Theory Reduction in Physics: A Model-Based, Dynamical Systems Approach

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Abstract

In 1973, Nickles identified two senses in which the term 'reduction' is used to describe the relationship between physical theories: namely, the sense based on Nagel's seminal account of reduction in the sciences, and the sense that seeks to extract one physical theory as a mathematical limit of another. These two approaches have since been the focus of most literature on the subject, as evidenced by recent work of Batterman and Butterfield, among others. In this paper, I discuss a third sense in which one physical theory may be said to reduce to another. This approach, which I call 'dynamical systems (DS) reduction,' concerns the reduction of individual models of physical theories rather than the wholesale reduction of entire theories, and specifically reduction between models that can be formulated as dynamical systems. DS reduction is based on the requirement that there exist a function from the state space of the low-level (more encompassing) model to that of the high-level (less encompassing) model that satisfies certain general constraints and thereby serves to identify quantities in the low-level model that mimic the behavior of those in the high-level model - but typically only when restricted to a certain domain of parameters and states within the low-level model. I discuss the relationship of this account of reduction to the Nagelian and limit-based accounts, arguing that it is distinct from both but exhibits strong parallels with a particular version of Nagelian reduction, and that the domain restrictions employed by the DS approach may, but need not, be specified in a manner characteristic of the limit-based approach. Finally, I consider some limitations of the account of reduction that I propose and suggest ways in which it might be generalised. I offer a simple, idealised example to illustrate application of this approach; a series of more realistic case studies of DS reduction is presented in another paper.

Contents

1 Introduction

Broadly speaking, 'reduction' in physics can be understood as a relationship that obtains between two theories describing the same physical system or class of systems, such that the success of one theory in describing the system can be accounted for on the basis of the other, which is taken to be the more encompassing and more accurate of the two descriptions. Probing further, we can ask what it is specifically about the relationship between the two theories that allows this subsumption to occur. Over the past 40 years, the literature on reduction in physics has tended to revolve around two ways of addressing this question, which were first distinguished by Nickles in his widely cited 1973 paper, 'Two Concepts of Intertheoretic Reduction': first, the concept of reduction built around Nagel's seminal account first spelled out in Chapter 11 of *The Structure of Science*, and second, the concept that regards reduction as a matter of extracting one physical theory as a mathematical limit of another.

In this paper, I elaborate a third sense in which one theory in physics may be said to 'reduce to' another, one which I argue resolves some of the vagueness that afflicts both the Nagelian and limit-based accounts. It is important to note from the outset that this account concerns the reduction of individual models of physical theories, rather than the wholesale reduction of entire theories, and moreover, that it applies specifically to the reduction of models that can be formulated as dynamical systems - i.e., models that can be specified by some mathematical state space and some deterministic rule prescribing the time evolution of points in that space. Dynamical systems theory, I claim, allows the formulation of particularly natural and simple conditions for one dynamical systems model to reduce to another, and because many theories in physics permit formulation of their models in terms of dynamical systems, provides a general mathematical framework for describing a wide range of inter-theory relations in physics. For this reason, I call this approach the 'dynamical systems,' or DS, approach to reduction. Insofar as one is inclined to speak of *theories* rather than models being reduced on this account, this occurs only piecemeal through the reduction of a theory's individual models.

In section 2, I briefly review the Nagelian and limit-based approaches to reduction in physics and consider various precisifications of these approaches. In section 3, I introduce the concept of a dynamical systems model and discuss some of its applications in physics. In section 4, I give conditions for the reduction of dynamical systems models in physics and provide a simple example to illustrate the application of these conditions. Section 5 reviews work by a number of authors that has served in one some way to anticipate or hint at the DS approach that I elaborate here. Section 6 provides a comparison between DS and Nagelian reduction, highlighting both parallels and differences. Section 7 comments on the role of limit-based results in DS reduction. Section 8 discusses some limitations of the DS view and ways in which it might be generalised. Section 9 is the conclusion.

2 Two Views of Reduction in Physics

In [26], Nickles distinguishes two uses of the term‘ reduction’ with regard to inter-theory relations in physics, one common to the philosophical literature and the other common to the physics literature on the subject. The ‘philosopher’s’ sense of reduction takes a high-level (i.e. less encompassing, less fundamental) theory T_h to ‘reduce to’ a low-level (i.e., more encompassing, more fundamental) theory T_l , while the ‘physicist’s’ sense takes T_l to ‘reduce to’ T_h. So on the philosopher’s usage, Newtonian mechanics ‘reduces to’ special relativity; on the physicist’s usage, special relativity ‘reduces to’ Newtonian mechanics; nevertheless, both usages presuppose that special relativity is the more accurate and more encompassing of the two theories, and that it is the success of Newtonian mechanics that must be accounted for on the basis of special relativity and not the other way around. Thus, the difference between the two usages is to some extent a matter of convention as to the direction in which ‘reduction’ is taken to go.

Yet the distinction between the two senses of reduction is not solely a matter of convention. Once the conventions are made to agree, there remains a substantive difference between the meaning of the term ‘reduction’ as it is most commonly employed in the physics literature and its meaning as it is most commonly employed in the philosophy literature. The philosopher’s notion is based on Nagel’s account of reduction in the sciences while the physicist’s notion views reduction in physics essentially as a matter of taking mathematical limits [25] [1]. These two concepts of reduction, which

[1]Of course, one may question whether it is entirely appropriate or fair to identify one sense of reduction as the physicist’s and the other as the philosopher’s. There are, after all, instances of physicists employing what is effectively reduction in the philosopher’s sense (arguably, textbook proofs of the Ideal Gas Law on the basis of statistical mechanics are

Nickles identifies as *reduction*$_1$ and *reduction*$_2$, respectively, can be defined as follows:

> **Nagelian Reduction:** T_h *reduces*$_1$ to T_l iff the laws of T_h can be derived from those of T_l along with auxiliary assumptions (known 'bridge laws') linking terms in T_h foreign to T_l with terms in T_l.
>
> **Limit-Based Reduction:** T_h *reduces*$_2$ to T_l iff there exists some set of parameters $\{\epsilon_i\}$ defined within T_l such that $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$. [26],[3] [2] [3]

For reasons that I discuss shortly, both of these definitions are problematic as descriptions of actual inter-theory relations in physics and have been subject to various refinements as a result.

Before moving on to discuss Nagelian and limit-based reduction in more detail, I should note that there also exist other influential philosophical accounts of reduction, such as Kim's functionalist model and Hooker's New Wave model [18], [17]. The relation of these accounts particularly to the Nagelian approach is a matter of some dispute; for example, Marras has argued that Kim's account is only superficially distinct from Nagel's, and Fazekas likewise has argued for a similar claim with regard to Hooker's New Wave model [23], [12]. Kim's and Hooker's accounts have been developed primarily within the context of discussions about reduction of the mental to the physical in philosophy of mind. Since it would take me too far afield to see how, if at all, they can be applied to reductions in physics, I will focus on the two approaches that Nickles discusses and that have been the main focus of the literature on theory reduction specifically in physics.

2.1 Nagelian Reduction

According to the account of reduction that Nagel sets out in *The Structure of Science*, reductions can be broadly classified into two categories: homoge-

examples of this [19]) and of philosophers employing reduction in the physicist's sense (see, for instance, [2]). Nevertheless, I will adhere to Nickles' terminology.

[2]Note that if one has $\lim_{\{\epsilon_i \to \infty\}} T_l = T_h$, or $\lim_{\{\epsilon_i \to a\}} T_l = T_h$ where $0 < a < \infty$, one can always redefine the parameters $\{\epsilon_i\}$ so that each ϵ_i approaches 0.

[3]In Nickles' original definition of *reduction*$_2$, the sense of reduction is the inverse of the one I give here, in that on Nickle's definition the superseding theory T_1 *reduces*$_2$ to the superseded theory T_2, rather than vice versa as in the definition that I provide. As discussed, this inversion merely reflects a difference of convention.

neous and inhomogeneous. In the former, the theory to be reduced contains no terms which are not contained in the reducing theory, while in the latter it does contain such terms. An example of a homogeneous reduction is the reduction of Kepler's theory of planetary motion to Newton's Theory of Gravitation, since Kepler's theory employs only concepts such as distance, time, and concepts derived from these, all of which are also contained in Newton's theory [9]. An example of a heterogeneous reduction is the reduction of thermodynamics, which employs the concept of temperature, to the Newtonian mechanics of microscopic particles, which contains no reference to temperature. Recognizing that 'no term can appear in the conclusion of a formal demonstration unless the term also appears in the premises,' Nagel asserts that in the case of an inhomogeneous reduction, additional assumptions connecting terms in the high-level theory (which Nagel refers to as the 'secondary science') with terms in the low-level theory (which Nagel calls the 'primary science') must be introduced. Thus, he proposes two formal conditions - the condition of 'connectability' and the condition of 'derivability' - that must be satisfied in order to effect an inhomogeneous reduction of the higher- to the lower- level theory:

> (1) Assumptions of some kind must be introduced which postulate suitable relations between whatever is signified by 'A' [a term in the secondary science] and traits represented by theoretical terms already present in the primary science. The nature of such assumptions remains to be examined; but without prejudging the outcome of further discussion, it will be convenient to refer to this condition as the 'condition of connectability.' (2) With the help of these additional assumptions, all the laws of the secondary science, including those containing the term 'A,' must be logically derivable from the theoretical premises and their associated coordinating definitions in the primary discipline. Let us call this the 'condition of derivability.' ([25], Ch. 11)

The additional premises furnished by the condition of connectability have come to be known as 'bridge laws,' though occasionally are also referred to as 'bridge principles,' 'bridge rules,' 'coordinating definitions,' and 'reduction functions.' Their purpose is to link those terms in the high-level theory that do not occur in the low-level theory with terms in the low-level theory and thereby to facilitate derivation of the high-level theory's laws from the low-level theory. The question as to the precise nature of these additional assumptions - whether they are in fact 'laws' of nature in the same sense that, say, Newton's laws are, or whether they are mere 'coordinating definitions' - remains a matter of controversy (see, for instance, [9] for further discussion of this point). I will abide by common usage here and refer to these additional assumptions as bridge laws, though the reader should not take this to entail any commitment to the view that they are laws of nature.

The central example that Nagel employs to illustrate his account of reduction is the relation of the Ideal Gas Law ($pV = nRT$), as understood in the context of classical thermodynamics, to the laws of Newtonian mechanics as applied to the microscopic constituents - hard sphere 'molecules' - of an idealised gas. He notes that while the term 'temperature' had an accepted meaning in the context of thermodynamics - given in terms of operational procedures employing thermometers and other devices - the term makes no appearance in the low-level theory, Newtonian mechanics. He then invokes a strategy employed by physicists in deriving the Ideal Gas Law from Newton's Law: namely, to associate the thermodynamical term 'temperature' with a quantity, average molecular kinetic energy, that is defined within the framework Newtonian mechanics. More precisely, from Newton's laws and certain auxiliary assumptions concerning the position and velocity distributions of the molecules in the gas (such as uniformity of the distribution in position and isotropy in velocity), one can deduce that

$$pV = \frac{2}{3}N\langle K.E.\rangle \tag{1}$$

where N is the number of molecules in the gas. Now the form of the Ideal Gas Law in thermodynamics is

$$pV = nRT, \tag{2}$$

where $n = \frac{N}{N_A}$ (with N_A Avogadro's constant) is the number of moles in the gas and $R = N_A k_B$ (with k_B Boltzmann's constant) is the universal gas constant. If one makes the additional assumption,

$$\underline{\text{Bridge ``Law''}} : T = \frac{2}{3}\frac{\langle K.E.\rangle}{k_B} \iff \langle K.E.\rangle = \frac{3}{2}k_B T \tag{3}$$

one can deduce (2) from (1). Equation (3) has come to serve as the primary exemplar of a bridge law in Nagelian reduction.

2.1.1 Nagel's Account, Refined

Schaffner, one of the first to expand on Nagel's account of reduction, observed that Nagel's account is, strictly speaking, too stringent since reductions in practice rarely if ever yield derivations of the higher level theory T_h, but rather of some modified or corrected version $T_h^{'}$ of T_h that employs the same vocabulary as T_h. $T_h^{'}$ is sometimes referred to as the 'analogue theory' of T_h and is required to be 'strongly analogous' to T_h in the sense of approximating it closely. According to Schaffner, bridge laws can then be understood as enabling the derivation of $T_h^{'}$ - rather than T_h - from T_l [32], [33].

