against one dimensional projections, or if you will, points of the Hilbert space.

Given a general representation $\mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$, however, our notion of a state is strictly more general then that of states that are obtained from tracing against positive operators. The difference mostly lies in the fact that our states are only finitely additive, while in physics complete additivity is convention, i.e. for every mutually orthogonal family of projections $P_i \in \mathcal{A}$ $\omega(\sum_i P_i) = \sum_i \omega(P_i)$. Although our concept of a state might thus be a bit more general than the states physicists use in practice, it does not differ much on a conceptual level. On the other hand, it is a lot easier to deal with, mathematically. This exhibits the physical relevance of the abstract algebraic concept of a state on a C*-algebra which we shall be using in the rest of this paper.

3.3 Propositions

3.3.1 Classical Propositions

To motivate the rest of this paper, which shall mostly be on quantum logic, we give a brief account of the logic associated to classical mechanics.

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observables to be bounded functions on our phase space. This is a reasonable thing to do since we only expect our classical physics to be applicable in a relatively small region of space. (On large distances general relativistic effects become relevant.) Similarly, we do not expect classical physics to hold for particles with very high momentum. In that domain one would also have to use a theory of relativity. One way to think of the relation between our compact phase space Σ and the usual space T^*Q from physics is that $C^*(T^*Q, \mathbb{C}) \cong C(\Sigma, \mathbb{C})$, or equivalently $\Sigma = \operatorname{Max}(C^*(T^*Q, \mathbb{C}))$, where $C^*(\Sigma, \mathbb{C})$ is the C*-algebra of bounded functions $\Sigma \longrightarrow \mathbb{C}$. Note that this is just the definition of the Stone-Čech compactification! Since T^*Q is locally compact Hausdorff it embeds in Σ as an open subspace (by the unit of the Stone-Čech-adjunction). [32]

[14]That is, finite regular measures.

[15]That we can choose to be separable if $\mathcal{A}$ is.

[16]We take the sum of the GNS-representations corresponding to all pure states.

Let us approach the matter naively. When we do experiments on a classical mechanical system, we formulate hypotheses about it. It is not uncommon to manipulate these hypotheses by logical operations such as negations, disjunctions, conjunctions and, very importantly, if we want to test physical theories, implications[17]. The idea is that we should be able to translate these logical operations in our metalanguage to operations in the mathematical framework of classical mechanics. In section 2, we argued that it is reasonable to expect to embed propositions in the powerset of our set of states. In fact, we will embed them in the powerset of the set $\mathscr{S}$ of pure states. Naively, we would expect to find a logical structure in classical mechanics that reflects our classical intuition: a Boolean algebra. One might hope to realise this structure as a Boolean subalgebra of $\mathcal{P}\mathscr{S}$.

To proceed, we take an operational point of view again. A hypothesis about a classical mechanical system should be verifiable, or, at least, falsifiable[18]. It would therefore typically be a statement of the form "If we measure observable a on our system, the outcome will be in $\Delta \subset \mathbb{R}$.". This can equivalently be put in terms of geometry (Gelfand-Naimark, Riesz-Markov) as "The state of our system has support in $a^{-1}(\Delta) \subset \Sigma$."

Now, a mathematician would ask: "How free are we in the choice of Δ?". One reasonable answer is that Δ should be open in $\mathbb{R}$. This gives us $\mathcal{O}(\Sigma)$ as the object representing our classical logic. However, in general, this is only a Heyting algebra: the rule of the excluded middle fails. Heunen, Landsman, Spitters suggest in [29] that it *almost* holds: we would never be able to establish its failure by performing experiments. Indeed, they reason, if P is a proposition and $U \in \mathcal{O}(\Sigma)$ is the corresponding open set, then $U \cup (X \setminus U)^o$ would be the open set representing $P \vee \neg P$ and this set is topologically big (dense) X. However, seeing how physical probabilities correspond with certain integrals on Σ w.r.t. some Borel measure, I would say that being 'big' in a measure theoretic sense is a better criterion. Note that $U \cup (X \setminus U)^o$ might not have full measure to conclude that this is not the most reasonable logic we could associate to classical mechanics. The law of the excluded middle could really fail in a physically detectable way in this logic.

If one wants to end up with a Boolean algebra, therefore, it might be better to allow Δ to be a general Borel measurable subset of $\mathbb{R}$ and to replace $\mathcal{A} = C(\Sigma, \mathbb{C})$ by the C*-algebra of essentially bounded functions $L^\infty((\Sigma, \mu), \mathbb{C})$, with respect to some Borel measure μ on Σ. Then our logic is given by the Borel σ-algebra (modulo null-sets)[19].

Note that we have a canonical map $C(\Sigma, \mathbb{C}) \hookrightarrow L^\infty((\Sigma, \mu), \mathbb{C},)$ that is injective if μ is non-degenerate. [42] Moreover, the logic of our classical mechanics does not only follow from the geometry. It can equivalently be described in terms of algebra as the Boolean algebra $\Pi(\mathcal{A})$ of self-adjoint idempotents[20] in $L^\infty((\Sigma, \mu), \mathbb{C})$. These results can be stated more generally as the following non-trivial theorem. (In our particular case $\mathcal{H} = L^2((\Sigma, \mu), C)$ and $L^\infty((\Sigma, \mu), \mathbb{C}) \subset \mathcal{B}(\mathcal{H})$ (by multiplication) is our Von Neumann algebra.)

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[17]As we shall see the lack of a suitable notion of an implication is one of the major drawbacks of Von Neumann's quantum logic.

[18]On the level of logic, one can argue that this difference is not that important. Indeed, verification of a proposition would coincide with the falsification of its negation.

[19]Another obvious way of getting a Boolean algebra, that is mentioned in some references such as [29], would to consider all subsets of Σ. However, this choice does not seem to reflect very well what can be established by experiments, according to the measure theoretic interpretation of states given by the Riesz-Markov theorem.

[20]One can check that these form a complete Boolean algebra, where the meet is given by the ring multiplication. [39] In this case thse are of course precisely the characteristic functions of measurable subsets of Σ.

Definition 2 (Von Neumann algebra)**.** A C*-subalgebra is called a Von Neumann algebra if it admits an embedding into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, such that it is equal to its double commutant. We will think of of Von Neumann algebras as a non-full subcategory[21] VNeu of CStar where the $*$-homomorphisms are additionally required to be continuous in the ultraweak topologies[22].

In this context it might be easiest to think of Von Neumann algebras as certain C*-algebras that have enough self adjoint idempotents. In fact, these generate a Von Neumann algebra in the sense that the double commutant of $\Pi(\mathcal{A})$ is $\mathcal{A}$. [42] This abundance of projections[23] will make our life easier in many ways[24].

The step of passing from $C(\Sigma, \mathbb{C})$ to $L^\infty((\Sigma, \mu), \mathbb{C})$ is a bit unsatisfactory, since it depends on the choice of a measure μ. The motivation for this construction was that we wanted to add self-adjoint idempotents (specifically, characteristic functions of measurable sets) to the algebra of observables, so we could obtain an interesting logic from the algebra. As it turns out, there is a more canonical way of doing this.

Theorem 3.5 ([34],[42])**.** *VNeu $\hookrightarrow$ CStar is a (non-full) reflective subcategory. Its reflector is called the universal enveloping Von Neumann algebra construction. It is given by applying the universal GNS-representation and taking the bicommutant, or equivalently, taking the second continuous dual space.*

The result for our particular case $\mathcal{A} = C(\Sigma, \mathbb{C})$ indeed reflects our intuition.

Theorem 3.6. *Recall that the Radon measures on Σ form a preorder under the relation of absolute continuity. $L^\infty((\Sigma, -), \mathbb{C})$ defines a functor from the opposite of this preorder to the category of complex Banach spaces. Then $C(\Sigma, \mathbb{C})^{**} \cong \varprojlim L^\infty((\Sigma, -), \mathbb{C})$. This is a Von Neumann algebra under the induced multiplication.*

Proof (sketch). It is well known from the construction of Riesz-Markov that $C(\Sigma, \mathbb{C})^* \cong \varinjlim L^1((\Sigma, -), \mathbb{C})$, where $L^1((\Sigma, -), \mathbb{C})$ is again a functor from the opposite of the preorder of Radon measures to the category of complex Banach spaces. Now, the continuous dual space functor sends colimits to limits (this is easy to see if one considers it as a contravariant endofunctor of the category of topological vector spaces, of which that of Banach spaces is a full subcategory) and therefore $C(\Sigma, \mathbb{C})^{**} \cong \varprojlim (L^1((\Sigma, -), \mathbb{C}))^*$. Now, according to [42], $(L^1((\Sigma, -), \mathbb{C}))^* \cong L^\infty((\Sigma, -), \mathbb{C})$. Finally, as a consequence of Hahn-Banach $C(\Sigma, \mathbb{C})$ is w*-dense in its bidual space. Therefore, by continuity the multiplication on $C(\Sigma, \mathbb{C})^{**}$, that we know to exist by the previous theorem, has to coincide with the one induced from the multiplications on each of the $L^\infty((\Sigma, \mu), \mathbb{C})$. $\square$

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[21] We will not think of a Hilbert space as being part of the data of a Von Neumann algebra. The reader should note that in literature these Von Neumann algebras without an embedding in some $\mathcal{B}(\mathcal{H})$ are also known as W*-algebras.

[22] This is the initial topology with respect to the family of maps $B(\mathcal{H}) \overset{\text{tr}(\rho-)}{\longrightarrow} \mathbb{C}$, for trace class operators $\rho \in B(\mathcal{H})$. It turns out that this topology does not depend on the choice of the Hilbert space[42]

[23] In the light of the universal GNS-representation, it is reasonable to call self-adjoint idempotents projections.

[24] To name a few things:

1. By the spectral theory for Von Neumann algebras, we can extend many results about projections to all normal elements, so in particular to all observables.

2. The projections will represent the propositions about our physical system. The abundance of projections tells us that this logic contains a lot of information about the system.

3. This logic will also be convenient from a calculational point of view: it is a complete lattice.

Note that this space contains $\bigcap_\mu L^\infty((\Sigma, \mu), \mathbb{C})$, so in particular all bounded measurable functions from Σ to $\mathbb{C}$.[25] Therefore, the characteristic functions define an order embedding of the Borel sigma-algebra into the self-adjoint idempotents of this Von Neumann algebra[26]. So effectively, what we have done is add those characteristic functions to $C(\Sigma, \mathbb{C})$ and embed the result in a C*-algebra. One might wonder how the abundance of projections in a Von Neumann algebra translates into properties of the spectrum.

Theorem 3.7 (Gelfand-Duality for commutative Von Neumann algebras). *The equivalence of theorem 3.1 that restricts to an equivalence between the opposite of the category of commutative Von Neumann algebras and the category of hyperstonean[27] spaces, where the maps are open continuous functions. [42]*

To accentuate the relation with measure theory, we also mention the following classification.

Let Σ be a locally compact Hausdorff space and μ a radon measure on Σ, then $L^\infty((\Sigma, \mu), \mathbb{C})$ is a Von Neumann algebra. Conversely, every Von Neumann algebra arises in this way. [40]

From the fact that the Gelfand spectrum of a Von Neumann algebra is a Stone space, one finds a second way of constructing a Boolean algebra out of it. Recall Stone's representation theorem for Boolean algebras.

Theorem 3.8 (Stone-Duality,[32]). *Write* Bool *for the category of Boolean algebras and homomorphisms and write* Stone *for the category of totally disconnected[28] compact Hausdorff spaces and continuous functions. Then the functors*

$$\mathrm{Bool}^{op} \overset{\mathcal{P}_{cl}}{\underset{\mathrm{Spec}}{\leftrightarrows}} \mathrm{Stone}$$

define an equivalence of categories, where $\mathcal{P}_{cl} \subset \mathcal{P}$ sends Σ to the Boolean algebra of clopen subsets and Spec *sends a Boolean algebra B to its prime spectrum, equipped with the Zariski topology and an algebra morphism $B \overset{f}{\longrightarrow} B'$ to a continuous map $\mathrm{Spec}(B') \overset{\mathrm{Spec}(f)}{\longrightarrow} \mathrm{Spec}(B)$ that sends an ideal to its inverse image under f.*

It turns out that this is this gives us $\Pi(\mathcal{A})$ again. We formulate this in terms of their spectra.

Theorem 3.9 ([4]). *The Gelfand spectrum of a Von Neumann algebra coincides with the Stone spectrum of the Boolean algebra of its self-adjoint idempotents:* $\mathrm{Spec} \circ \Pi = \mathrm{Max} : \mathrm{VNeu}^{op} \longrightarrow \mathrm{Stone}$

In this sense, (commutative) Von Neumann algebras are suitable algebras of observables for classical mechanics, from a logical point of view. We can reconstruct the observables from their associated logic: $C(\mathrm{Spec}(\Pi(\mathcal{A})), \mathbb{C}) \cong \mathcal{A}$.

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[25]Of course, it contains a lot more. For instance, objects that do not have an interpretation as a function.

[26]The self-adjoint idempotents of a Von Neumann algebra are a complete lattice, while the Borel sigma-algebra may only be countably complete. So this embedding is some sort of completion (by objects that do not have an interpretation as subsets of Σ).

[27]i.e. extremally disconnected and admitting a perfect measure. What is important here is that it is a stronger condition that being Stone (see below).

[28]Spaces that have no non-trivial connected subspaces.

3.3.2 Von Neumann's Quantum Propositions

The construction in the previous paragraph was a contrived such that it would yield a logic for classical mechanics that agrees with our classical (Boolean) intuition. Things get more interesting if we try to do something similar for quantum mechanics, as it is not a priori clear that our classical logic is suitable at all to deal with statements about quantum mechanical systems. The first thorough account of such a quantum logic was given by Birkhoff and Von Neumann in [5]. The approach that is now standard is only a minor alteration of their original one[29]. [25]

Inspired by the discussion of classical logic, we might postulate that the poset $\Pi(\mathcal{A})$ of self-adjoint idempotents of our C*-algebra of quantum observables $\mathcal{A}$ should model quantum logic, playing the role of a Lindenbaum-Tarski algebra. Mimicking the situation in classical mechanics, we define a partial order on $\Pi(\mathcal{A})$ by $a \leq b$ iff $ab = a$. (It then follows that $ba = b^*a^* = (ab)* = a* = a$.)[30] However, in general, $\mathcal{A}$ might not contain enough projections to give rise to an interesting logic, just like $C(\Sigma, \mathbb{C})$ did not in the classical case. Therefore, we should embed $\mathcal{A}$ in some Von Neumann algebra, the quantum analogue of $L^\infty(\Sigma, \mathbb{C})$, if you will.

One way of doing is this, if $\mathcal{A}$ arises as an subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is just taking its double commutant. A canonical method would be using the universal GNS-representation for $\mathcal{A}$ to embed it in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, thus computing the universal enveloping Von Neumann algebra. Then we have the following.

Theorem 3.10 ([39]). *The lattice $\Pi(\mathcal{A})$ of projections in a Von Neumann algebra $\mathcal{A}$ is a complete orthomodular[31] lattice, w.r.t. its natural order. Moreover, it is distributive if and only if $\mathcal{A}$ is commutative. In particular, in the non-commutative case, the adjoint functor theorem tells us that it does not have an implication.*

Proof. All properties are straightforward verifications, except the completeness. This is an elementary consequence of the (non-trivial) bicommutant theorem, which I have therefore used to define what a Von Neumann algebra is. □

Note that a faithful representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ defines an order embedding $\Pi(\mathcal{A})$ into the Grassmannian $\Pi(\mathcal{H})$ of closed linear subspaces of $\mathcal{H}$. We see that this quantum logic resembles the situation in classical mechanics quite closely in terms of geometry: the propositions are given by the closed linear subspaces of $\mathcal{H}$ (rather than the measurable subsets). The meet is given by intersection and the join by closed linear span. Although an implication is lacking, Birkhoff and Von Neumann did define a negation on $\Pi(\mathcal{A})$, given by the orthocomplement (or, $p \mapsto 1 - p$ in terms of algebra).

Like we anticipated in our informal syntactical discussion in section 2, this logic should play the role of a Lindenbaum-Tarski algebra and its propositions should arise as (in some sense provable) equivalence classes of sequents $\top \vdash a\epsilon\Delta$, for variables a of sort $\mathcal{A}$ and Δ of sort $\mathcal{PR}$. In practice this goes as follows.

