MAURO AVON

A different approach to logic

Mauro Avon
born 1967 in Spilimbergo, Italy;
holds a Master's degree in Computer Science from the University of Udine, Italy.
E-mail: mauro.avon@alice.it

Contents

Abstract

The paper is about an approach to logic that differs from the standard first-order logic and other known approaches. It should be a new approach the author has created proposing to obtain a general and unifying approach to logic and a faithful model of human mathematical deductive process. We list the most relevant features of the system. In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. The set-builder notation is enclosed as an expression-building pattern. In our system we can easily express second-order and all-order conditions (the set to which a quantifier refers is explicitly written in the expression). The meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'. Our deductive system is based on a very simple definition of proof and provides a good model of human mathematical deductive process. The soundness and consistency of the system are proved, as well as the fact that our system is not affected by the most known types of paradox. The paper provides both the theoretical material and two fully documented examples of deduction. The author believes his aims have been achieved, obviously the reader is free to examine the system and get his own opinion about it.

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1. Introduction

This paper outlines a system or approach to mathematical logic which is different from the standard one. By 'the standard approach to logic' I mean the one presented in chapter 2 of Enderton's book [2] and there named 'First-Order Logic'. The same approach is also outlined in chapter 2 of Mendelson's book [4], where it is named 'Quantification Theory'.

We now list the features of our system, pointing out the differences and improvements with respect to standard logic.

In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. Each expression is referred to a certain 'context'. A context can be seen as a (possibly empty) sequence of ordered pairs (x, φ), where x is a variable and φ is itself an expression. Given a context $k = (x_1, \varphi_1) \dots (x_m, \varphi_m)$ we call a 'state on k' a function which assigns 'allowable values' (we'll explain this later) to the variables $x_1, \dots, x_m$. If t is an expression with respect to context k and σ is a state on k, we'll be able to define the meaning of t with respect to k and σ, which we'll denote by $\#(k, t, \sigma)$.
Our approach requires to build all at the same time, contexts, expressions, states and meanings. We'll call sentences those expressions which are related to an empty context and whose meaning is true or false. The meaning of a sentence depends solely on the meaning of the symbols it contains, it doesn't depend on external 'structures'.

In first-order logic we have terms and formulas and we cannot apply a predicate to one or more formulas, this seems a clear limitation. With our system we can apply predicates to formulas. We'll see this allows in principle to give a rigorous construction of something similar to the liar paradox, but we can also give a fairly simple explanation of such paradox, which in the end is not a paradox (see chapter 7).

When we specify a set in mathematics we often use the 'set-builder notation'. Examples of sets defined with this notation are $\{x \in \mathbb{N} | \ \exists y \in \mathbb{N} : x = 2y\}$, $\{x \in \mathbb{R} | x = x^2\}$, and so on. In our system the set-builder notation is enclosed as an expression-building pattern, and this is an advantage over standard logic.

Of course in our approach we allow connectives and quantifiers, but unlike first-order logic these are at the same level of other operators, such as equality, membership and more. While the set-builder notation is necessarily present with its role, connectives and quantifiers as 'operators' are not strictly mandatory and are part of a broader category. For instance the universal quantifier simply applies an operation of logical conjunction to a set of conditions, and so it can be classified as an operator.

In first-order logic variables range over individuals, but in mathematics there are statements in which both quantifiers over individuals and quantifiers over sets of individuals occur. One simple example is the following condition:

for each subset X of $\mathbb{N}$ and for each $x \in \mathbb{N}$ we have $x \in X$ or $x \notin X$.

Another example is the condition in which we state that every bounded, non empty set of real numbers has a supremum. Formalisms better suited to express such conditions are second-order logic and type theory, but these systems have a certain level of complexity and are based on different types of variable. In our system we can express the conditions we mentioned above, and we absolutely don't need different types of variables, the set to which the quantifier refers is explicitly written in the expression, this ultimately makes things easier and allows a more general approach. If we read the statement of a theorem in a mathematics book, usually in this statement some variables are introduced, and when introducing them often the set in which they are varying is explicitly specified, so from this point of view our approach is consistent with the actual processes of mathematics.

Let's examine how our system behaves when giving a meaning and possibly a truth value to expressions. Standard logic doesn't plainly associate meanings and truth values to formulas. It introduces some related notion as the concepts of 'structure' (defined in section 2.2 of Enderton's book), truth in a structure, validity, satisfiability. Within first-order logic a structure is used, first of all, to define the collection of things to which a quantifier refers to. Moreover, some symbols such as connectives and quantifiers have a fixed meaning, while for other symbols the meaning is given by the structure. In first-order logic there is a certain level of independence between the meaning of symbols and the language's set of formulas. For instance, if P is a 2-places predicate symbol and t_1, t_2 are terms then Pt_1t_2 is always a formula, and this doesn't depend on the meaning of P, t_1 and t_2. Anyway, what if P was a 3-places predicate? In this case Pt_1t_2 couldn't be a formula. This is just an example to show that the independence between the meaning of symbols and the set of formulas isn't absolute.

In our approach we do not ask, as a requirement, to have independence between the meaning of symbols and the set of expressions, nor do we take care to investigate what happens when changing the meaning of symbols. It wouldn't be easy to deal with this because, for example, you should determine the desired level of independence and variability. Also, I could not say whether trying to deal with this matters would produce any result or added value. For a first presentation of our approach, this topic doesn't seem a priority, it could be a subject of future studies.

Therefore if a symbol is in our system it has his own meaning, and we don't feature a notion of structure like the one of first-order logic. Also, the set of expressions in our language depends on the meaning of symbols. We'll simply speak of the meaning of an expression and when possible of the truth value of that meaning. As we've already said, the meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'.

Our deductive system seeks to provide a good model of human mathematical deductive process. The concept of proof we'll feature is probably the most simple and intuitive that comes to mind, we try to anticipate some of it.

If we define S as the set of sentences then an axiom is a subset of S, an n-ary rule is a subset of S^{n+1}. If φ is a sentence then a proof of φ is a sequence $(\psi_1, \ldots, \psi_m)$ of sentences such that

- there exists an axiom A such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \ldots m$ one of the following holds
  - there exists an axiom A such that $\psi_j \in A$,
  - there exists an n-ary rule R and $i_1, \ldots, i_n < j$ such that $(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

Our deductive system, in order to do its job, needs to track the various hypotheses we have introduced along our proof. In a fixed moment of our reasoning we have a sequence of active hypotheses, and we need to be able to apply one of our rules. To this end our axioms and rules need to be properly constructed.

As regards the soundness of the system, it is proved at the beginning of chapter 5. Consistency is a direct consequence of soundness. We also discuss (in chapter 7) how the system relates with some well known paradoxes, it comes out that our system doesn't lead to this kind of inconsistencies. Actually (and obviously) I'm not aware of inconsistencies to which it would lead.

We have examined the main features of the system. If the reader will ask what is the basic idea behind a system of this type, in agreement with what I said earlier I could say that the principle is to provide something like a general and unifying approach to logic and a faithful model of human mathematical deductive process.

This statement about our system of course is not a mathematical statement, so I cannot give a mathematical proof of it. On the other hand, logic exists with the specific primary purpose of being a model to human deduction. In general, suppose we want to provide a mathematical model of some process or reality. The fairness of the model can be judged much more through experience than through mathematics. In fact, mathematics always has to do with models and not directly with reality.

This paper's purpose is to present an approach to logic, but clearly we cannot provide here all possible explanations and comparisons in any way related to the approach itself. The author believes that this paper provides a fairly comprehensive presentation of the approach in question, this introduction includes significant elements of explanation, justification and comparison with the standard approach to logic. Other material in this regard is presented in the subsequent sections (for example in chapter 7).

First-order logic has been around for many decades, but to date no absolute evidence has been found that first-order logic is the best possible logic system. In this regard I may quote a stronger statement at the beginning of Josè Ferreirós' paper 'The road to modern logic an interpretation' ([3]).

> It will be my contention that, contrary to a frequent assumption (at least among philosophers), First-Order Logic is *not* a 'natural unity', i.e. a system the scope and limits of which could be justified solely by rational argument.

Honestly, in my opinion, the approach to logic I propose seems to be a 'natural unity' much more than first-order logic is, and I did what I thought was reasonable to explain this.

Further investigations on this approach will be conducted, in the future, if and when possible, by the author and/or other people. If any claim of this introduction would seem inappropriate, the author is ready to reconsider and possibly fix it. In any case he believes the most important part of this paper is not in the introduction, but in the subsequent chapters.

The paper is quite long but the time required to get an idea of the content is not very high. In fact, the author has chosen to include all the proofs, but quite often these are simple proofs. In addition, the most complex parts are the two definitions 2.7 and 4.16. These have a certain complexity, but at first reading it is not necessary to care of all the details.

2. Language: symbols, expressions and sentences, and their meaning

We begin to describe our language and then the expressions that characterize it. In the process of defining expressions we also define their meaning and the context to which the expression refers. The expressions of our language are constructed from some set of symbols according to certain rules. Expressions are sequences of symbols with meaning, 'sentences' are specific expression whose meaning has the property of being true or false. We begin by describing the sets of symbols we need.

First we need a set of symbols $\mathcal{V}$. $\mathcal{V}$ members are also called variables and just play the role of variables in the construction of our expressions (this implies that $\mathcal{V}$ members have no meaning associated).

In addition we need another set of symbols $\mathcal{C}$. $\mathcal{C}$ members are also called 'constants' and have a meaning. For each $c \in \mathcal{C}$ we denote by $\#(c)$ the meaning of c.

Let f be a member of $\mathcal{C}$. Being f endowed with meaning, f is always an expression of our language. However, the meaning of f could also be a function. In this case f can also play the role of an 'operator' in the construction of expressions that are more complex than the simple constant f.

Not all the operators that we need, however, are identifiable as functions. Think to the logical connectives (logical negation, logical implication, quantifiers, etc..), but also to the membership predicate '$\in$' and to the equality predicate '$=$'. The meaning of these operators cannot be mapped to a precise mathematical object, therefore these operators won't have a precise meaning in our language, but we'll need to give meaning to the application of the operator to objects, where the operator is applicable.

In mathematics and in the real world objects can have properties, such as having a certain color, or being true, or being false. A property is therefore something that can be assigned to an object, no object, more than one object. For example, with reference to color, one or more objects are red or have the property 'to be of red color'. But more generally one or more objects have a color. Suppose to indicate, for objects x that have a color, the color of x with $C(x)$. So we can say that C is a property applicable to a class of objects. On the same object class we can indicate with $R(x)$ the condition 'x has the red color'. R is in turn a property applicable to a class of objects, with the characteristic that for all x $R(x)$ is true or false. A property with this additional feature can be called a 'predicate'.

The class of objects to which a property may be assigned may be called the domain of the property. The members of that domain may be individual objects or sequences of objects, for example, if x is an object and X is a set, the condition '$x \in X$' involves two

objects, and then the domain of the membership property consists of the ordered pairs (x, X), where x is an object and X is a set.
Generally we are dealing with properties such that the objects of their domain are all individual objects, or all ordered pairs. Theoretically there may also be properties such that the objects of their domain are sequences of more than two items or even the number of items in sequence may be different in different elements of the domain.

As mentioned above the concept of 'property' is similar to the concept of function, but in mathematics there are properties that are not functions. For example, the condition '$x \in X$' just introduced can be applied to an arbitrary object and an arbitrary set, so the 'membership property' has not a well determined domain and cannot be considered a function in a strict sense.

So to build our language we need another set of symbols $\mathcal{F}$, where each f in $\mathcal{F}$ represents a property P_f. Symbols in $\mathcal{F}$ are also called operators or 'property symbols'. We will not assign a meaning to operators, because a property cannot be mapped to a consistent mathematical object (function or other). However, for each f

- for each positive integer n and $x_1, \ldots, x_n$ arbitrary objects we must know the condition $A_f(x_1, \ldots, x_n)$ that indicates if P_f is applicable to $x_1, \ldots, x_n$;
- for each positive integer n and $x_1, \ldots, x_n$ arbitrary objects such that $A_f(x_1, \ldots, x_n)$ holds we must know the value of $P_f(x_1, \ldots, x_n)$.

Since the concept might be unclear we immediately explain it by specifying what are the most important operators that we may include in our language, providing for each of them the conditions $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ (in general $P_f(x_1, \ldots, x_n)$ is a generic value, but in these cases it is a condition, i.e. its value can be true or false).

- Logical conjunction: it's the symbol $\wedge$ and we have
  for $n \neq 2$ $A_\wedge(x_1, \ldots, x_n)$ is false ,
  $A_\wedge(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\wedge(x_1, x_2) =$ both x_1 and x_2 are true ;

- Logical disjunction: it's the symbol $\vee$ and we have
  for $n \neq 2$ $A_\vee(x_1, \ldots, x_n)$ is false ,
  $A_\vee(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\vee(x_1, x_2) =$ at least one between x_1 and x_2 is true ;

- Logical implication: it's the symbol $\rightarrow$ and we have
  for $n \neq 2$ $A_\rightarrow(x_1, \ldots, x_n)$ is false ,
  $A_\rightarrow(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\rightarrow(x_1, x_2) = x_1$ is false or x_2 is true ;

- Double logical implication: it's the symbol $\leftrightarrow$ and we have
  for $n \neq 2$ $A_\leftrightarrow(x_1, \ldots, x_n)$ is false ,

$A_{\leftrightarrow}(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
$P_{\leftrightarrow}(x_1, x_2) = P_{\rightarrow}(x_1, x_2)$ and $P_{\rightarrow}(x_2, x_1)$;

- Logical negation: it's the symbol $\neg$ and we have
  for $n > 1$ $A_{\neg}(x_1, \ldots, x_n)$ is false ,
  $A_{\neg}(x_1)$ is true,
  $P_{\neg}(x_1) = x_1$ is false ;

- Universal quantifier: it's the symbol $\forall$ and we have
  for $n > 1$ $A_{\forall}(x_1, \ldots, x_n)$ is false ,
  $A_{\forall}(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_{\forall}(x_1) =$ for each x in x_1 (x is true) .