For example, a more precise treatment of the reduction just considered would take into account the fact that equation (1) holds only approximately (given the approximate nature of the assumptions concerning the distribution of molecules in position and velocity):

$$\underline{\text{Image Theory}} : pV \approx \frac{2}{3}N\langle K.E.\rangle. \tag{4}$$

This relation, which is formulated in the language of the low-level theory, is an example of what is sometimes called an 'image theory,' in the sense that it provides an 'image' of laws in the high-level theory formulated in the language of the low-level theory (Hooker also employs the notion of an 'image theory' in his New Wave account of reduction). Equations (4) and (3) jointly imply that

$$\underline{\text{Analogue Theory}} : pV \approx nRT, \tag{5}$$

which is 'strongly analogous' to the Ideal Gas Law.

The refinement of Nagel's account that I consider here, dubbed the Generalized Nagel-Schaffner (GNS) model by Dizadji-Bahmani, Frigg and Hartmann in (see [9]), consolidates both Schaffner's and Nagel's insights. On this model, reduction can be understood as a three-step process, starting with the basic ingredients of a low-level theory T_l, a high-level theory T_h, and a set of bridge laws:

1. Derive the image theory T_h^* for some restricted boundary or initial conditions within the low level theory T_l, without employing bridge laws.

2. Use bridge laws to replace terms in T_h^*, which belong to the vocabulary of the low level theory, with corresponding terms belonging to the high level theory. This yields the analogue theory T_h'.

3. If T_h' is 'strongly analogous' to the high-level theory T_h, the high-level theory has been reduced to T_l. The 'strong analogy' relation is sometimes also characterised as 'approximate equality,' 'close agreement,' or 'good approximation.'

Henceforth, when I speak of Nagelian reduction, I will construe it according to the GNS model, unless explicitly stated otherwise. Moreover, note that Nagel's connectability condition on this refinement consists of two 'connections': first, the bridge laws that link the image theory T_h^* and the analogue theory $T_h^{'}$, and second, the rather vaguely defined 'strong analogy' relation that connects the analogue theory $T_h^{'}$ to the high-level theory T_h. In what, precisely, does the approximate agreement typically used to characterise the strong analogy relation consist? Specifically which quantities between these two descriptions must approximately agree, and in what contexts? I argue below that the DS approach to reduction that I develop in this paper exhibits certain strong parallels with the GNS approach but also serves to resolve certain of its ambiguities, particularly relating to the notion of strong analogy and the nature of the linkages that must be established between the concepts of the reduced and reducing theories.

2.2 Limit-Based Reduction

Unlike Nagelian reduction, the notion of a limit-based approach to reduction, as first explicitly identified by Nickles, seems to arise not from any clear-cut statement of the general conditions for this kind of reduction to take place, but rather from a plethora of highly suggestive mathematical results all of which involve or somehow imply the use of mathematical limits, and also from a certain manner of speaking that is often employed in talk about inter-theory relations in physics, as exemplified by references to the 'nonrelativistic limit' of special relativity, the 'classical limit' of quantum mechanics, the 'thermodynamic limit' of statistical mechanics, the 'geometric optics limit' of wave optics, and so on. Batterman, Butterfield, Rohrlich, Berry, Ehlers, and Scheibe, among others, all have explored various facets of this approach to reduction from a general philosophical and methodological point of view as well as in the context of particular case studies [7], [2], [30], [4], [11], [34], [35]. Talk of reduction in terms of limits also pervades discussion of inter-theory relations in many physics textbooks and journals. Despite the popularity of the limit-based approach to reduction, the vague and schematic relation $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$ appears to be as close to a statement of general criteria for limit-based reduction as has been given in the literature.

In this subsection, I will attempt to clarify the intended meaning of claims that one theory is a limit, or limiting case, of another by proposing a series of refinements to the definition given above that suggest general criteria for limit-based reduction. But first it will prove instructive to consider particular examples of the sort of result that motivates the idea of a limit-based approach to reduction in physics, of which there are many.

Perhaps the simplest and most widely known example of such a result is given by the relativistic expressions for time dilation and length contraction between inertial reference frames:

$$\begin{aligned} t' &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\, t \\ l' &= \sqrt{1-\frac{v^2}{c^2}}\, l, \end{aligned} \tag{6}$$

where t is the time between two events at the same location in some intertial 'lab' frame and t' the time between these same events as measured from an inertial frame moving with constant velocity v with respect to the lab frame, and likewise l is the length of an object at rest in the lab frame and l' its length as measured from an inertial frame moving parallel to the line along which l is measured with constant velocity v relative to the lab frame. From these relations, it follows that

$$\begin{aligned} \lim_{c\to\infty} t' &= t \\ \lim_{c\to\infty} l' &= l \end{aligned} \tag{7}$$

so that in the limit $c \to \infty$, time and length in the lab frame are equal, respectively, to time and length in the moving frame, as is the case in Newtonian mechanics.

As another example of limiting relations between theories in physics, this time concerning the quantum-classical correspondence, consider the quantum mechanical equation of motion for the Wigner function (a representation of the quantum state on phase space):

$$\frac{\partial}{\partial t}W(q,p,t) = \frac{2i}{\hbar}\sin\left(\frac{\hbar}{2i}\{\circ,\circ\}\right)(H,W), \tag{8}$$

where $\{H,W\}$ is the classical Poisson bracket. From the expansion of the Moyal bracket, $\frac{2i}{\hbar}\sin\left(\frac{\hbar}{2i}\{\circ,\circ\}\right)(H,W) = \{H,W\} + \frac{\hbar^2}{24}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 W}{\partial p^3} + \mathcal{O}(\hbar^4) + ...$, it follows that

$$\lim_{\hbar\to 0}\frac{2i}{\hbar}\sin\left(\frac{\hbar}{2i}\{H,\circ\}\right)W = \{H,W\}, \tag{9}$$

so that in the limit $\hbar \to 0$, we retrieve the relation

$$\frac{\partial}{\partial t}W(q,p,t) = \{H,W\}, \tag{10}$$

the classical Liouville equation for the evolution of a probability distribution on phase space [4].

There are many examples throughout physics similar to the two just described, where an equation of the form that occurs in one theory is retrieved from an equation of another theory by means of some limiting process. On the basis of results like these, classical mechanics is sometimes also characterised as the limit as $N \to \infty$ of quantum mechanics, where N is the energy quantum number, thermodynamics as the $N \to \infty$ of statistical mechanics, where N is now the number of degrees of freedom, and geometric optics as the $\lambda \to 0$ limit of wave optics, where λ is wavelength.

The prevalence throughout physics of limiting relations like the ones just described initially seems to make a compelling case for the notion that reduction in physics is essentially a matter of taking mathematical limits. But if we are to countenance Nickles' *reduction*$_2$ as furnishing a *bona fide* set of criteria for one physical theory to account for the success of another, we should be able to state precisely what these criteria are, rather than allowing them to remain implicit in the wide range of results that are taken to exemplify this kind of reduction. If the limit-based approach to reduction is to be more than a vague and merely suggestive manner of speaking about inter-theory

[4] This example is drawn from Chapter 3 of [39].

relations in physics, it is necessary first to clarify what is meant *generally* when one theory is characterised as a limit or limiting case of another, and then to explain why this relation enables the low-level theory to incorporate the successes of the high-level theory.

What, then, is meant by the claim that one theory is a limit or limiting case of another? Let us start by considering an example before attempting to give a general answer: what, for instance, is meant by the claim that classical mechanics is the limit as $\hbar \to 0$ of quantum mechanics (which is taken to encompass more than simply the result quoted above)? On the most naive construal, one might interpret this as meaning that if one takes any quantity or relation in quantum mechanics and considers its $\hbar \to 0$ limit, one retrieves a corresponding quantity or relation in classical mechanics. Yet such a claim would be obviously false in this case (as well as in other purported cases of $reduction_2$), for if one considers the $\hbar \to 0$ limit of Schrodinger's equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(x,t) + V(x)\psi(x,t). \tag{11}$$

one obtains the non-sensical - and certainly not classical - result,

$$0 = V(x)\psi(x,t). \tag{12}$$

So we need to be more precise about what we mean by 'the $\hbar \to 0$ limit of quantum mechanics' (where quantum mechanics is a theory which we might naturally regard as being embodied in some respects by the Schrodinger equation) and more generally what is meant by $\lim_{\epsilon\to 0} T_l$, since taking the limit of *just any* part of the low-level theory is unlikely to yield an element of the high-level theory.

Another potential pitfall in interpreting the relation $\lim_{\{\epsilon_i\to 0\}} T_l = T_h$, already noted widely by a number of authors, occurs in cases (such as those discussed so far) where one of the parameters ϵ_i is taken to be a constant of nature, as in the claims $CM = \lim_{\hbar\to 0} QM$ and $NM = \lim_{c\to\infty} SR$ (read CM: classical mechanics, QM: quantum mechanics, NM: Newtonian mechanics, SR: special relativity). Reduction, however precisely it is to be defined, requires that the domain of physical systems that the high-level theory describes well should be a subset of the domain of systems that the low-level theory describes well, and that in the domain of the high-level theory, the low-level theory provides a still more accurate approximation. So an account of the reduction of the high-level theory to the low-level theory should furnish a means of identifying that subset of theoretical descriptions within the low-level theory that approximately agree with the high-level theory in cases where the high-level theory is approximately valid. Thus, it might seem natural at first to read the relation $CM = \lim_{\hbar \to 0} QM$ as implying that those quantum systems which exhibit approximately classical behavior are those for which $\hbar = 0$ or for which $\hbar$ 'approaches' zero, or that those relativistic systems which exhibit approximately Newtonian behavior are those for which $c = \infty$ or for which c 'approaches' infiinity. But of course, all real physical systems possess the same fixed values for c and $\hbar$, so the physical significance of results that involve taking the limit $\hbar \to 0$ or $c \to \infty$ is unclear. Results that require us to posit values for $\hbar$ and c other than those possessed by actual physical systems contribute an unhelpful layer of obscurity to the analysis of inter-theory relations in physics in that they fail to clearly identify the domain of *real* physical systems, as characterised by the low-level theory, for which the high-level theory provides an approximately accurate description. These considerations suggest the following revision to Nickles' $reduction_2$:

> **Limit-Based Reduction (Second Pass):** T_h $reduces_2$ to T_l iff there exists some set of parameters $\{\epsilon_i\}$, defined within T_l, and which are *not* constants of nature, such that $\lim_{\epsilon_i \to 0} T_l = T_h$.

Given this restriction, in the NM/SR case we may replace the limit $c \to \infty$ with the limit $v \to 0$, where v is some appropriate velocity characterising the system in question. With this revision, one retains the equality of times and of lengths in different reference frames in the relevant limit. Likewise, with regard to the CM/QM relation, a number of authors have tried to address this concern by claiming that strictly speaking the proper limit to take is not the $\hbar \to 0$ limit, but the limit $\frac{\hbar}{S_{cl}} \to 0$, or equivalently the limit $S_{cl} \to \infty$, where S_{cl} is some measure of the 'typical classical action' of the quantum system in question. With this revised understanding of these limits, we now have a hope of extracting some physical, rather than merely mathematical, significance for them since the relevant parameter now involves a quantity that varies from system to system rather than being the same for all systems.

Yet once we have made made this last refinement to $reduction_2$ there remain still other potential pitfalls in the interpretation of the relation $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$. Consider the oft-cited result relating relativistic kinetic energy $(\gamma - 1)mc^2$ to Newtonian kinetic energy $\frac{1}{2}mv^2$:

$$(\gamma - 1)mc^2 = \frac{1}{2}mv^2 + \frac{3}{4}m\frac{v^4}{c^2} + ... \quad (13)$$

While it is true that the Newtonian kinetic energy approximates the relativistic kinetic energy when $v << c$, the *limit* of this expression as the dimensionless parameter $v \to 0$ (or as $\frac{v}{c} \to 0$) is *zero*, not $\frac{1}{2}mv^2$. More generally, $\lim_{v\to 0} SR$ (or $\lim_{\frac{v}{c}\to 0} SR$), if we are to understand the limit in terms of the *mathematical* notion of a limit, is a theory in which *nothing moves*, not Newtonian mechanics. Thus, when interpreting claims that one theory is a limit of another, in this and many other cases *we must be careful not to understand 'limit' literally to mean 'limit' in the mathematical sense.*

There are also other important pitfalls that come with taking claims that one theory is a limit of another literally to mean that one theory is a *mathematical* limit of another. As Berry and Batterman have emphasised repeatedly, certain mathematical limits may fail to yield anything resembling the high-level theory because they are singular. However, they do not interpret this as a problem with the limit-based approach to reduction, but rather take such singularities to signify a failure of reducibility between the theories in question [2], [4]. As both Butterfield and Norton have observed, even in cases where the relevant limit is not singular, it still may yield unrealistic idealisations that do not correspond to any actual physical system, or may fail to yield any idealisation at all, as in certain classical- and statistical-mechanical examples where the number of degrees of freedom is taken to infinity. In cases where the limit leads to unrealistic idealisations, Butterfield suggests stopping 'before' the limit in order to facilitate application of limit-based results to real physical systems. Norton is concerned to highlight the distinction between approximation and idealisation, and argues that in some cases where an infinite limit of well-behaved finite systems is taken, the limit system may fail to provide an idealisation of any real physical system [7], [28].