Let $a \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $a^* = a$ be a quantum observable and let Δ be a Borel measurable subset of $\mathbb{R}$. Then we can form a proposition $[a\epsilon\Delta] \in \Pi(\mathcal{A})$ as the projection in the spectral decomposition of a corresponding to $\Delta \cap \sigma(a) \subset \sigma(a)$, where $\sigma(a)$ denotes the spectrum of a. Conversely, every $p \in \Pi(\mathcal{A})$ is of this form. Indeed, take $a = p$ and $\Delta = \mathbb{R}$.

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[29]It is now generally accepted that quantum logic should be only orthomodular, not modular, i.e. distributivity fails in an even stronger sense.

[30]Geometrically speaking, i.e. if $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, this order is the inclusion of subspaces.

[31]This weakend law of distributivity says that if $A \leq B$ and $A^\perp \leq C$, then $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$, where we write $A^\perp$ for $1 - A$ (in the algebraic sense). Note that is not a Heyting implication.

A first indication for the physical interpretation of this logic is found in the following nice but quite non-trivial analogy[32] with the Riesz-Markov representation of states from classical mechanics.

Theorem 3.11 (Mackey-Gleason,[8, 7, 9] (original references),[26] (complete proof))**.** *Let $\mathcal{A}$ be a Von-Neumann algebra. Every state ω on $\mathcal{A}$ restricts to a finitely additive measure on $\Pi(\mathcal{A})$. That is, a map $\Pi(\mathcal{A}) \xrightarrow{\mu_\omega} [0,1]$ such that*

1. $\mu_\omega(\top) = 1$;

2. $\mu_\omega(x) + \mu_\omega(y) = \mu_\omega(x \vee y)$,   whenever $x \wedge y = \bot$.

If $\mathcal{A}$ has no type I_2-summands, this assignment defines a bijection between states[33] on $\mathcal{A}$ and finitely additive measures on $\Pi(\mathcal{A})$.

We see that the choice of a state associates a number to every quantum proposition. The Born interpretation (one of the cornerstones of quantum mechanics that gives the formalism its empirical content) precisely states that these numbers should be interpreted as (frequentist) probabilities and that[34], for a state $\mathcal{A} \xrightarrow{\omega} \mathbb{C}$, the probability of measuring a value in $\Delta \subset \mathbb{R}$ for the observable a is

$$\mathrm{Prob}_\omega(a\epsilon\Delta) = \mu_\omega([a\epsilon\Delta]) = \omega([a\epsilon\Delta]).$$

As it turns out, we identify a quantum proposition with all the pure states for which it is true with certainty. The Born rule tells us that, for a pure state, which we shall represent by a unit vector $|\psi\rangle \in \mathcal{H}$, this probability is

$$\mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = \langle\psi|[a\epsilon\Delta]|\psi\rangle.$$

We see that

$$[a\epsilon\Delta]\mathcal{H} = \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = 1\}.$$

Similarly,

$$[a\epsilon\Delta]^\perp\mathcal{H} = \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = 0\}$$
$$= [a \not\epsilon\Delta]\mathcal{H} = \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a \not\epsilon\Delta) = 1\}.$$

Note that the negation $p^\perp$ does not say "p is not true", but rather "p is false!". (In jargon: we have a choice negation, rather than an exclusion negation.)[35] We see that pure states do not define morphisms $\Pi(\mathcal{A}) \longrightarrow \{0,1\}$, as one might expect from experience with classical mechanics[36]. Some propositions are neither true nor false for a state $|\psi\rangle$. We will later see that even more is true: even if we replace $\{0,1\}$ by some other Boolean algebra (or a general distributive lattice, for that matter) and lower our expectations we still can't assign truth values to the propositions. (Jargon: quantum logic fails to admit Boolean-valued models.) We see that an ignorance interpretation of quantum probabilities is an unacceptable point of view.

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[32]One should immediately note however, that a big difference is that quantum pure states do not correspond to the two valued measures: even pure states exhibit non-deterministic behaviour.

[33]In fact, it always defines a bijection between quasi-states and these finitely additive measures. (See definition 3)

[34]This is a rather reasonable assumption, since this is precisely our interpretation in the case of classical mechanics.

[35]All propositions thus are of the form $\mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = \lambda$, where λ is 0 or 1. Other probabilities are not included in the logic. One might argue that this makes this quantum logic a bit crude, as propositions with other values for λ would be just as valid from an operational point of view. However, there still is no consensus as to how to include propositions of this form in the logic. [39]

[36]Indeed, there pure states define a homomorphism of complete Boolean algebras from the logic to the two-value Boolean algebra.

(See theorem 4.3.) Surprisingly, quantum logic does satisfy the law of the excluded middle: $p \vee p^{\perp} = \top$.

One sees the origin of this curiosity, not by investigating at the meet, which is perfectly well-behaved,

$$[a\epsilon\Delta] \wedge [a'\epsilon\Delta']\mathcal{H} = \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = 1 \text{ and } \mathrm{Prob}_{|\psi\rangle}(a'\epsilon\Delta') = 1\}$$
$$= \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta \text{ and } a'\epsilon\Delta') = 1\}),$$

but rather the join

$$[a\epsilon\Delta] \vee [a'\epsilon\Delta']\mathcal{H} = \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta \text{ or } a'\epsilon\Delta') = 1\})$$
$$\neq \{|\psi\rangle \in \mathcal{H} \,|\, \mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta) = 1 \text{ or } \mathrm{Prob}_{|\psi\rangle}(a'\epsilon\Delta') = 1\}.$$

Even if $\mathrm{Prob}_{|\psi\rangle}(a\epsilon\Delta)$ is neither 0 nor 1, i.e. $|\psi\rangle \notin [a\epsilon\Delta]\mathcal{H}$ and $|\psi\rangle \notin \neg[a\epsilon\Delta]\mathcal{H}$, then still $|\psi\rangle \in ([a\epsilon\Delta] \vee \neg[a\epsilon\Delta])\mathcal{H} = \mathcal{H}$. We see that there is some friction between this quantum concept of truth and our intuition.

Now, we should stop to think about the issue of epistemology versus ontology: can we give an operational meaning to these propositions? Since measurements in general destroy a quantum state, this is a subtle issue, most notably the interpretation of the conjunction and disjunction.

The principal problem is that we cannot perform many measurements in a successive fashion to verify or falsify our proposition, as measurements change our quantum state. Ideally, therefore, we would like to prepare many copies of the same state and perform parallel experiments. Unfortunately, due to various no cloning theorems this is still not possible in principle. [36] We can however prepare almost identical states[37] and perform experiments on them. (See for instance [24] for the theoretical principle and [12] for a recent experimental realisation of the procedure.) In this way, we can make sense of a probabilistic interpretation of quantum mechanics.

However, the issue of conjunctions and disjunctions remains: which proposition do we verify first? (In quantum mechanics, this will make a big difference[38]!) Of course, $\wedge$ and $\vee$ should be symmetric, so there seems to be a problem. The way out of this, proposed by Piron and Jauch, is to interpret $p \wedge p'$ operationally as "Randomly choose to perform an experiment for either p or p'.". By executing this procedure many times on almost identical states we can hope to verify or falsify this conjunction. We interpret the disjunction in a similar way. [2]

Finally, one might wonder if we can again reconstruct $\mathcal{A}$ from $\Pi(\mathcal{A})$. Put bluntly, does this logic contain all the interesting information about our physics? The answer is yes[39], if we start out with $\mathcal{A}$ as a Von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and we remember the embedding $\Pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$. Then, a non-trivial result from functional analysis tells us that $\mathcal{A}$ is isomorphic to the double commutant of $\Pi(\mathcal{H})$. [42] However, without this extra information, it is not immediately clear how one would reconstruct $\mathcal{A}$. In that sense, the result is a bit unsatisfactory when one compares it to its classical analogue.

Summarising, we have found that quantum logic differs from the classical logic (that we can derive from classical mechanics) in the following respects.

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[37]To an arbitrary precision, in theory.

[38]I am talking about the case that of two propositions that involve non-commuting observables. Unless we are in a common eigenstate by coincidence, these observables are not comeasurable.

[39]This is also true in another sense. If we know the truth value of all propositions, we know the support of our state. This means that it uniquely determines our state if and only if it is pure, exactly like in classical kinematics.

1. $\lor$ and $\land$ do not distribute over each other.

2. Therefore, there exists no Heyting implication.

3. The interpretation of $(-)^\perp$ is not what we are used to: for a state $|\psi\rangle$, truth of $p^\perp$ should be interpreted as "p is false", rather than as "p is not true".

4. Pure states do not determine the truth of all quantum propositions. Because the notion of truth is too crude, they do not determine an interesting[40] map from our quantum lattice to some set of truth values (rather than probabilities). Jargon: there is no satisfactory state-proposition pairing.

5. To say more, there cannot be such a map to a Boolean algebra of truth values if we require it to preserve the logical structure on the Boolean sublogics.

6. The interpretation of $\lor$ is not what we are used to: for a state $|\psi\rangle$, $p \lor p'$ can be true, while neither p nor p' is true. In particular, the law of the excluded middle holds.

7. The operational interpretation of various propositions is a bit subtle.

8. It is not obvious how one can reconstruct the algebra of observables from this quantum logic.

Of course, this does not say that quantum logic is not correct! One has to be careful though, when interpreting quantum propositions.

4 Topos Quantum Kinematics (Semantics)

4.1 A Topos for Quantum Kinematics

4.1.1 Bohr's Doctrine of Classical Contexts

We hope that this discussion of conventional quantum logic has convinced the reader that a new notion of quantum kinematics that is closer to our classical intuition might lead to new insights. Why would one expect that topos theory could be of any use here? Probably, judging by the introductions given in [10],[20] and [29] the primary motivation can be found in the need for a suitable object of truth values for the quantum logic. The sentiment seems to be that the all or nothing distinction that is made is Birkhoff-Von Neumann quantum logic does not do justice to the subtleties of quantum probabilities. The hope is of course, that this can be realised as the subobject classifier of a suitable topos.

As I will argue, in pursuit of this goal, the topos approaches to quantum kinematics build a framework in a topos that presents the interplay between observables, states and propositions in a way that stays very close to what we know from classical mechanics. Put bluntly, the distinction between classical and quantum kinematics is reduced to the choice of a topos. Therefore, the next question to address is: "If quantum mechanics does not take place in Set, which topos should we choose?".

The papers of both Butterfield and Isham and those of the Landsman et al. motivate this choice from a vague philosophy known as *Bohr's doctrine of classical contexts*. Bohr once phrased this as follows. [6]

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[40] Of course, one can choose to call the propositions 'true' that are true with probability 1, while one says the rest is 'false'.

> However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. (...) The argument is simply that by the word experiment we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangements and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics.

We obtain information about a quantum system by investigating it in various classical contexts. In the mathematical formalism, these classical contexts are represented by commutative subalgebras of our algebra of observables. The idea that these classical contexts should contain a lot of information about a quantum system (at least in the case $\dim \mathcal{H} > 2$!) is made rigourous by the following much more recent results[41].

Apparently, assigning a value to all observables in such a way that it is consistent in each classical context is already too much to ask for:

Theorem 4.1 (Kochen-Specker (Observable version),[16], Theorem 2.4). *Suppose $\mathcal{A}$ is a Von Neumann algebra with no type I_1 and I_2 summands. Then, there does not exist a map $\mathcal{A} \longrightarrow \mathbb{C}$ that restricts to a *-homomorphism on each commutative Von Neumann subalgebra.*

Similarly, as a corollary of the solution to the Mackey-Gleason problem (theorem 3.11), we see that a state is immediately determined if we know it in all classical contexts.

Definition 3 (Quasi-State). Let $\mathcal{A}$ be a Von Neumann algebra. Then we understand a quasi-state on $\mathcal{A}$ to be a map $\mathcal{A} \xrightarrow{\omega} \mathbb{C}$ with the properties that its restriction to each commutative Von Neumann subalgebra is a state and that $\omega(a + ia') = \omega(a) + i\omega(a')$ for all self-adjoint $a, a' \in \mathcal{A}$.

Theorem 4.2 (Gleason,[8]). *Let $\mathcal{A}$ be a Von Neumann algebra without type I_2 summand. Then any quasi-state on $\mathcal{A}$ is actually a state.*

Finally, we have a similar phenomenon on the logical level:

Theorem 4.3 (Kochen-Specker (Logical version[42]), easy corollary of results in [16]). *Suppose $\mathcal{A}$ is a Von Neumann algebra with no type I_1 and I_2 summands. Then, there does not exist a map $\Pi(\mathcal{A}) \longrightarrow L$ into a non-trivial distributive lattice L that is a lattice homomorphism when restricted to each Boolean subalgebra of $\Pi(\mathcal{B}(\mathcal{H}))$.*

Proof. Suppose we do have such a map $\Pi(\mathcal{A}) \xrightarrow{q} L$. Then, we have a lattice homomorphism $L \xrightarrow{ay} \mathrm{Spec}(L)$ that embeds the lattice (as the principal ideals) in its frame of ideals. Seeing that this is a frame, it has a natural structure of a (complete) Heyting algebra. This means that we have a homomorphism of Heyting algebras $\mathrm{Spec}(L) \xrightarrow{\neg\neg} \mathrm{Compl}(\mathrm{Spec}(L)) =: B$ to its Boolean subalgebra B of complemented elements. Then, pick a point p of B (a homomorphism to $\{\bot, \top\}$). (We can do this since non-triviality of L implies that B is non-trivial and every non-trivial Boolean

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[41]One should note that the Kochen-Specker theorem actually emphasises a fundamental difference between the classical and the quantum world. Paradoxically enough, it will serve as an important motivation for the topos approach to quantum kinematics that tries to stress analogies with classical kinematics.

[42]To make this sound more like a statement about logic, the usual, weaker statement of this theorem is that there does not exist a $\Pi(\mathcal{A}) \longrightarrow B$ into a non-trivial Boolean algebra B that is a morphism of Boolean algebras when restricted to each Boolean subalgebra of $\Pi(\mathcal{B}(\mathcal{H}))$.

algebra has points (the points of its Stone space). Then, $p \circ \neg\neg \circ ay \circ q$ defines a two-valued finitely additive probability measure on $\Pi(\mathcal{A})$. According to Lemma 2.7 in [16] this extends to a *-homomorphism $\mathcal{A} \longrightarrow \mathbb{C}$, which we know not to exist by theorem 4.1. $\qquad\square$

The motivating idea for topos quantum kinematics will therefore be that we only investigate quantum systems by probing them in these classical contexts. Recall that Set is the topos we use to describe classical mechanics. The idea rises that the 'quantum topos' might therefore be related to Set. Its objects should represent 'things that we can probe by classical contexts to obtain something in Set'. Phrasing the doctrine of classical contexts this suggestive way strongly suggests that our topos should be a topos of certain presheaves[43] over some category of classical contexts.

We have reached the point where the two approaches to topos quantum theory part ways. Butterfield, Isham and Doering model a quantum system by a (non-commutative) Von Neumann algebra $\mathcal{A}$ and consider the category of presheaves over the poset $\mathcal{V}(\mathcal{A})$ of commutative Von Neumann subalgebras (ordered by inclusion) as their quantum topos[44]. Heunen, Landsman, Spitters et al. in contrast start out with a general C*-algebra $\mathcal{A}$ and construct their quantum topos as the category of presheaves over $\mathcal{C}(\mathcal{A})^{op}$, where $\mathcal{C}(\mathcal{A})$ denotes the poset of commutative C*-subalgebras of $\mathcal{A}$[45].

4.1.2 Which subalgebras?

We should immediately ask how much information we lose by passing from the non-commutative algebra of observables to the poset of its commutative subalgebras. In the case of Von Neumann algebras a partial answer has been given by Doering:

Theorem 4.4 ([19]). *Suppose $\mathcal{A}, \mathcal{A}'$ are Von-Neumann algebras without type I_2 summands[46]. Then, for each order-isomorphism $\mathcal{V}(\mathcal{A}) \xrightarrow{f} \mathcal{V}(\mathcal{A}')$ there exists a unique Jordan *-isomorphism[47] $\mathcal{A} \xrightarrow{g} \mathcal{A}'$ such that for all $A \in \mathcal{V}(\mathcal{A})$ $f(A) = g(A)$.*

One might expect that we do not preserve the commutator and with that, by Heisenberg's equation, the dynamics. Fortunately, we do not lose more than that[48]! We can therefore hope to formulate a good theory of quantum kinematic in this topos framework.