- Existential quantifier: it's the symbol $\exists$ and we have
  for $n > 1$ $A_{\exists}(x_1, \ldots, x_n)$ is false ,
  $A_{\exists}(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_{\exists}(x_1) =$ there exists x in x_1 such that (x is true) .

- Membership predicate: it's the symbol $\in$ and we have
  for $n \neq 2$ $A_{\in}(x_1, \ldots, x_n)$ is false ,
  $A_{\in}(x_1, x_2) = x_2$ is a set,
  $P_{\in}(x_1, x_2) = x_1$ is a member of x_2 ;

- Equality predicate: it's the symbol $=$ and we have
  for $n \neq 2$ $A_{=}(x_1, \ldots, x_n)$ is false ,
  $A_{=}(x_1, x_2)$ is true,
  $P_{=}(x_1, x_2) = x_1$ is equal to x_2 .

We can think and use also other operators, for instance operations between sets such as union or intersection can be represented through an operator, etc. .

In the standard approach to logic, quantifiers are not treated like the other logical connectives, but in this system we mean to separate the operation of applying a quantifier from the operation whereby we build the set to which the quantifier is applied, and therefore the quantifier is treated as the other logical operators (altogether, the universal quantifier is simply an extension of logical conjunction, the existential quantifier is simply an extension of logical disjunction).

With regard to the operation of building a set, we need a specific symbol to indicate that we are doing this, this symbol is the symbol '{}' which we will consider as a unique symbol.

Besides the set builder symbol, we need parentheses and commas to avoid ambiguity in the reading of our expressions; to this end we use the following symbols: left parenthesis '(', right parenthesis ')', comma ',' and colon ':'. We can indicate this further set of symbols with $\mathcal{Z}$.

To avoid ambiguity in reading our expressions we require that the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{Z}$ are disjoint. It's also requested that a symbol does not correspond to any chain of more symbols of the language. More generally, given $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ symbols of our language, and using the symbol '$\|$' to indicate the concatenation of characters and strings, we assume that equality of the two chains $\alpha_1 \| \ldots \| \alpha_n$ and $\beta_1 \| \ldots \| \beta_m$ is achieved when and only when $m = n$ and for each $i = 1 \ldots n$ $\alpha_i = \beta_i$.

While the set $\mathcal{Z}$ will be always the same, the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ may change according to what is the language that we describe. To describe our language it is required to know the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and the function $\#$ which associates a meaning to every element of $\mathcal{C}$. In other words, our language is identified by the 4-tuple $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$. Since the 'meaning' of an operator is not a mathematical object, operators must be seen as symbols which are tightly coupled with their meaning.

Before we can describe the process of constructing expressions we still need to introduce some notation. In fact in that process we'll use the notion of 'context' and the notion of 'state'. Context and states have a similar form, and here we want to define their common form.

We define $\mathcal{D} = \{\emptyset\} \cup \{\{1, \ldots, m\} |\ m \text{ is a positive integer}\}$.

Suppose x is a function whose domain $dom(x)$ belongs to $\mathcal{D}$. Suppose $C \in \mathcal{D}$ is such that $C \subseteq dom(x)$. Then we define $x_{/C}$ as a function whose domain is C and such that for each $j \in C$ $x_{/C}(j) = x(j)$.

Suppose x and φ are two functions with the same domain D, and $D \in \mathcal{D}$. Then we say that (x, φ) is a '*state-like pair*'.

Given a state-like pair $k = (x, \varphi)$ the domain of x will be also called the *domain of k*. Therefore $dom(k) = dom(x) = dom(\varphi)$.

Furthermore $dom(k) \in \mathcal{D}$ and given $C \in \mathcal{D}$ such that $C \subseteq dom(k)$ we can define $k_{/C} = (x_{/C}, \varphi_{/C})$. Clearly $k_{/C}$ is a state-like pair.

We define $\mathcal{R}(k) = \{k_{/C} |\ C \in \mathcal{D}, C \subseteq dom(k)\}$.

Given another state-like pair h we say that $h \sqsubseteq k$ if and only if $h \in \mathcal{R}(k)$.

Suppose $h \in \mathcal{R}(k)$, then there exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C} = (x_{/C}, \varphi_{/C})$. Therefore $dom(h) = C$ and $k_{/dom(h)} = k_{/C} = h$.

Suppose $h \in \mathcal{R}(k)$ and $g \in \mathcal{R}(h)$. This means there exist $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C}$, and there exist $D \in \mathcal{D}$ such that $D \subseteq dom(h)$, $g = h_{/D}$. So $D \subseteq dom(h) = C \subseteq dom(k)$, $g = (k_{/C})_{/D} = (x_{/C}, \varphi_{/C})_{/D} = (x_{/D}, \varphi_{/D}) = k_{/D}$. Therefore $g \in \mathcal{R}(k)$.

Suppose $k = (x, \varphi)$ is a state-like pair whose domain is D. Suppose (y, ψ) is an ordered pair. Then we can define the 'addition' of (y, ψ) to k.
Suppose $D = \{1, \ldots, m\}$, then we define $D' = \{1, \ldots, m+1\}$. We define x' as a function whose domain is D' such that for each $\alpha = 1 \ldots m$ $x'(\alpha) = x(\alpha)$, and $x'(m+1) = y$. We define φ' as a function whose domain is D' such that for each $\alpha = 1 \ldots m$ $\varphi'(\alpha) = \varphi(\alpha)$, $\varphi'(m+1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\{1,\ldots,m\}} = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
If $D = \emptyset$ then clearly $D' = \{1\}$. We define x' as a function whose domain is D' such that $x'(1) = y$. We define φ' as a function whose domain is D' such that $\varphi'(1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\emptyset} = \emptyset = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
In both cases $k + (y, \psi)$ is a state-like pair, and $k \in \mathcal{R}(k + (y, \psi))$.

We also define $\epsilon = (\emptyset, \emptyset)$, so ϵ is a state-like pair.

In the next lemma we prove that, when a state-like pair is obtained as $k + (y, \psi)$, then k, y, and ψ are univocally determined.

Lemma 2.1. *Suppose $k_1 = (x_1, \varphi_1)$ is a state-like pair whose domain is D_1, and (y_1, ψ_1) is an ordered pair. Suppose $k_2 = (x_2, \varphi_2)$ is a state-like pair whose domain is D_2, and (y_2, ψ_2) is an ordered pair. Finally suppose $k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Under these assumptions we can prove that $k_1 = k_2, y_1 = y_2, \psi_1 = \psi_2$.*

Proof.
We define $h = k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Since $h = k_1 + (y_1, \psi_1)$ we can have two possibilities:

- $D_1 = \emptyset, D'_1 = \{1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$;
- there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}, D'_1 = \{1, \ldots, m_1+1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$.

Similarly, since $h = k_2 + (y_2, \psi_2)$ we can have two possibilities:

- $D_2 = \emptyset, D'_2 = \{1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$;
- there exists a positive integer m_2 such that $D_2 = \{1, \ldots, m_2\}, D'_2 = \{1, \ldots, m_2+1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$.

It follows that $(x'_1, \varphi'_1) = h = (x'_2, \varphi'_2)$, so $x'_1 = x'_2$ and $\varphi'_1 = \varphi'_2$, and $D'_1 = D'_2$.

Suppose $D_1 = \emptyset$. This implies that $D'_2 = D'_1 = \{1\}$, thus $D_2 = \emptyset$.
In this case $k_1 = \epsilon = k_2$, $y_1 = x'_1(1) = x'_2(1) = y_2$, $\psi_1 = \varphi'_1(1) = \varphi'_2(1) = \psi_2$.

Suppose there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}$. This implies that $D'_2 = D'_1 = \{1, \ldots, m_1 + 1\}$, thus $D_2 = \{1, \ldots, m_1\}$.
In this case for each $\alpha = 1 \ldots m_1$ $x_1(\alpha) = x'_1(\alpha) = x'_2(\alpha) = x_2(\alpha)$, $\varphi_1(\alpha) = \varphi'_1(\alpha) = \varphi'_2(\alpha) = \varphi_2(\alpha)$. So $k_1 = (x_1, \varphi_1) = (x_2, \varphi_2) = k_2$; and moreover $y_1 = x'_1(m_1 + 1) = x'_2(m_1 + 1) = y_2$, $\psi_1 = \varphi'_1(m_1 + 1) = \varphi'_2(m_1 + 1) = \psi_2$. ∎

Other useful results are the following.

LEMMA 2.2. *Suppose* $h = (x, \varphi)$, $k = (z, \psi)$ *are state-like pairs such that* $h \in \mathcal{R}(k)$ *and for each* $i, j \in dom(k)$ $i \neq j \rightarrow z_i \neq z_j$. *Then, for each* $i \in dom(k)$, $j \in dom(h)$ $z_i = x_j \rightarrow \psi_i = \varphi_j$.

Proof. Let $i \in dom(k)$, $j \in dom(h)$ and $z_i = x_j$. Clearly $j \in dom(k)$, $x_j = z_j$, thus $z_i = z_j$, $i = j$, $\varphi_j = \psi_j = \psi_i$. ■

LEMMA 2.3. *Suppose* $h = (x, \varphi)$ *is a state-like pair,* (y, ϕ) *is an ordered pair and define* $k = h + (y, \phi)$. *Suppose* $g \in \mathcal{R}(k)$ *is such that* $g \neq k$. *Then* $g \in \mathcal{R}(h)$.

Proof.

Let $D = dom(h)$.

Suppose m is a positive integer and $D = \{1, \dots, m\}$. Then $k = (x', \varphi')$ has a domain $\{1, \dots, m+1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \dots, m+1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C \subseteq \{1, \dots, m\}$. We have

$$g = k_{/C} = (x'_{/C}, \varphi'_{/C}) = ((x'_{/D})_{/C}, (\varphi'_{/D})_{/C}) = (x_{/C}, \varphi_{/C}) = h_{/C} \ .$$

Now suppose $D = \emptyset$. Then $k = (x', \varphi')$ has a domain $\{1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C = \emptyset$ and $g = (\emptyset, \emptyset) = h$.

In both cases $g \in \mathcal{R}(h)$. ■

LEMMA 2.4. *Let* x *be a function such that* $dom(x) \in \mathcal{D}$, *let* $C, D \in \mathcal{D}$ *such that* $C \subseteq D \subseteq dom(x)$. *Then we can define* $x_{/C}$ *and* $(x_{/D})_{/C}$, *and we have* $(x_{/D})_{/C} = x_{/C}$.

Proof. Define $y = x_{/D}$, we have $dom(y) = D$ and for each $j \in D$ $y(j) = x(j)$. Moreover $dom(y_{/C}) = C = dom(x_{/C})$ and for each $j \in dom(C)$ $y_{/C}(j) = y(j) = x(j) = x_{/C}(j)$. ■

LEMMA 2.5. *Let* g, h *and* $k = (x, \varphi)$ *be state-like pairs such that* $g, h \in \mathcal{R}(k)$, $dom(g) \subseteq dom(h)$. *Then* $g \in \mathcal{R}(h)$.

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $g = k_{/C}$. And there exists $D \in \mathcal{D}$ such that $D \subseteq dom(k)$, $h = k_{/D}$. It results $C = dom(g) \subseteq dom(h) = D$. Then, clearly

$$g = (x, \varphi)_{/C} = (x_{/C}, \varphi_{/C}) = ((x_{/D})_{/C}, (\varphi_{/D})_{/C}) = (x_{/D}, \varphi_{/D})_{/C} = h_{/C} \ .$$

■

We also need some notation referred to generic strings, this notation will be useful when applied to our expressions, which are non-empty strings. If t is a string we can indicate with $\ell(t)$ t's length, i.e. the number of characters in t. If $\ell(t) > 0$ then for each $\alpha \in \{1, \dots, \ell(t)\}$ at position α within t there is a character, this symbol will be indicated with $t[\alpha]$. We call 'depth of α within t' (briefly $d(t, \alpha)$) the number which is obtained by subtracting the number of right round brackets ')' that occur in t before position α from

the number of left round brackets '(' that occur in t before position α .
The following lemma will be useful later within proofs of unique readability. Its proof is so simple that we feel free to omit it.