How are to we interpret the relation $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$, if not literally as a mathematical limit? Considering the example of Eq. (13), we might take the $\lim_{\{\epsilon_i \to 0\}}$ simply to signify the requirement that we restrict the values of ϵ_i (in this case v) to be 'small.' Such an interpretation is compatible with viewing T_h as a first- or higher-order approximation rather than as the mathematical limit (which, in cases where the relevant function is continuous, is equal to a zeroth-order approximation). These considerations suggest the following further refinement of $reduction_2$:

Limit-Based Reduction (Third Pass): T_h *reduces*$_2$ to T_l iff there exists some set of parameters $\{\epsilon_i\}$ defined within T_l which are not constants of nature, such that when $\{\epsilon_i\}$ all are 'small,' T_h approximates T_l.

Once we adopt this refinement of limit-based reduction, it is important to keep in mind that *reduction*$_2$ is no longer strictly speaking about taking *limits* in the mathematical sense of the term, but about a certain way of specifying a particular *domain* within the low-level theory. Yet there still remain difficulties with this construal of *reduction*$_2$ in that if $\{\epsilon_i\}$ are dimensionful quantities, the question of whether their numerical values are 'small' will depend on an arbitrary choice of units. For this reason, when specifying that a dimensionful quantity is small, it is necessary to say small in comparison to *what*. For example, it is meaningless to characterise the velocity v as small unless one provides some other measure of velocity against which to compare it; for instance, in the NM/SR case, we may insist that $v << c$. Likewise in the CM/QM case we may insist that $\hbar << S_{cl}$. Equivalently, we may recast these conditions by restricting the domain of an appropriate *dimensionless* parameter ϵ_i so that $\epsilon_i << 1$; for instance, $\frac{v}{c} << 1$, and $\frac{\hbar}{S_{cl}} << 1$. On this basis, we may propose one final revision to our definition of *reduction*$_2$:

Limit-Based Reduction (Fourth Pass): T_h *reduces*$_2$ to T_l iff there exists some set of dimensionless parameters $\{\epsilon_i\}$ defined within T_l, which are not constants of nature, such that when $\{\epsilon_i << 1\}$ for all i, T_h approximates T_l.

Again, it is important to emphasise that imposing the restriction $\epsilon_i << 1$ is not the same thing as taking the limit $\epsilon_i \to 0$. Henceforth, I shall construe limit-based reduction, or *reduction*$_2$, in the sense provided here. Note moreover, that this concept of limit-based reduction includes cases where the intended approximation is literally the mathematical limit or zeroth-order approximation, though it is not restricted to such cases as a literal interpretation of the limit is.

Yet it is also important to recognise the significant ambiguity that remains in the definition of limit-based reduction just provided. In particular, the meaning of the phrase 'T_h approximates T_l' is still quite vague. How are we to assess in any given case whether T_h approximates T_l in the prescribed domain, given that T_h and T_l may be formulated in radically different conceptual and mathematical frameworks? Precisely which quantities in T_l and in T_h ought we to be comparing in order to judge whether one theory approximates another? Simply restricting to the domain of T_l in which $\epsilon_i << 1$ does not of itself return T_h or an approximation to T_h; the resulting description is still formulated in the mathematical and conceptual framework of T_l, not that of T_h. For example, if we examine the quantum commutator in the domain of quantum theory where $\frac{\hbar}{S_{cl}} << 1$, the resulting quantity is an operator on Hilbert space, with no straightforward classical interpretation (it is not, for instance, approximately equal to the classical Poisson bracket, which is often cited as the classical counterpart to the commutator, since the Poisson bracket is a function on classical phase space, not an operator on any Hilbert space). Some means of translating between the framework of the high-level theory and that of the low-level theory - akin to Nagel's bridge laws - is needed before we can judge whether one theory approximates the other in the specified domain.

Acknowledging this observation, I could continue in the attempt to resolve the various additional points of vagueness and ambiguity that afflict the limit-based approach to reduction. At this point, though, one can reasonably question whether the limit-based approach, in spite of its popularity, ever existed as anything more than a vague intuition (as oppposed a well-defined approach to reduction) given that we must do all the work of formulating $reduction_2$ ourselves in order to figure out precisely what is *meant* by it. Rather than continue in the attempt to give a precise meaning to Nickles' $reduction_2$, I will simply state in the next two sections what I believe to be an appropriate set of general criteria for characterising the relationship between a wide range of physical theories, and specifically between models of these theories.

In this subsection, I have highlighted a number of points of ambiguity in the limit-based approach to reduction and argued that in many purported instances of this type of reduction, a literal interpretation of the claim that one theory is a limit of another - i.e., in terms of mathematical limits - is simply not viable. As a consequence, the general appropriateness of approaching reduction in terms of limits, rather than in terms of *domains* - as one does for instance when one imposes the restriction $\epsilon_i << 1$ - should be called into question. If, on the other hand, we choose to associate limit-based reduction with conditions of the form $\epsilon_i << 1$ for some dimensionless parameters

ϵ_i (though, again, the term 'limit' in this case is arguably more misleading than descriptive), then this sort of of condition is naturally incorporated into various applications of the dynamical systems (DS) approach to reduction that I develop here, as I demonstrate in another paper where I consider numerous case studies of DS reduction. All of this is not to say that limits bear *no* relevance to reduction (for instance, they do play a useful role in the discussion of the thermodynamic limit), but only that the centrality of their role in reduction of physical theories has in places been both overstated and not made sufficiently precise.

3 Dynamical Systems in Physics

Before laying out the conditions for dynamical systems reduction in the next section, I briefly introduce the concept of a dynamical system and discuss its relevance to physics, where it is has a very wide range of applicability.

3.1 Definition of a Dynamical System

A dynamical systems model M consists of a state space S and a dynamical map D that prescribes the time evolution of states in S; for brevity, I will write $M = (S, D)$. I assume here that S is endowed at least with the structure of a differentiable manifold and a norm, and that D is a differentiable function both of x, the state in S, and of the time t such that for every t, D specifies a one-to-one function from S onto itself, and such that D is the identity map on S when $t = 0$:

$$D : \mathbb{R} \times S \longrightarrow S, \tag{14}$$

$$x(t) = D(x_0, t), \tag{15}$$

$$x_0 = D(x_0, 0). \tag{16}$$

The requirement that the dynamical map at fixed time be one-to-one ensures that the dynamics are deterministic.

In most, and possibly all, cases of interest in physics, the dynamical map for the model is associated with the set of solutions to some set of first-order differential equations:

$$\frac{dx}{dt} = f(x,t), \tag{17}$$

where $f(x,t)$ is an arbitrary continuous function of the state x and of time t. In many cases, there is no explicit time dependence in f, so $f(x,t) = f(x)$ (however, f may still depend on time indirectly through the time dependence of the state x). Since $x(t) = D(x_0, t)$ is a solution to the above differential equation, we have the following relation between the functions f and D: $f(x(t), t) = \frac{\partial}{\partial t} D(x_0, t)$. [5]

3.2 Symmetries of Dynamical Systems

The concept of a symmetry has numerous definitions depending on the particular context under consideration. Broadly speaking, it is a transformation on a system that leaves some relevant aspect of the system - which depends strongly on the particular context - unchanged (see, for instance, [37] for an extended philosophical discussion of symmetries and their relation to physical laws). In the context of a dynamical system, a symmetry may be understood as a one-to-one transformation of both the dependent and independent variables (i.e., x and t, respectively) in the equation of motion (17) that leaves the form of this equation invariant. The most general such transformation is one that transforms the variables (x,t) into some other variables (x', t') - that is, that involves transformations of the time parameter as well as of the state x. A map $x' = s_x(x,t)$, $t' = s_t(x,t)$ is a symmetry of the model M with dynamical equation $\frac{dx}{dt} = f(x,t)$ if it is differentiable, one-to-one and satisfies the condition

$$\frac{dx'}{dt'} = f(x', t'), \tag{18}$$

[5] While conventional applications of dynamical systems models in physics involve equations whose solutions are deterministic, as a caveat it is important to note the work of Earman, Norton and others, which has served to underscore the fact that for certain forms of f, equations of this form do no always yield deterministic solutions, in the sense that they may admit solutions with the same initial condition that diverge in time, or solutions that differ at an earlier time but pass through the same point in state space at some later time [27], [10]. Whether this occurs will depends on the function $f(x,t)$ obeying particular mathematical constraints that I do not discuss here. In the analysis that follows, I will assume that the function f obeys the constraints necessary for the dynamics to respect determinism; in particular, this will prove to be the case in the examples that I consider.

where f is the same function appearing in the untransformed equation (17). Henceforth, the reader should understand the term 'symmetry' in this sense when I use it (see, for instance, [5] for further discussion of symmetries of dynamical systems in classical physics).

It will prove worthwhile here to highlight certain subclasses of symmetries of dynamical systems. First, some symmetries, such as rotations, translations, and Galilean transformations, do not transform the time parameter, so that $x' = s_x(x,t)$, $t' = t$; let us designate such symmetries as 'invariant-time' symmetries and all others as 'variant-time' symmetries. A sub-class of invariant-time symmetries is the set of symmetries for which the state transformation does not depend on time, so that $x' = s_x(x)$, $t' = t$; this includes rotations and translations but not Galilean transformations; call these 'time-independent, invariant-time' symmetries.

Let us now examine two examples of dynamical systems in physics: first, a model of Hamiltonian classical mechanics, and second a model of non-relativistic quantum mechanics in the Schrodinger picture.

3.3 Example 1: Hamiltonian Classical Mechanics

A system of N particles in non-relativistic Hamiltonian classical mechanics can be modelled as a dynamical system whose state space is given by

$$S = \Gamma_N, \tag{19}$$

where Γ_N is the phase space of N particles moving in 3-dimensional space, which is a $6N$-dimensional symplectic manifold whose points (x,p) consist of the spatial positions x and canonically conjugate momenta p of all N particles (see, for instance, [16], or any other graduate-level text of classical mechanics, for detailed definition and discussion of phase space). The dynamical map D of this model furnishes solutions to the first-order equations,

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p} = \{x, H\} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x} = \{p, H\}, \end{aligned} \tag{20}$$

also known as Hamilton's equations, so that $(x(t), p(t)) = D[(x_0, p_0), t]$, where (x_0, p_0) are the initial conditions (specified at $t = 0$), represents a distinct solution for each (x_0, p_0). Such a solution is given formally by the expression

$$D\big[(x_0,p_0),t\big] = \left(e^{\{\circ,H\}t}x\big|_{x_0,p_0}, e^{\{\circ,H\}t}p\big|_{x_0,p_0}\right), \tag{21}$$

where $e^{\{\circ,H\}t}f(x,p) \equiv f(x,p)+\{f(x,p),H\}t+\frac{1}{2!}\{\{f(x,p),H\},H\}t^2+\frac{1}{3!}\{\{\{f(x,p),H\},H\},H\}t^3+$ ... , and $\{,\}$ denotes the Poisson bracket, defined by $\{f,g\} \equiv \partial_x f \partial_p g - \partial_x g \partial_p f$, with f and g some arbitrary differentiable functions on phase space.

Assume now that the Hamiltonian takes the form $H = \sum_i^N \frac{p_i^2}{2m_i} + \frac{1}{2}\sum_{i\neq j} V(x_i - x_j)$. Then it can be shown, for instance, that among other transformations, the transformation

$$\begin{aligned}(x', p') = s_x((x,p);t) = (x - vt, p - mv)\\ t' = s_t((x,p),t) = t,\end{aligned} \tag{22}$$

for some constant velocity v, is a symmetry of the dynamics.