If we start out with a general C*-algebra $\mathcal{A}$, however, one might suspect that the Von Neumann subalgebras do not contain enough information. This can be seen from the following result of Heunen. Indeed, an arbitrary C*-algebra need not even have any projections.

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[43]Indeed, one common way of interpreting the Yoneda lemma is as the statement that presheaves over a category $\mathcal{C}$ are entities modeled on $\mathcal{C}$, entities that we can get to know by probing them by objects of $\mathcal{C}$.

[44]It should be noted that these authors have introduced many different approaches in their fairly broad programme of topos theoretic descriptions of physics. We shall, however, restrict to their account of quantum mechanics that connects best with that of Heunen, Landsman and Spitters.

[45]The reader might object that in this way the topos we use for quantum theory depends on the algebra $\mathcal{A}$ and therefore on the particular system we are considering. On the other hand Set is used for *all* classical mechanical systems. This is indeed a strange distinction. However, one might say that '*the* quantum topos' should be the one where we take $\mathcal{A}$ to be the algebra of observables corresponding to all measurable quantities in the universe.

[46]i.e summands of the form $\mathcal{B}(\mathbb{C}^2)$.

[47]A $*$-algebra homomorphism that need only preserve the symmetric product, rather than the whole product.

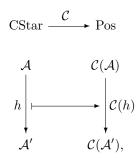[48]Of course, this theorem does not say that we indeed lose the commutator. In fact, we do remember if two elements commute or not. It is not obvious, however, how one should reconstruct the precise value of the commutator if elements do not commute.

Theorem 4.5 (Practically[49], theorem 4 in [27])**.** *Let $\mathcal{A}$ be a C*-algebra. Then $\mathcal{V}(\mathcal{A})$, together with the inclusion functions rather than just the order relation[50], and $\Pi(\mathcal{A})$ contain the same information. That is, on the one hand the Boolean algebras of projections corresponding to elements of $\mathcal{V}(\mathcal{A})$ unite into one non-distributive lattice, using Kalmbach's Bundle lemma:*

$$\Pi(\mathcal{A}) \cong \varinjlim_{A \in \mathcal{V}(\mathcal{A})} \Pi(A), \quad (\text{in Pos})$$

and on the other hand, we can retrieve each commutative Von Neumann subalgebra of $\mathcal{A}$ (up to isomorphism) as the continuous functions on the Stone spectrum of a Boolean subalgebra of $\Pi(A)$.

Apart from showing us that we can recover the Birkhoff-Von Neumann quantum logic from our new topos framework of quantum logic, we also see that it is natural to consider the poset of all C*-subalgebras in case we are dealing with an arbitrary C*-algebra, since the Von Neumann subalgebras simply do not contain enough information. A recent result by Nuiten leads us to believe that the C*-subalgebras actually contain a serious amount of information about the C*-algebra we started out with. To state this result, we first note that our definition of $\mathcal{C}$ on objects extends to a functor

$$\text{CStar} \xrightarrow{\;\mathcal{C}\;} \text{Pos}$$

$$
\begin{array}{ccc}
\mathcal{A} & & \mathcal{C}(\mathcal{A}) \\
h \downarrow & \longmapsto & \downarrow \mathcal{C}(h) \\
\mathcal{A}' & & \mathcal{C}(\mathcal{A}'),
\end{array}
$$

where $\mathcal{C}(h)$ takes the direct image of subalgebras under h. This is again a C*-algebra by theorem 4.1.9 in [33]. Let us adopt the convention that 0 is not called a C*-algebra. Then we have the following. (This convention will also help avoid difficulties caused by the fact that 0 has an empty Gelfand spectrum.)

Theorem 4.6 ([37])**.** *The functor $\text{CStar} \xrightarrow{\;\mathcal{C}\;} \text{Pos}$ is faithful and reflects all isomorphisms.*

This shows that the poset of commutative C*-subalgebras should contain enough information to set up a formalism of physics, even if we are dealing with an arbitrary C*-algebra. These three theorems validate the choices of sites[51] for the quantum toposes made by the two different approaches to quantum kinematics.

In this paper, we shall be assuming that our quantum observables form a Von Neumann algebra, perhaps by constructing a universal enveloping Von Neumann algebra, if the reader would like to think of it that way. Therefore, it will suffice to work with the poset of commutative Von Neumann subalgebras in our description of the contravariant approach as well as in that of the covariant approach. This will enable us to associate an interesting logic to the physics.

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[49]Heunen proves this for a Von Neumann algebra $\mathcal{A}$. However, the same proof holds for a general C*-algebra.

[50]This can also be described as the pair $(\mathcal{V}(\mathcal{A}), \underline{\mathcal{A}})$, where $\underline{\mathcal{A}}$ is the tautological commutative C*-algebra object in $\text{Set}^{\mathcal{V}(\mathcal{A})}$ (see description of covariant approach).

[51]Of course, one can still argue about Grothendieck topologies.

4.2 The Contravariant Approach

As we will see, the more recent covariant approach to topos quantum theory begins by defining an internal commutative C*-algebra in its 'quantum topos', starting from the data of the original non-commutative C*-algebra in Set, and then proceeds along the lines of section 3, setting up the whole framework (of observables, states and propositions) for kinematics both in terms of algebra and geometry, using internal versions of the duality theorems of this section. By contrast, the contravariant approach was less theory driven and more ad hoc, skipping the stepping stone of algebra and immediately proceeding to the construction of a geometry in the topos. In this section, we will discuss the basic definitions of this formalism.

Recall that, in this approach, we assume that we start out with a Von Neumann algebra $\mathcal{A}$ of quantum observables. All of our quantum mechanics will be taking place in the topos $\text{Set}^{\mathcal{V}(\mathcal{A})^{op}}$.

In the contravariant approach, it is proposed to replace the Hilbert space by the so-called *spectral presheaf*, as the fundamental object representing the geometry of a quantum system. To define it, we should note the following:

Theorem 4.7 (Maximal ideal in a commutative C*-algebra, [41]). *There is a one to one correspondence between maximal ideals in a C*-algebra $\mathcal{A}$ and ring *-homomorphisms $\mathcal{A} \longrightarrow \mathbb{C}$.*

Proof (sketch). Obviously, every such *-morphisms has a maximal ideal as its kernel. The converse fact that every maximal ideal comes from a morphism to $\mathbb{C}$ is a direct consequence of the Gelfand-Mazur theorem. □

Definition 4 (Spectral presheaf). The spectral presheaf[52] $\underline{\Sigma} \in \text{Set}^{\mathcal{V}(\mathcal{A})^{op}}$ is defined as

1. On objects $A \in \mathcal{V}(\mathcal{A})$, $\underline{\Sigma}(A) := \text{Max}(A)$ is the Gelfand spectrum of A.

2. On morphisms $A \subset A'$, $\underline{\Sigma}(A \subset A') := (\lambda \mapsto \lambda|_A)$.

The idea is that this object represents the geometries associated to all the classical contexts bundled together. [21] continuous to associate a kind of logical structure to this 'geometry'. The intuition it behind the construction seems to be that the spectral presheaf is a presheaf of Stone spaces[53] and therefore its subpresheaves of clopen subsets should represent some sort of logic. This leads to the definition of an internal frame[54] $\underline{\mathcal{P}_{cl}\Sigma} \subset \underline{\mathcal{P}\Sigma}$, such that

$$\text{Sub}_{cl}\underline{\Sigma} := \Gamma(\mathcal{P}_{cl}\Sigma)^{55} \subset \text{Sub}(\underline{\Sigma})$$

consists precisely of the subpresheaves $\underline{P}$ of $\underline{\Sigma}$ such that $\underline{P}(A)$ is clopen in $\underline{\Sigma}(A)$ for all $A \in \mathcal{V}(\mathcal{A})$.

In a recent paper by Wolters, [43], another internal frame[56] $\underline{\mathcal{O}\Sigma^*} \subset \underline{\mathcal{P}\Sigma}$ was introduced trying to make the role of the spectral presheaf as representing the quantum geometry in the topos more explicit. One should think of it as a realisation of the spectral presheaf as an internal locale. Although it might not have been entirely clear from [43], these two subframes of $\underline{\mathcal{P}\Sigma}$ are closely related. The constructions

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[52]I will adhere to the convention of underlining objects of presheaf categories, that is common in the literature on topos quantum logic.

[53]The Gelfand spectrum of a commutative Von Neumann algebra is even hyperstonean. [42]

[54]It might be better to speak about a complete Heyting algebra here rather than a frame, because the interpretation will be mainly a logical one.

[55]Here Γ denotes the global sections functor.

[56]The * is used in the notation since we want to distinguish the corresponding internal locale $\underline{\Sigma^*}$ from the spectral presheaf $\underline{\Sigma}$.

boil down to the following.

Note that $\underline{y_A} \times \underline{\Sigma} \subset \underline{\Sigma}$ and therefore

$$\underline{\mathcal{P}\Sigma}(A) = \mathrm{Hom}(\underline{y_A} \times \underline{\Sigma}, \underline{\Omega})$$
$$\cong \mathrm{Sub}(\underline{y_A} \times \underline{\Sigma})$$
$$\subset \mathrm{Sub}(\underline{\Sigma}).$$

We define a pair of subfunctors

Definition 5 $(\underline{\mathcal{P}_{cl}\Sigma} \subset \underline{\mathcal{O}\Sigma^*} \subset \underline{\mathcal{P}\Sigma})$**.** We define

$$\underline{\mathcal{P}_{cl}\Sigma}(A) := \left\{ \underline{P} \subset \underline{y_A} \times \underline{\Sigma} \mid \text{for all } A' \in \mathcal{V}(\mathcal{A}): \quad \underline{P}(A') \text{ is clopen (as a subset of } \underline{\Sigma}(A')) \right\}$$

and

$$\underline{\mathcal{O}\Sigma^*}(A) := \left\{ \underline{P} \subset \underline{y_A} \times \underline{\Sigma} \mid \text{for all } A' \in \mathcal{V}(\mathcal{A}): \quad \underline{P}(A') \text{ is open (as a subset of } \underline{\Sigma}(A')) \right\}.$$

Since $\underline{\mathcal{O}\Sigma^*}$ will play a very important role in later constructions, it is worthwhile lingering for a bit on its definition. To make this more explicit, we define the following bundle of topological spaces.

Definition 6 ((Contravariant) spectral bundle)**.** Let Σ^* be the topological space with underlying set $\{(A, \lambda) \mid A \in \mathcal{V}(\mathcal{A}), \ \lambda \in \underline{\Sigma}(A)\}$ and opens $U \subset \Sigma^*$ such that, when we write $U_A := U \cap \underline{\Sigma}(A)$,

1. $\forall A \in \mathcal{V}(\mathcal{A}) : U_A \in \mathcal{O}\underline{\Sigma}(A)$;

2. If $\lambda \in U_A$ and $A' \subset A$, then $\lambda|_{A'} \in U_{A'}$.

Let us endow $\mathcal{V}(\mathcal{A})$ with the anti-Alexandroff topology (consisting of the lower sets). Then it is straightforwardly verified that the projection map

$$\Sigma^* \xrightarrow{\ \pi\ } \mathcal{V}(\mathcal{A})$$

is continuous. We shall call this map the spectral bundle.

Theorem 4.8 ([43])**.** $\underline{\mathcal{O}\Sigma^*}$ *is an internal frame in* $\mathrm{Set}^{\mathcal{V}(\mathcal{A})^{op}}$. *We shall write* $\underline{\Sigma^*}$ *for it, if we want to think of it as an internal locale.*

Proof. It is almost tautological that

$$\underline{\mathcal{O}\Sigma^*}(A) = \mathcal{O}(\pi^{-1}(\downarrow A)) = \mathcal{O}(\Sigma^*|_{\downarrow A}). \qquad (*)$$

Now, we recall that by the comparison lemma (see e.g. [31] C2.2.3) we have an equivalence of categories between the presheaves over a poset and the sheaves over that poset endowed with the anti-Alexandroff topology. In particular,

$$\mathrm{Set}^{\mathcal{V}(\mathcal{A})^{op}} \xrightarrow{\ \cong\ } \mathrm{Sh}(\mathcal{V}(\mathcal{A}))$$

$$\underline{P} \longmapsto \overline{P}^{57}, \ \overline{P}(\downarrow A) = \underline{P}(A)$$

which restricts to an equivalence of categories between the categories of internal locales. Moreover, by theorem C1.6.3 in [31], we have an equivalence of categories

$$\mathrm{Loc}(\mathrm{Sh}(X)) \xrightarrow{\ \cong\ } \mathrm{Loc}/X$$

$$\overline{L} \longmapsto (\overline{L}(X) \longrightarrow X),$$

where the total space of the bundle of locales corresponding to an internal locale $\overline{L}$ is $\overline{L}(X)$. We apply this with $X = \mathcal{V}(\mathcal{A})$. We had already found $(*)$ that

$$\overline{\Sigma^*}(\mathcal{V}(\mathcal{A})) = \varinjlim_{A \in \mathcal{V}(\mathcal{A})} \overline{\Sigma^*}(\downarrow A) = \varinjlim_{A \in \mathcal{V}(\mathcal{A})} \underline{\Sigma}^*(A) = \Sigma^* \xrightarrow{\pi} \mathcal{V}(\mathcal{A})$$

was a bundle of locales (in Set). This makes $\overline{\Sigma^*}$ and therefore $\underline{\Sigma}^*$ into an internal locale. $\square$

Similarly, $\underline{\mathcal{P}_{cl}\Sigma}$ is a complete Heyting algebra. (We think of it this way rather than a locale.)

Theorem 4.9. $\underline{\mathcal{P}_{cl}\Sigma}$ *is an internal complete Heyting algebra.*

Proof. Note that, by spectral theory, the lattice of clopen subsets of $\underline{\Sigma}(A)$ is isomorphic to the frame of projectors. (See also theorem 4.13.) Since A is a Von Neumann algebra, the frame of projectors is complete. Consequently, $\mathcal{P}_{cl}(\underline{\Sigma}(A))$ is a frame. The argument of theorem 4.8 show that $\underline{\mathcal{P}_{cl}\Sigma}$ is an internal frame. Finally, we invoke lemma 1 to conclude that it is an internal Heyting algebra. $\square$

One can wonder if the internal locale $\underline{\Sigma}^*$ can be constructed from an internal commutative C*-algebra, like in the covariant approach. The answer turns out to be negative in all interesting cases, as was proved by Wolters.

Theorem 4.10 ([43])**.** *The internal locale $\underline{\Sigma}^*$ is compact. However, if $\mathcal{A}$ is such that $\mathcal{V}(\mathcal{A}) \neq \{\mathbb{C} \cdot 1\}$, then it fails to be regular. In particular, it is not the internal Gelfand spectrum of some internal commutative C*-algebra.*

We will see that the spectral presheaf and this internal locale associated to it are the fundamental objects representing the geometry in the contravariant approach. That is, we will define the propositions, observables and states in terms of them.

In particular, we will later see that, using a process called daseinisation, we can construct an embedding of sup-lattices

$$\Pi(\mathcal{A}) \lhook\joinrel\longrightarrow \Gamma(\underline{\mathcal{P}_{cl}\Sigma}) = \mathrm{Sub}_{cl}(\underline{\Sigma}) \subset \Gamma(\mathcal{O}\underline{\Sigma}*) = \mathrm{Sub}_{open}(\underline{\Sigma}).$$

Note that (in general) the meet cannot be preserved as well, since $\Pi(\mathcal{A})$ is not distributive as a lattice, while $\mathrm{Sub}_{cl}(\underline{\Sigma})$ is. We gain distributivity at the cost of the rule of the excluded middle.

Moreover, it turns out that the construction of daseinisation naturally extends to self-adjoint elements of $\mathcal{A}$. Using this construction, observables take the form of arrows $\underline{\Sigma} \longrightarrow \underline{R}$, where $\underline{R}$ is a kind of real numbers object in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})^{op}}$. We will have an embedding $\mathcal{A}_{sa} \lhook\joinrel\longrightarrow \mathrm{Hom}(\underline{\Sigma}, \underline{R})$.

Finally, one can also express quantum states in terms of the spectral presheaf. Given this geometry, one might try to mimic classical mechanics or Hilbert space quantum mechanics, where the pure states are represented by points of the geometry, and hope that the points of this internal locale give a good notion of state of our quantum system. It turns out, as was first noted by Butterfield and Isham, that this analogy cannot possibly hold. Indeed, they proved the following.