LEMMA 2.6. *Let ϑ, φ, η be strings with $\ell(\vartheta) > 0$, $\ell(\varphi) > 0$, and let $t = \vartheta\|\varphi\|\eta$; let also $\alpha \in \{1, \ldots, \ell(\varphi)\}$. The following result clearly holds:*

$$d(t, \ell(\vartheta) + \alpha) = d(t, \ell(\vartheta) + 1) + d(\varphi, \alpha).$$

■

We can now describe the process of constructing expressions for our language $\mathcal{L}$. This is an inductive process in which not only we build expressions, but also we associate them with meaning, and in parallel also define the fundamental concept of 'context'. This process will be identified as 'Definition 2.7' although actually it is a process in which we give the definitions and prove properties which are needed in order to set up those definitions.

2.1. Definition process. This section contains only definition 2.7. This definition is an inductive definition process within which we have assumptions, lemmas etc.. Symbols like ■ within this definition are not intended to terminate the definition, they just terminate an assumption or lemma etc. which is internal to the definition. Within the definition there are also internal tasks in which we verify some expected condition. We'll use the symbol $\diamond$ to mark the end of each of those tasks.

DEFINITION 2.7. Since this is a complex definition, we will first try to provide an informal idea of the entities we'll define in it. The definition is by induction on positive integers, we now introduce the sets and concepts we'll define for a generic positive integer n (this first listing is not the true definition, it's just to introduce the concepts, to enable the reader to understand their role).

$K(n)$ is the set of 'contexts' at step n. A context k is a state-like pair of the form (x, φ) where x and φ have the same domain $D = \{1, \ldots, m\} \in \mathcal{D}$, and for each $i = 1 \ldots m$ x_i is a variable and φ_i is an expression.

For each $k \in K(n)$ $\Xi(k)$ is the set of 'states' bound to context k. If $n > 1$ and $k \in K(n-1)$ then $\Xi(k)$ has already been defined at step $n-1$ or formerly, otherwise it will be defined at step n.

If $k = (x, \varphi)$ is a context, a state on k is a state-like pair $\sigma = (x, s)$ where (roughly speaking) for each i in the domain of x, φ and s s_i is a member of the meaning of the corresponding expression φ_i .

For each $k \in K(n)$ $E(n, k)$ is the set of expressions bound to step n and context k.

$E(n)$ is the union of $E(n, k)$ for $k \in K(n)$ (this will not be explicitly recalled on each iteration in the definition).

For each $k \in K(n)$, $t \in E(n,k)$, $\sigma \in \Xi(k)$ we'll define $\#(k,t,\sigma)$ which stays for 'the meaning of t bound to k and σ'. If $n > 1$, $k \in K(n-1)$ and $t \in E(n-1,k)$ then $\#(k,t,\sigma)$ has already been defined at step $n-1$ or formerly, otherwise it will be defined at step n.

For each $k \in K(n)$, $t \in E(n,k)$
$V_b(t)$ is the set of the variables that occur within t, bound to a quantifier ;
$V_f(t)$ is the set of the variables that occur within t, not bound to a quantifier ;
$V(t)$ is the set of the variables that occur within t (of course $V(t) = V_b(t) \cup V_f(t)$, so $V(t)$ will not be explicitly defined each time).
If $n > 1$, $k \in K(n-1)$ and $t \in E(n-1,k)$ then $V_b(t)$ and $V_f(t)$ have already been defined at step $n-1$ or formerly, otherwise they will be defined at step n.

We'll also use some sets that will be defined in the same way at each step, we put here their definition and we'll avoid to repeat those definitions each time.
For each $k \in K(n)$ we define $E_s(n,k) = \{t | t \in E(n,k), \forall\sigma \in \Xi(k)\, \#(k,t,\sigma) \text{ is a set}\}$.
For each $k \in K(n)$, $t \in E_s(n,k)$ we define $M(k,t) = \bigcup_{\sigma\in\Xi(k)} \#(k,t,\sigma)$.
For each $k \in K(n)$ we define $M(n,k) = \bigcup_{t\in E_s(n,k)} M(k,t)$.
We finally define $M(n) = \bigcup_{k\in K(n)} M(n,k)$.

We have seen that some entities may have been defined before step n, and in this case we are not to define them at step n, however at step n we need to check the definition that has been given is consistent with what we would expect.

We are now are ready to begin the actual definition process, so we perform the simple initial step of our inductive process.

We define $K(1) = \{\epsilon\}$, $\mathit{\Xi}(\epsilon) = \{\epsilon\}$, $E(1,\epsilon) = \mathcal{C}$.
For each $t \in E(1,\epsilon)$ we define $\#(\epsilon,t,\epsilon) = \#(t)$, $V_b(t) = \emptyset$, $V_f(t) = \emptyset$.

The inductive step is much more complex. Suppose all our definitions have been given at step n and let's proceed with step $n+1$. In this inductive step we'll need several assumptions which will be identified with a title like 'Assumption 2.1.x'. Each assumption is a statement that must be valid at step 1, we suppose is valid at step n and needs to be proved true at step $n+1$ at the end of our definition process.

The first assumption we need is the following.

Assumption 2.1.1. For each $k \in K(n)$ such that $k \neq \epsilon$ and for each $\sigma \in \mathit{\Xi}(k)$ there exist a positive integer m, a function $x\colon \{1,\ldots,m\} \to \mathcal{V}$, a function $\varphi\colon \{1,\ldots,m\} \to E(n)$, a function $s\colon \{1,\ldots,m\} \to M(n)$ such that

- for each $i,j \in \{1\ldots m\}$ $(i \neq j \to x_i \neq x_j)$
- $k = (x,\varphi)$
- $\sigma = (x,s)$

∎

This assumption ensures that for each $k \in K(n)$ such that $k \neq \epsilon$ k is a state-like pair, and for each $\sigma \in \mathit{\Xi}(k)$ σ is a state-like pair.

Given $k = (x, \varphi) \in K(n)$ we define $var(k)$ as the image of the function x. In other words if $k = \epsilon$ then $x = \emptyset$, so $var(k) = \emptyset$, otherwise x has a domain $\{1, \ldots, m\}$ and $var(k) = \{x_i | i = 1 \ldots m\}$.

We can go on with the inductive step and define

$$K(n)^+ = \{h + (y, \phi) |\, h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))\}\,,$$

$$K(n+1) = K(n) \cup K(n)^+\,.$$

Let $k \in K(n)^+$. Then there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. By lemma 2.1 we know that h, ϕ, y are *univocally determined.*

We can assume that $\Xi(k)$ is defined for $k \in K(n)$, and we need to define this for $k \in K(n+1) - K(n)$. If $k \in K(n+1) - K(n)$ then clearly $k \in K(n)^+$ and so there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$; and h, ϕ, y are univocally determined. So we can define

$$\Xi(k) = \{\sigma + (y, s) |\, \sigma \in \Xi(h), s \in \#(h, \phi, \sigma)\}\,.$$

We need to prove that this definition of $\Xi(k)$ holds for all $k \in K(n)^+$. To prove this we need a second assumption.

Assumption 2.1.2. For each $k \in K(n)$

$$\begin{aligned}&(k = \epsilon)\\ &\vee ((n > 1) \wedge \exists g \in K(n-1), z \in \mathcal{V} - var(g), \psi \in E_s(n-1, g) :\\ &\qquad\qquad k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\ \sigma \in \Xi(g),\ s \in \#(g, \psi, \sigma)\})\end{aligned}$$

∎

Thanks to this assumption we can prove the following lemma.

Lemma 2.1.3. For each $k \in K(n)^+, h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$ we have

$$\Xi(k) = \{\sigma + (y, s) |\, \sigma \in \Xi(h), s \in \#(h, \phi, \sigma)\}\,.$$

Proof. If $k \notin K(n)$ this is true by definition. If $k \in K(n)$ we can apply the former lemma. Since $k \neq \epsilon$ we have $n > 1$ and there exist $g \in K(n-1), z \in \mathcal{V} - var(g), \psi \in E_s(n-1, g)$ such that $k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\, \sigma \in \Xi(g), s \in \#(g, \psi, \sigma)\}$.

Since $k = h + (y, \phi)$ we have $g = h, z = y, \psi = \phi$, and therefore

$$\Xi(k) = \{\sigma + (y, s) |\, \sigma \in \Xi(h), s \in \#(h, \phi, \sigma)\} .$$

■

Another consequence of lemma 2.1 is the following: for each $k \in K(n)^+$ and $\sigma + (y, s)$ in $\Xi(k)$, σ, y and s are *univocally determined.*

To ensure the unique readability of our expressions we need the following assumption (which is clearly satisfied for $n = 1$).

Assumption 2.1.4. For each $t \in E(n)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t, \ell(t)) = 1$, else $d(t, \ell(t)) = 0$;
- for each $\alpha \in \{1, \ldots, \ell(t)\}$ if $(t[\alpha] = \text{':'}) \vee (t[\alpha] = \text{','}) \vee (t[\alpha] = \text{')'})$ then $d(t, \alpha) \geqslant 1$.

■

It is time to define $E(n+1, k)$, for each k in $K(n+1)$. Then for each t in $E(n+1, k)$ and σ in $\Xi(k)$ we need to define $\#(k, t, \sigma)$, and also we need to define $V_b(t)$ and $V_f(t)$.

We have to warn that the definition of $\#(k, t, \sigma)$, $V_b(t)$ and $V_f(t)$ is not an easy matter.

In fact, $E(n+1, k)$ will be defined as the union of different sets. Consider for instance the situation where $k \in K(n)$. One of these sets is $E(n, k)$, the old set of expressions bound to context k. But of course there are also new sets. If an expression t belongs just to $E(n, k)$, and not to the new sets, then we don't need to reason about $\#(k, t, \sigma)$, because simply it has already been defined.
However, if t belongs both to $E(n, k)$ and to one or more of the new sets, we'll have a proposed definition of $\#(k, t, \sigma)$ for each of the new sets, and we'll have to check that this proposed definition is equal to the real definition.
If t belongs to just one new set and not to $E(n, k)$ then we'll simply define $\#(k, t, \sigma)$ with the proposed definition of $\#(k, t, \sigma)$ for the new set.
If t belongs to more than one new set, and not to $E(n, k)$, we'll need to check that the proposed definitions of $\#(k, t, \sigma)$ for each new set are equal to each other, and then we'll be authorized to set $\#(k, t, \sigma)$ with one of these proposed definitions.

When $k \notin K(n)$ the discussion is simpler: it cannot be $t \in E(n, k)$, so we just have to consider the other situations. For the definition of $V_b(t)$ and $V_f(t)$ the reasoning is similar but slightly different.

At this point we can proceed to formally define the new sets of expressions bound to context k, and for expressions in each of them we define the proposed values of $\#(k, t, \sigma)$,

$V_b(t)$ and $V_f(t)$.

For each $k = h + (y, \phi) \in K(n)^+$ we define

$$E_a(n+1, k) = \{t | t \in E(n, h) \wedge y \notin V_b(t)\}.$$

For each $t \in E_a(n+1, k)$, $\sigma = \rho + (y, s) \in \Xi(k)$ we define the proposed values of $\#(k, t, \sigma)$, $V_b(t)$ and $V_f(t)$:

$$\#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho);$$
$$V_f(t)_{(n+1,k,a)} = V_f(t);\ V_b(t)_{(n+1,k,a)} = V_b(t).$$

For each $k = h + (y, \phi) \in K(n)^+$ we define

$$E_b(n+1, k) = \{y\}.$$

For each $t \in E_b(n+1, k)$, $\sigma = \rho + (y, s) \in \Xi(k)$ we define:

$$\#(k, t, \sigma)_{(n+1,k,b)} = s;$$
$$V_f(t)_{(n+1,k,b)} = \{y\};\ V_b(t)_{(n+1,k,b)} = \emptyset.$$

As a premise to the following definition of $E_c(n+1, k)$, we specify that, given a positive integer m and a set D, we call D^m the set $D \times \cdots \times D$ where D appears m times (when $m = 1$ of course $D^1 = D$), and a function whose domain is a subset of D^m is called a *function with m arguments*.

For each $k \in K(n)$ we define $E_c(n+1, k)$ as the set of the strings $(\varphi)(\varphi_1, \ldots, \varphi_m)$ such that:

- m is a positive integer;
- $\varphi, \varphi_1, \ldots, \varphi_m \in E(n, k)$;
- for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is a function with m arguments and $(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ is a member of its domain.

This means that for each $t \in E_c(n+1, k)$ there exist a positive integer m and $\varphi, \varphi_1, \ldots, \varphi_m \in E(n)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$. In the following lemma we'll show that $m, \varphi, \varphi_1, \ldots, \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 2.6 and assumption 2.1.4, they will not be quoted each time they are used.

Lemma 2.1.5. Let $t \in E_c(n+1, k)$ and suppose

- there exist a positive integer m and $\varphi, \varphi_1, \ldots, \varphi_m \in E(n)$: $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$.

- there exist a positive integer p and $\psi, \psi_1, \ldots, \psi_p \in E(n)$: $t = (\psi)(\psi_1, \ldots, \psi_p)$.

Then $p = m$, $\psi = \varphi$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

Proof.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = (\varphi)(\varphi_1, \varphi_2)$, if $m = 3$ then $t = (\varphi)(\varphi_1, \varphi_2, \varphi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. There are explicit occurrences of ',' only when $m > 1$. We indicate with q the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

In the same way, if we know p we can provide another 'explicit representation' of t. In fact if $p = 2$ then $t = (\psi)(\psi_1, \psi_2)$, if $p = 3$ then $t = (\psi)(\psi_1, \psi_2, \psi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. There are explicit occurrences of ',' only when $p > 1$. We indicate with r the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $p > 1$ we indicate with $r_1, \ldots, r_{p-1}$ the positions of the explicit occurrences of ','.