3.4 Example 2: Non-Relativistic Quantum Mechanics (NRQM) in the Schrodinger Picture

A system of N spinless particles in non-relativistic quantum mechanics can be modelled as a dynamical system whose state space is given by

$$S = \mathcal{H}_N, \tag{23}$$

the Hilbert space of N spinless particles moving in three dimensions. The dynamics of the system are furnished by the Schrodinger equation,

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{24}$$

where $|\psi\rangle \in \mathcal{H}_N$. If $\hat{H}$ does not depend explicitly on time, as for example when it takes the common form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, the solution to Schrodinger's equation can be written formally as

$$D\big[|\psi_0\rangle, t\big] = e^{-i\hat{H}t}|\psi_0\rangle, \tag{25}$$

where $|\psi_0\rangle$ is some arbitrary initial condition [6].

If the potential in the Hamiltonian takes the form $\frac{1}{2}\sum_{i\neq j}^{N} V(\hat{x}_i - \hat{x}_j)$, where V is a function only of inter-particle distance, then it can be shown that, among other transformations, the transformation

$$\begin{aligned} |\psi'\rangle &= s_x(|\psi\rangle) = e^{-i\hat{\vec{L}}\cdot\hat{n}\theta}|\psi\rangle \\ t' &= s_t(|\psi\rangle, t) = t, \end{aligned} \tag{26}$$

a rotation about the axis $\hat{n}$ (the hat here denotes a unit vector, not an operator as with all other quantities) by angle θ, with $\hat{\vec{L}}$ the quantum angular momentum operator, is a symmetry of the dynamics.

3.4.1 Other Examples of Dynamical Systems in Physics

Other examples of dynamical systems models in physics include: other Hamiltonian models of Newtonian mechanics, other models of nonrelativistic quantum mechanics in the Schrodinger picture, Hamiltonian models of relativistic classical mechanics, relativistic quantum mechanics in the Schrodinger picture, the Schrodinger picture formulation of quantum field theories, the ADM or 3+1 formulation of general relativity (which applies only in the special case of a globally hyperbolic spacetime and describes the evolution of a 3-metric with a parameter that functions as 'time'), and the Liouville equation model

[6] Note that this model is deterministic, despite the oft-cited indeterminism of quantum mechanics. This is of course, a result of the fact that collapse of the wave function, where the indeterminism of quantum mechanics arises, is not incorporated in this model. The question as to whether the collapse process is actual or merely effective, and of whether it is genuinely indeterministic or merely apparently so, is a highly interpretation-dependent matter. The Everett or Many Worlds interpretation, as well as de-Broglie Bohm or pilot wave interpretation, both posit fully deterministic dynamics, and treat the probabilistic aspects of quantum theory as merely effective or apparent; effective wave function collapse on both views is intimately tied up with the process of decoherence. The GRW-Pearle (Ghirardi, Rimini, Weber, Pearle) interpretation, a particular variety of 'spontaneous collapse' model, posits a dynamics that is indeterministic and takes the probabilistic aspects of quantum theory to reflect a fundamental indeterminism of nature. Non-realist interpretations such as the Copenhagen Interpretation likewise relinquish determinism in a sense simply by their failure to specify exactly when or how wave function collapses occur. In my analysis here, I restrict myself to considering the deterministic, unitary evolution of the wave function without collapse.

for the evolution of a phase space probabilitiy distribution in classical statistical mechanics. Thus, the framework of dynamical systems theory offers a very general and encompassing mathematical framework in which to consider issues of reduction in physics.

4 Dynamical Systems (DS) Reduction

The framework of dynamical systems theory facilitates the formulation of an especially natural and intuitive concept of reduction in physics. Because of the wide range of applicability of dynamical systems models in physics, this concept of reduction succeeds at describing a very wide range of inter-theory relations in physics, and for this reason, I claim, should be regarded as an alternative to the limit-based and Nagelian approaches. Nevertheless, as I argue in Sections 6 and 7, it incorporates elements of both approaches. Note, however, that dynamical systems reduction concerns reduction between individual models of physical theories rather than the wholesale reduction of entire theories; reduction of theories on the DS approach occurs only piecemeal via reduction between individual models or classes of models of the theories in question.

In this section, I set out formal criteria for DS reduction after some preliminary definitions and remarks. I then consider a relatively simple, idealised example in order to illustrate this approach. Discussion of more realistic examples is deferred to another paper.

4.1 Bridge Maps

Define a *bridge map* between two dynamical systems models to be a differentiable, time-independent function B from the low-level state space S_l to the high-level state space S_h that satisfies certain added conditions to be specified in Section 4.5:

$$B : S_l \longrightarrow S_h \tag{27}$$

$$B : x^l \longmapsto B(x^l), \tag{28}$$

where $x^l \in S_l$. The function B will typically be many-one; its domain may be the whole of S_l or a subset of S_l, and its range the whole of S_h or a subset of S_h. As we will see shortly, as a consequence of satisfying the conditions specified in 4.5, the bridge map will serve to identify structures in the low-level model that approximately emulate the behavior of states in the high-level model to within a certain margin of error and on a certain timescale.

4.2 Induced Dynamics

Given a low-level model $M_l = (S_l, D_l)$, a high-level model $M_h = (S_h, D_h)$, and a bridge map $B : S_l \longrightarrow S_h$, the dynamical map $D_l : S_l \to S_l$ of the low-level model induces, via the bridge map, a trajectory on the state space S_h:

$$x'^h(t) = B(D_l(x_0^l, t)). \tag{29}$$

Generally, the trajectory $x'^h(t)$ will depend on the particular choice of initial condition x_0^l, not just on the image $x_0'^h \equiv B(x_0^l)$ to which x_0^l maps under B.

4.3 Reducing Dynamics

The evolution of the quantity $B(x^l(t))$, determined by the dynamics D_l of M_l, will mimic the evolution prescribed by the dynamics D_h of the high-level model M_h if the following condition holds:

$$B(D_l(x_0^l, t)) \approx D_h(B(x_0^l), t), \tag{30}$$

where the norm on S_h furnishes the sense of approximate equality (see Figure 2). Note that the left-hand side of (30) corresponds to the dynamics induced on S_h by D_l through B, with initial condition x_0^l, while the right hand side corresponds to the dynamics D_h applied to $x_0'^h \equiv B(x_0^l)$, the image of x_0^l under B. In applications of this requirement to reductions of realistic models in physics, approximate equality will only hold for x_0^l in some restricted domain d of states and over some limited timescale τ. With this in mind, we can state the condition more precisely as the requirement that

$$\left| B\big(D_l(x_0^l, t)\big) - D_h\big(B(x_0^l), t\big) \right|_h < \delta, \tag{31}$$

for x_0^l in some domain d of states in S_l and $0 \le t \le \tau$, where $| \; |_h$ designates the norm on S_h, δ is a prescribed margin of error characterising the accuracy

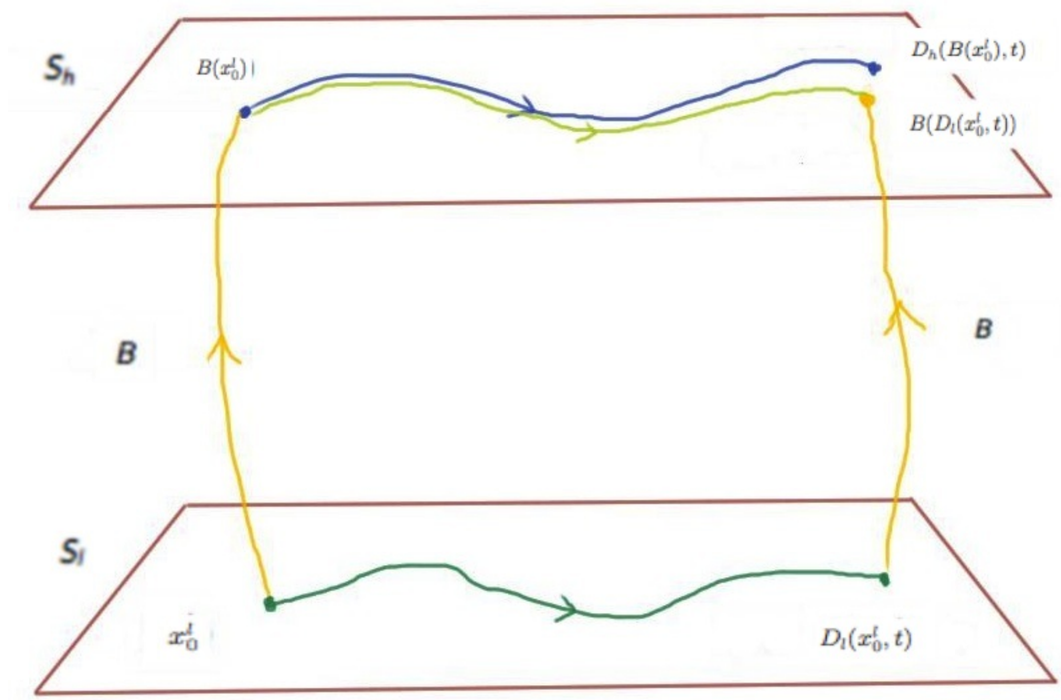

Figure 1: Dynamical systems reduction requires that, for some domain of low-level states and on some limited timescale, the result of applying some bridge map followed by an applcation of the high-level dynamics for some time t yield approximately the same result as applying the low-level dynamics for time t followed by an application of the bridge map - in short, that dynamics and the bridge map approximately 'commute.'

of the approximation, and τ the timescale over which the approximation holds.

If no constraints were imposed on the bridge map other than the basic requirements of continuity and differentiability, then it would always be possible to find a function B satisfying the condition (31) between any two models, so long as the cardinality of the low-level state space is greater than or the same as that of the high-level theory; the reason for this is that one can simply absorb any differences of dynamical structure between the models into the time dependence of the bridge map itself [7]. To avoid such triviality, we should further require that the bridge map function not depend explicitly on the time t (though it may depend implicitly on time via the time dependence of the low-level state).

4.4 Reducing Symmetries

Beyond the requirements already imposed on the bridge map, we also should require that it respect a certain kind of compatibility with the symmetries of the high- and low-level models. I restrict my attention here to invariant-time symmetries - that is, symmetries in which the time-coordinate is not transformed - since variant-time symmetries pose complications relating to the fact that DS reduction assumes a common time parameter between the two models (I leave the incorporation of variant-time symmetries as a subject for future investigations). Let us then consider symmetries of the form $t' = t$, $x' = s(x, t)$. Notably, this class of symmetries includes rotations, spatial translations, parity transformations, Galilean boosts, and gauge transformations, to name a few.

The specific symmetry-related condition that I impose on the bridge map has two parts. Consider an arbitrary invariant-time symmetry s_h of M_h such that $s_h(x^h) \in B(d)$ for some $x^h \in B(d)$, where $d \subset S_l$ is the domain of states satisfying the dynamical commutation condition for some fixed δ and τ and $B(d)$, which I call the 'image domain,' is its image under B. The first part of the symmetry-related condition requires that for any such symmetry s_h, there exist a corresponding symmetry s_l of the low-level model such that

$$s_h(B(x^l), t) \approx B(s_l(x^l, t)) \tag{32}$$

for all $x^l \in d$ such that $s_h(B(x^l), t) \in B(d)$. This condition serves to ensure

[7] Thanks to Christopher Timpson and Jeremy Butterfield for pointing this out.