Theorem 4.11 (Contravariant internal Kochen-Specker,[11, 16])**.** *Suppose $\mathcal{A}$ is a Von Neumann algebra with no type I_1 and I_2 summands. Then, the spectral presheaf $\underline{\Sigma}$ does not have any global sections.*

Proof. This is just a reformulation of theorem 4.1. One only has to note the correspondence between maximal ideals in a commutative C*-algebra a *-homomorphisms to $\mathbb{C}$. $\square$

Remark 4.12. In [43], it is claimed that, as a consequence of this, the internal locale $\underline{\Sigma}^*$ has no points[58]. However, I have trouble following the arguments that are given. For instance, it seems to be implicitly assumed that Σ^* and $\mathcal{V}(\mathcal{A})$ are sober. Indeed, Wolters assumes that the internal locale would have a point and tries to derive a contradiction, by stating (after definition 2.1) that this would imply that *the bundle of topological spaces* $\Sigma^* \longrightarrow \mathcal{V}(\mathcal{A})$ *has a section*. At the same time he goes through a lot a trouble to obtain a partial result of sobriety of Σ^* later on in the paper (Lemma 2.26). Moreover, it is not hard to see that $\mathcal{V}(\mathcal{A})$ is sober if and only if it is well-founded as a poset (every non-empty subset has a minimal element), which is certainly not true for all infinite dimensional $\mathcal{A}$. (As an easy consequence, Σ^* also is sober iff $\mathcal{V}(\mathcal{A})$ is well-founded.) If this were the only problem, the proof would at least still hold for finite dimensional $\mathcal{A}$. However, there seems to be a crucial mistake in the point set topology that follows (top of page 14). (This seems to be a consequence of an attempt to mimic the proof of the corresponding theorem in the covariant approach, where the topology of the spectral bundle is defined 'in reverse', because one replaces $\mathcal{V}(\mathcal{A})$ by $\mathcal{V}(\mathcal{A})^{op}$.) I have not yet had the time to find an alternative proof or counter example (as these are necessarily rather complicated).

The construction of states in the contravariant approach unfortunately is a bit more intricate, but still resembles the situation of classical kinematics quite closely if one takes the right point of view: states are represented by certain maps $\mathrm{Sub}_{cl}(\underline{\Sigma}) \longrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A})^{op}, [0,1])$[59]. Note that in this quantum logic, therefore, states indeed pair with propositions (elements of $\mathrm{Sub}_{cl}(\underline{\Sigma})$) to yield truth values in $\mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A})^{op}, [0,1])$, which in their turn give rise to maps $\underline{1} \longrightarrow \underline{\Omega}$, truth values in a topos theoretic sense.

4.2.1 Propositions

Our goal in this section is to construct an injection $\Pi(\mathcal{A}) \overset{\delta}{\hookrightarrow} \mathrm{Sub}_{cl}(\underline{\Sigma})$. As we investigate $\mathcal{A}$ by performing experiments in all the classical contexts $A \in \mathcal{V}(\mathcal{A})$, the intuition is that $\underline{\Sigma}$ should represent the idea of the 'Gelfand spectrum of $\mathcal{A}$'. (We take the spectrum in each context.) Recall that for a commutative Von Neumann algebr the Gelfand spectrum coincides with the Stone spectrum of the Boolean algebra of its self-adjoint idempotents. Therefore, one hopes that it can act as a sort of Stone space for quantum logic.

Let us proceed with the construction. First, note that for $p \in A \in \mathcal{V}(\mathcal{A})$, treating A as an algebra of classical observables, it is reasonable from the point of view of our philosophy to set

$$\underline{\delta}(p)(A) := \{\lambda \in \underline{\Sigma}(A) \mid \lambda(p) = 1\}.$$

What should $\underline{\delta}(p)(A)$ be for A that do not contain p however? According to our philosophy, we should construct $\underline{\delta}(p)(A)$ out of the data that someone in classical context A has available about p. We should somehow try to approximate the information captured by p as good as we can by information that is accesible to us from our classical point of view A. The key idea will be that we approximate p by some projection in A. There are two natural ways of doing this, either we take

$$\delta^o(p)_A := \bigwedge \{q \in \Pi(A) \mid q \geq p\},$$

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[58]That is internal locale morphisms from the terminal internal locale Ω. Using the equivalence $\mathrm{Loc}(\mathrm{Sh}(X)) \cong \mathrm{Loc}/X$, one could equivalently say that $\Sigma^* \longrightarrow \mathcal{V}(\mathcal{A})$ has no global sections *in the sense of maps of locales*.

[59]Originally [21] proposed two different notions of a state. As it turns out, however, both are special cases of the definition of a state we shall be using, which is due to [43].

the approximation from above or we approximate from below:

$$\delta^i(p)_A := \bigvee \{ q \in \Pi(A) \,|\, q \leq p \,\}.$$

We would then like to set

$$\underline{\delta}(p)(A) := \{ \lambda \in \underline{\Sigma}(A) \,|\, \lambda(\delta(p)_A) = 1 \,\}.$$

(Note that this agrees with our previous definition.) This should define a subfunctor of $\underline{\Sigma}$. In the case we take $\delta = \delta^o$ it indeed does, while this fails for $\delta = \delta^i$. This motivates our choice for $\underline{\delta} := \underline{\delta}^{o}$[60]. These maps δ^o and $\underline{\delta}^o$ are both referred to as *outer daseinisation*.

A physical interpretation of these approximation procedures is the following. The outer daseinisation of a Birkhoff-Von Neumann quantum proposition represents in some sense its strongest consequence in our classical context, while the inner daseinisation would stand for the weakest antecedent in our classical context that would imply it.

We now have the following. (The proof is not too illuminating.)

Theorem 4.13 ([17])**.**

$$\Pi(\mathcal{A}) \overset{\delta^o}{\hookrightarrow} \mathrm{Sub}_{cl}(\underline{\Sigma})$$

$$p \longmapsto \{ \lambda \in \underline{\Sigma}(A) \,|\, \lambda(\delta^o(p)_A) = 1 \,\}$$

defines an embedding[61] *of complete distributive sup-lattices (preserving $\leq, 0, 1, \vee$). Note that it does not also preserve $\wedge$ in general, since it is a map from a non-distributive lattice to a distributive one. However, $\mathrm{Sub}_{cl}(\underline{\Sigma})$ does have small meets and*

$$\underline{\delta}^o(p \wedge q) \leq \underline{\delta}^o(p) \wedge \underline{\delta}^o(q).$$

Moreover, $\underline{\delta}^o$ does not preserve the negation and $\mathrm{Sub}_{cl}(\underline{\Sigma})$ does in general not satisfy the law of the excluded middle.
Finally, for each $A \in \mathcal{V}(\mathcal{A})$, it restricts to an order isomorphism

$$\Pi(A) \xrightarrow[\underline{\delta}^o(A)]{\cong} \mathrm{Sub}_{cl}(\underline{\Sigma}(A)).$$

Summarising, we have embedded the Birkhoff-Von Neumann quantum logic into a complete Heyting algebra, sacrificing our meet, negation and with that the law of the excluded middle to gain distributivity.

4.2.2 Observables

As we will see quantum observables inject into $\mathrm{Hom}(\underline{\Sigma}, \underline{\mathbb{R}}^{\leftrightarrow})$, where $\underline{\mathbb{R}}^{\leftrightarrow}$ is some sort of object in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})^{op}}$ related to the real numbers. One might expect that quantum propositions, like classical propositions, then arise as the pullback of certain sub-objects $\underline{\Delta} \subset \underline{\mathbb{R}}^{\leftrightarrow}$ along observables $\underline{\Sigma} \xrightarrow{\delta(a)} \underline{\mathbb{R}}^{\leftrightarrow}$. This is indeed the approach that is argued one should take in [22] and [23]. As far as I know, however, this approach has never been fully worked out.

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[60]As the whole situation is mirrored, we will be using the inner daseinisation in the covariant approach.
[61]Note that an injective (finite) sup-preserving morphisms is automatically an order embedding.

Instead, the contravariant approach uses the ordinary Birkhoff-Von Neumann quantum propositions[62], but embedded in $\mathrm{Sub}_{cl}(\underline{\Sigma})$, as in theorem 4.13. The realisation as observables as maps $\underline{\Sigma} \longrightarrow \underline{\mathbb{R}}^{\leftrightarrow}$) therefore mostly serves to emphasise the role of $\underline{\Sigma}$ a quantum analogue of the classical phase space. A second reason for us to present it here is because it originally inspired the covariant approach, where one does construct quantum propositions like one does in classical mechanics, as inverse images.

The first thing that has to be done is extend the definition of the outer and inner daseinisation from projections to all observables. This is done by extending the order on $\Pi(\mathcal{A})$ to the so-called *spectral order* $\leq_s$ on $\mathcal{A}_{sa}$. Let $a, a' \in \mathcal{A}_{sa}$ and let $(e_\lambda \in \Pi(\mathcal{A}))_{\lambda \in \mathbb{R}}$ and $(e'_\lambda \in \Pi(\mathcal{A}))_{\lambda \in \mathbb{R}}$ be their respective resolutions. Then we say that $a \leq_s a'$ if and only if $e_\lambda \leq e'_\lambda$ for all $\lambda \in \mathbb{R}$. This order turns the self-adjoint elements into a conditionally complete lattice[63]. [38] Note that this is not the usual order on $\mathcal{A}_{sa}$. Then, for $a \in \mathcal{A}_{sa}$ and $A \in \mathcal{V}(\mathcal{A})$, we define the definitions of outer and inner daseinisations,

$$\delta^o(a)_A := \bigwedge \{a' \in \mathcal{A}_{sa} \mid a' \geq_s a\} \quad \text{and}$$

$$\delta^i(a)_A := \bigvee \{a' \in \mathcal{A}_{sa} \mid a' \leq_s a\}.$$

Effectively, what we are doing is replacing the projections in the spectral resolution of a by their daseinisation as a projection.

Then, $\underline{\mathbb{R}}^{\leftrightarrow}$ is defined as the subpresheaf[64] of $\mathrm{Hom}_{\mathrm{Pos}}(\downarrow -, \mathbb{R}) \times \mathrm{Hom}_{\mathrm{Pos}}((\downarrow -)^{op}, \mathbb{R})$ where $\underline{\mathbb{R}}^{\leftrightarrow}$ consists of the (μ, ν) such that pointwise $\mu \leq \nu$. The the *contravariant daseinisation of observables* is defined to be the map

$$\mathcal{A}_{sa} \overset{\check{\delta}}{\longleftrightarrow} \mathrm{Hom}(\underline{\Sigma}, \underline{\mathbb{R}}^{\leftrightarrow})$$

$$a \longmapsto \left(\underline{\Sigma} \overset{\check{\delta}(a)}{\longrightarrow} \underline{\mathbb{R}}^{\leftrightarrow} \right)$$

$$\lambda \longmapsto (\lambda(\delta^i(a)_A), \lambda(\delta^o(a)_A)).$$

It is not difficult to see that this is injective. See for instance [23].

Trying to mirror classical kinematics, one might hope that $\check{\delta}(a)$ would be a continuous map or a measurable map in some sense. Then, one would have a reasonable way of constructing propositions as inverse images of open or measurable subobjects of $\underline{\mathbb{R}}^{\leftrightarrow}$. Indeed, in the covariant approach, one can indeed show that a similar construction helps realise observables as continuous maps or internal locales. In [43], Wolters showns that if one replaces $\underline{\Sigma}$ by the internal locale $\underline{\Sigma}^*$ and if one chooses the most obvious frame of opens for $\underline{\mathbb{R}}^{\leftrightarrow}$, then the map defined above is not continuous.

4.2.3 States

In their original papers Isham and Doering use two notions of a state which they call 'pseudo-states' and 'truth objects'. The definition of state we will use here, first introduced in [18], is a more general one that incorporates both of these notions.

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[62]Recall that these in turn were constructed from an observable $a \in \mathcal{A}_{sa}$ and a Borel subset $\Delta \subset \mathbb{R}$

[63]i.e. every set of elements bounded that has an upper bound also has a join and every set of elements that has a lower bound has a meet.

[64]$\mathrm{Hom}_{\mathrm{Pos}}(\downarrow (A \leq A'), \mathbb{R}) = \mathrm{Hom}_{\mathrm{Pos}}(\downarrow A', \mathbb{R}) \longrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\downarrow A, \mathbb{R})$, $\mu \mapsto \mu|_{\downarrow A}$ and similarly for $\mathrm{Hom}_{\mathrm{Pos}}((\downarrow -)^{op}, \mathbb{R})$.

It has the added advantage that it can also deal with mixed states and that it resembles the convention in the covariant approach more closely.

Definition 7 (Measure on an internal lattice L). We call a map $L \xrightarrow{\mu} [0,1]_l$[65] a (finitely additive) measure if

1. $\mu(\top) = 1$;

2. $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$.

Definition 8 (Contravariant state). States in the contravariant approach are represented by internal measures on the spectral presheaf, natural transformations

$$\underline{\mathcal{P}_{cl}\Sigma} \xrightarrow{\mu} \underline{[0,1]_l}.$$

They are uniquely determined by $\mu = \Gamma\underline{\mu}$. Then they are characterised as functions

$$\Gamma(\underline{\mathcal{P}_{cl}\Sigma}) = \mathrm{Sub}_{cl}(\underline{\Sigma}) \xrightarrow{\mu} \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A})^{op}, [0,1]) = \Gamma(\underline{[0,1]_l})^{66}$$

s.t. for every $A \in \mathcal{V}(\mathcal{A})$ and for every $\underline{S}_1, \underline{S}_2 \in \mathrm{Sub}_{cl}(\underline{\Sigma})$

1. $\mu(\underline{\Sigma})(A) = 1$;

2. $\mu(\underline{S}_1)(A) + \mu(\underline{S}_2)(A) = \mu(\underline{S}_1 \wedge \underline{S}_2)(A) + \mu(\underline{S}_1 \vee \underline{S}_2)(A)$;

3. $\mu(\underline{S})(A)$ only depends on $\underline{S}_A$.

The first thing we should check is that this definition of a state indeed has some relation with the conventional notion of a quantum state. We have the following.

Theorem 4.14 ([18]). *It is easy to check that a quasi-state $\mathcal{A} \xrightarrow{\omega} \mathbb{C}$ defines a measure μ_ω on the spectral presheaf by*

$$\mu_\omega(\underline{S})(A) := \omega\left(\underline{\delta}^o(A)^{-1}(\underline{S}(A))\right),$$

where $\underline{\delta}^o(A)$ is the isomorphism $\Pi(A) \longrightarrow \mathrm{Sub}_{cl}(\underline{\Sigma}(A))$ of theorem 4.13. Less trivially, it can be proved that this defines a bijection between quasi-states on $\mathcal{A}$ and measures on $\mathcal{P}_{cl}\underline{\Sigma}$.

Therefore, by theorem 4.2, if $\mathcal{A}$ has no summands of type I_2, the injection of states into measures on the spectral presheaf is also surjective.

Of course, seeing the motivation of our topos framework by theorems 4.1, 4.2 and 4.3, we could never really hope that it would work out for $\mathcal{A}$ with type I_2 summands. This result therefore basically tells us that measures on the spectral presheaf are precisely the right notion of state in the contravariant approach.

Again, the reader should note that this notion of state is the general notion of a positive normalised functional on $\mathcal{A}$. One can also characterise the states used by physicists, those that come from trace class operators, in this framework. These are precisely those measures that are in some sense σ-additive, instead of just finitely additive. The details can be found in [18].

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[65]Here $[0,1]_l$ denotes the unit interval in the lower reals (in the topos).

[66]Explicitly $\underline{[0,1]_l}(A) = \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(A)^{op}, [0,1])$.

4.2.4 State-Proposition Pairing

Recall that one of the issues with Birkhoff-Von Neumann quantum logic was that it did not have a satisfactory state-proposition pairing. According to theorem 4.3 quantum logic did not have any non-trivial Boolean-valued models, not even of the weaker kind where we demand that the logical structure was respected only on Boolean subalgebras of the quantum logic. In theorem 4.13 we already found that the contravariant topos quantum logic provided some consolation: outer daseinisation defines a Heyting-valued model of this weaker kind.