We have $d(t, q-1) = d(t, 1 + \ell(\varphi)) = d(t, 1+1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] =$ ')' then $d(t, q) = d(t, q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.
Else $t[q-1] = \varphi[\ell(\varphi)] \notin \{\text{'('}, \text{')'}\}$, so $d(t, q) = d(t, q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

Suppose $q < r$. Obviously $q > 1$, $q - 1 \geqslant 1$, $q - 1 \leqslant r - 2 = \ell(\psi)$; $\psi[q-1] = t[q] =$ ')'.
So $d(t, q) = d(t, 1 + (q-1)) = d(t, 2) + d(\psi, q-1) = 1 + d(\psi, q-1) \geqslant 2$.
This is a contradiction, because we have proved $d(t, q) = 1$. Thus $q \geqslant r$.

In the same way we can prove that $r \geqslant q$, so we have $r = q$.

Clearly $\ell(\psi) = r-2 = q-2 = \ell(\varphi)$, and for each $\alpha = 1 \ldots \ell(\varphi)$ $\varphi[\alpha] = t[\alpha+1] = \psi[\alpha]$. In other words $\psi = \varphi$.

Of course we have also $d(t, r) = d(t, q) = 1$, $d(t, r+2) = d(t, r) - 1 + 1 = 1$,
$d(t, q+2) = d(t, q) - 1 + 1 = 1$.

We still need to show that $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

First we examine the case where $m = 1$. We want to show that $p = 1$.

Suppose $p > 1$. In this situation we have

$$
\begin{aligned}
d(t, r_1 - 1) = d(t, r + 1 + (r_1 - 1 - (r+1))) = d(t, r + 1 + \ell(\psi_1)) = \\
= d(t, r+2) + d(\psi_1, \ell(\psi_1)) = 1 + d(\psi_1, \ell(\psi_1)).
\end{aligned}
$$

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$.
Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

Moreover we have to consider that

$$\begin{aligned}
\ell(\varphi_1) &= \ell(t) - 1 - (q+1) = \ell(t) - q - 2,\\
r_1 &\leqslant \ell(t) - 1,\\
r_1 - (q+1) &\leqslant \ell(t) - 1 - (q+1) = \ell(t) - q - 2 = \ell(\varphi_1),\\
r_1 &\geqslant q+2,\\
r_1 - (q+1) &\geqslant 1,\\
\varphi_1[r_1 - (q+1)] &= t[r_1] = \text{‘,’},\\
1 = d(t, r_1) &= d(t, q+2) + d(\varphi_1, r_1 - (q+1)) = 1 + d(\varphi_1, r_1 - (q+1)).
\end{aligned}$$

This causes $d(\varphi_1, r_1 - (q+1)) = 0$, but by assumption 2.1.4 we must have $d(\varphi_1, r_1 - (q+1)) \geqslant 1$. So it must be $p = 1$.

Of course

$$\ell(\psi_1) = \ell(t) - 1 - (r+1) = \ell(t) - r - 2 = \ell(t) - q - 2 = \ell(\varphi_1).$$

For each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[q+1+\alpha] = t[r+1+\alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let's discuss the case where $m > 1$.

First we want to prove that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Let's show that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$.

If $p = 1$ of course $m = 1$, so $p > 1$ holds.

We have that

$$d(t, q_1 - 1) = d(t, q+1+\ell(\varphi_1)) = d(t, q+1+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] = \text{‘)’}$ then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{‘(’}, \text{‘)’}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Suppose $q_1 < r_1$, we have

$$\begin{aligned}
\ell(\psi_1) &= r_1 - 1 - (r+1) = r_1 - r - 2,\\
q_1 - r - 1 &< r_1 - r - 1,\\
q_1 - r - 1 &\leqslant \ell(\psi_1),\\
q_1 &> q+1,\\
q_1 &> r+1,\\
q_1 - r - 1 &\geqslant 1,
\end{aligned}$$

and then

$$\begin{aligned}
1 = d(t, q_1) &= d(t, r+1+(q_1 - r - 1)) = d(t, r+2) + d(\psi_1, q_1 - r - 1) =\\
&= 1 + d(\psi_1, q_1 - r - 1).
\end{aligned}$$

So $d(\psi_1, q_1 - r - 1) = 0$. But since $\psi_1[q_1 - r - 1] = t[q_1] = \text{‘,’}$, by assumption 2.1.4 we must have $d(\psi_1, q_1 - r - 1) \geqslant 1$, so we have a contradiction.

Hence $q_1 \geqslant r_1$ and in the same way we can show that $r_1 \geqslant q_1$, therefore $r_1 = q_1$.

At this point we observe that

$$\ell(\varphi_1) = q_1 - 1 - (q+1) = q_1 - q - 2 = r_1 - r - 2 = \ell(\psi_1).$$

Moreover, for each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[q+1+\alpha] = t[r+1+\alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

We have proved that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$, and if $m = 2$ we have also shown that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Now suppose $m > 2$, let $i = 1 \dots m-2$, suppose we have proved $p > i$, $d(t, q_i) = 1$, $r_i = q_i$, $\psi_i = \varphi_i$, we want to show that $p > i+1, d(t, q_{i+1}) = 1, r_{i+1} = q_{i+1}, \psi_{i+1} = \varphi_{i+1}$.

First of all

$$\begin{aligned} d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = \\ = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i+1$. We have $i \leqslant m-2$, $i+2 \leqslant m$, $t[q_{i+1}] =$ ','. And we have also

$$\begin{aligned} \ell(\psi_p) &= \ell(t) - 1 - r_i, \\ q_{i+1} &\leqslant \ell(t) - 1, \\ q_{i+1} - r_i &\leqslant \ell(t) - 1 - r_i = \ell(\psi_p), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_p[q_{i+1} - r_i] &= t[q_{i+1}] = \text{‘,’}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) = \\ = 1 + d(\psi_p, q_{i+1} - r_i). \end{aligned}$$

So $d(\psi_p, q_{i+1} - r_i) = 0$ and this contradicts assumption 2.1.4. Therefore $p > i+1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$$\begin{aligned} \ell(\psi_{i+1}) &= r_{i+1} - 1 - r_i, \\ q_{i+1} &\leqslant r_{i+1} - 1, \\ q_{i+1} - r_i &\leqslant r_{i+1} - 1 - r_i = \ell(\psi_{i+1}), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_{i+1}[q_{i+1} - r_i] &= t[q_{i+1}] = \text{‘,’}, \end{aligned}$$

and

$$1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) =$$
$$= 1 + d(\psi_{i+1}, q_{i+1} - r_i).$$

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$ and this contradicts assumption 2.1.4. Therefore $q_{i+1} \geqslant r_{i+1}$.

In the same way we can prove that $q_{i+1} \leqslant r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved.

Moreover

$$\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1}),$$

and for each $\alpha = 1 \ldots \ell(\psi_{i+1})$

$$\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha].$$

We have proved that for each $i = 1 \ldots m - 1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

So $p \geqslant m$, and in the same way we could prove $m \geqslant p$, therefore $p = m$.

We have seen that $r_{m-1} = q_{m-1}$, it follows

$$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m),$$

and for each $\alpha = 1 \ldots \ell(\psi_m)$

$$\psi_m[\alpha] = t[r_{m-1} + \alpha] = t[q_{m-1} + \alpha] = \varphi_m[\alpha],$$

therefore $\psi_m = \varphi_m$.

So also in the case $m > 1$ it is shown that $p = m$ and for each $i = 1 \ldots m$ $\psi_i = \varphi_i$. ∎

For each $t = (\varphi)(\varphi_1, \ldots, \varphi_m) \in E_c(n + 1, k)$ we define

$$\begin{aligned} \#(k, t, \sigma)_{(n+1,k,c)} &= \#(k, \varphi, \sigma)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)), \\ V_f(t)_{(n+1,k,c)} &= V_f(\varphi) \cup V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m), \\ V_b(t)_{(n+1,k,c)} &= V_b(\varphi) \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m). \end{aligned}$$

For each $k \in K(n)$ we define $E_d(n + 1, k)$ as the set of the strings $(f)(\varphi_1, \ldots, \varphi_m)$ such that:

- f belongs to $\mathcal{F}$
- m is a positive integer;
- $\varphi_1, \ldots, \varphi_m \in E(n, k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ is true.

For instance, this means that if the 'logical conjunction' symbol '$\wedge$' belongs to $\mathcal{F}$, φ_1, φ_2 belong to $E(n, k)$ and for each $\sigma \in \Xi(k)$ both $\#(k, \varphi_1, \sigma)$ and $\#(k, \varphi_2, \sigma)$ are true or false, then $(\wedge)(\varphi_1, \varphi_2)$ belongs to $E_d(n + 1, k)$.

This implies that for each $t \in E_d(n+1,k)$ there are f in $\mathcal{F}$, a positive integer m and $\varphi_1, \ldots, \varphi_m \in E(n)$ such that $t = (f)(t_1, \ldots, t_m)$. We will now show that $f, m, \varphi_1, \ldots, \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 2.6 and assumption 2.1.4, they will not be quoted each time they are used.

LEMMA 2.1.6. Let $t \in E_d(n+1,k)$ and suppose

- there exist $f \in \mathcal{F}$, a positive integer m and $\varphi_1, \ldots, \varphi_m \in E(n)$: $t = (f)(\varphi_1, \ldots, \varphi_m)$.
- there exist $g \in \mathcal{F}$, a positive integer p and $\psi_1, \ldots, \psi_p \in E(n)$: $t = (g)(\psi_1, \ldots, \psi_p)$.

Then $g = f$, $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

Proof.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = (f)(\varphi_1, \varphi_2)$, if $m = 3$ then $t = (f)(\varphi_1, \varphi_2, \varphi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. There are explicit occurrences of ',' only when $m > 1$. The explicit occurrences of ')' are clearly in positions 3 and $\ell(t)$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

In the same way, if we know p we can provide another 'explicit representation' of t. In fact if $p = 2$ then $t = (g)(\psi_1, \psi_2)$, if $p = 3$ then $t = (g)(\psi_1, \psi_2, \psi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. There are explicit occurrences of ',' only when $p > 1$. The explicit occurrences of ')' are clearly in positions 3 and $\ell(t)$. If $p > 1$ we indicate with $r_1, \ldots, r_{p-1}$ the positions of the explicit occurrences of ','.

It is immediate to see that $g = t[2] = f$.

We still need to show that $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

First we examine the case where $m = 1$. We want to show that $p = 1$.

Suppose $p > 1$. In this situation we have

$$\begin{aligned} d(t, r_1 - 1) &= d(t, 4 + (r_1 - 1 - 4)) = d(t, 4 + \ell(\psi_1)) = \\ &= d(t, 4+1) + d(\psi_1, \ell(\psi_1)) = 1 + d(\psi_1, \ell(\psi_1)). \end{aligned}$$

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$. Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin \{$'(', ')'$\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

Moreover we have to consider that

$$\begin{aligned}
\ell(\varphi_1) &= \ell(t) - 1 - 4 = \ell(t) - 5,\\
r_1 &\leqslant \ell(t) - 1,\\
r_1 - 4 &\leqslant \ell(t) - 1 - 4 = \ell(t) - 5 = \ell(\varphi_1),\\
r_1 &\geqslant 4 + 1,\\
r_1 - 4 &\geqslant 1,\\
\varphi_1[r_1 - 4] &= t[r_1] = \text{`},\text{'},\\
1 &= d(t, r_1) = d(t, 4+1) + d(\varphi_1, r_1 - 4) = 1 + d(\varphi_1, r_1 - 4).
\end{aligned}$$

This causes $d(\varphi_1, r_1 - 4) = 0$, but by assumption 2.1.4 we must have $d(\varphi_1, r_1 - 4) \geqslant 1$. So it must be $p = 1$.

Of course

$$\ell(\psi_1) = \ell(t) - 1 - 4 = \ell(\varphi_1).$$

For each $\alpha = 1 \ldots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[4 + \alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let's discuss the case where $m > 1$.

First we want to prove that for each $i = 1 \ldots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Let's show that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$.

If $p = 1$ of course $m = 1$, so $p > 1$ holds.

We have that

$$d(t, q_1 - 1) = d(t, 4 + \ell(\varphi_1)) = d(t, 4+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] = $ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{`('}, \text{`)'}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Suppose $q_1 < r_1$, we have

$$\begin{aligned}
\ell(\psi_1) &= r_1 - 1 - 4 = r_1 - 5,\\
q_1 - 4 &< r_1 - 4,\\
q_1 - 4 &\leqslant \ell(\psi_1),\\
q_1 &> 4,\\
q_1 - 4 &\geqslant 1,
\end{aligned}$$

and then

$$\begin{aligned}
1 = d(t, q_1) = d(t, 4 + (q_1 - 4)) = d(t, 4+1) + d(\psi_1, q_1 - 4) =\\
= 1 + d(\psi_1, q_1 - 4).
\end{aligned}$$

So $d(\psi_1, q_1 - 4) = 0$. But since $\psi_1[q_1 - 4] = t[q_1] = $ ',', by assumption 2.1.4 we must have $d(\psi_1, q_1 - 4) \geqslant 1$, so we have a contradiction.