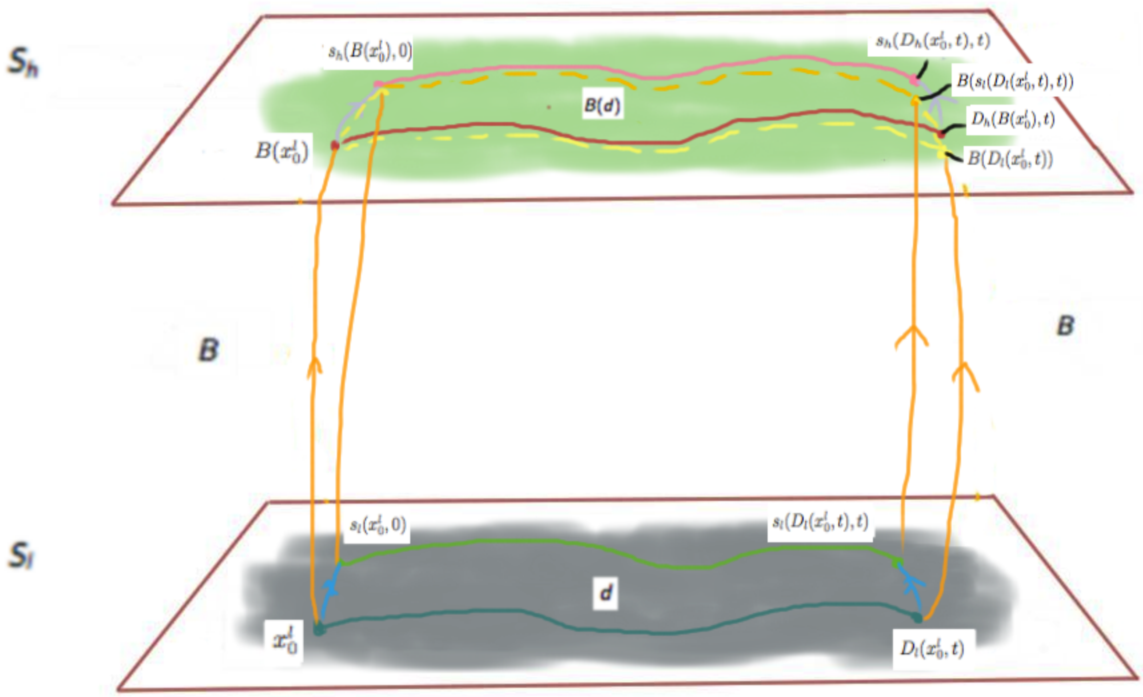

Figure 2: The symmetry condition of DS reduction entails that if two D_h-trajectories lying in $B(d)$ are related by a high-level symmetry, then there is a corresponding low-level symmetry relating the two low-level trajectories that approximate these high-level trajectories under the bridge map.

that the action of any high-level symmetry within the image domain is approximately mimicked by the action in $B(d)$ induced via the bridge map by some low-level symmetry.

The second part of the symmetry-related condition requires that the group structure of the high-level symmetries acting within the image domain $B(d)$ should be approximately mimicked by the action in $B(d)$ induced via the bridge map by the group structure of the corresponding low-level symmetries. So if the action of s_h^1 in $B(d)$ is approximated by the induced action of s_l^1 and the action of s_h^2 in $B(d)$ is approximated by the induced action of s_l^2, then the action of the product $s_h^2 \circ s_h^1$ should be approximated by the induced action of $s_l^2 \circ s_l^1$. More precisely, we should require that if

$$s_h^1(B(x^l),t) \approx B(s_l^1(x^l,t)) \text{ for all } x^l \in d \text{ such that } s_h^1(B(x^l),t) \in B(d), \quad (33)$$

and

$$s_h^2(B(x^l),t) \approx B(s_l^2(x^l,t)) \text{ for all } x^l \in d \text{ such that } s_h^2(B(x^l),t) \in B(d), \quad (34)$$

and $s_l^1 \circ s_l^2(x^l,t) \in d$, and $s_h^1 \circ s_h^2(B(x^l),t) \in B(d)$, then

$$s_h^1 \circ s_h^2(B(x^l),t) \approx B(s_l^1 \circ s_l^2(x^l,t)). \quad (35)$$

In this sense, the bridge map serves to identify that quantity $B(x^l)$ constructed within the low-level model that emulates both the dynamical and symmetry transformation behavior of a high-level state when x^l is restricted to lie in the domain d.

4.5 Formal Criteria for DS Reduction

Having made these motivating remarks, we are now in a position to state formal conditions for dynamical systems (DS) reduction:

DS Reduction:

A model M_h=(S_h, D_h) describing some physical system reduces over time scale τ and to within margin of error δ to a model M_l =(S_l, D_l) of describing the same system only if there exists a differentiable function $B : S_l \to S_h$ that does not depend explicitly on time, and a nonempty subset $d \subset S_l$, such that

- **DSR1:** for any $x_0^l \in d$

$$\boxed{\left| B\big(D_l(x_0^l,t)\big) - D_h\big(B(x_0^l),t\big) \right|_h < \delta,} \quad (36)$$

  for all $0 \leq t \leq \tau$;

- **DSR2:**

- a) for every invariant-time symmetry s_h of M_h such that $s_h(x^h, t) \in B(d)$ for some $x^h \in B(d)$, there exists a symmetry s_l of M_l such that

$$\boxed{s_h(B(x^l), t) \approx B(s_l(x^l, t))} \text{ for all } x^l \in d \text{ such that } s_h(B(x^l), t) \in B(d); \tag{37}$$

- b) if

$$s_h^1(B(x^l), t) \approx B(s_l^1(x^l, t)) \text{ for all } x^l \in d \text{ such that } s_h^1(B(x^l), t) \in B(d), \tag{38}$$

and

$$s_h^2(B(x^l), t) \approx B(s_l^2(x^l, t)) \text{ for all } x^l \in d \text{ such that } s_h^2(B(x^l), t) \in B(d), \tag{39}$$

and $s_l^1 \circ s_l^2(x^l, t) \in d$, and $s_h^1 \circ s_h^2(B(x^l), t) \in B(d)$, then

$$\boxed{s_h^1 \circ s_h^2(B(x^l), t) \approx B(s_l^1 \circ s_l^2(x^l, t)).} \tag{40}$$

These conditions should be understood as necessary conditions for one dynamical system to reduce to another. Whether they are sufficient depends on the possibility of finding a trivialising counterexample - i.e., an example of two dynamical systems models satisfying these conditions such that we cannot reasonably regard the low-level model as accounting for the success of the high-level model. If such examples can be found, then further conditions must be imposed on the bridge map B. What the above conditions are meant to capture are two of the most salient requirements that must be satisfied for a mathematical structure defined in a low-level model - specified by the bridge map - to emulate, or approximately instantiate the physically salient dynamical and transformation properties of the state in the high-level model. I leave it to future work to ascertain whether any further conditions need be placed on the bridge map, and if so, what these conditions are.

It is important to distinguish the notion of reduction of dynamical systems just described from another that has received a significant amount of attention in the mathematics literature. This alternate sense of reduction of dynamical systems has been spelled out clearly by Marmo, Saletan, and Simoni:

> A *dynamical system*, a dynamics, is a vector field Δ on a manifold M of finite dimension n [8]. The problem of mechanics is to integrate the dynamics, i.e., to obtain the integral curves of Δ. Such integral curves for a dynamical system can sometimes be obtained by integrating related dynamical systems on manifolds whose dimensions are lower than n. The object of this paper is to discuss ways of finding such dynamical systems of lower dimension, that is of *reducing* the original dynamics [14].

The sense of reduction of dynamical systems specified in this quotation is intended primarily as a method for solving high-dimensional systems of differential equations by first solving some related lower-dimensional system of equations; the sense of reduction that I elaborate in this paper, on the other hand, is intended primarily as a framework for describing the relationship between different models in physics that describe some common set of physical systems. But there are also more pointed differences. First, simply as a matter of convention, on Marmo *et al*'s definition, the higher-dimensional system, which typically corresponds to the low-level model on my approach, is said to reduce to the lower-dimensional system; on the sense of reduction that I propose, the reverse is typically the case. Second, while on Marmo *et al*'s definition a solution to the lower-dimensional system will typically suffice to specify a solution to the higher-dimensional system, on the definition I elaborate here, this is not the the case (given the many-one nature of the bridge map on my approach). Third, on the sense of reduction that Marmo *et al* consider, the low-dimensional model is usually obtained from the high-dimensional model through the process of quotienting out the high-dimensional state space by the action of some symmetry group; this need not be, and often is not, the case in the definition of reduction that I propose. For further discussion of reduction of dynamical systems in the alternate sense described in the above quotation, see for example [14], [13], [29], [6].

[8] On the definition of a dynamical system that I have given, the dynamical map D can be understood as specifying the integral curves of a vector field, whose components are given by the function $f(x, t)$ that appears in the first-order equation of motion of the dynamical system. The state space S in the definition of dynamical system that I employ here furnishes the manifold M to which Marmo *et al* refer.

4.6 Condition DSR1 and Equations of Motion

It is often more convenient to specify the dynamics of a DS model in the form of first-order differential equations, rather than in the form of a dynamical map. Let us examine how the condition DSR1 may be formulated when the dynamics of the high- and low-level models are prescribed in this way. As already discussed, the dynamical map of M_h specifies the solutions $x^h(t) = D_h(x_0^h, t)$ to the differential equation

$$\frac{dx^h}{dt} = f_h(x^h, t) \tag{41}$$

where $f_h(x^h(t), t) = \frac{\partial}{\partial t} D_h(x_0^h, t)$, and likewise the dynamical map of M_l specifies the solutions $x^l(t) = D_l(x_0^l, t)$ to the differential equation

$$\frac{dx^l}{dt} = f_l(x^l, t), \tag{42}$$

where $f_l(x^l(t), t) = \frac{\partial}{\partial t} D_l(x_0^l, t)$.

At the level of differential equations, condition DSR1 will be satisfied if the induced trajectory $x'^h(t) \equiv B(x^l(t))$ approximately satisfies the equation of motion of M_h,

$$\frac{dx'^h}{dt} \approx f_h(x'^h, t) \tag{43}$$

over any time interval less than or equal to τ. More explicitly, this equation can be written,

$$\frac{d}{dt} B\big(x^l(t)\big) \approx f_h\Big(B\big(x^l(t)\big), t\Big), \tag{44}$$

or, even more explicitly,

$$\boxed{\frac{d}{dt} B\big(D_l(x_0, t)\big) \approx f_h\Big(B\big(D_l(x_0, t)\big), t\Big).} \tag{45}$$

Note that while this relation is a sufficient condition for DSR1 to hold, it is not a necessary condition. We can see that it is a sufficient condition by integrating both sides of (44) with respect to time, up to some t such that $0 \leq t \leq \tau$:

$$\begin{aligned}&\int_0^t dt' \, \frac{d}{dt'} B(x^l(t')) \approx \int_0^t dt' \, f_h\Big(B\big(x_l(t')\big), t'\Big) \\ &B(x^l(t)) - B(x_0^l) \approx D_h(B(x_0^l), t) - D_h(B(x_0^l), 0) \\ &B(D_l(x_0^l, 0)) \approx D_h(B(x_0^l), 0)\end{aligned} \tag{46}$$

where in going from the first line to the second line I have used that $f_h(x^h, t) = \frac{\partial}{\partial t} D_h(x_0^h, t)$, and in going from the second to the third I have used that $B(x_0^l) = D_h(B(x_0^l), 0)$. Note that for the condition (44) to be sustained over time period τ, the domain d should be such that it is preserved by the low-level dynamics approximately over timescale τ; that is, the low-level dynamics should carry states in d to other states in d over time periods less than τ.

While (44) is sufficient for the DSR condition to hold, it is not mathematically necessary insofar as there may exist induced trajectories on the high-level state space that remain close (in the sense of the S_h's norm) to the trajectory prescribed by the high-level model but such that the time derivative of these trajectories does not remain close in value to the derivatives prescribed by (41). For example, consider a trajectory rapidly oscillating with small amplitude around the trajectory prescribed by the high-level dynamics; the values of the states will be close, so that condition DSR1 is satisfied, but the time derivatives will differ drastically so that (45) is not. In all reductions considered in later chapters, the stronger condition (45) will be proven, rather than merely the condition (36).

4.7 A Simple Example of DS Reduction

To illustrate the application of DS reduction let us consider a case in which the high-level model is the classical Hamiltonian model described in section 3.3 and the low-level model the model of an isolated quantum system described in section 3.4. While the high-level classical model in this case may serve as an approximate description of some real physical systems - macroscopic centers of mass interacting through some time-independent potential - the low-level model is in some respects ill-equipped to describe these systems because it fails to take into account the effects of environmental decoherence, as a result of which the quantum model predicts coherence lengths for the centers of mass that are incompatible with empirical observation. Nevertheless, the relation between the two models serves to illustrate the basic conditions of DS reduction. A reduction of the classical model to a quantum model that does take into account effects of decoherence is considered in another paper.