However, this is not exactly what we would hope for yet. In classical mechanics, each pure state defines a homomorphism from the associated logic to the two-value Boolean algebra. One might hope that pure states also define homomorphisms from to some Heyting algebra of subterminals in this topos framework. This turns out to be too much to hope for. Although we do have a natural map from $\Pi(\mathcal{A})$ to this Heyting algebra, it is not a homomorphism, not even for pure states. This yet again indicates a non-deterministic nature of quantum mechanics. In this respect both pure and mixed quantum states behave like mixed states in classical mechanics. The situation is as follows.

We note that a quantum state ω represented by a measure μ_ω on the spectral presheaf determines a map

$$\Pi(\mathcal{A}) \xhookrightarrow{\underline{\delta}^o} \mathrm{Sub}_{cl}(\underline{\Sigma}) \xrightarrow{\mu_\omega} \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A})^{op}, [0,1])$$

$$[a\epsilon\Delta] \longmapsto \underline{\delta}^o([a\epsilon\Delta]) \longmapsto \left(A \mapsto \mu_\omega(\underline{\delta}^o([a\epsilon\Delta])(A)) = \omega(\delta^o([a\epsilon\Delta])_A) \right).$$

According to the Born interpretation, we thus assign the set of probabilities of that the approximation of $[a\epsilon\Delta]$ in each classical context is true. If we do not want to deal with probabilities but one might now apply the map that forgets all the probabilistic information and sends every non-1-probability to 0. This defines a map[67]

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A})^{op}, [0,1]) = \Gamma(\underline{[0,1]}_l) \xrightarrow{\Gamma\underline{\mathrm{Dich}}} \Gamma\underline{\Omega} \cong \mathrm{Sub}\underline{1} \cong \{S \subset \mathcal{V}(\mathcal{A}) \,|\, S \text{ is a downset}\,\}$$

$$\nu \longmapsto \{A \in \mathcal{V}(\mathcal{A}) \,|\, \nu(A) = 1\,\}.$$

We conclude that we have found a map[68]

$$\Pi(\mathcal{A}) \xrightarrow{\mathrm{truth}_\omega} \Gamma\underline{\Omega} = \mathrm{Sub}(\underline{1})$$

$$p \longmapsto \Gamma\underline{\mathrm{Dich}}(\mu_\omega(\underline{\delta}^o(p))).$$

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[67] Again, this comes from a map

$$\underline{[0,1]}_l \xrightarrow{\underline{\mathrm{Dich}}} \underline{\Omega}$$

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(A)^{op}, [0,1]) = \underline{[0,1]}_l(A) \xrightarrow{\underline{\mathrm{Dich}}_A} \underline{\Omega}(A) = \mathcal{P}(\downarrow A)$$

$$\nu \longmapsto \{A' \in\downarrow A \,|\, \nu(A') = 1\,\}.$$

.

[68] The reader might have been expecting this to be defined as a map $\mathrm{Sub}_{cl}\underline{\Sigma} \longrightarrow \mathrm{Sub}\underline{1}$, hoping that this would be a lattice (or even Heyting) homomorphism for pure states ω. (The pure states define homomorphisms to the truth values in classical kinematics.) Unfortunately, this does not work out. As one can check, this map indeed preserves $\wedge$, but not $\vee$. This follows since, if it would, we would have found a lattice homomorphism from $\Pi(\mathcal{A})$ to $\{\bot, \top\}$, which we know not to exist by theorem 4.3. (Indeed, $\underline{\delta}^o$ preserves $\vee$.)

4.2.5 Interpretation of Truth

Two questions that spring to mind are:

1. How much information can we infer about ω from knowing truth_ω?

2. How should we interpret these truth values?

It is not difficult to see that truth_ω contains the same information as the support of ω (all the possible senses are equivalent: either as linear functional on $\mathcal{A}$, or as measure, or as operator on $\mathcal{H}$). Therefore, in this respect these 'nuanced' truth values are no better than those of the Birkhoff-Von Neumann logic. (Recall that there, a state defined a truth value $\Pi(\mathcal{A}) \overset{\text{truth}_\omega}{\longrightarrow} \{0,1\}, p \mapsto \text{Dich}(\omega(p))$.) The difference is therefore to be sought in the truth value of one particular proposition.

By definition[69] $\text{truth}_\omega(p)(A) = 1$ iff $\omega(\delta^o(p)_A) = 1$, i.e. if with certainty the proposition $\delta^o(p)_A$ is true for ω (in the Birkhoff-Von Neumann sense). Seeing that $\delta^o(p)_A$ is the strongest logical consequence of p that we can measure in our classical context A, we can immediately say that p is true for ω if and only if $\text{truth}_\omega(p)(A) = 1$ for all A. The way one should therefore think of these propositions is with the intension of falsification in mind.

Indeed, truth of $\delta^o(p)_A$ guarantees nothing about the truth of p. However, assume proposition p is not true for state ω and we want to exhibit this with the restriction that we are only allowed to perform measurements in classical context A (because we want to avoid disturbing our system to much, say). Then, our best option is to examine $\delta^o(p)_A$. If that turns out to be not true, neither was p. The way to interpret $\text{truth}_\omega(p)$, therefore, is that it has a value 0 at context A if and only if we can falsify p by measuring observables from A. Since the logical operations on the subobject lattice are computed pointwise, so truth of $\underline{\delta}^o(p) \wedge \underline{\delta}^o(p')$ at A should be interpreted as possibility of falsifying p and q by measuring observables from A and similarly for $\vee$. In particular, note that there is no ambiguity as to which of the two experiments should be performed first, since the propositions commute.

4.2.6 Summary

Summarising, we have set up a framework for quantum kinematics in the topos $\text{Set}^{\mathcal{V}(\mathcal{A})^{op}}$. This was built on the definition of the spectral presheaf $\underline{\Sigma}$, or the corresponding internal locale $\underline{\Sigma}^*$. In terms of this 'geometry'[70], we defined generalised sets of observables, states and propositions, in which the corresponding objects of ordinary quantum mechanics were embedded. This went as follows.

Observables:

$$\mathcal{A}_{sa} \overset{\breve{\delta}}{\hookrightarrow} \text{Hom}(\underline{\Sigma}, \underline{\mathbb{R}}^{\leftrightarrow}),$$

(also, recall that this failed to be a map into $C(\underline{\Sigma}^*, \underline{\mathbb{IR}})$), states:

$$\mathscr{S}(\mathcal{A}) \hookrightarrow \mathscr{S}_{quasi}(\mathcal{A}) \cong \text{``finitely additive measures''}(\underline{\mathcal{P}_{cl}\Sigma}),$$

propositions:

$$\Pi(\mathcal{A}) \overset{\delta^o}{\hookrightarrow} \text{Sub}_{cl}\underline{\Sigma} \hookrightarrow \Gamma\underline{\mathcal{O}\Sigma}^* \cong \mathcal{O}\Sigma^*$$

---

[69] We will occasionality denote the values this maps takes at each context respectively by 0 and 1, rather than by $\emptyset$ and $\{*\}$.

[70] Also, recall that there was no corresponding algebra in the topos, as $\underline{\Sigma}^*$ failed to be regular and could therefore not be interpreted as the spectrum of an internal C*-algebra.

. This last map defines a sup-embedding of orders and it restricts to an order-isomorphism between each $\Pi(A)$ and $\text{Sub}_{cl}(\underline{\Sigma}(A))$, $A \in \mathcal{V}(\mathcal{A})$. This enlarged 'quantum logic' has the following properties.

1. It is a complete Heyting algebra.

2. The interpretation of the (Heyting) negation is "p is not true", rather than "p is false".

3. *Each* state determines the truth-value of all quantum propositions. This is a consequence of our dichotomy-map that transformed probabilities into true-false[71]-judgements in each classical context. The truth values at different classical contexts have the immediate operational interpretation as possibility of falsification: a 0 means that falsification is possible.

4. The pure (and mixed) states do not determine lattice homomorphisms $\text{Sub}_{cl}\underline{\Sigma} \longrightarrow$ $\text{Sub}\underline{1}$.

5. We have a sup-embedding of $\Pi(\mathcal{A})$ into the complete Heyting algebra $\text{Sub}_{cl}(\underline{\Sigma})$, that, when restricted to any Boolean subalgebra of $\Pi(\mathcal{A})$, defines an isomorphism of Boolean algebras to $\text{Sub}_{cl}(\text{Max}(A))$[72].

6. The interpretation of $\vee$ is what we are used to: for a state $|\psi\rangle$, $p \vee p'$ can only be true, if either p or p' is true. However, the law of the excluded middle fails.

7. The propositions have a clear operational interpetation. The logical operations are defined pointwise so we are only combining commuting propositions in conjunctions and disjunctions. The truth value of such combined propositions therefore does not depend on the order in which we verify the truth value of the building blocks of this proposition.

8. It is not obvious how one can reconstruct the algebra of observables from this quantum logic.

By using outer daseinisation we can form a truth value for each classical context. A proposition if precisely not true in a context for a certain state, if it can be falsified by performing measurements from that context.

4.3 The Covariant Approach

Conventionally, the covariant approach to topos quantum logic starts out with a general C*-algebra $\mathcal{A}$. As discussed in section 4.1.2 it not enough in this case to consider only its commutative Von Neumann subalgebras. Instead, one should take into account the poset $\mathcal{C}(\mathcal{A})$ of all commutative C*-subalgebras. However, recently, in particular in [28], the covariant approach too has started to consider more specific kinds of C*-algebras[73]. The reason to do this is that, although one can formulate a nice theory of observables and states when dealing with an arbitrary C*-algebra, the logical side of the story is not very nice. This is a consequence of the lack of projections that a C*-algebra might have.

With the ideas in mind that one can embed any C*-algebra in its universal enveloping Von Neumann algebra and that these are in fact the only algebras of operators mathematical physicists seem to use in practice, I have made the choice to restrict to the case of Von Neumann algebras, also in my description of the covariant

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[71] Maybe, 'not true' might better describe the situation here than 'false'.

[72] Note that this does not contradict theorem 4.3.

[73] In particular, from general to specific, it considered spectral, Rickart, AW* and Von Neumann algebras.

approach. This will also make a comparison with the contravariant approach more easy. The reader should bear in mind, however, that most of the results hold in greater generality.

In this approach we will therefore start out with a Von Neumann algebra $\mathcal{A}$ and set up a formalism of quantum kinematics in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$. This time, however, we take a more theory driven approach. We have a naturally have an internal commutative C*-algebra in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$. We mirror the constructions we performed in classical kinematics to obtain from this both a fully algebraic formalism for quantum kinematics in the topos and a geometric counterpart to this. This internal duality between algebra and geometry will rely on recent constructive versions of Gelfand-Duality, due to Banaschewski and Mulvey (e.g. [1]), and of Riesz-Markov-Duality, due to Coquand and Spitters ([14]) as well as on a less recent constructive version of Stone duality.

4.3.1 Algebra

4.3.1.1 Observables Given a Von Neumann algebra $\mathcal{A}$, we would like to define an object in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$ that will represent this algebra. Note that we have a tautological object[74]

$$\mathcal{V}(\mathcal{A}) \xrightarrow{\;\underline{\mathcal{A}}\;} \mathrm{CStar}$$

$$A \leq A' \longmapsto A \subset A'.$$

Recall the following elementary result from categorical model theory.

Lemma 1 (e.g. [31], Corollary D1.2.14). *Let $\mathbb{T}$ be a geometric theory and let $\mathcal{C}$ be a small category. Then*

$$\mathbb{T} - \mathrm{Mod}(\mathrm{Set}^{\mathcal{C}^{op}}) \longrightarrow \mathbb{T} - \mathrm{Mod}(\mathrm{Set})^{\mathcal{C}^{op}}$$

$$P \longmapsto \mathrm{ev}_-(P)$$

defines an isomorphism of categories.

Proof. The notation in the lemma for what is basically the identity functor is supposed to suggestive. Indeed, note that for each $C \in \mathcal{C}^{op}$ we have a geometric morphism $\mathrm{Set}^{\mathcal{C}^{op}} \xrightarrow{\mathrm{ev}_C} \mathrm{Set}$ with inverse image evaluation at C and right adjoint $S \mapsto S^{\mathcal{C}(C,-)}$. This shows that evaluation induces a functor $\mathbb{T} - \mathrm{Mod}(\mathrm{Set}^{\mathcal{C}^{op}}) \longrightarrow \mathbb{T} - \mathrm{Mod}(\mathrm{Set})^{\mathcal{C}^{op}}$ (which is obviously injective on objects and faithful). Moreover, the set of points $(\mathrm{ev}_C)_{C \in \mathcal{C}}$ is separating. This implies that it is surjective on objects and full. $\qquad\square$

Since we have a geometric (even algebraic) theory of commutative rings, $\underline{\mathcal{A}}$ is an internal commutative ring in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$. We would actually like to say that this is an internal C*-algebra[75], so we can apply an internal version of Gelfand-Duality later. However, since metric completeness is typically a second order property, a theory of C*-algebras will be second order, hence we cannot use the above lemma to conclude this. To prove this, one has to get one's hands dirty and write out the sheaf semantics. This was done in [29].

---

[74]Recall from theorem 4.5 that the pair $(\mathcal{V}(A), \underline{\mathcal{A}})$ contains the same information as $\Pi(\mathcal{A})$ and therefore, by our discussion in paragraph 3.3.2, one should be able to reconstruct $\mathcal{A}$ from it.

[75]We might even want to say that it is an internal Von Neumann algebra. However, it is very involved to work out what that would mean. In section 4.3.1.3 we shall see that the internal Boolean algebra associated to $\underline{\mathcal{A}}$ is complete and in section 4.3.2.3 we see that the internal Gelfand spectrum of $\underline{\mathcal{A}}$ is in fact the internal Stone spectrum of this Boolean algebra. This strongly suggest that $\underline{\mathcal{A}}$ is at least an internal AW*-algebra.

Theorem 4.15 (Essentially [29], theorem 5). *Any presheaf of commutative C*-algebras in* cCStar$^{\mathcal{C}^{op}}$, *for some category* $\mathcal{C}$, *is an internal C*-algebra in* Set$^{\mathcal{C}^{op}}$, *under the operations inherited from the pointwise ones. In particular, it is an internal vector space over the constant functor* $\underline{\Delta}_{\mathbb{C}} : c \mapsto \mathbb{C}$ *(the Cauchy complex numbers).*

The process of passing from the non-commutative C*-algebra $\mathcal{A}$ to this internal commutative one is known as *Bohrification* in literature. Again, as one might expect, the observables are given by the self-adjoint elements.

Theorem 4.16. *The object of observables* $\underline{\mathcal{A}}_{sa}$ *is given by* $\underline{\mathcal{A}}_{sa}(A) = A_{sa}$.

Proof. We define $\underline{\mathcal{A}}_{sa} := \{a \in \underline{\mathcal{A}} \mid a^* = a\} \subset \underline{\mathcal{A}}$. This is immediate from the interpretation of sheaf semantics in a presheaf category. $\qquad\square$

4.3.1.2 States Now we have given this internal representation of the algebra of observables, we would like to do something similar for the states.

Definition 9 (Internal State). We define an internal state on an internal C*-algebra $\mathcal{A}$ in a topos to be a $\mathbb{C}$ [76]-linear map $\mathcal{A} \xrightarrow{I} \mathbb{C}$, such that $I(1) = 1$ and $I(aa*) \geq 0$ for all generalised elements $a \in \underline{\mathcal{A}}$.

We have the following correspondence.

Theorem 4.17 ([29]). *There is a natural bijection between external quasi-states on* $\mathcal{A}$ *and internal states on* $\underline{\mathcal{A}}$. *Therefore, if* $\mathcal{A}$ *does not have summands to type* I_2 *external states on* $\mathcal{A}$ *are in natural bijection with internal states on* $\underline{\mathcal{A}}$.

Proof. The first statement is almost tautological. Indeed, any quasi-state defines an internal state by restriction to commutative subalgebras. Conversely, each internal state $\underline{\omega}$ defines a quasi-state ω by setting $\omega(a) = \underline{\omega}_A(a)$, where A is some commutative subalgebra containing a. This is well-defined by naturality of $\underline{\omega}$. Naturality of this bijection is immediate. The second statement is just theorem 4.2. $\qquad\square$

4.3.1.3 Propositions Since we already have an internal structure representing our observables, it is easy to also obtain one for the quantum propositions.