Hence $q_1 \geqslant r_1$ and in the same way we can show that $r_1 \geqslant q_1$, therefore $r_1 = q_1$.

At this point we observe that $\ell(\varphi_1) = q_1 - 1 - 4 = r_1 - 1 - 4 = \ell(\psi_1)$.

Moreover, for each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[4 + \alpha] = \psi_1[\alpha]$.
Therefore $\psi_1 = \varphi_1$.

We have proved that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$, and if $m = 2$ we have also shown that for each $i = 1 \dots m - 1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Now suppose $m > 2$, let $i = 1 \dots m - 2$, suppose we have proved $p > i$, $d(t, q_i) = 1$, $r_i = q_i$, $\psi_i = \varphi_i$, we want to show that $p > i + 1, d(t, q_{i+1}) = 1, r_{i+1} = q_{i+1}, \psi_{i+1} = \varphi_{i+1}$.

First of all

$$\begin{aligned} d(t, q_{i+1} - 1) &= d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = \\ &= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i + 1$. We have $i \leqslant m - 2$, $i + 2 \leqslant m$, $t[q_{i+1}] =$ ','. And we have also

$$\begin{aligned} \ell(\psi_p) &= \ell(t) - 1 - r_i, \\ q_{i+1} &\leqslant \ell(t) - 1, \\ q_{i+1} - r_i &\leqslant \ell(t) - 1 - r_i = \ell(\psi_p), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_p[q_{i+1} - r_i] &= t[q_{i+1}] = \text{','}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, q_{i+1}) &= d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) = \\ &= 1 + d(\psi_p, q_{i+1} - r_i). \end{aligned}$$

So $d(\psi_p, q_{i+1} - r_i) = 0$ and this contradicts assumption 2.1.4. Therefore $p > i + 1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$$\begin{aligned} \ell(\psi_{i+1}) &= r_{i+1} - 1 - r_i, \\ q_{i+1} &\leqslant r_{i+1} - 1, \\ q_{i+1} - r_i &\leqslant r_{i+1} - 1 - r_i = \ell(\psi_{i+1}), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_{i+1}[q_{i+1} - r_i] &= t[q_{i+1}] = \text{','}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, q_{i+1}) &= d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) = \\ &= 1 + d(\psi_{i+1}, q_{i+1} - r_i). \end{aligned}$$

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$ and this contradicts assumption 2.1.4. Therefore $q_{i+1} \geqslant r_{i+1}$.

In the same way we can prove that $q_{i+1} \leqslant r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved.

Moreover

$$\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1}),$$

and for each $\alpha = 1 \dots \ell(\psi_{i+1})$

$$\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha].$$

We have proved that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

So $p \geqslant m$, and in the same way we could prove $m \geqslant p$, therefore $p = m$.

We have seen that $r_{m-1} = q_{m-1}$, it follows

$$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m),$$

and for each $\alpha = 1 \dots \ell(\psi_m)$

$$\psi_m[\alpha] = t[r_{m-1} + \alpha] = t[q_{m-1} + \alpha] = \varphi_m[\alpha],$$

therefore $\psi_m = \varphi_m$.

So also in the case $m > 1$ it is shown that $p = m$ and for each $i = 1 \dots m$ $\psi_i = \varphi_i$. ∎

For each $t = (f)(\varphi_1, \dots, \varphi_m) \in E_d(n+1, k)$ we define

$$\begin{aligned} \#(k, t, \sigma)_{(n+1,k,d)} &= P_f(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)), \\ V_f(t)_{(n+1,k,d)} &= V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m), \\ V_b(t)_{(n+1,k,d)} &= V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m). \end{aligned}$$

Let $k \in K(n)$, m a positive integer, x a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \to x_i \neq x_j$, φ a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$, and finally let $\phi \in E(n)$. We write

$$\mathcal{E}(n, k, m, x, \varphi, \phi)$$

to indicate the following condition (where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$):

- $\varphi_1 \in E_s(n, k)$;
- if $m > 1$ then for each $i = 1 \dots m-1$ $k'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, k'_i)$;
- $k'_m \in K(n) \wedge \phi \in E(n, k'_m)$.

For each $k \in K(n)$ we define $E_e(n+1, k)$ as the set of the strings

$$\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$$

such that:

- m is a positive integer;

- x is a function whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1\ldots m$ $i \neq j \rightarrow x_i \neq x_j$;
- φ is a function whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\mathcal{E}(n,k,m,x,\varphi,\phi)$.

This implies that for each $t \in E_e(n+1,k)$ there exist a positive integer m, a function x whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $x_i \in \mathcal{V}$, a function φ whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $\varphi_i \in E(n)$, and $\phi \in E(n)$ such that $t = \{\}(x_1 : \varphi_1,\ldots,x_m : \varphi_m,\phi)$. We will now show that m,x,φ,ϕ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 2.6 and assumption 2.1.4, they will not be quoted each time they are used.

Lemma 2.1.7. Let $t \in E_e(n+1,k)$ and suppose

- there exist a positive integer m, a function x whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $x_i \in \mathcal{V}$, a function φ whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $\varphi_i \in E(n)$, and $\phi \in E(n)$ such that $t = \{\}(x_1 : \varphi_1,\ldots,x_m : \varphi_m,\phi)$;
- there exist a positive integer p, a function y whose domain is $\{1,\ldots,p\}$ such that for each $i = 1\ldots p$ $y_i \in \mathcal{V}$, a function ψ whose domain is $\{1,\ldots,p\}$ such that for each $i = 1\ldots p$ $\psi_i \in E(n)$, and $\vartheta \in E(n)$ such that $t = \{\}(y_1 : \psi_1,\ldots,y_p : \psi_p,\vartheta)$;

Then $p = m, y = x, \psi = \varphi$ and $\vartheta = \phi$.

Proof.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = \{\}(x_1 : \varphi_1, x_2 : \varphi_2,\phi)$, if $m = 3$ then $t = \{\}(x_1 : \varphi_1, x_2 : \varphi_2, x_3 : \varphi_3,\phi)$, and so on. In this explicit representation of t we can see explict occurrences of the symbols ',' and ':'. We indicate with $q_1,\ldots,q_m$ the positions of the explicit occurrences of ':' and with $r_1 \ldots r_m$ the positions of the explicit occurrences of ','.

In the same way, if we know p we can provide another 'explicit representation' of t. In fact if $p = 2$ then $t = \{\}(y_1 : \psi_1, y_2 : \psi_2,\vartheta)$, if $p = 3$ then $t = \{\}(y_1 : \psi_1, y_2 : \psi_2, y_3 : \psi_3,\vartheta)$, and so on. In this explicit representation of t we can see explict occurrences of the symbols ',' and ':'. We indicate with $q'_1,\ldots,q'_m$ the positions of the explicit occurrences of ':' and with $r'_1 \ldots r'_m$ the positions of the explicit occurrences of ','.

We want to show that for each $i = 1\ldots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t,r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

The first step is to show that $y_1 = x_1, q'_1 = q_1, d(t,r_1) = 1, r'_1 = r_1, \psi_1 = \varphi_1$.

Of course $y_1 = t[3] = x_1$, $q'_1 = 4 = q_1$. Moreover

$$d(t, r_1 - 1) = d(t, q_1 + (r_1 - 1 - q_1)) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = \\ = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Now suppose $r_1 < r'_1$. This would mean that

$$\begin{aligned} \ell(\psi_1) &= r'_1 - 1 - q'_1, \\ r_1 - q'_1 &\leqslant r'_1 - 1 - q'_1 = \ell(\psi_1), \\ r_1 - q'_1 &= r_1 - q_1 \geqslant 1, \\ \psi_1[r_1 - q'_1] &= t[r_1] = \text{`,'}, \end{aligned}$$

and

$$1 = d(t, r_1) = d(t, q'_1 + (r_1 - q'_1)) = d(t, q'_1 + 1) + d(\psi_1, r_1 - q'_1) = 1 + d(\psi_1, r_1 - q'_1).$$

So $d(\psi_1, r_1 - q'_1) = 0$ and this contradicts assumption 2.1.4. Hence $r_1 \geqslant r'_1$, and in the same way we can show that $r'_1 \geqslant r_1$, therefore $r_1 = r'_1$.

At this point we observe that $\ell(\varphi_1) = r_1 - 1 - q_1 = \ell(\psi_1)$.
Moreover, for each $\alpha = 1 \ldots \ell(\psi_1)$ $\psi_1[\alpha] = t[q'_1 + \alpha] = t[q_1 + \alpha] = \varphi_1[\alpha]$, hence $\psi_1 = \varphi_1$.

We have proved that $y_1 = x_1, q'_1 = q_1, d(t, r_1) = 1, r'_1 = r_1, \psi_1 = \varphi_1$. As a consequence, if $m = 1$ we have proved that for each $i = 1 \ldots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

Consider the case where $m > 1$. Let $i = 1 \ldots m - 1$, we suppose

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i,$$

and want to show that

$$p \geqslant i + 1, y_{i+1} = x_{i+1}, q'_{i+1} = q_{i+1}, d(t, r_{i+1}) = 1, r'_{i+1} = r_{i+1}, \psi_{i+1} = \varphi_{i+1}.$$

We can immediately show that $d(t, r_{i+1}) = 1$. In fact $d(t, q_{i+1} + 1) = d(t, r_i) = 1$,

$$d(t, r_{i+1} - 1) = d(t, q_{i+1} + (r_{i+1} - 1 - q_{i+1})) = d(t, q_{i+1} + \ell(\varphi_{i+1})) = \\ = d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})).$$

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i$. In this case

$$\begin{aligned}
\ell(\vartheta) &= \ell(t) - 1 - r'_i,\\
r_{i+1} - r'_i &\leqslant \ell(t) - 1 - r'_i = \ell(\vartheta),\\
r_{i+1} - r'_i &= r_{i+1} - r_i \geqslant 1,\\
\vartheta[r_{i+1} - r'_i] &= t[r_{i+1}] = \text{‘,’},
\end{aligned}$$

and

$$\begin{aligned}
1 &= d(t, r_{i+1}) = d(t, r'_i + (r_{i+1} - r'_i)) = d(t, r'_i + 1) + d(\vartheta, r_{i+1} - r'_i) =\\
&= 1 + d(\vartheta, r_{i+1} - r'_i).
\end{aligned}$$

So $d(\vartheta, r_{i+1} - r'_i) = 0$, and this contradicts assumption 2.1.4. Therefore $p \geqslant i + 1$. It follows immediately that $y_{i+1} = t[r'_i + 1] = t[r_i + 1] = x_{i+1}$ and $q'_{i+1} = r'_i + 2 = q_{i+1}$.

Now we suppose $r_{i+1} < r'_{i+1}$. This would mean that

$$\begin{aligned}
\ell(\psi_{i+1}) &= r'_{i+1} - 1 - q'_{i+1},\\
r_{i+1} - q'_{i+1} &\leqslant r'_{i+1} - 1 - q'_{i+1} = \ell(\psi_{i+1}),\\
r_{i+1} - q'_{i+1} &= r_{i+1} - q_{i+1} \geqslant 1,\\
\psi_{i+1}[r_{i+1} - q'_{i+1}] &= t[r_{i+1}] = \text{‘,’},
\end{aligned}$$

and

$$\begin{aligned}
1 &= d(t, r_{i+1}) = d(t, q'_{i+1} + (r_{i+1} - q'_{i+1})) = d(t, q'_{i+1} + 1) + d(\psi_{i+1}, r_{i+1} - q'_{i+1}) =\\
&= 1 + d(\psi_{i+1}, r_{i+1} - q'_{i+1}).
\end{aligned}$$

So $d(\psi_{i+1}, r_{i+1} - q'_{i+1}) = 0$ and this contradicts assumption 2.1.4. Hence $r_{i+1} \geqslant r'_{i+1}$. In the same way we can show that $r_{i+1} \leqslant r'_{i+1}$, therefore $r_{i+1} = r'_{i+1}$.

At this point we observe that $\ell(\varphi_{i+1}) = r_{i+1} - 1 - q_{i+1} = \ell(\psi_{i+1})$. Furthermore, for each $\alpha = 1 \ldots \ell(\varphi_{i+1})$ $\psi_{i+1}[\alpha] = t[q'_{i+1} + \alpha] = t[q_{i+1} + \alpha] = \varphi_{i+1}[\alpha]$, hence $\psi_{i+1} = \varphi_{i+1}$.

It is shown that for each $i = 1 \ldots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

So $p \geqslant m$. In the same way we could prove that $m \geqslant p$, so $p = m$. At this stage we have shown that $y = x$ and $\psi = \varphi$, we just need a final step, which is proving that $\vartheta = \phi$. This clearly holds because of

$$\ell(\vartheta) = \ell(t) - 1 - r'_p = \ell(t) - 1 - r_m = \ell(\phi),$$

and for each $\alpha = 1 \ldots \ell(\vartheta)$

$$\vartheta[\alpha] = t[r'_p + \alpha] = t[r_m + \alpha] = \phi[\alpha].$$

■

For every $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \in E_e(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,e)} = \{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$.