DSR1: Dynamics

Take the Hamiltonians in the classical and quantum models, respectively, to be $H = \frac{p^2}{2m} + V(x)$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. Then the condition (45), which suffices to prove condition DSR1, takes the form,

$$\begin{aligned} \frac{d}{dt}\langle\hat{x}\rangle &\approx \{x, H(x,p)\}\big|_{\langle\hat{x}\rangle,\langle\hat{p}\rangle} = \frac{1}{m}\langle\hat{p}\rangle \\ \frac{d}{dt}\langle\hat{p}\rangle &\approx \{p, H(x,p)\}\big|_{\langle\hat{x}\rangle,\langle\hat{p}\rangle} = -\frac{\partial V(\langle\hat{x}\rangle)}{\partial\langle\hat{x}\rangle}, \end{aligned} \tag{47}$$

where the subscript $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$ on the Poisson brackets indicates that the Poisson bracket is to be evaluated at $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$. Employing the bridge map substitutions $x' \equiv \langle\hat{x}\rangle$, $p' \equiv \langle\hat{p}\rangle$, this can be written in a form more closely resembling the original classical equations of the high-level model:

$$\begin{aligned} \frac{dx'}{dt} &\approx \{x', H(x',p')\}_{x',p'} = \frac{1}{m}p' \\ \frac{dp'}{dt} &\approx \{p', H(x',p')\}_{x',p'} = -\frac{\partial V(x')}{\partial x'}. \end{aligned} \tag{48}$$

Relation (47), and hence (48), can be shown to hold in the domain of states that are wave packets narrowly peaked both in position and momentum. This fact is proven using Ehrenfest's Theorem, which states that for any state of a quantum system with the above-specified Hamiltonian, the following relation holds:

$$\frac{d\langle\hat{p}\rangle}{dt} = -\left\langle\frac{\partial\hat{V}}{\partial x}\right\rangle. \tag{49}$$

(see for instance [24] for a proof of Ehrenfest's Theorem). Note however that this relation does not suffice to ensure that expectation values of position and momentum evolve approximately according to Newtonian equations. For this, it is necessary that the stronger (though approximate) condition,

$$\frac{d\langle\hat{p}\rangle}{dt} \approx -\frac{\partial V(\langle\hat{x}\rangle)}{\partial\langle\hat{x}\rangle} = \frac{\partial V}{\partial x}\bigg|_{\langle\hat{x}\rangle}, \tag{50}$$

also holds. It can be shown that when we restrict to the domain of narrow wave packets, relation (49) implies (50). It is crucial to note at this point that the domain of narrow wave packets is not generally preserved under the Schrodinger dynamics since wave packets tend to spread. However, one can expect wave packets to remain narrow at least on some limited timescale, which depends on how rapidly spreading occurs. Broadly speaking, the larger the mass of the system, the more slowly wave packets spread, and the stronger the effects of chaos (as gauged for example by the size of the Lyapunov exponent associated with the system's Hamiltonian) the more rapidly wave packets tend to spread (see, for instance, [45] and [39] for discussion of the role of chaos in quantum wave packet spreading). Thus, the time scale over which the domain of narrow wave packets is preserved by the Schrodinger dynamics will depend on these factors as well as on the upper bound that one sets for what is to be counted as 'narrow;' the width beyond which (50) ceases to he a good approximation will depend on a particular measure of the length scale on which the potential V varies.

The relation (50) suffices to ensure the validity of condition DSR1 over some timescale τ determined by the factors just mentioned. In this particular case this condition takes the form,

$$\left|\langle\psi_0|e^{i\hat{H}t}\hat{x}\; e^{-i\hat{H}t}|\psi_0\rangle - e^{\{\circ,H\}t}x\big|_{\langle\hat{x}\rangle_0,\langle\hat{p}\rangle_0}\right| < \delta_x,$$

and (51)

$$\left|\langle\psi_0|e^{i\hat{H}t}\hat{p}\; e^{-i\hat{H}t}|\psi_0\rangle - e^{\{\circ,H\}t}p\big|_{\langle\hat{x}\rangle_0,\langle\hat{p}\rangle_0}\right| < \delta_p,$$

where $\langle\hat{x}\rangle_0 \equiv \langle\psi_0|\hat{x}|\psi_0\rangle$ and $\langle\hat{p}\rangle_0 \equiv \langle\psi_0|\hat{p}|\psi_0\rangle$, for $0 \le t \le \tau$, where τ is timescale on which wave packets become widely spread out on spatial dimensions characteristic of the variation of the potential $V(x)$ (for a more precise characterisation of this length scale, see for instance [1]). The norm employed on phase space is simply the difference of the positions and of the momenta. Less formally, we can write this condition as

$$\langle\psi_0|e^{i\hat{H}t}\hat{x}\ e^{-i\hat{H}t}|\psi_0\rangle \approx e^{\{\circ,H\}t}x\big|_{\langle\hat{x}\rangle_0,\langle\hat{p}\rangle_0}$$

$$\text{and} \tag{52}$$

$$\langle\psi_0|e^{i\hat{H}t}\hat{p}\ e^{-i\hat{H}t}|\psi_0\rangle \approx e^{\{\circ,H\}t}p\big|_{\langle\hat{x}\rangle_0,\langle\hat{p}\rangle_0},$$

where, again, the approximation should be understood as being relative to some specified margins of error δ_x and δ_p.

DSR2: Symmetries

Here I will demonstrate the validity of condition DSR2, concerning the relation between the symmetries of the models, with regard to rotations and Galilean boosts in classical mechanics. In principle, these conditions should be shown to hold for all symmetries and states of the high-level model such that both the states and their mappings under the symmetry are in the image domain $B(d)$, which here consists of the entire classical phase space Γ. While I limit myself here to considering these two symmetries, following these examples it should be straightforward for the reader to demonstrate these conditions for other symmetries of the given classical model, such as translations and parity transformations.

Symmetry 1: Rotation

In the case of a Hamiltonian system with spherically symmetric potential $V(r)$, the rotations about the origin constitute a group of dynamical symmetries. Condition 2a) for rotations is ensured by the fact that

$$\left(\langle\psi|e^{i\vec{L}\cdot\hat{n}\theta}\ \hat{x}\ e^{-i\vec{L}\cdot\hat{n}\theta}|\psi\rangle, \langle\psi|e^{i\vec{L}\cdot\hat{n}\theta}\ \hat{p}\ e^{-i\vec{L}\cdot\hat{n}\theta}|\psi\rangle\right) \approx e^{\{\circ,\vec{L}\cdot\hat{n}\theta\}}\left(\langle\psi|\hat{x}|\psi\rangle, \langle\psi|\hat{p}|\psi\rangle\right) \tag{53}$$

for $|\psi\rangle \in d$; in fact, the equality holds exactly, and for arbitrary states, not just narrow wave packets. So, rotations on phase space are induced via the bridge map by rotations on Hilbert space. That this condition is satisfied can be shown using the Baker-Hausdorff Lemma, which states that

$$e^{i\lambda \hat{B}} \hat{A} e^{-i\lambda \hat{B}} = \hat{A} + i\lambda \left[\hat{B}, \hat{A}\right] + \frac{(i\lambda)^2}{2!}\left[\hat{B}, \left[\hat{B}, \hat{A}\right]\right] + \frac{(i\lambda)^3}{3!}\left[\hat{B}, \left[\hat{B}, \left[\hat{B}, \hat{A}\right]\right]\right] + ... \tag{54}$$

$$\equiv e^{\left[i\lambda \hat{B}, \circ\right]} \hat{A} \tag{55}$$

(see, for instance [31], p.96).

Condition 2 b) for rotations takes the form

$$\left(\langle\psi| e^{i\vec{\hat{L}}\cdot\hat{m}\phi} e^{i\vec{\hat{L}}\cdot\hat{n}\theta}\ \hat{x}\ e^{-i\vec{\hat{L}}\cdot\hat{n}\theta} e^{-i\vec{\hat{L}}\cdot\hat{m}\phi}|\psi\rangle, \langle\psi| e^{i\vec{\hat{L}}\cdot\hat{m}\phi} e^{i\vec{\hat{L}}\cdot\hat{n}\theta}\ \hat{p}\ e^{-i\vec{\hat{L}}\cdot\hat{n}\theta} e^{-i\vec{\hat{L}}\cdot\hat{m}\phi}|\psi\rangle\right)$$
$$\approx e^{\{\circ, \vec{L}\cdot\hat{n}\theta\}} e^{\{\circ, \vec{L}\cdot\hat{m}\phi\}} (\langle\psi|\hat{x}|\psi\rangle, \langle\psi|\hat{p}|\psi\rangle). \tag{56}$$

Again, the equality in this case actually holds exactly and for arbitrary $|\psi\rangle$, not just for narrow wave packet states, and can be proven in much the same fashion as condition 2 a) for rotations.

Symmetry 2: Galilean Boosts

The dynamical map associated with a one-particle classical Hamiltonian $H = \frac{p^2}{2m} + V(x)$ above will not generally commute with a boost by some velocity v, which therefore will not serve as a dynamical symmetry of the model. However, if we consider the two-particle case in which the potential depends only on the spatial distance between the particles, so that $H = \frac{p_1^2}{2m_1} + \frac{p_1^2}{2m_1} + V(|x_1 - x_2|)$, then a boost of both particles by the same velocity v will commute with the dynamical map associated with this Hamiltonian. Thus, a Galilean boost in this case will count as a symmetry of the model. A Galilean boost by velocity v takes the form

$$x_1' = x_1 - vt \quad (57)$$
$$x_2' = x_2 - vt \quad (58)$$
$$p_1' = p_1 - m_1 v \quad (59)$$
$$p_2' = p_2 - m_2 v. \quad (60)$$

In the quantum mechanical model, there is likewise a symmetry of the dynamics that typically also goes under the name of a Galilean boost. As in the classical model, these transformations are parametrised by a velocity v; under such a transformation, the wave function $\psi(x_1, x_2, t)$ transforms to $\psi'(x_1', x_2', t')$, given by

$$\psi'(x_1', x_2', t') = e^{-i(m_1 v \cdot x_1 + m_2 v \cdot x_2 - \frac{1}{2} m_1 v^2 t - \frac{1}{2} m_2 v^2 t)} \psi(x_1, x_2, t) \quad (61)$$

with $x_1' = x_1 - vt$, $x_2' = x_2 - vt$ and $t' = t$ (see, for instance [24], p.75). It is straightforward to see that under the bridge map given by the expectation value, the quantum mechanical Galiliean boost induces a classical Galilean boost:

$$(\langle\psi'|\hat{x}_1|\psi'\rangle, \langle\psi'|\hat{x}_2|\psi'\rangle; \langle\psi'|\hat{p}_1|\psi'\rangle, \langle\psi'|\hat{p}_2|\psi'\rangle) \quad (62)$$
$$= (\langle\psi|\hat{x}_1|\psi\rangle - vt, \langle\psi|\hat{x}_2|\psi\rangle - vt; \langle\psi|\hat{p}_1|\psi\rangle - m_1 v, \langle\psi|\hat{p}_2|\psi\rangle - m_2 v), \quad (63)$$

thereby satisfying condition 2a). Thus, for any Galilean boost on phase space, there exists a corresponding transformation on Hilbert space that induces it via the expectation value. To satisfy condition 2b), though, it is necessary that the composition of two Galilean boosts on phase space , by v and then by v', agree approximately with the transformation induced under the bridge map by the composition of the corresponding boosts on Hilbert space. The composition of two boosts on Hilbert space gives

$$\psi''(x_1'', x_2'', t'') = e^{-i\left[m_1(v+v')\cdot x_1 + m_2(v+v')\cdot x_2 - \frac{1}{2} m_1\left(v^2+v'^2\right)t - \frac{1}{2} m_2\left(v^2+v'^2\right)t\right]} \psi(x_1, x_2, t) \quad (64)$$

with $x_i'' = x_i - (v + v')t$ for $i = 1, 2$ and $t'' = t$. Note that this is equal to a single boost by $v + v'$ up to a global time-dependent phase factor $(m_1 + m_2)(v \cdot v')t$, which does not make a difference to any of the amplitudes of the theory, or to rays in projective Hilbert space. Under the composed boosts, it is straightforward to see that

$$\left(\langle\psi''|\hat{x}_1|\psi''\rangle, \langle\psi''|\hat{x}_1|\psi''\rangle; \langle\psi''|\hat{p}_1|\psi''\rangle, \langle\psi''|\hat{p}_2|\psi''\rangle\right) \tag{65}$$

$$= \left(\langle\psi|\hat{x}_1|\psi\rangle - (v+v')\,t, \langle\psi|\hat{x}_2|\psi\rangle - (v+v')\,t; \langle\psi|\hat{p}_1|\psi\rangle - m_1\,(v+v')\,, \langle\psi|\hat{p}_2|\psi\rangle - m_2\,(v+v')\right) \tag{66}$$

thereby ensuring the validity of condition 2b) with respect to classical Galilean symmetry.