Definition 10. $\mathcal{V}(\mathcal{A}) \xrightarrow{\Pi(A)} \mathrm{Set}$ is the subfunctor of $\underline{\mathcal{A}}$ that sends a commutative C*-algebra A to its Boolean algebra $\Pi(A)$ of self adjoint idempotents.

Note that this is an internal Boolean algebra in Set$^{\mathcal{V}(A)}$, by lemma 1, since the theory of Boolean algebras is geometric (even algebraic). Its operations are induced by the those on $\Pi(A)$, for $A \in \mathcal{V}(\mathcal{A})$. Moreover, combining the fact that each $\Pi(A)$ is a complete Boolean algebra with an argument along the lines of 4.8, one shows that $\underline{\Pi(\mathcal{A})}$ is in fact complete.

4.3.1.4 State-Proposition Pairing Let $\underline{\mathcal{A}} \xrightarrow{\omega} \underline{\Delta}_{\mathbb{C}}$ be an internal state. We note that this restricts to a map $\underline{\Pi(\mathcal{A})} \xrightarrow{\omega} \underline{\Delta}_{[0,1]}$. As one might expect, we can recover our state from this. For comparison with the contravariant approach, we interpret this map $\underline{\Pi(\mathcal{A})} \xrightarrow{\omega} \underline{\Delta}_{[0,1]}$ a bit differently.

Theorem 4.18 ([28]). *There is a natural bijection between:*

    1. External quasi-states on $\mathcal{A}$;

---

[76]The Cauchy complexes.

2. *Internal states on $\underline{\mathcal{A}}$;*

3. *External finitely additive[77] measures on $\Pi(\mathcal{A})$.*

4. *Internal finitely additive measures on $\underline{\Pi(\mathcal{A})}$.*

Proof. The equivalence between 1. and 2. has already been established in theorem 4.17. The equivalence between 1. and 3. is the affirmative answer to the quite non-trivial Mackey-Gleason problem. (Theorem 3.11.) The equivalence between 3. and 4. is a consequence of the fact that an internal valuation on a Boolean algebra takes values in the Dedekind reals $\underline{[0,1]} \subset \underline{[0,1]}_l$ (i.e. the constant presheaf $[0,1]$). This assertion is lemma 24 in [28]. Then, note that naturality of an internal measure $\underline{\mu}$ precisely means that $\underline{\mu}_A(p) = \underline{\mu}_{A'}(p)$ if $p \in \Pi(A) \cap \Pi(A')$. $\square$

Like in the contravariant approach, we can define a pairing between states and propositions that will yield truth values in $\mathrm{Sub}(\underline{1}) \cong \Gamma\underline{\Omega}$. We simply mirror what we do there. For each (quasi-)state ω, we have a map

$$\Pi(\mathcal{A}) \xrightarrow{\;\omega \circ \delta^i\;} \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A}), [0,1])$$

$$p \longmapsto \Big(A \mapsto \omega(\delta^i(p)_A) \Big).$$

Again, we introduce a 'dichotomy map' that will regard the probabilities that quantum mechanics normally produces (by the Born rule) from an all or nothing point of view. We define

$$\underline{[0,1]}_l \xrightarrow{\qquad\qquad \mathrm{Dich} \qquad\qquad} \underline{\Omega}$$

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(A),[0,1]) = \underline{[0,1]}_l(A) \xrightarrow{\;\mathrm{Dich}_A\;} \underline{\Omega}(A) = \mathcal{P}(\uparrow A)$$

$$\nu \longmapsto \{\, A' \in\, \uparrow A \mid \nu(A') = 1 \,\}.$$

Taking the composition of these maps we find our truth assignment

$$\Pi(\mathcal{A}) \xrightarrow{\;\mathrm{truth}_\omega := \Gamma\underline{\mathrm{Dich}} \circ \omega \circ \delta^i\;} \Gamma\underline{\Omega} \cong \mathrm{Sub}(\underline{1}).$$

4.3.2 Geometry

4.3.2.1 Observables Seeing that we have an internal commutative C*-algebra in our topos, one might hope to realise this algebra as an algebra of functions of some 'internal space' by using Gelfand-Naimark duality. However, the conventional proof of the theorem relies on the axiom of choice[78]. Since our topos is (generally) non-Boolean[79], we see that the internal axiom of choice fails.

Not all is lost, however, since Banaschewski and Mulvey (and finally Coquand and Spitters) gave a constructive proof of the theorem in a series of recent papers. Their version of the theorem constructs the Gelfand spectrum not as a compact Hausdorff space, but as a compact completely regular locale. To make sense of this, recall that the categories CHaus of compact Hausdorff spaces and of KRegLoc of compact completely regular locales are equivalent in presence of the axiom of choice.

---

[77][28] states this result with finitely addtive replaced by countably additive and their proof skips over many details. I could see that this would be true if a strong version of Gleason's theorem holds. However, one would certainly have to exclude I_2 summands to have such a result at one's disposal.

[78]For instance, the Stone-Weierstrass theorem, that is invoked in the proof, does so.

[79]Recall that a presheaf topos is Boolean if and only if its site is a groupoid. Our site is a (generally) non-trivial poset.

Theorem 4.19 (Constructive Gelfand-Naimark,[1, 13]). *For any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories*

$$\text{cCStar}(\mathcal{E})^{op} \overset{\underline{C}(-,\underline{\mathbb{C}})}{\underset{\underline{\text{Max}}}{\leftrightarrows}} \text{KRegLoc}(\mathcal{E}),$$

where $\underline{\mathbb{C}}$ denotes the internal locale[80] of Dedekind complex numbers, $\underline{C}(-,-)$ denotes the internal continuous Hom (a subfunctor of the internal Hom, defined in terms of the internal language), and $\underline{\text{Max}}$ is the functor that sends an internal commutative C*-algebra to the locale of maximal ideals (defined in terms of the internal language of $\mathcal{E}$)[81]. In particular, we can view

$$\underline{\mathcal{A}}_{sa} \cong \underline{C}(\underline{\text{Max}}(\underline{\mathcal{A}}), \underline{\mathbb{R}}),$$

where $\underline{\mathbb{R}}$ is the locale[82] of Dedekind real numbers.

At the moment, this internal spectrum is a rather abstract thing at the moment. If it should be of any use for physicists, we had better give a more explicit description of it. For this purpose, we shall view it as a bundle of locales (in Set). For convenience, we shall write $\underline{\Sigma}_* := \underline{\text{Max}}(\mathcal{A})$. The reader is encouraged to compare it with the internal locale $\underline{\Sigma}^*$ from the contravariant approach.

Analogous to what we did for the contravariant approach, referring to the comparison lemma, we will make the identification $\text{Set}^{\mathcal{V}(\mathcal{A})} \cong \text{Sh}(\mathcal{V}(\mathcal{A}))$, where $\mathcal{V}(\mathcal{A})$ is equipped with the Alexandroff topology[83] here. We do this since we have a nice desciption of locales internal to localic toposes, namely,

$$\text{Loc}(\text{Sh}(X)) \cong \text{Loc}/X.$$

As was computed in [30], these identifications give us the following bundle of locales, representing $\underline{\Sigma}_*$.

Definition 11 ((Covariant) spectral bundle). Let Σ_* be the topological space with underlying set $\{(A, \lambda) | A \in \mathcal{V}(\mathcal{A}), \ \lambda \in \text{Max}(A)\}$ and opens $U \subset \Sigma_*$ such that, when we write $U_A := U \cap \text{Max}(A)$,

1. $\forall A \in \mathcal{V}(\mathcal{A}) : U_A \in \mathcal{O}\text{Max}(A)$;

2. If $A \subset A'$, then $\lambda' \in U_{A'}$ whenever $\lambda'|_A \in U_A$.

Let us endow $\mathcal{V}(\mathcal{A})$ with the Alexandroff topology (consisting of the upper sets). Then it is straightforwardly verified that the projection map

$$\Sigma^* \overset{\pi}{\longrightarrow} \mathcal{V}(\mathcal{A})$$

is continuous. We shall call this map the spectral bundle.

Writing out the identifications we made above, we get the following simple description of the internal Gelfand spectrum, in terms of the spectral bundle.

Corollary 4.20. *As an object of $\text{Sh}(\mathcal{V}(\mathcal{A}))$, $\overline{\Sigma_*}$ has the frame of opens*

$$U \overset{\overline{\mathcal{O}\Sigma_*}}{\longmapsto} \mathcal{O}\Sigma_*|_U$$

$$U \subset V \longmapsto (W \mapsto W \cap U).$$

---

[80]In our specific case, its corresponding frame is $\underline{\mathcal{O}(\mathbb{C})}$ is given by $\underline{\mathcal{O}(\mathbb{C})}(A) = \mathcal{O}(\uparrow A \times \mathbb{C})$, where $\uparrow A \subset \mathcal{V}(\mathcal{A})$ is equipped with the subspace topology. [29]

[81]The interested reader can find an account of its action on morphisms in [37].

[82]In our case $\underline{\mathcal{O}(\mathbb{R})}(A) = \mathcal{O}(\uparrow A \times \mathbb{R}) \cong \text{Hom}_{\text{Pos}}(\uparrow A, \mathcal{O}(\mathbb{R}))$.

[83]This consists of the upper sets.

As an object of $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$, $\underline{\Sigma}_*$ *has the frame of opens*

$$A \xmapsto{\quad \mathcal{O}\Sigma_* \quad} \mathcal{O}\Sigma_*|_{\uparrow A}$$

$$A \leq A' \longmapsto (W \mapsto W \cap \uparrow A').$$

Note that by interpreting the internal spectra in both approaches as bundles of locales, we are able to compare them, even though they were internal locales in different toposes to start out with.

We may have a reasonable idea of what the internal spectrum looks like now, but the whole operation seems a tad pointless if we do not know how actual physical observables should relate to it. It is somewhat unfortunate that there seems to be no obvious way of interpreting observables as global sections of $\underline{C}(\underline{\mathrm{Max}}(\mathcal{A}), \underline{\mathbb{R}})$, i.e. as elements of the external continuous homset $C(\underline{\mathrm{Max}}(\mathcal{A}), \underline{\mathbb{R}})$. In [29], the authors were able, however, to find a reasonable injection $\mathcal{A}_{sa} \hookrightarrow C(\underline{\mathrm{Max}}(\mathcal{A}), \underline{\mathbb{IR}})$, where $\underline{\mathbb{IR}}$ denotes the internal Scott interval domain, another kind of real numbers object. In analogy with the contravariant approach, they dubbed this map 'daseinisation' as well.
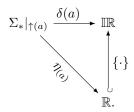
This interval domain $\mathbb{IR}$ (in Set) as a poset is defined as the set of all non-empty compact intervals of real numbers ordered by inclusion. Like any preorder, this can be endowed with the Scott topology. The closed subsets are the lower sets that are closed under suprema of directed subsets. The collection

$$(p, q)_S := \{[r, s] \mid p < r \leq s < q\}, \quad p, q \in \mathbb{Q}, p < q,$$

is a basis for this topology. We have a continuous embedding

$$\mathbb{R} \xhookrightarrow{\;\{\cdot\}\;} \mathbb{IR}$$

$$r \mapsto [r, r].$$

We shall be using a variant of the covariant daseinsisation map that was introduced by Wolters in [43]. This is obtained from the map in [29] by replacing the (usual) order on $\mathcal{A}_{sa}$ by the spectral order. We have made this choice on the one hand because it is compatible with the ordinary Gelfand transform $A \xrightarrow{\eta_A} C(\mathrm{Max}(A), \mathbb{C})$ in the sense that for all $a \in \mathcal{A}$, if we write (a) for the (commutative) subalgebra generated by a,

$$\begin{array}{ccc}
\Sigma_*|_{\uparrow(a)} & \xrightarrow{\;\delta(a)\;} & \mathbb{IR} \\
& \searrow{\scriptstyle \eta_{(a)}} & \big\uparrow{\scriptstyle \{\cdot\}} \\
& & \mathbb{R}.
\end{array}$$

On the other hand, it will turn out that by making this choice, the resulting way of constructing propositions out of observables and intervals coincides with the one described in section 3.3.2. (See theorem 4.30.)

Definition 12. We define the *covariant daseinisation map* to be the function (which is easily seen to be injective, by injectivity of the (inner and outer) da-

seinisation)

$$\mathcal{A}_{sa} \xrightarrow{\ \delta\ } C(\Sigma_*, \mathbb{IR})$$

$$a \longmapsto \left(\Sigma_* \xrightarrow{\ \delta(a)\ } \mathbb{IR}\right)$$

$$(A, \lambda) \longmapsto [\lambda(\delta^i(a)_A), \lambda(\delta^o(a)_A)].$$

One can also define the interval domain internal to $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$, for instance as an internal locale, by using the internal language. (See [43] for an explicit description.) The specifics do not matter too much here. The result is, however, that we also find a map $\mathcal{A}_{sa} \xrightarrow{\ \delta\ } C(\underline{\Sigma}_*, \underline{\mathbb{IR}})$.

We will use this covariant daseinisation map to construct elementary propositions out of the combination of an observable and a subset of the outcomes.

4.3.2.2 States   As might be expected, there is also a geometric interpretation to be given to states. This will precisely reflect the one given by Riesz-Markov duality in classical kinematics. However, to avoid having to introduce the pointless extra definition of an 'internal Borel sigma-algebra associated to an internal locale', we note that finite regular measures on the Borel sigma algebra correspond precisely with completely additive measures on the frame of open sets (where regularity of the measure corresponds with the complete additivity). This motivates the following definition.

Definition 13 (Probability valuation on an internal frame F). We call a map $F \xrightarrow{\ \mu\ } [0,1]_l$[84] a probability valuation or a completely additive measure[85] if

1. $\mu(\top) = 1$;

2. $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$;

3. $\mu(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} \mu(x_i)$, for any directed set $(x_i)_{i \in I}$.

As was recently proved by Coquand and Spitters, the Riesz-Markov theorem holds in any (elementary) topos. We formulate a weak version of it[86].

Theorem 4.21 (Constructive Riesz-Markov,[14]). *For any commutative C*-algebra $\underline{\mathcal{A}}$ internal to a Grothendieck topos $\mathcal{E}$, there is a natural bijection between internal states on $\underline{\mathcal{A}}$ and probability valuations on $\underline{\mathcal{O}(\mathrm{Max}(\mathcal{A}))}$.*

This gives yet another characterisation of states in the covariant approach (c.f. theorem 4.18): as internal probability valuations $\underline{\mathcal{O}(\mathrm{Max}(\mathcal{A}))} \xrightarrow{\ \mu\ } [0,1]_l$. Again, these are uniquely determined by their global sections $\mathcal{O}(\Sigma_*) \cong \mathcal{O}(\underline{\mathrm{Max}(\mathcal{A})}) \xrightarrow{\ \mu = \Gamma\mu\ } \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A}), [0,1])$. The reader is encouraged to note the similarities with both classical kinematics and the contravariant approach.

However, the analogy with classical mechanics fails in the sense that, here, pure states are not derived from points of the space. Indeed, we have the following reformulation of the Kochen-Specker theorem.

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[84]Here $[0,1]_l$ denotes the unit intervals in the lower reals (in the topos).

[85]c.f. definition 7.

[86]i.e. one not taking into account the structure of the space of valuations as a locale and immediately combining it with Gelfand-Naimark.

Theorem 4.22 ([16], [29]). *Suppose $\mathcal{A}$ is a Von Neumann algebra with no type I_1 and I_2 summands. Then, the internal Gelfand spectrum $\underline{\mathrm{Max}(\mathcal{A})}$ in $\mathrm{Set}^{\mathcal{V}(\mathcal{A})}$ does not have any points as an internal locale.*

Proof (sketch). The idea is that a point[87] $\underline{*} \xrightarrow{\rho} \underline{\mathrm{Max}(\mathcal{A})}$ defines a map $\mathcal{A} \longrightarrow \mathrm{pt}(\underline{\mathbb{C}}) \cong \mathbb{C}$, that turns out to restrict to a *-homomorphism on each commutative subalgebra, i.e. a map that cannot exist by the Kochen-Specker theorem. (The converse is easy to see.) Indeed, such a point defines a map $\underline{C}((\mathrm{Max}\mathcal{A}), \underline{\mathbb{C}}) \longrightarrow \underline{C}(\underline{*}, \underline{\mathbb{C}})$. Recall that $\underline{\mathcal{A}} \cong \underline{C}((\mathrm{Max}\mathcal{A}), \underline{\mathbb{C}})$, to see that we get a map $\underline{\mathcal{A}} \longrightarrow \underline{C}(\underline{*}, \underline{\mathbb{C}})$. That is, in components, compatible maps $A \longrightarrow \mathbb{C}$ for all $A \in \mathcal{V}(\mathcal{A})$. It can be shown that these are *-homomorphisms. (This is quite non-trivial.) As we know, these cannot exist by the Kochen-Specker theorem. $\square$

We also have a reformulation of the Kochen-Specker theorem in terms of the spectral bundle.