Notice that the set $\{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}$ is specified using a standard mathematical notation. We could specify it using a notation closer to the one of our formulas, in this case it could have been written as $\{\}(\sigma'_m \in \Xi(k'_m) : \sigma \sqsubseteq \sigma'_m, \#(k'_m, \phi, \sigma'_m))$.

It might still be a bit unclear what is the intended meaning of the expression

$$\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi).$$

This is the same meaning that the expression

$$\{\phi|\ x_1 \in \varphi_1, \ldots, x_m \in \varphi_m\}$$

is intended to have when used in most mathematics books.

If $m = 1$ we also define

$$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}),$$
$$V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi).$$

If $m > 1$ we define

$$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup \\ \cup (V_f(\phi) - \{x_1, \ldots, x_m\}),$$
$$V_b(t)_{(n+1,k,e)} = \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi).$$

We have terminated the definition of the 'new sets' (of expressions bound to context k) and the related work, we are now ready to define $E(n+1, k)$.

If $k \in K(n)^+$ we define $E'_a(n+1, k) = E_a(n+1, k)$, $E'_b(n+1, k) = E_b(n+1, k)$, else we define $E'_a(n+1, k) = \emptyset$, $E'_b(n+1, k) = \emptyset$.

If $k \in K(n)$ we define $E'(n, k) = E(n, k)$, $E'_c(n+1, k) = E_c(n+1, k)$, $E'_d(n+1, k) = E_d(n+1, k)$, $E'_e(n+1, k) = E_e(n+1, k)$,
else $E'(n, k) = \emptyset$, $E'_c(n+1, k) = \emptyset$, $E'_d(n+1, k) = \emptyset$, $E'_e(n+1, k) = \emptyset$.

Finally we define

$$E(n+1, k) = E'(n, k) \cup E'_a(n+1, k) \cup E'_b(n+1, k) \cup E'_c(n+1, k) \cup E'_d(n+1, k) \cup E'_e(n+1, k).$$

For every $k \in K(n+1)$, $t \in E(n+1, k)$ and $\sigma \in \Xi(k)$ we need that $\#(k, t, \sigma)$ is defined. But we also need that the definition is such that for each $k \in K(n+1)$, $w \in \{a, b, c, d, e\}$, $t \in E'_w(n+1, k)$ and $\sigma \in \Xi(k)$ $\#(k, t, \sigma)_{(n+1,k,w)} = \#(k, t, \sigma)$.

Given $k \in K(n+1)$, $t \in E(n+1,k)$ and $\sigma \in \Xi(k)$ there are three possibilities.

1. t is in $E'(n,k)$: then $\#(k,t,\sigma)$ is already defined; if t is in one or more of the sets $E'_w(n+1,k)$ then for each w we need to verify that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$.
2. t is not in $E'(n,k)$ and t is in just one of the sets $E'_w(n+1,k)$: then we just define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$.
3. t is not in $E'(n,k)$ and t is in more than one of the sets $E'_w(n+1,k)$: in this case we choose $\bar{w}$ such that $t \in E'_{\bar{w}}(n+1,k)$ and define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,\bar{w})}$. We also need to verify that for each w such that $t \in E'_w(n+1,k)$ $\#(k,t,\sigma)_{(n+1,k,w)} = \#(k,t,\sigma)_{(n+1,k,\bar{w})}$.

By point 1. we are required to verify that for each $k \in K(n+1)$, $w \in \{a,b,c,d,e\}$, $t \in E'(n,k) \cap E'_w(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$.

By point 3. we are required to verify that for each $k \in K(n+1)$, $w_1, w_2 \in \{a,b,c,d,e\}$: $w_1 \neq w_2$, $t \in E'_{w_1}(n+1,k) \cap E'_{w_2}(n+1,k)$ and $\sigma \in \Xi(k)$

$$\#(k,t,\sigma)_{(n+1,k,w_1)} = \#(k,t,\sigma)_{(n+1,k,w_2)}.$$

It's easy to see that if these properties are verified then we can state that for each $k \in K(n+1)$, $w \in \{a,b,c,d,e\}$, $t \in E'_w(n+1,k)$, $\sigma \in \Xi(k)$ $\#(k,t,\sigma)_{(n+1,k,w)} = \#(k,t,\sigma)$.

With respect to the definitions of $V_b(t)$ and $V_f(t)$ we can make a similar argument. For every $k \in K(n+1)$ and $t \in E(n+1,k)$ we need that $V_b(t)$ and $V_f(t)$ are defined. But we also need that the definition is such that for each $k \in K(n+1)$, $w \in \{a,b,c,d,e\}$ and $t \in E'_w(n+1,k)$ $V_b(t)_{(n+1,k,w)} = V_b(t)$ and $V_f(t)_{(n+1,k,w)} = V_f(t)$.

Given $t \in E(n+1)$ there are three possibilities.

1. t is in $E(n)$: then $V_b(t)$ and $V_f(t)$ are already defined; if t is in one or more of the sets $E'_w(n+1,k)$ then for each k and w we need to verify that $V_b(t) = V_b(t)_{(n+1,k,w)}$ and $V_f(t) = V_f(t)_{(n+1,k,w)}$.
2. t is not in $E(n)$ and t is in just one of the sets $E'_w(n+1,k)$: then we just define $V_b(t) = V_b(t)_{(n+1,k,w)}$ and $V_f(t) = V_f(t)_{(n+1,k,w)}$.
3. t is not in $E(n)$ and there are more than one $k \in K(n+1)$ and $w \in \{a,b,c,d,e\}$ such that t is in $E'_w(n+1,k)$. In this case we arbitrarily choose $\bar{k}$ and $\bar{w}$ such that t is in $E'_{\bar{w}}(n+1,\bar{k})$ and define $V_b(t) = V_b(t)_{(n+1,\bar{k},\bar{w})}$; $V_f(t) = V_f(t)_{(n+1,\bar{k},\bar{w})}$. Here we need to verify that for each k,w such that t is in $E'_w(n+1,k)$ $V_b(t)_{(n+1,k,w)} = V_b(t)_{(n+1,\bar{k},\bar{w})}$, $V_f(t)_{(n+1,k,w)} = V_f(t)_{(n+1,\bar{k},\bar{w})}$.

By point 1. we are required to verify that for each $k \in K(n+1)$, $w \in \{a,b,c,d,e\}$, $t \in E(n) \cap E'_w(n+1,k)$ $V_b(t) = V_b(t)_{(n+1,k,w)}$ and $V_f(t) = V_f(t)_{(n+1,k,w)}$.

By point 3. we are required to verify that for each $k_1, k_2 \in K(n+1)$, $w_1, w_2 \in \{a, b, c, d, e\}$, $t \in E'_{w_1}(n+1, k_1) \cap E'_{w_2}(n+1, k_2)$ such that $t \notin E(n)$ we have

$$V_b(t)_{(n+1,k_1,w_1)} = V_b(t)_{(n+1,k_2,w_2)},$$
$$V_f(t)_{(n+1,k_1,w_1)} = V_f(t)_{(n+1,k_2,w_2)}.$$

It's easy to see that if these properties are verified then we can state that for each $k \in K(n+1)$, $w \in \{a, b, c, d, e\}$ and $t \in E'_w(n+1, k)$ $V_b(t)_{(n+1,k,w)} = V_b(t)$ and $V_f(t)_{(n+1,k,w)} = V_f(t)$.

We now have to perform the required verifications. These verifications require a further set of assumptions. We'll list those assumptions, and also significant consequences to them and other results that will in turn be used in our verification process.

Assumption 2.1.8. if $n > 1$ then $K(n-1) \subseteq K(n)$.

Assumption 2.1.9. Let $k = (x, \varphi), h = (y, \psi) \in K(n)$ such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. Let $t \in E(n, k) \cap E(n, h)$. Let $\sigma = (x, s) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to s_i = r_j$. Then $\#(k, t, \sigma) = \#(h, t, \rho)$.

∎

The next assumption has a central role in our verification process.

Assumption 2.1.10. For each $k \in K(n)$, $t \in E(n,k)$ one and only one of these five alternative situations is verified:

a.

$$t \in \mathcal{C},\ \forall \sigma \in \Xi(k)\ \#(k,t,\sigma) = \#(t),\ V_f(t) = \emptyset,\ V_b(t) = \emptyset.$$

b.

$$\begin{array}{l} n > 1, \\ \text{if we set } k = (x,\varphi) \text{ then } \exists i \in dom(k) : (t = x_i,\ \forall \sigma = (x,s) \in \Xi(k)\ \#(k,t,\sigma) = s_i), \\ V_f(t) = \{t\},\ V_b(t) = \emptyset. \end{array}$$

c.

$$\begin{array}{l} n > 1, \\ \exists h \in K(n-1) : h \sqsubseteq k, \exists m \text{ positive integer }, \varphi, \varphi_1, \dots \varphi_m \in E(n-1,h) : \\ \qquad t = (\varphi)(\varphi_1, \dots, \varphi_m),\ t \in E(n,h), \\ \qquad \forall \rho \in \Xi(h)\ (\ \#(h,\varphi,\rho) \text{ is a function with } m \text{ arguments}, \\ \qquad\qquad (\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho)) \text{ is a member of the domain of } \#(h,\varphi,\rho), \\ \qquad\qquad \#(h,t,\rho) = \#(h,\varphi,\rho)(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho))\ ), \\ \qquad V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m), \\ \qquad V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m), \\ \qquad \forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma \text{ it results } \#(k,t,\sigma) = \#(h,t,\rho). \end{array}$$

d.

$$\begin{array}{l} n > 1, \\ \exists h \in K(n-1) : h \sqsubseteq k, \exists f \in \mathcal{F},\ m \text{ positive integer }, \varphi_1, \dots \varphi_m \in E(n-1,h) : \\ \qquad t = (f)(\varphi_1, \dots, \varphi_m),\ t \in E(n,h), \\ \qquad \forall \rho \in \Xi(h)\ (\ A_f(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho)), \\ \qquad\qquad \#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho))\ ), \\ \qquad V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m), \\ \qquad V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m), \\ \qquad \forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma \text{ it results } \#(k,t,\sigma) = \#(h,t,\rho). \end{array}$$

e.

$n > 1$,

there exist

$h \in K(n-1) : h \sqsubseteq k$,

a positive integer m,

a function x whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$

$x_i \in \mathcal{V} - var(h)$, and for each $i, j = 1 \dots m$ $i \neq j \to x_i \neq x_j$,

a function φ whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$

$\varphi_i \in E(n-1)$,

$\phi \in E(n-1)$

such that

$\mathcal{E}(n-1, h, m, x, \varphi, \phi)$,

$t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$, $t \in E(n, h)$,

for each $\rho \in \Xi(h)$ $\#(h, t, \rho) = \{\#(h'_m, \phi, \rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}$

(where $h'_1 = h + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$

$h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$),

if $m = 1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\})$, $V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,

if $m > 1$

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \dots \cup (V_f(\varphi_m) - \{x_1, \dots, x_{m-1}\}) \cup$

$\cup (V_f(\phi) - \{x_1, \dots, x_m\})$,

$V_b(t) = \{x_1, \dots, x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi)$,

$\forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma$ it results $\#(k, t, \sigma) = \#(h, t, \rho)$.

Assumption 2.1.11. Let $n > 1$, $k \in K(n)$, $h \in \mathcal{R}(k) : h \neq k$. Then $h \in K(n-1)$, for each $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)}$ then $\rho \in \Xi(h)$.

Assumption 2.1.12. If $n > 1$ then for each $g \in K(n-1)$ $E(n-1, g) \subseteq E(n, g)$.

Lemma 2.1.13. Suppose $h, k \in K(n)$, $y \in \mathcal{V} - var(h)$, $\phi \in E_s(n, h)$, $k = h + (y, \phi)$. Moreover let $\rho \in \Xi(h)$, $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$. Then there is $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$.

Proof.

We can apply our assumption 2.1.2 and get

$$(n > 1) \wedge \exists g \in K(n-1), z \in \mathcal{V} - var(g), \psi \in E_s(n-1, g) :$$
$$k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\ \sigma \in \Xi(g), s \in \#(g, \psi, \sigma)\}$$

So $h + (y, \phi) = k = g + (z, \psi)$ and by lemma 2.1 $h = g$, $y = z$, $\phi = \psi$.

Therefore $\Xi(k) = \{\rho' + (y, s) |\, \rho' \in \Xi(h), s \in \#(h, \phi, \rho')\}$.

Hence there exist $\rho' \in \Xi(h),\ s \in \#(h, \phi, \rho')$ such that $\sigma = \rho' + (y, s)$.

Now $dom(\rho) = dom(h) = dom(\rho')$, and since both $\rho, \rho' \in \mathcal{R}(\sigma)$ we have

$$\rho = \sigma_{/dom(\rho)} = \sigma_{/dom(\rho')} = \rho'.$$

Therefore there is $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$. ■

LEMMA 2.1.14. Suppose $h, k \in K(n),\ y \in \mathcal{V} - var(h),\ \phi \in E_s(n, h),\ k = h + (y, \phi)$. Moreover let $\rho \in \Xi(h)$ and $\sigma = \rho + (y, s)$ with $s \in \#(h, \phi, \rho)$. Then $\sigma \in \Xi(k)$, and clearly $\rho \sqsubseteq \sigma$.