Limitations of the Quantum Model w/o Decoherence

Note that the quantum models to which the classical models considered so far have been reduced make no mention of environmental decoherence, and thus allow for arbitrary coherent superpositions of the degrees of freedom in question. Moreover, in chaotic systems, the quantum models predict that initially narrow wave packets will spread on fairly short time scales beyond the coherence lengths that typically characterise macroscopic or mesoscopic systems that are known to exhibit approximately Newtonian behavior (see [39] Ch.3 for detailed discussion of this point). Thus, although the classical model considered here may serve as an effective (if only approximate) description of such systems, the quantum model does not insofar as it will, on relatively short timescales, predict coherence lengths that disagree dramatically with those observed in these systems. Thus, it is necessary to replace the quantum model considered here with a more sophisticated one that takes account of environmental degrees of freedom and thereby continually suppresses the coherence length of the system in question; this is done in another paper. Nevertheless, the reduction involving the quantum model without environmental decoherence serves to provide a simplified illustration the basic components of DS reduction, if we momentarily allow ourselves to overlook its shortcomings as a description of real, approximately Newtonian systems.

5 Precursors to DS Reduction

The starting point of the DS approach to reduction, the commutation of high- and low-level dynamics with some bridge map, is a variation on an old idea. While applications of this idea in physics have thus far been restricted primarily to the context of reductions in statistical mechanics - where the bridge map usually consists of some sort of coarse-graining function - I claim that, suitably refined, this pattern of reasoning can be applied with a great deal more generality to reductions across many branches of physics. In this section, I discuss the work of a number of authors who propose different variants of the dynamical commutation condition, highlighting differences from the DS approach where they occur. It is worth noting from the outset that none of these other variants imposes the additional condition requiring compatibility of the bridge map (though they do not call it that) with the symmetries of the models, nor do any explicitly require the bridge map to be time-independent. Some of these other variants of the dynamical commutation condition also impose requirements on the bridge map that I do not, conditions which preclude their application to a number of inter-theory relations to which DS reduction does apply. To distinguish the general idea that dynamics should commute with some function between state spaces of the high- and low- level models from its formulation specifically within the context of DS reduction, I will refer to the general idea as the 'dynamical commutation' condition, and to my own particular formulation of it as condition DSR1.

I was first introduced to the idea of approaching reduction in terms dynamical commutation through discussions with my doctoral thesis supervisor, David Wallace, who has for some time been advocating use of the dynamical commutation approach to reduction both in writing and informally in conversation. Wallace's [39], Ch. 9 proposes a version of this condition in the context of attempting to explain the direction of wave function branching in the Everett interpretation of quantum mechanics. In Ch. 2 he also suggests a generalisation of this condition formulated in the mathematical language of histories rather than of states evolving in time; this approach thus involves a map not between state spaces but rather between the history spaces, though Wallace does not impose any precise constraints on this map. Finally, I also encountered a variant of the dynamical commutation condition in David Albert's Columbia University course on the foundations of statistical mechanics.

Both Giunti and Yoshimi have suggested their own variants of the dynamical commutation condition with regard to the reduction of dynamical systems generally, though their formulations of this condition do not accommodate many reductions within physics, where (I claim) it is especially salient [15], [44]. Giunti, for example, requires that his version of the bridge map, which he calls an 'emulation,' be an injective, or one-to one, function. As we have seen, the bridge map of the DS approach imposes no such requirement, and may be many-one; in fact, as will be shown in another paper where I consider a range of particular applications of the DS approach, bridge maps typically are not one-to-one functions. Yoshimi, on the other hand, develops his approach primarily to accommodate reductions involving mental phenomena, and requires that his counterpart to the bridge map, which he calls a 'supervenience function,' be an onto function between state spaces that also furnishes a partitioning of the low-level state space; again, the bridge map of DS reduction is subject no such requirements. The version of the dynamical commutation condition that Yoshimi proposes is very close to the 'meshing' condition proposed by Butterfield which I discuss below, and many of the differences that I highlight between the DS approach and Butterfield's approach apply also to the contrast between DS reduction Yoshimi's approach. To highlight yet another difference between the DS approach and those of Giunti and Yoshimi, neither author demands compatibility of their bridge map counterparts with the symmetries of the models, nor do they explicitly insist on time-independence of these maps.

While much of his work on reduction and emergence focuses on limit-based and Nagelian approaches, Butterfield also discusses inter-level relations in physics in terms of dynamical systems. Like condition DSR1, the core condition for reduction that Butterfield's analysis draws on, which he calls 'meshing' of 'macro-' and 'micro-' level dynamics, involves the commutation of some 'coarse-graining' function between micro- and macro-level state spaces with the time-evolution on those spaces. The macro-level state space is identified with a partition of the micro-level state space, and the coarse-graining function simply maps an element of the micro-level space into the cell of the partition to which it belongs. On Butterfield's account, the closest analogue to what I call the high-level model is the macro-level model; to what I call call the low-level model, the micro-level model; and to what I call the bridge map, the coarse-graining function [8]. Note that Butterfield's terminology draws heavily on examples of reduction in statistical mechanics.

Butterfield characterises the dynamics of a macro- and a micro- model as 'meshing' relative to a particular partitioning $\mathcal{P} = \{C_i\}$ of the micro-level state space $\mathbb{S}$ when the set obtained by applying the micro-evolution law $T : \mathbb{S} \to \mathbb{S}$ to an element of $\mathcal{P}$ is itelf an element of $\mathcal{P}$, so that for any i, $T(C_i) = C_j$ for some j. Thus, the micro-level dynamics $T : \mathbb{S} \to \mathbb{S}$ induces, via the coarse-graining, some macro-level dynamics $\bar{T} : \mathcal{P} \to \mathcal{P}$. This will not be the case for an arbitrary partition of $\mathbb{S}$ since two microstates in the same partition may evolve under the microdynamics into separate elements of the partition, so that micro-level determinism gives rise to macro-level indeterminism (where the macro state space corresponds to the partitioning of the micro state space).

However, Butterfield acknowledges that this concept of meshing may not apply to many realistic cases in which one dynamical system purportedly reduces to another - such as the reduction of models involving the Boltzmann, Navier-Stokes and diffusion equations to some micro-physical mechanical model - and so suggests that the following modifications and allowances to his notion of meshing may be required before these realistic examples can be counted as instances of it (I quote directly from Butterfield here):

- 'the meshing may not last for all times;
- the meshing may apply, not for all micro-states s, but only for all except a "small" class;
- the coarse-graining may not be so simple as paritioning $\mathbb{S}$; and indeed
- the definition of the micro-state space $\mathbb{S}$ may require approximation and-or idealisation, especially by taking a limit of a parameter: in particular, by letting the number of microscopic contitutents tend to infinity, while demanding of course that other quantities, such as mass and density, remain constant or scale appropriately.' [8]

Indeed, all of the first three of these considerations are already built into the definition of DS reduction. DS reduction is defined only relative to a particular timescale and margin of error and for a particular, potentially restricted, domain d of states in the low-level state space; moreover, the bridge-map of DS reduction need not yield a partitioning of the low-level space (that is, the inverse images under B of points in S_h need not form a partition of S_l; indeed, it will not necessarily be the case that every point in S_h even *has* an inverse image since it may not be mapped to under the bridge map). Butterfield's fourth concern only comes into play in certain special cases, for example in reductions where quantum field theory or statistical mechanics furnishes the reducing model, since both of these theories typically involve taking limits as the number of degrees of freedom in the theory goes to infinity. In the case of quantum field theory, this fourth concern of Butterfield's is averted by taking a 'cut-off' approach to quantum field theory and thereby treating the QFT model in question as a model of a large-but-finite, rather than an infinite, number of degrees of freedom (see, for instance, [38] for a development of the cut-off approach to QFT).

While the modifications to his meshing condition that Butterfield suggests anticipate a number of differences between meshing and DS reduction, it will be worthwhile to explore these differences in a bit more detail. One essential difference, just noted, is that while Butterfield's meshing condition requires that the coarse-graining function (his counterpart of the bridge map) be associated with some partition of the low-level state space, the bridge map need not take as its domain the whole of S_l, and therefore need not prescribe a partitioning of S_l; moreover, the bridge map need not take the whole of S_h as its image, providing still another reason why the high-level state space cannot in general be regarded on the DS approach as a partition of the low-level space.

Furthermore, if a micro-level system obeys Butterfield's meshing condition with respect to some partition, then for any macro-level initial condition - i.e., some partition cell - it must be the case that the deterministic dynamics induced on the partition by the micro-level dynamics yield the same result irrespective of the microcondition that instantiates that initial macrocondition. Since any element of the partition can serve as the macro- initial condition, and since every element of the micro-level state space belongs to some element of the partition, Butterfield's meshing condition requires that the *whole* micro-level state space (or at least all but a very small subset of this space) serve as the domain that approximates the macro-level dynamics under coarse-graining; by contrast, in DS reduction, the domain of S_l whose induced dynamics under the bridge map approximates the high-level dynamics is not required to be the entirety of the low-level space.

Finally, on Butterfield's approach, the coarse-graining function associated with a partition that respects the meshing condition is not required to respect the symmetries of the low-level model insofar as it does not require that for any symmetry of the deterministic macrodynamics, there will be some symmetry of the micro-level dynamics that induces it under coarse-graining - nor does it entail that the group structure of the micro-level symmetries induce the group structure of the macro-level symmetries on the partition.

A final, though potentially less substantive, difference between Butterfield's account of dynamical commutation and the DS approach is that while the inspiration for the DS approach comes from examples of reduction in statistical mechanics, on the DS approach the reduced and reducing models need not correspond, respectively, to models of macroscopic and microscopic phenomena, nor does the bridge map need to correspond to a 'coarse-graining' in any sense other than its being a many-one function (certainly, it is not required to furnish a partition of S_l, nor is it required to map onto the whole of S_h). Of course, if Butterfield is using the terms 'macro-' and 'micro-' merely to suggest some analogy with statistical mechanical reductions, and not by way of restricting this approach to reductions in which high- and low- level descriptions correspond respectively to 'macro-' and 'micro-' level phenomena, then this distinction collapses into one merely of terminology.

Within statistical mechanics, Lanford's Theorem provides an explicit instance of the dynamical commutation [9] (see, for instance, [20], [21], [22], [36]). Lanford's Theorem shows that the Boltzmann equation, which describes the behavior of the distribution $f_t(\vec{x}, \vec{p})$ in 6-dimensional μ-space of particles in a dilute gas (and assumes the molecules in the gas are modelled as solid spheres), can be derived from the formalism of classical Hamiltonian mechanics, which prescribes via the Liouville equation the time evolution of a probability distribution $\rho_t(\vec{x}_1, \vec{p}_1, ..., \vec{x}_N, \vec{p}_N)$. The theorem establishes a particular bridge or correspondence between probability distributions ρ on phase space and distributions f on μ-space, such that to any probability distribution ρ there corresponds a unique f, but such that there are in general many ρ that may yield the same f under this correspondence. The theorem then shows that provided certain constraints are imposed on the initial phase space probability distribution ρ_0 at some time $t = 0$, the evolution of f induced by the evolution of ρ via this correspondence approximately satisfies Boltzmann's equation for some time scale τ (what this time scale turns out to be depends on the strength of the assumptions made about the evolution of ρ). Thus, Lanford's Theorem shows that, applied to some domain of possible initial probability distributions ρ_0, the low-level dynamics (The Liouville Equation) for some time t followed by an application of the bridge map or coarse-graining yields approximately the same final distribution f_t as

[9] Thanks to Jeremy Butterfield for pointing me to this example.

does the bridge map followed by an application of the high-level dynamics (The Boltzmann equation) for the same time t, thus satisying the dynamical commutation condition.