Theorem 4.23 ([16],[29]). *Suppose $\mathcal{A}$ is a Von Neumann algebra with no type I_1 and I_2 summands. Then, the bundle $\Sigma_* \longrightarrow \mathcal{V}(\mathcal{A})$ does not admit global sections[88] (as maps of locales even).*

Proof. The idea is that under the identification $\mathrm{Loc}(\mathrm{Sh}(X)) \cong \mathrm{Loc}/X$ points of an internal locale on the left hand side correspond with sections of the bundle of locales on the right hand side. Indeed, a point of an internal locale L in $\mathrm{Sh}(X)$ is a map of internal frames $\Omega \longrightarrow \mathcal{O}L$ and Ω corresponds to the frame map $\mathcal{O}(X) \xrightarrow{\mathrm{id}} \mathcal{O}(X) = \Omega(X)$ under this correspondence. We have already shown that the internal spectrum is an internal locale with no points. $\square$

Remark 4.24. As far as we know, this result appeared first in [30]. However, that proof rests on the claim (that is not explained further) that every section of the bundle as a map of locales comes from a section as a map of topological spaces. Seeing that the spaces $\mathcal{V}(\mathcal{A})$ and Σ_* are sober if and only if $\mathcal{V}(\mathcal{A})$ is co-well-founded (every non-empty subset has a maximal element), this claim is not immediately obvious to the author. Note however that it does arise as a corollary of this result.

4.3.2.3 Propositions The following result of [28] gives us to hope that we will be able to give a construction of the internal quantum phase space only using the quantum logic, as opposed to the original construction as the internal Gelfand spectrum of the internal commutative C*-algebra of quantum observables.

Theorem 4.25. *The Gelfand spectrum $\mathcal{O}(\mathrm{Max}(A))$ of a commutative Rickart C*-algebra[89] A is isomorphic to the frame $\mathrm{Spec}(\Pi(A))$ of ideals of $\Pi(A)$.*

We would like to go one step further than [28] and interpret the consequences of this result internally in our topos. To be precise we are going to apply internal Stone duality[90] to the internal Boolean algebra $\underline{\Pi(\mathcal{A})}$ to obtain another internal locale,

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[87]Here $\underline{*}$ denotes the terminal internal locale, i.e. the one with corresponding frame the subobject classifier.

[88]A mathematician might wonder what the added value is of stating the result like this. Such a form of the result is appealing to a physicist since it resembles a very fundamental phenomenon from (Yang-Mills) gauge theory, called Gribov ambiguity. There, the fact that a certain principal bundle does not admit global sections (i.e. is non-trivial) results in the impossibility of making a global choice of gauge (that would fix the value of observables). We have a similar phenomenon here: we cannot globally fix the value of all observables. Regardless of the (mathematical) structural similarity, however, the physical origin of the effects are very different.

[89]For our purposes the following is a practical definition: a commutative C*-algebra whose Gelfand spectrum is Stone and whose Boolean algebra of projections is countably complete. [3]

[90]Stone duality holds constructively if one is satisfied with the Stone space as a locale. (Recall that one needs the Boolean prime ideal theorem, a weak form of AC, to construct the points of the Stone spectrum.)

which, fortunately, coincides with our internal Gelfand spectrum. This means that we can recover $\underline{\mathcal{A}}$ from $\underline{\Pi(\mathcal{A})}$, as the object of continuous functions to $\underline{\mathbb{C}}$ on its internal Stone spectrum.

Theorem 4.26 (Constructive Stone-Duality,[32]). *For any (elementary) topos $\mathcal{E}$, we have an equivalence of categories*

$$\mathrm{Bool}(\mathcal{E})^{op} \overset{\mathrm{Compl}}{\underset{\mathrm{Spec}}{\rightleftarrows}} \mathrm{Stone}(\mathcal{E}),$$

where we write $\mathrm{Bool}(\mathcal{E})$ for the category of internal Boolean algebras and homo- morphisms and $\mathrm{Stone}(\mathcal{E})$ for the category of internal Stone locales[91] and continuous functions and where $\underline{\mathrm{Compl}}$ sends Σ to the Boolean algebra of complemented ele- ments of $\mathcal{O}\Sigma$ and $\underline{\mathrm{Spec}}$ sends a Boolean algebra B to its internal frame of ideals.

Proof. Noting that the finite elements of a regular frame (0-dimensionality implies regularity) are precisely the complemented ones, this is an easy corollary of Corollary II.3.3 in [32]. $\qquad\square$

Corollary 4.27. *The internal Gelfand spectrum $\underline{\mathrm{Max}(\mathcal{A})}$ of $\underline{\mathcal{A}}$ coincides with the in- ternal Stone spectrum[92] $\underline{\mathrm{Spec}(\Pi(\mathcal{A}))}$ of $\underline{\Pi(\mathcal{A})}$. Put differently[93], $\underline{\Pi(\mathcal{A})} \cong \underline{\mathrm{Compl}(\mathrm{Max}(\mathcal{A}))} = \{U \in \underline{\mathcal{O}\mathrm{Max}(\mathcal{A})} \,|\, U \vee \neg U = \top\}$.*

Proof. Let $A \in \mathcal{V}(\mathcal{A})$. Then, we already know that $\underline{\mathrm{Max}(\mathcal{A})}(A) = \mathrm{Max}(A)$. [30] Moreover, by noting that every Von Neumann algebra is a Rickart C*-algebra ([28]) we see from theorem 4.25 that $\mathcal{O}(\mathrm{Max}(A)) \cong \mathrm{Spec}(\Pi(A))$. So we have to show that the internal Stone spectrum of $\underline{\mathrm{Spec}(\Pi(\mathcal{A}))}(A) \cong \mathrm{Spec}(\Pi(A))$. The following lemma therefore does the rest. (We do not have to worry about their actions on morphisms, as both are subfunctors of $\underline{\mathcal{P}(\mathcal{A})}$.) $\qquad\square$

Lemma 2. *Let L be an internal lattice in a topos $\mathrm{Set}^{P^{op}}$ of presheaves over a poset P. Then the internal lattice of ideals is*

$$(\mathrm{Spec}(L))(C) = \mathrm{Spec}(L(C)).$$

Proof. We compute $\mathrm{Spec}(L)$ by using the sheaf semantics of $\mathrm{Set}^{P^{op}}$. (We use the notation of [35].) Note $\mathrm{Spec}(L) := \{I \in \mathcal{P}(L) \,|\, \phi(I) \text{ and } \psi(I) \text{ and } \chi(I)\}$, where $\phi(I)$ and $\psi(I)$ are formulae

$$\phi(I) := \forall i \in I \, \forall j \in L \, \exists k \in I : i \wedge j = k,$$

$$\psi(I) := \forall i \in I \, \forall j \in I \, \exists k \in I : i \vee j = k$$

and

$$\chi(I) := \exists k \in I : \bot = k.$$

Now, $I \in (\mathrm{Spec}(L))(C)$ iff $C \Vdash \phi(I) \wedge \psi(I) \wedge \chi(I)$ iff

$$\forall D \leq C \in P \, \forall i \in I(D) \, \forall E \leq D \in P \, \forall j \in L(E) \, \exists k \in (I|_E)(E) : (i|_E) \wedge_E j = k,$$

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[91]i.e. zero dimensional compact locales

[92]This shows that probably (at least, if we are working with a Von Neumann algebra) the internal Gelfand spectrum should not be viewed as an analogue of the classical phase space. It is a Stone space and therefore 0-dimensional. (So in particular, we hardly expect to formulate any kind of dynamics on it.) We should rather view it as the quantum analogue of the Gelfand spectrum of the universal enveloping Von Neumann algebra of the continuous functions on our classical phase space, i.e. its hyperstonean cover. If we are really looking for a 'quantum phase space', we might want to look for a characterisation of $C(X)$ in its universal enveloping Von Neumann algebra, find the analogous C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and study its internal Gelfand spectrum.

[93]Here, $\underline{\mathrm{Compl}(\mathrm{Max}(\mathcal{A}))}$ denotes the internal Boolean subalgebra of $\underline{\mathcal{O}\mathrm{Max}(\mathcal{A})}$ consisting of the complemented elements.

$$\forall D \leq C \in P \, \forall i \in I(D) \, \forall E \leq D \in P \, \forall j \in (I|_D)(E) \, \exists k \in (I|_E)(E) : (i|_E) \vee_E j = k,$$

and

$$\exists k \in I(C) : \perp_C = k.$$

This requires some explanation. Here $I \in \mathcal{P}L(C)$, so by the Yoneda lemma it corresponds to a unique arrow $y_C \longrightarrow \mathcal{P}L$ and by the exponential adjunction to a unique arrow $y_C \times L \longrightarrow \Omega$. This arrow in turn corresponds with a unique subobject of $L|_C = y_C \times L$, which we also denote I. Note that since P is a poset $y_C(C') = *$ if $C' \leq C$ and empty otherwise. This explains our notation $L|_C$. Similarly, with $I|_D$ we mean $I \times y_D$. Therefore $I|_D(E) = I|_E(E) = I(E)$. So our conditions reduce to

$$\forall E \leq D \leq C \in P \, \forall i \in I(D) \, \forall j \in L(E) \, \exists k \in I(E) : (i|_E) \wedge_E j = k,$$

$$\forall E \leq D \leq C \in P \, \forall i \in I(D) \, \forall j \in I(E) \, \exists k \in I(E) : (i|_E) \vee_E j = k$$

and

$$\exists k \in I(C) : \perp_C = k.$$

One easily verifies that these are equivalent to respectively

$$\forall C \in P \, \forall i \in I(C) \, \forall j \in L(C) \, \exists k \in I(C) : i \wedge_E j = k,$$

$$\forall C \in P \, \forall i \in I(C) \, \forall j \in I(C) \, \exists k \in I(C) : i \vee_E j = k$$

and

$$\exists k \in I(C) : \perp_C = k.$$

i.e. precisely the conditions stating that $I(C)$ is an ideal of $L(C)$. $\qquad \square$

One might wonder if this object $\underline{\mathrm{Compl}(\Sigma_*)}$ could play an analogous role to that of $\underline{\mathcal{P}_{cl}\Sigma}$ in the contravariant approach. The answer is "no". Indeed, note that $\Gamma\underline{\mathrm{Compl}(\Sigma_*)}$ is a Boolean algebra, since $\underline{\mathrm{Compl}(\Sigma_*)}$ is an internal Boolean algebra and the theory of Boolean algebras is algebraic. This means that, by theorem 4.3, we cannot have a morphism $\Pi(\mathcal{A}) \longrightarrow \Gamma\underline{\mathrm{Compl}(\Sigma_*)}$ that preserves anywhere near as much structure as the embedding of theorem 4.13 does.

The analogy does actually hold, but one has to choose the correct[94] object of 'clopen subobjects' of $\underline{\Sigma}_*$. Indeed, $\underline{\mathcal{P}_{cl}\Sigma}$ is not the internal Boolean subalgebra of $\underline{\mathcal{O}\Sigma^*}$ of complemented elements either. To correctly mirror the situation of the contravariant approach, we define the following subfunctor of $\underline{\mathcal{O}\Sigma_*}$:

$$\underline{\mathrm{Clopen}(\Sigma_*)}(A) := \left\{ D \in \underline{\mathcal{O}\Sigma_*}(A) \,|\, \text{for all } A' \in \mathcal{V}(\mathcal{A}) : D \cap \mathrm{Max}(A') \text{ is closed} \right\}.$$

In particular, $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ shall play the role of $\mathrm{Sub}_{cl}\underline{\Sigma}$:

$$\Gamma\underline{\mathrm{Clopen}(\Sigma_*)} = \left\{ D \in \mathcal{O}\Sigma_* \,|\, \text{for all } A \in \mathcal{V}(\mathcal{A}) : D \cap \mathrm{Max}(A) \text{ is closed} \right\}.$$

Mimicking theorem 4.13, we define

$$\Pi(\mathcal{A}) \longrightarrow \Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$$

$$p \longmapsto \{\lambda \in \mathrm{Max}(A) \,|\, \lambda(\delta^i(p)_A) = 1\}.$$

Seeing that $p \in A$ for some $A \in \mathcal{V}(\mathcal{A})$ and therefore $\delta^i(p)_A = p$ for this A, we see that this map is an injection and (literally) mirroring the proof of theorem 4.13 in [17], we have

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[94]The reader should note that the two options coincide in Set.

Theorem 4.28.

$$\Pi(\mathcal{A}) \xhookrightarrow{\underline{\delta}^i} \Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$$

$$p \longmapsto \{\lambda \in \mathrm{Max}(A) \,|\, \lambda(\delta^i(p)_A) = 1\,\}$$

defines an embedding[95] of complete distributive inf-lattices (preserving $\leq, 0, 1, \wedge$). Note that it does not also preserve $\vee$ in general, since it is a map from a non-distributive lattice to a distributive one. However, $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ does have small meets and

$$\underline{\delta}^i(p \vee q) \geq \underline{\delta}^i(p) \vee \underline{\delta}^i(q).$$

Moreover, $\underline{\delta}^i$ does not preserve the negation and $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ does in general not satisfy the law of the excluded middle.
Finally, for each $A \in \mathcal{V}(\mathcal{A})$, it restricts to an order isomorphism

$$\Pi(A) \xrightarrow[\underline{\delta}^i(A)]{\cong} \mathrm{Sub}_{cl}(\mathrm{Max}(A)).$$

One may wonder if we can define our states, like in theorem 4.14 of the contravariant approach, as finitely additive measures this set $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ of clopen subobjects. This indeed works out and it even agrees with the restriction of our probability valuations.

Theorem 4.29. *For each quasi-state ω, one can define a finitely additive measure μ_ω on $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ by*

$$\mu_\omega(\underline{S})(A) := \omega\left(\underline{\delta}^i(A)^{-1}(\underline{S}(A))\right),$$

where $\underline{\delta}^i(A)$ is the isomorphism $\Pi(A) \longrightarrow \Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ of theorem 4.28. In fact, we can equivalently define states to be finitely additive measures on $\Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$. This abuse of notation is acceptable since this definition precisely coincides with the restriction of the probability valuation μ_ω on $\mathcal{O}\Sigma_*$ defined by internal Riesz-Markov duality.

Proof. We leave verification of the first statement to the reader. The second claim follows by exactly the same proof as one uses in the contravariant approach. This can be found in [18]. We examine the last claim.

We write μ_ω for the probability valuation defined by the Riesz-Markov theorem and derive that for $\underline{S} \in \Gamma\underline{\mathrm{Clopen}(\Sigma_*)}$ the formula of the theorem holds. According to Lemma 4.6 in [43], we have, for $A \in \mathcal{V}(\mathcal{A})$,

$$\mu_\omega(\underline{S})(A) = \sup\{\omega(p) \,|\, p \in \Pi(A), X_p^A \subset \underline{S}(A)\},$$

where $X_p^A = \{\lambda \in \mathrm{Max}(A) \,|\, \lambda(p) > 0\,\}$. Now, writing $p_{\underline{S}(A)} := \underline{\delta}^i(A)^{-1}(\underline{S}(A))$, note that $\underline{S}(A) = \{\lambda \in \mathrm{Max}(A) \,|\, \lambda(p_{\underline{S}(A)}) = 1$. Therefore, for each p over we take the supremum of ω,

$$\forall \lambda \in \mathrm{Max}(A) : \lambda(p) > 0 \Rightarrow \lambda(p_{\underline{S}(A)}) = 1.$$

We see that the supremum is attained at $p = p_{\underline{S}(A)}$, by noting that for all $\lambda \in \mathrm{Max}(A)$ there always exists a $p \in \Pi(A)$ such that $\lambda(p) > 0$ and that for all $\lambda \in \mathrm{Max}(A)$ and $p \in \Pi(A)$ $\lambda(p) \leq 1$.