Proof.

We can apply our assumption 2.1.2 and get

$$(n > 1) \wedge \exists g \in K(n-1), z \in \mathcal{V} - var(g), \psi \in E_s(n-1, g) :$$
$$k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\, \sigma \in \Xi(g), s \in \#(g, \psi, \sigma)\}$$

So $h + (y, \phi) = k = g + (z, \psi)$ and by lemma 2.1 $h = g,\ y = z,\ \phi = \psi$.

Therefore $\Xi(k) = \{\rho' + (y, s) |\, \rho' \in \Xi(h), s \in \#(h, \phi, \rho')\}$.

It follows immediately that $\sigma \in \Xi(k)$, and clearly $\rho \sqsubseteq \sigma$. ■

LEMMA 2.1.15. Let $g = (y, \vartheta),\ h = (z, \psi) \in K(n)$; m a positive integer; x a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in (\mathcal{V} - var(g)) \cap (\mathcal{V} - var(h))$, and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$; φ a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$; $\phi \in E(n)$. Let also $\mathcal{E}(n, g, m, x, \varphi, \phi),\ \mathcal{E}(n, h, m, x, \varphi, \phi)$.

Moreover we suppose that for each $i \in dom(g),\ j \in dom(h),\ y_i = z_j \to \vartheta_i = \psi_j$. Let also $\rho = (y, r) \in \Xi(g),\ \sigma = (z, u) \in \Xi(h)$ be such that for each $i \in dom(\rho),\ j \in dom(\sigma)$, $y_i = z_j \to r_i = u_j$. If as usual we define

- $g'_1 = g + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$ $g'_{i+1} = g'_i + (x_{i+1}, \varphi_{i+1})$,
- $h'_1 = h + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$ $h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$,

then we have

$$\{\#(h'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(h'_m), \sigma \sqsubseteq \sigma'_m\} = \{\#(g'_m, \phi, \rho'_m) |\ \rho'_m \in \Xi(g'_m), \rho \sqsubseteq \rho'_m\}.$$

Proof.

By assumption 2.1.1 $h = \epsilon$ or there exists a positive integer p such that $dom(h) = \{1, \ldots, p\}$. In the case where $h = \epsilon$ we define $p = 0$. At this point we can notice that for each $i = 1 \ldots m$ $dom(h'_i) = \{1, \ldots, p + i\}$. In fact $dom(h'_1) = \{1, \ldots, p + 1\}$. If $m > 1$ we need an inductive step. Let $i = 1 \ldots m - 1$, suppose $dom(h'_i) = \{1, \ldots, p + i\}$. Then $dom(h'_{i+1}) = \{1, \ldots, p + i + 1\}$.

Let $u \in \{\#(h'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(h'_m), \sigma \sqsubseteq \sigma'_m\}$, we want to show that

$$u \in \{\#(g'_m, \phi, \rho'_m) |\ \rho'_m \in \Xi(g'_m), \rho \sqsubseteq \rho'_m\}.$$

There exists $\sigma'_m \in \Xi(h'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ and $u = \#(h'_m, \phi, \sigma'_m)$.

If $m > 1$ then for each $i = 1 \dots m-1$, since $dom(\sigma'_m) = dom(h'_m) = \{1, \dots, p+m\}$, we can define $\sigma'_i = (\sigma'_m)_{/dom(h'_i)}$.

We also define $h'_0 = h,\ \sigma'_0 = \sigma$.

We can prove that for each $i = 1 \dots m$ $\sigma'_i \in \Xi(h'_i)$ and there is $s_i \in \#(h'_{i-1}, \varphi_i, \sigma'_{i-1})$ such that $\sigma'_i = \sigma'_{i-1} + (x_i, s_i)$.

We'll prove this by induction on i. Let us perform the initial step.

If $m = 1$ then $\sigma'_1 = \sigma'_m \in \Xi(h'_1)$. Otherwise $\sigma'_1 = (\sigma'_m)_{/dom(h'_1)}$. Since $h'_1 \in \mathcal{R}(h'_m)$ we can apply assumption 2.1.11 and determine that $\sigma'_1 \in \Xi(h'_1)$.

At this point we need to show that $\sigma'_0 \in \mathcal{R}(\sigma'_1)$. We have that $\sigma \sqsubseteq \sigma'_m$.
If $m = 1$ this means precisely that $\sigma'_0 \in \mathcal{R}(\sigma'_1)$.
Otherwise there exists $C \in \mathcal{D}$ such that $C \subseteq dom(\sigma'_m)$ and $\sigma = (\sigma'_m)_{/C}$. We have $C = dom(\sigma) = dom(h) \subseteq dom(h'_1) = dom(\sigma'_1)$. Suppose $\sigma'_m = (z'_m, s'_m)$, then

$$\begin{aligned}(\sigma'_1)_{/C} &= ((\sigma'_m)_{/dom(h'_1)})_{/C} = ((z'_m)_{/dom(h'_1)}, (s'_m)_{/dom(h'_1)})_{/C} = \\ &= (((z'_m)_{/dom(h'_1)})_{/C}, ((s'_m)_{/dom(h'_1)})_{/C}) = ((z'_m)_{/C}, (s'_m)_{/C}) = (\sigma'_m)_{/C} = \sigma.\end{aligned}$$

Therefore $\sigma'_0 \in \mathcal{R}(\sigma'_1)$.

We observe that $h, h'_1 \in K(n),\ x_1 \in \mathcal{V} - var(h),\ \varphi_1 \in E_s(n, h),\ h'_1 = h + (x_1, \varphi_1)$, and also $\sigma'_0 \in \Xi(h),\ \sigma'_1 \in \Xi(h'_1),\ \sigma'_0 \sqsubseteq \sigma'_1$ as already seen. By lemma 2.1.13 there is $s_1 \in \#(h'_0, \varphi_1, \sigma'_0)$ such that $\sigma'_1 = \sigma'_0 + (x_1, s_1)$.

If $m > 1$ we need an inductive step. Let $i = 1 \dots m-1$. We suppose $\sigma'_i \in \Xi(h'_i)$ and there is $s_i \in \#(h'_{i-1}, \varphi_i, \sigma'_{i-1})$ such that $\sigma'_i = \sigma'_{i-1} + (x_i, s_i)$.

If $i+1 = m$ then $\sigma'_{i+1} = \sigma'_m \in \Xi(h'_{i+1})$. Otherwise $\sigma'_{i+1} = (\sigma'_m)_{/dom(h'_{i+1})}$. Since $h'_{i+1} \in \mathcal{R}(h'_m)$ we can apply assumption 2.1.11 and determine that $\sigma'_{i+1} \in \Xi(h'_{i+1})$.

At this point we need to show that $\sigma'_i \in \mathcal{R}(\sigma'_{i+1})$. Consider that $dom(\sigma'_{i+1}) = dom(h'_{i+1}) = \{1, \dots, p+i+1\}$. We have

$$\begin{aligned}(\sigma'_{i+1})_{/\{1,\dots,p+i\}} &= ((\sigma'_m)_{/dom(h'_{i+1})})_{/\{1,\dots,p+i\}} = \\ &= ((z'_m)_{/dom(h'_{i+1})}, (s'_m)_{/dom(h'_{i+1})})_{/\{1,\dots,p+i\}} = \\ &= (((z'_m)_{/dom(h'_{i+1})})_{/\{1,\dots,p+i\}}, ((s'_m)_{/dom(h'_{i+1})})_{/\{1,\dots,p+i\}}) = \\ &= ((z'_m)_{/\{1,\dots,p+i\}}, (s'_m)_{/\{1,\dots,p+i\}}) = (\sigma'_m)_{/\{1,\dots,p+i\}} = \sigma'_i.\end{aligned}$$

This proves $\sigma'_i \in \mathcal{R}(\sigma'_{i+1})$.

We then observe that $h'_i, h'_{i+1} \in K(n),\ x_{i+1} \in \mathcal{V} - var(h'_i),\ \varphi_{i+1} \in E_s(n, h'_i),\ h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$, and also $\sigma'_i \in \Xi(h'_i),\ \sigma'_{i+1} \in \Xi(h'_{i+1}),\ \sigma'_i \sqsubseteq \sigma'_{i+1}$ as already seen. By lemma 2.1.13 there is $s_{i+1} \in \#(h'_i, \varphi_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s_{i+1})$.

We have proved that for each $i = 1 \dots m$ $\sigma'_i \in \Xi(h'_i)$ and there is $s_i \in \#(h'_{i-1}, \varphi_i, \sigma'_{i-1})$ such that $\sigma'_i = \sigma'_{i-1} + (x_i, s_i)$.

We now define $\rho'_1 = \rho + (x_1, s_1)$, and, if $m > 1$, for each $i = 1 \dots m-1$ $\rho'_{i+1} = \rho'_i + (x_{i+1}, s_{i+1})$.

By assumption 2.1.1 $g = \epsilon$ or there exists a positive integer q such that $dom(g) = \{1, \dots, q\}$. In the case where $g = \epsilon$ we define $q = 0$.
For each $i = 1 \dots m$ we define y'_i, ϑ'_i, r'_i as functions whose domain is $\{1, \dots, q+i\}$ such that

- for each $j = 1 \dots q$ $y'_i(j) = y(j),\ \vartheta'_i(j) = \vartheta(j),\ r'_i(j) = r(j)$;
- for each $j = 1 \dots i$ $y'_i(q+j) = x_j,\ \vartheta'_i(q+j) = \varphi_j,\ r'_i(q+j) = s_j$.

For each $i = 1 \dots m$ we also define z'_i, ψ'_i, u'_i as functions whose domain is $\{1, \dots, p+i\}$ such that

- for each $j = 1 \dots p$ $z'_i(j) = z(j),\ \psi'_i(j) = \psi(j),\ u'_i(j) = u(j)$;
- for each $j = 1 \dots i$ $z'_i(p+j) = x_j,\ \psi'_i(p+j) = \varphi_j,\ u'_i(p+j) = s_j$.

We now prove that for each $i = 1 \dots m$

$$g'_i = (y'_i, \vartheta'_i),\ h'_i = (z'_i, \psi'_i),\ \rho'_i = (y'_i, r'_i),\ \sigma'_i = (z'_i, u'_i).$$

We see that

$$\begin{aligned}
g'_1 &= g + (x_1, \varphi_1) = (y, \vartheta) + (x_1, \varphi_1) = (y'_1, \vartheta'_1),\\
h'_1 &= h + (x_1, \varphi_1) = (z, \psi) + (x_1, \varphi_1) = (z'_1, \psi'_1),\\
\rho'_1 &= \rho + (x_1, s_1) = (y, r) + (x_1, s_1) = (y'_1, r'_1),\\
\sigma'_1 &= \sigma + (x_1, s_1) = (z, u) + (x_1, s_1) = (z'_1, u'_1).
\end{aligned}$$

If $m > 1$ our proof needs an inductive step. In this case, given $i = 1 \dots m-1$, we see that

$$\begin{aligned}
g'_{i+1} &= g'_i + (x_{i+1}, \varphi_{i+1}) = (y'_i, \vartheta'_i) + (x_{i+1}, \varphi_{i+1}) = (y'_{i+1}, \vartheta'_{i+1}),\\
h'_{i+1} &= h'_i + (x_{i+1}, \varphi_{i+1}) = (z'_i, \psi'_i) + (x_{i+1}, \varphi_{i+1}) = (z'_{i+1}, \psi'_{i+1}),\\
\rho'_{i+1} &= \rho'_i + (x_{i+1}, s_{i+1}) = (y'_i, r'_i) + (x_{i+1}, s_{i+1}) = (y'_{i+1}, r'_{i+1}),\\
\sigma'_{i+1} &= \sigma'_i + (x_{i+1}, s_{i+1}) = (z'_i, u'_i) + (x_{i+1}, s_{i+1}) = (z'_{i+1}, u'_{i+1}).
\end{aligned}$$

We also can prove that for each $\alpha = 1 \dots m$

- for each $i \in dom(g'_\alpha),\ j \in dom(h'_\alpha)$ $(y'_\alpha)_i = (z'_\alpha)_j \to (\vartheta'_\alpha)_i = (\psi'_\alpha)_j$;
- for each $i \in dom(\rho'_\alpha),\ j \in dom(\sigma'_\alpha)$ $(y'_\alpha)_i = (z'_\alpha)_j \to (r'_\alpha)_i = (u'_\alpha)_j$.

In fact, let $\alpha = 1 \dots m$. We notice that

$$\begin{aligned}
dom(g'_\alpha) &= \{1, \dots, q+\alpha\} = dom(\rho'_\alpha),\\
dom(h'_\alpha) &= \{1, \dots, p+\alpha\} = dom(\sigma'_\alpha).
\end{aligned}$$

Let $i \in \{1, \dots, q+\alpha\},\ j \in \{1, \dots, p+\alpha\}$.

If $q > 0,\ i \leqslant q,\ p > 0,\ j \leqslant p$ and $y_i = (y'_\alpha)_i = (z'_\alpha)_j = z_j$ then $(\vartheta'_\alpha)_i = \vartheta_i = \psi_j = (\psi'_\alpha)_j$ and $(r'_\alpha)_i = r_i = u_j = (u'_\alpha)_j$.