Werndl has shown that for every deterministic dynamical system, there is an indeterministic model that reproduces the same empirical predictions to within some given margin of error, and also that for every indeterministic dynamical model, there is a deterministic one that is observationally indistinguishable from it, again to within some margin of error [40], [41], [42], [43]. All of the models considered in this thesis are deterministic, although it is possible (particularly in the case of the quantum theories I consider) that observationally equivalent stochastic models can be chosen in place of these; in such a case, it would be necessary to extend the account of reduction among deterministic models that I provide to reductions among indeterministic models, as well as to reductions of deterministic to indeterministic models, and reductions of indeterministic to deterministic models.

6 DS Reduction and Nagelian Reduction

Dynamical systems reduction incorporates a number of basic insights from Nagelian reduction, though there are also a number of crucial differences between the two approaches.

6.1 DS Reduction and Nagelian Reduction: Parallels

Perhaps the most salient parallel between DS and Nagelian reduction is that both make use of special correspondences between the elements of the high- and low-level descriptions of a particular system. More specifically, the bridge maps of DS reduction serve much the same purpose as the bridge laws of GNS reduction, insofar as they identify those elements of the low-level description that approximately mimic the behavior of particular elements in the high-level description.

However, the analogy between the two approaches extends further than this. Recall that the GNS account of theory reduction distinguishes four 'theories': the low level theory T_l, the high level theory T_h, the image theory T_h^*, and the analogue theory T_h'. On the GNS approach, T_h^* is formulated in the language of T_l and deduced from T_l without the use of bridge laws;

T'_h is then obtained from T^*_h by straightforward bridge law substitution, and is formulated in the language of T_h; if the reduction is successful, T'_h will be 'strongly analogous' to T_h. It is in this sense that a high level theory T_h may be reduced to a low level theory T_l on the GNS account. On the DS approach to reduction, I claim, the portion of a reduction that involves demonstrating that condition DSR1 is satisfied proceeds much according to the same basic outline, with a major revision being that it is *models* rather than whole theories that are reduced. Let us exhibit these parallels more explicitly.

6.1.1 Image Models, Bridge Laws, Analogue Models and 'Strong Analogy'

On the DS account of the reduction of a high-level model M_h to a low level model M_l, one can, by analogy with the GNS approach, identify an image model M^*_h and an analogue model M'_h. It is the analogue model that approximates, or is 'strongly analogous' to, the high-level model M_h.

The image model M^*_h is formulated using elements of the model M_l - that is, in terms of the mathematical structures defined on M_l's state space - and can be deduced from M_l solely on the basis of a restriction to a particular domain of states in S_l. Its dynamics are given by the relation,

Image Model Dynamics:

$$\frac{d}{dt}B\big(x^l(t)\big) \approx f_h\big(B\big(x^l(t)\big), t\big) \tag{67}$$

which is assumed to hold for x^l in some nonempty set d, where d is preserved under the dynamical evolution over some limited timescale τ. Note that this relation approximately takes the same form as the high-level equation of motion $\frac{dx^h}{dt} = f_h(x^h, t)$, but with x^h replaced by its counterpart $B(x^l)$ in the low-level model. Recall, moreover, that satisfaction of image model dynamics for some such domain d suffices to ensure satisfaction of the condition DSR1.

By further analogy with the GNS account, the analogue model is obtained from the image model through the bridge map substitution,

Bridge Map Substitution:

$$x'^h \equiv B(x^l) \tag{68}$$

and its dynamics are specified by the equation of motion:

Analogue Model Dynamics:

$$\frac{dx'^h}{dt} \approx f_h(x'^h, t). \tag{69}$$

The domain of applicability of this model within S_h is the image domain $B(d)$. Note that the expression $B(x^l)$, which occurs in the image model, is an expression built from structures within the low level model M_l - in this sense the image model is formulated in the mathematical 'language' of the low-level model. On the other hand, the more condensed notation of the analogue model conceals the detailed construction of x'^h from quantities in the low-level model M_l, regarding x'^h simply as a point in S_h rather than as a quantity constructed from elements of M_l; in this sense one may view the analogue model as formulated in the mathematical 'language' of the high-level model.

For a reduction to take place in the GNS account, the analogue model M'_h must be 'strongly analogous' to the high level model M_h. Within the context of the GNS model the condition of strong analogy is ambiguous, though is intended to include some requirement of approximate agreement between M'_h and M_h. On the DS approach, the relation of strong analogy is unambiguous, and specifically requires that

'Strong Analogy':

$$\left|x'^h(t) - x^h(t)\right| < \delta \ \forall \ 0 \leq t \leq \tau, \tag{70}$$

where τ again is the reduction timescale. Note that this 'strong analogy' claim is just the condition DSR1 rewritten using bridge map substitution $x'^h(t) \equiv B\big(D_l(x^l_0, t)\big)$ and the definition $x^h(t) \equiv D_h\big(B(x^l_0), t\big)$.

6.2 DS Reduction and Nagelian Reduction: Disanalogies

The first and most general distinction between DS and Nagelian reduction is that the former concerns the reduction of individual models while the latter concerns the reduction of theories. Nagelian reduction, moreover, specifically requires the derivation of the *laws* of the high-level theory from those of the low-level theory. In the case of DS reduction, to be sure, it is also necessary that the laws of the high-level model - which I take it are most naturally associated in the DS picture with the equations of motion of the model - be derivable from those of the low-level model in the sense that it is possible to derive some image laws from the low-level model, which serve to approximate the laws of the high-level theory via bridge map substitution and the strong analogy relation.

Yet models of physical theories involve more structure than simply their dynamics - for example, the structures associated with their state spaces and symmetries on those state spaces. In models of classical Hamiltonian mechanics, for example, the dynamical equations, as expressed in terms of Poisson brackets with the Hamiltonian, are but a portion of the larger symplectic structure of the phase space manifold, which serves as a unified geometrical framework in which to understand not only the dynamics but the symmetries of the theory as well as the whole formalism of canonical transformations. In models of non-relativistic quantum mechanics, likewise, the dynamical law specified by the Schrodinger equation is but a portion of the larger mathematical apparatus associated with Hermitian operators, unitary transformations, and the like. Unlike Nagelian reduction, which focuses on the derivation of the high-level theory's *laws*, DS reduction more generally seeks to identify substructures of the low-level model that approximately instantiate the structures of the high-level model in some domain. While the dynamical laws of the high-level model certainly represent one crucial piece of the high-level model's structure that must be instantiated by the low-level model (and the fact of this instantiation is one that must be derived from the low-level model), they do not exhaust it.

Another difference between DS and Nagelian reduction is that, while the bridge maps of DS reduction and the bridge laws of Nagelian reduction do fulfill similar roles, DS reduction is framed in terms of the *existence* of a mathematical function (the bridge map) satisfying certain criteria, while the bridge laws of Nagelian reduction are understood as separate assumptions made independently of the high-and low- level theories, which are necessary to derive the appropriate analogue to the high-level laws. That is, the DS approach takes it as a given that the high- and low- level models succeed in describing the behavior of some system, and a reduction is said to occur only if a certain mathematical relationship obtains between these models - namely, the existence of a function between the state space satisfying the necessary mathematical conditions stated above. The Nagelian approach, on the other hand, treats bridge laws as independent auxiliary assumptions that supplement the low-level theory to facilitate the derivation of an analogue to the laws of the high-level theory.

On a final and perhaps more controversial point - which I do not have space to defend in detail here but instead make in order to provoke further discussion - in the cases where DS reduction applies, the mathematically precise nature of its conditions and of its definitions serves to expel much of the ambiguity that afflicts corresponding notions in the Nagelian acount: in particular, the concepts of bridge law and 'strong analogy.' The discussion of Nagelian bridge laws continues to be fraught with controversy over a variety of issues, such as whether bridge laws need to be treated as identity claims, the question as to whether they deserve to be called empirical 'laws' in the same sense that equations of motion are or are merely conventions, and various questions surrounding the issue of multiple realisability. Likewise, the meaning of 'strong analogy' in the GNS account is also highly ambiguous in the sense that it is not clear precisely which elements of T_h' need to approximate corresponding elements of T_h for the two theories to be strongly analogous. In the mathematically precise context of dynamical systems reduction, I suggest, much of this ambguity is avoided.

7 DS Reduction and Limits

Within the framework of DS reduction, the signficance of limits, understood not in the literal sense of taking a mathematical limit (which, as I argued in section 2.2, is problematic), but rather as a restriction on the values of some appropriately chosen dimensionless parameters ϵ_i, is to specify constraints on parameters specifying the state space and dynamics of the low-level model, as well as on the domain d of states in the low-level model for which the quantity specified by the bridge map satisfies the DSR conditions to within some given margin of approximation and over some given time scale. However, nothing in the definition of DS reduction *requires* that these restrictions be imposed in the manner specified by limit-based reduction (construed according to our last definition). Indeed, as will become more apparent in another paper where I present case studies of DS reduction, there are a number of cases of DS reduction where these restrictions are not most perspicuously imposed by requiring $\epsilon_i << 1$ for some dimensionless parameters ϵ_i.

In many cases, results that involve 'taking the limit' (again, on our updated construal) are strongly relevant to the reduction of models in the DS framework, but their role is secondary in that they serve as a particular means of satisfying the DSR conditions, which themselves make no reference to limits. Note, moreover, that while in the limit-based approach the sense in which 'T_h approximates T_l' is left vague, on the DS approach the sense in which a model M_h approximates another model M_l is made precise: bridge maps explicitly identify the quantities in the low-level model that approximate the dynamics and symmetry transformation properties of states in the high-level model, where the sense of approximation or 'strong analogy' is specified exactly by the norm on the high-level state space.

8 Limitations of the DS Approach

While the DS approach encompasses a very wide range of reductions in physics, there are some that it does not include. In particular, cases in which:

- one of the two models involved in the reduction is not describable as a dynamical system: for instance, because one of the models is not deterministic; or because (for example in the context of general relativity) no global foliation of the solution space into state spaces at different times is possible (in general relativity this is only possible if the spacetime is globally hyperbolic); or because the laws of the theory are not specified as first-order differential equations, but as constraints of some other form (as is the case with the Ideal Gas Law);

- the models do not naturally share some common time parameter; for example, special relativity and general relativity both admit many parameterisations of time; a reduction of the former to the latter would first require some clear correspondence between a given time parameter in SR to some time parameter in GR; similar considerations apply to the reduction of GR to speculative models of quantum gravity (for instance, models of string theory or loop quantum gravity).

While these cases serve to highlight the limitations of the DS approach, they also suggest ways in which we might try to generalise it - a task I leave for future work. The DS approach offers a promising starting point from which to develop frameworks for reduction that are even more inclusive and preserve the DS account's successes in the cases where it does work.

9 Conclusion

In the preceding sections, I have spelled out the basic elements of an alternative concept of reduction in physics, one that I have argued is distinct from the popular limit-based and Nagelian approaches, although it bears some significant parallels with Nagelian reduction relating to the use of bridge laws, and may incorporate limit-based results as a means of specifiying the domain of low-level models and states that satisfy the DSR conditions. One of the crucial distinctions between this approach and the limit-based and Nagelian approaches is that it concerns reduction only at the level of individual models of physical theories, not the wholesale reduction of entire theories; on the DS view, reduction of theories proceeds piecemeal, via the reduction of individual models of the high-level theory. Moreover, rather than seeking to provide a completely general account of reduction across all of the sciences, or all of physics, this account is specialised to the framework of dynamical systems models in physics. While this restriction limits the applicability of the DS view relative to the intended scope of other approaches to reduction, the specialisation to the precisely defined context of dynamical systems models permits us to analyse reduction in terms that are in some respects more precise and less ambiguous than the terms in which other approaches have attempted to cast the conversation about reduction in physics.

Moreover, the very wide range of models in physics that can be treated as dynamical systems should make this of interest as a general framework for reduction in physics. In a separate paper, I demonstrate the application of the DS approach to reduction in a wide range of cases, including the reduction of 1) 'Center of Mass' Newtonian mechanics to Newtonian mechanics of constituent particles, 2) Newtonian mechanics to special relativistic mechanics, 3) non-relativistic classical mechanics to unitary non-relativistic quantum mechanics, 4) nonrelativistic quantum mechanics of a spin-1/2 particle to the Dirac theory of a relativistic spin-1/2 particle, 5) non-relativistic quantum mechanics of a free spinless particle to relativistic quantum field theory of a free scalar particle, 6) master equation descriptions of quantum mixed-state dynamics for some subsystem to quantum pure-state dynamics of system+environment combination. Finally, I have hinted at some ways in which this approach might be generalized in order to accommodate a wider range of reductions in physics, though leave elaboration of the details to future work.

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