Finally, note that, according to Mackey-Gleason, every such measure also defines a quasi-state by restricting along $\underline{\delta}^i$. $\qquad\square$

---

[95]Note that an injective (finite) inf-preserving morphisms is automatically an order embedding.
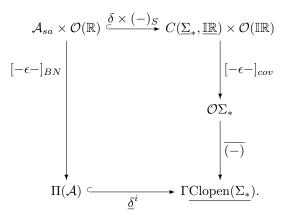
Although this parallel with the contravariant approach is nice enough, the situation does not resemble classical mechanics one bit and that was the purpose of this whole endeavour after all. The framework of [29] does not start with the Birkhoff-Von Neumann quantum propositions from section 3.3.2 and embed them into some topos-related logic. Rather, it builds propositions from the data of observables and subsets of outcomes directly in their topos framework, imitating what we did in the classical kinematics of section 3.3.1.

For an $a \in C(\Sigma_*, \mathbb{IR})$ and $\Delta \in \mathcal{O}\mathbb{IR}$, they define a 'proposition'

$$[a\epsilon\Delta]_{cov} := a^{-1}\Delta \in \mathcal{O}\Sigma_*.$$

If we are to regard this as a proposition, there had better be some relation with our Birkhoff-Von Neumann quantum propositions that had an actual motivation from quantum physics and did not just arise in an attempt to push some analogy with classical physics. Let us write $[a\epsilon\Delta]_{BN}$ for the Birkhoff-Von Neumann quantum propositions from now on to make a clear distinction between the two notions. Then, in [43], Wolters proved the following surprising result. (The proof is not particularly complicated, but it is a rather long exercise in unraveling all the definitions.)

Theorem 4.30 ([43], essentially theorem 4.9). *The way propositions are formed in the covariant approach to quantum kinematics is compatible with the way the Born rule dictates one should do it for Birkhoff-Von Neumann quantum logic:*

$$
\begin{array}{ccc}
\mathcal{A}_{sa} \times \mathcal{O}(\mathbb{R}) & \xrightarrow{\delta \times (-)_S} & C(\underline{\Sigma}_*, \underline{\mathbb{IR}}) \times \mathcal{O}(\mathbb{IR}) \\
\Big\downarrow {\scriptstyle [-\epsilon-]_{BN}} & & \Big\downarrow {\scriptstyle [-\epsilon-]_{cov}} \\
& & \mathcal{O}\Sigma_* \\
& & \Big\downarrow {\scriptstyle \overline{(-)}} \\
\Pi(\mathcal{A}) & \xhookrightarrow{\underline{\delta}^i} & \underline{\Gamma\mathrm{Clopen}}(\Sigma_*).
\end{array}
$$

Here $\mathcal{O}\Sigma_* \xrightarrow{\overline{(-)}} \underline{\Gamma\mathrm{Clopen}}(\Sigma_*)$ denotes the topological closure - the closure of an open is again open as a consequence of the fact that the spectrum of a commutative Von Neumann algebra is extremally connected, by theorem 3.7 - and $(-)_S$ denotes is the restriction of the injection[96]

$$\mathcal{P}\mathbb{R} \xhookrightarrow{(-)_S} \mathcal{P}\mathbb{IR}$$

$$X \longmapsto \{[r,s] \mid [r,s] \subset X\}.$$

Although this is a rather nice and surprising mathematical relation between the two notions of proposition. We should verify that they indeed represent the same idea physically. To do this, we study the state-proposition pairing and show that they yield the same truth values when combined with an arbitrary state.

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[96]This is a reasonable map to consider since it restricts to the map $(r,s) \mapsto (r,s)_S$ between the bases of the topologies. Also $(\{x\})_S = \{\{x\}\}$, so it embodies the idea of the map $\mathbb{R} \xrightarrow{\{\cdot\}} \mathbb{IR}$.

4.3.2.4 State-Proposition Pairing

From a geometric point of view, the state-proposition pairing the covariant approach is very close to the one in the contravariant approach. Recall our dichotomy map $[0,1]_l \overset{\mathrm{Dich}}{\Longrightarrow} \underline{\Omega}$.

Each internal valuation $\underline{\mu}$ on $\underline{\mathcal{O}(\mathrm{Max}(\mathcal{A}))}$ defines a map $\underline{\mathcal{O}(\mathrm{Max}(\mathcal{A}))} \overset{\mathrm{Dich} \circ \mu}{\longrightarrow} \underline{\Omega}$. Seeing that our quantum propositions were maps $\underline{1} \overset{p}{\longrightarrow} \underline{\mathcal{O}(\mathrm{Max}(\mathcal{A}))}$, in this framework, we can take their composition to yield a truth value

$$\underline{1} \xrightarrow{\mathrm{Dich} \circ \underline{\mu} \circ p} \underline{\Omega}$$

or equivalently

$$1 \xrightarrow{\Gamma \underline{\mathrm{Dich}} \circ \mu \circ p} \Gamma \underline{\Omega} = \mathrm{Sub}(\underline{1}).$$

We say that a proposition p is true at context A for a state μ if $\Gamma \underline{\mathrm{Dich}} \circ \mu \circ p(A) = 1$. Using this notion of truth, we have the following result.

Theorem 4.31 ([43],lemma's 4.7 and 4.8 and theorem 4.9)**.** *As far as truth values go, the two ways $[-\epsilon-]_{BN}$ and $[-\epsilon-]_{cov}$ of building propositions are equivalent. The same holds for the algebraic and geometric ways of pairing states and propositions.*

Let μ be a valuation $\mathcal{O}\Sigma_* \longrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathcal{V}(\mathcal{A}), [0,1])$ (one of our generalised quantum states). By theorems 4.21 and 4.18, we know that μ corresponds to some quasi-state ω_μ on $\mathcal{A}$. Then, for $A \in \mathcal{V}(\mathcal{A})$, the following are equivalent.

1. $\omega_\mu(\delta^i([a\epsilon\Delta]_{BN})_A) = 1$;

2. $\mu(\underline{\delta}^i([a\epsilon\Delta]_{BN}))(A) = 1$;

3. $\mu(\overline{[\delta(a)\epsilon\Delta_S]_{cov}})(A) = 1$;

4. $\mu([\delta(a)\epsilon\Delta_S]_{cov})(A) = 1$.

Proof. Wolters only proves this for the case that ρ is a state. However, I see no reason why his proof should not work for the general case. $\qquad\square$

If one likes to put it this way, for each (quasi-)state ω and each proposition, we have a map[97]

$$\Pi(\mathcal{A}) \xrightarrow{\mathrm{truth}_\omega} \Gamma \underline{\Omega} \cong \mathrm{Sub}(\underline{1}),$$

where $\mathrm{truth}_\omega(p)(A)$ is Dich[98] applied to the left hand side of any of the equations in the above enumeration (and we have picked some description $[a\epsilon\Delta]_{BN} = p$).

4.3.3 Interpretation of Truth

Let us ask the same questions as in the contravariant approach.

1. How much information can we infer about ω from knowing truth_ω?

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[97]The reader might have been expecting this to be defined as a map $\Gamma \underline{\mathrm{Clopen}\Sigma_*} \longrightarrow \mathrm{Sub}\underline{1}$, hoping that this would be a lattice (or even Heyting) homomorphism for pure states ω. (The pure states define homomorphisms to the truth values in classical kinematics.) Unfortunately, this does not work out. As one can check, this map indeed preserves $\wedge$, but not $\vee$. This is a consequence of the fact that pure states do not define a one-point measure. (In the GNS-representation the measure of a state $|\psi\rangle$ is defined by the inner product $p \mapsto \langle\psi|\delta^i(p)|\psi\rangle$.)

[98]$[0,1] \overset{\mathrm{Dich}}{\Longrightarrow} \{0,1\}$, $(x < 1) \mapsto 0, 1 \mapsto 1$.

2. How should we interpret these truth values?

Again truth$_\omega$ contains the same information as the support of ω. Therefore, in this respect these 'nuanced' truth values are no better than those of the Birkhoff-Von Neumann logic. The difference is to be sought in the truth value of one particular proposition.

By definition truth$_\omega(p)(A) = 1$ iff $\omega(\delta^i(p)_A) = 1$, i.e. if with certainty the proposition $\delta^i(p)_A$ is true for ω (in the Birkhoff-Von Neumann sense). Seeing that $\delta^i(p)_A$ is the weakest antecedent that we can investigate in our classical context A that would imply p, we can immediately say that p is true for ω if and only if there exists an A such that truth$_\omega(p)(A) = 1$. The way one should therefore think of these propositions is with the intension of verification in mind.

Indeed, if $\delta^i(p)_A$ is not true, this tells us nothing about the truth of p. However, assume proposition p is true for state ω and we want to exhibit this with the restriction that we are only allowed to perform measurements in classical context A (because we want to avoid disturbing our system to much, say). Then, our best option is to examine $\delta^i(p)_A$. If that turns out to be true, so was p. The way to interpret truth$_\omega(p)$, therefore, is that it has a value 1 at context A if and only if we can verify p by measuring observables from A.

Since the logical operations on the subobject lattice are computed pointwise, so truth of $\underline{\delta}^i(p) \vee \underline{\delta}^i(p')$ at A should be interpreted as possibility of verifying either p or q by measuring observables from A and similarly for $\wedge$. In particular, note that there is no ambiguity as to which of the two experiments should be performed first, since the propositions commute.

4.3.4 Summary

Summarising, we have set up a framework for quantum kinematics in the topos Set$^{\mathcal{V}(\mathcal{A})}$. This was built on the definition of the tautological internal C*-algebra $\underline{A}$ and its internal Gelfand spectrum $\underline{\Sigma}_*$. In terms of these objects, we defined respectively both an algebraic and a geometric formalism internal to the topos that deals with generalised kinds of observables, states and propositions, in which the corresponding objects of ordinary quantum mechanics were embedded. This went as follows.

Observables:

$$\mathcal{A}_{sa} \overset{\delta}{\hookrightarrow} C(\underline{\Sigma}_*, \underline{\mathbb{R}}) \cong C(\Sigma_*, \mathbb{R}),$$

states:

$$\mathscr{S}(\mathcal{A}) \hookrightarrow \mathscr{S}_{quasi}(\mathcal{A}) \cong \quad \text{"internal states"}(\underline{A}) \cong \text{"finitely additive measures"}(\Pi(\mathcal{A}))$$

$$\cong \qquad \text{"finitely additive measures"}(\underline{\Pi(\mathcal{A})})$$

$$\cong \text{"finitely additive measures"}(\Gamma\underline{\text{Clopen}(\Sigma_*)}) \cong \text{"valuations"}(\underline{\Sigma}_*),$$

propositions:

$$\Pi(\mathcal{A}) \overset{\delta^i}{\hookrightarrow} \Gamma\underline{\text{Clopen}(\Sigma_*)} \hookrightarrow \Gamma\underline{\mathcal{O}\Sigma_*} \cong \mathcal{O}\Sigma_*.$$

This last map defines a inf-embedding of orders and it restricts to an order-isomorphism between each $\Pi(A)$ and $\text{Sub}_{cl}(\text{Max}(A))$, $A \in \mathcal{V}(\mathcal{A})$. This enlarged 'quantum logic' has the following properties.

1. It is a complete Heyting algebra.

2. The interpretation of the (Heyting) negation is "p is not true", rather than "p is false".

3. *Each* state determines the truth-value of all quantum propositions. This is a consequence of our dichotomy-map that transformed probabilities into true-false-judgements in each classical context. The truth values at different classical contexts have the immediate operational interpretation as possibility of verification: a 1 means that verification is possible.

4. For pure states (and mixed states) the truth values fail to define lattice homomorphisms $\underline{\text{Clopen}\Sigma_*} \longrightarrow \text{Sub}\underline{1}$.

5. We have an inf-embedding of $\Pi(\mathcal{A})$ into the complete Heyting algebra $\text{Sub}_{cl}(\underline{\Sigma})$, that, when restricted to any Boolean subalgebra of $\Pi(\mathcal{A})$, defines an isomorphism of Boolean algebras to $\text{Sub}_{cl}(\text{Max}(A))$

6. The interpretation of $\vee$ is what we are used to: for a state $|\psi\rangle$, $p \vee p'$ can only be true, if either p or p' is true. However, the law of the excluded middle fails.

7. The propositions have a clear operational interpetation. The logical operations are defined pointwise so we are only combining commuting propositions in conjunctions and disjunctions. The truth value of such combined propositions therefore does not depend on the order in which we verify the truth value of the building blocks of this proposition.

8. We have a rather nice way of reconstructing the internal algebra $\mathcal{A}$ of observables from the 'quantum logic' $\underline{\Pi(\mathcal{A})}$. Namely, as the continuous functions on its internal Stone spectrum.

4.4 Discussion

Recall that the primary motivation[99] for attempting to give a topos theoretic description of quantum kinematics was to provide the ordinary quantitative probabilistic formalism with a qualitative logical counterpart that was not so blunt in its notion of truth as the Birkhoff-Von Neumann logic. Have these ambitions been realised? Partly, I would say. Possibly more important is the byproduct[100] of this search for a suitable quantum logic: a framework for quantum kinematics that is surprisingly similar to that of classical kinematics, except for the fact that it is set up in a different topos than Set.

Let us recap what we have done in setting up the framework. Essentially, we have restored the commutativity of the algebra and the distributivity of the logic by giving our objects under consideration a more interesting internal structure. Non-classical aspects of quantum kinematics are hidden in the indexing of objects by the poset of classical contexts. In many cases, we were able to show that the constructions from classical kinematics performed in the topos resulted in (slightly) generalised notions of quantum states, observables and propositions. In the covariant approach, the topos contained a tautological internal commutative C*-algebra

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[99]I should admit that [29] and [28] give an entirely different motivation for using toposes: a case is made that we should expect quantum logic to be intuitionistic, since the law of excluded middle cannot possibly hold, seeing that ignorance interpretation of quantum probabilities is unacceptable. I am afraid to say that I do not entirely follow the reasoning and it leaves me with the impression that the subtleties in the physical interpretation of the logical operations in the different logics are not being fully appreciated.

[100]Here, of course, I am mostly speaking of the covariant approach.

from which we managed to develop the whole framework in a way that was entirely analogous to classical kinematics.

By applying daseinisation maps to approximate propositions in other classical contexts, we managed to embed the Birkhoff-Von Neumann logic into a complete Heyting algebra, regaining distributivity at the cost of the law of the excluded middle and either the meet or the join.

We defined a new, more subtle notion of truth on these logics, giving a separate judgement in each classical context, the object of truth values being $\Gamma\underline{\Omega} \cong \mathrm{Sub}(\underline{1})$. In my opinion the best interpretation of truth values of propositions in the contravariant and covariant approach is that they respectively contain the (in)possibility of falsification and verification in different contexts. I have not seen this interpretation of the truth of quantum propositions in the two approaches in literature. The treatment of the issue is usually limited to a remark along the following lines. Our new quantum propositions relate as follows to the Birkhoff-Von Neumann ones. Contravariant propositions (not truth values) represent truths about a system. Covariant propositions represent statements about the system that one might try to verify (but that are not necessarily true) in each context to establish truth of the corresponding Birkhoff-Von Neumann quantum proposition. In particular, truth of a proposition in the covariant sense in some context immediately implies truth in every context in the contravariant approach[101].

Unfortunately, the truth value maps corresponding to pure states did not preserve the logical operations, thereby breaking the parallel with classical kinematics. This, again, shows that pure quantum states are fundamentally different from pure states in classical mechanics. They do not seem to be very susceptible to deterministic descriptions.

We conclude with a brief comparison of aspects of the two approaches to the subject. One should note that the intensions of the two approaches were initially very different: where the contravariant approach seemed mostly to be looking for a good notion of quantum logic (within their broader framework of topos approaches to physics), the covariant approach focussed on stressing the parallel with classical kinematics.

We can still see that this parallel is much more pronounced in the covariant approach, mostly because there is both an algebraic and a geometric side to the story, while the contravariant approach so far has resisted attempts to associate a geometry to it. However, in recent work, notably [28] and [43], the logical side of the covariant approach has been explored, mirroring all constructions from the contravariant approach. As far as I can tell, this has been a big succes. One could even say that the covariant approach now encompasses the contravariant one, with the caveat that the logic has an interpretation in terms of verification rather than falsification.

Interaction clearly has taken place between the two approaches. Even with that in mind, however, it is quite striking that the two produce such similar results, given their completely different origins. This might show that, even though the constructions employed are a bit uncommon for the standard of physics, they can hardly be complete non-sense.

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