If $q > 0,\ i \leqslant q,\ j > p$ then $(y'_\alpha)_i = y_i \in var(g),\ (z'_\alpha)_j = x_{j-p} \in \mathcal{V} - var(g)$ so $(y'_\alpha)_i \neq (z'_\alpha)_j$.

If $i > q,\ p > 0,\ j \leqslant p$ then $(y'_\alpha)_i = x_{i-q} \in \mathcal{V} - var(h),\ (z'_\alpha)_j = z_j \in var(h)$ so $(y'_\alpha)_i \neq (z'_\alpha)_j$.

If $i > q, j > p$ and $x_{i-q} = (y'_\alpha)_i = (z'_\alpha)_j = x_{j-p}$ then $i - q = j - p$, so $(\vartheta'_\alpha)_i = \varphi_{i-q} = \varphi_{j-p} = (\psi'_\alpha)_j$ and $(r'_\alpha)_i = s_{i-q} = s_{j-p} = (u'_\alpha)_j$.

We'll now show that for each $i = 1 \dots m$ $\rho'_i \in \Xi(g'_i)$.

We begin by showing that $\rho'_1 \in \Xi(g'_1)$. We intend to use assumption 2.1.9 to show that $s_1 \in \#(g, \varphi_1, \rho)$.

We consider that $g, h \in K(n)$, for each $i \in dom(g),\ j \in dom(h),\ y_i = z_j \to \vartheta_i = \psi_j$, $\varphi_1 \in E(n,g) \cap E(n,h),\ \rho \in \Xi(g),\ \sigma \in \Xi(h)$, for each $i \in dom(\rho),\ j \in dom(\sigma),\ y_i = z_j \to r_i = u_j$. By assumption 2.1.9 $\#(g, \varphi_1, \rho) = \#(h, \varphi_1, \sigma)$, so $s_1 \in \#(g, \varphi_1, \rho)$.

We can now use lemma 2.1.14 to show that $\rho'_1 \in \Xi(g'_1)$. In fact $g, g'_1 \in K(n)$, $x_1 \in \mathcal{V} - var(g),\ \varphi_1 \in E_s(n,g),\ g'_1 = g + (x_1, \varphi_1),\ \rho \in \Xi(g),\ \rho'_1 = \rho + (x_1, s_1)$, $s_1 \in \#(g, \varphi_1, \rho)$. So by 2.1.14 we get $\rho'_1 \in \Xi(g'_1)$.

If $m > 1$ we need an inductive step. Let $\alpha = 1 \dots m-1$, we suppose that $\rho'_\alpha \in \Xi(g'_\alpha)$ and want to show that $\rho'_{\alpha+1} \in \Xi(g'_{\alpha+1})$. First we intend to use assumption 2.1.9 to show that $s_{\alpha+1} \in \#(g'_\alpha, \varphi_{\alpha+1}, \rho'_\alpha)$.

We consider that $g'_\alpha, h'_\alpha \in K(n)$, for each $i \in dom(g'_\alpha),\ j \in dom(h'_\alpha),\ (y'_\alpha)_i = (z'_\alpha)_j \to (\vartheta'_\alpha)_i = (\psi'_\alpha)_j,\ \varphi_{\alpha+1} \in E(n, g'_\alpha) \cap E(n, h'_\alpha),\ \rho'_\alpha \in \Xi(g'_\alpha),\ \sigma'_\alpha \in \Xi(h'_\alpha)$, for each $i \in dom(\rho'_\alpha),\ j \in dom(\sigma'_\alpha)$ $(y'_\alpha)_i = (z'_\alpha)_j \to (r'_\alpha)_i = (u'_\alpha)_j$. By assumption 2.1.9 $\#(g'_\alpha, \varphi_{\alpha+1}, \rho'_\alpha) = \#(h'_\alpha, \varphi_{\alpha+1}, \sigma'_\alpha)$, so $s_{\alpha+1} \in \#(g'_\alpha, \varphi_{\alpha+1}, \rho'_\alpha)$.

We can now use lemma 2.1.14 to show that $\rho'_{\alpha+1} \in \Xi(g'_{\alpha+1})$. In fact $g'_\alpha, g'_{\alpha+1} \in K(n),\ x_{\alpha+1} \in \mathcal{V} - var(g'_\alpha),\ \varphi_{\alpha+1} \in E_s(n, g'_\alpha),\ g'_{\alpha+1} = g'_\alpha + (x_{\alpha+1}, \varphi_{\alpha+1})$, $\rho'_\alpha \in \Xi(g'_\alpha),\ \rho'_{\alpha+1} = \rho'_\alpha + (x_{\alpha+1}, s_{\alpha+1}), s_{\alpha+1} \in \#(g'_\alpha, \varphi_{\alpha+1}, \rho'_\alpha)$. So by 2.1.14 we get $\rho'_{\alpha+1} \in \Xi(g'_{\alpha+1})$.

We can conclude that $\rho'_m \in \Xi(g'_m)$. By 2.1.9 we can derive that $\#(g'_m, \phi, \rho'_m) = \#(h'_m, \phi, \sigma'_m)$. In fact $g'_m, h'_m \in K(n)$, for each $i \in dom(g'_m)$, $j \in dom(h'_m),\ (y'_m)_i = (z'_m)_j \to (\vartheta'_m)_i = (\psi'_m)_j,\ \phi \in E(n, g'_m) \cap E(n, h'_m)$, $\rho'_m \in \Xi(g'_m),\ \sigma'_m \in \Xi(h'_m)$, for each $i \in dom(\rho'_m),\ j \in dom(\sigma'_m)$ $(y'_m)_i = (z'_m)_j \to (r'_m)_i = (u'_m)_j$. Therefore $\#(g'_m, \phi, \rho'_m) = \#(h'_m, \phi, \sigma'_m)$.

It follows that $u = \#(g'_m, \phi, \rho'_m)$, and since $\rho'_m \in \Xi(g'_m),\ \rho \sqsubseteq \rho'_m$ also hold, we have proved that

$$u \in \{\#(g'_m, \phi, \rho'_m)|\ \rho'_m \in \Xi(g'_m), \rho \sqsubseteq \rho'_m\}.$$

With a perfectly analogous proof we can show the converse implication i.e. that if $u \in \{\#(g'_m, \phi, \rho'_m)|\ \rho'_m \in \Xi(g'_m), \rho \sqsubseteq \rho'_m\}$ then

$$u \in \{\#(h'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(h'_m), \sigma \sqsubseteq \sigma'_m\}.$$

■

LEMMA 2.1.16. Let $h = (x, \varphi) \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V} - var(h)$ and $k = h + (y, \phi)$. Let $\rho = (x, s) \in \Xi(h)$, $r \in \#(h, \phi, \rho)$ and $\sigma = \rho + (y, r)$. Then k is a state-like pair (x', φ') and σ is a state-like pair (x', s'). Moreover

- for each $i \in dom(k)$, $j \in dom(h)$ $x'_i = x_j \to \varphi'_i = \varphi_j$.
- for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x'_i = x_j \to s'_i = s_j$.

Proof. If $dom(h) = \emptyset$ then $dom(\rho) = \emptyset$ and the statements are trivially satisfied.

Otherwise there exists a positive integer m such that $dom(\rho) = dom(h) = \{1, \ldots, m\}$, and $dom(\sigma) = dom(k) = \{1, \ldots, m+1\}$.

Let $i \in dom(k)$, $j \in dom(h)$. If $i = m+1$ then $x'_i = y \notin var(h)$, so $x'_i \neq x_j$. Else $x'_i = x_i$, so $x'_i = x_j$ implies $x_i = x_j$, which implies $i = j$ and $\varphi'_i = \varphi_i = \varphi_j$.

Let $i \in dom(\sigma)$, $j \in dom(\rho)$. If $i = m+1$ then $x'_i = y \notin var(h)$, so $x'_i \neq x_j$. Else $x'_i = x_i$, so $x'_i = x_j$ implies $x_i = x_j$, which implies $i = j$ and $s'_i = s_i = s_j$. ■

We now start with the verifications required to define $\#(k, t, \sigma)$. There we have to verify that

- for each $k \in K(n+1)$, $w \in \{a, b, c, d, e\}$, $t \in E'(n, k) \cap E'_w(n+1, k)$ and $\sigma \in \Xi(k)$ $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1,k,w)}$;
- for each $k \in K(n+1)$, $w_1, w_2 \in \{a, b, c, d, e\} : w_1 \neq w_2$, $t \in E'_{w_1}(n+1, k) \cap E'_{w_2}(n+1, k)$ and $\sigma \in \Xi(k)$ $\#(k, t, \sigma)_{(n+1,k,w_1)} = \#(k, t, \sigma)_{(n+1,k,w_2)}$.

We begin by verifying the first statement.

Suppose $t \in \mathbf{E}'(\mathbf{n}, \mathbf{k}) \cap \mathbf{E}'_{\mathbf{a}}(\mathbf{n}+\mathbf{1}, \mathbf{k})$, and so $t \in E(n, k) \cap E_a(n+1, k)$. As a consequence of $t \in E_a(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t \in E(n, h)$. Given $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$, and $\#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho)$.

We want to apply assumption 2.1.9. We observe that $h = (x, \varphi)$, $k = (x', \varphi') \in K(n)$, for each $i \in dom(k)$, $j \in dom(h)$ $x'_i = x_j \to \varphi'_i = \varphi_j$. Moreover $t \in E(n, k) \cap E(n, h)$, $\rho = (x, r) \in \Xi(h)$, $\sigma = (x', r') \in \Xi(k)$, for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x'_i = x_j \to r'_i = r_j$. At this point by assumption 2.1.9 we have $\#(k, t, \sigma) = \#(h, t, \rho) = \#(k, t, \sigma)_{(n+1,k,a)}$.

Suppose $t \in \mathbf{E}'(\mathbf{n}, \mathbf{k}) \cap \mathbf{E}'_{\mathbf{b}}(\mathbf{n}+\mathbf{1}, \mathbf{k})$, and so $t \in E(n, k) \cap E_b(n+1, k)$. As a consequence of $t \in E_b(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t = y$. Given $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$, and $\#(k, t, \sigma)_{(n+1,k,b)} = s$.

Suppose $h = (x, \varphi)$ and $\rho = (x, r)$, then we can also set $k = (x', \varphi')$, $\sigma = (x', r')$. By assumption 2.1.10 $\exists i \in dom(k)$ such that $t = x'_i$, $\#(k, t, \sigma) = r'_i$. There exists an integer $m \geqslant 0$ such that $dom(k) = \{1, \ldots, m+1\}$. Since $y = t = x'_i$ it must be $i = m+1$, so $\#(k, t, \sigma) = r'_{m+1} = s = \#(k, t, \sigma)_{(n+1,k,b)}$.

Let's examine the situation in which $t \in \mathbf{E}'(\mathbf{n},\mathbf{k}) \cap \mathbf{E}'_{\mathbf{c}}(\mathbf{n}+\mathbf{1},\mathbf{k})$, and then t belongs to $E(n,k) \cap E_c(n+1,k)$. As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$ and

$$\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,\varphi,\sigma)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)).$$

Since $t \in E(n,k)$ we can apply assumption 2.1.10 and obtain that $n > 1$, there exists $h \in K(n-1)$: $h \sqsubseteq k$, $\varphi, \varphi_1, \ldots \varphi_m \in E(n-1,h)$, $t \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\varphi,\rho)$ is a function with m arguments, $(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho))$ is a member of the domain of $\#(h,\varphi,\rho)$, $\#(h,t,\rho) = \#(h,\varphi,\rho)(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho))$.
Moreover, given $\sigma \in \Xi(k)$ and $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results $\#(k,t,\sigma) = \#(h,t,\rho)$.

Given $\sigma \in \Xi(k)$ we have $dom(h) \in \mathcal{D}$ and $dom(h) \subseteq dom(k) = dom(\sigma)$, so we can define $\rho = \sigma_{/dom(h)}$. If $h = k$ then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.11 we still get $\rho \in \Xi(h)$. Therefore

$$\#(k,t,\sigma) = \#(h,t,\rho) = \#(h,\varphi,\rho)(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho)).$$

We want to apply assumption 2.1.9. We observe that $k = (x,\vartheta)$, $h = (y,\psi) \in K(n)$, $h \in \mathcal{R}(k)$, for each $i,j \in dom(k)$ $i \neq j \rightarrow x_i \neq x_j$. Then (by lemma 2.2) for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \rightarrow \vartheta_i = \psi_j$. Moreover $\sigma = (x,s) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$, $\rho \in \mathcal{R}(\sigma)$, for each $i,j \in dom(\sigma)$ $i \neq j \rightarrow x_i \neq x_j$. Then for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \rightarrow s_i = r_j$. Since $\varphi \in E(n,k) \cap E(n,h)$ and for each $\alpha = 1 \ldots m$ $\varphi_\alpha \in E(n,k) \cap E(n,h)$ by 2.1.9 we obtain

$$\#(h,\varphi,\rho)(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho)) = \#(k,\varphi,\sigma)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)),$$

and therefore $\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,t,\sigma)$.

Let's examine the situation in which $t \in \mathbf{E}'(\mathbf{n},\mathbf{k}) \cap \mathbf{E}'_{\mathbf{d}}(\mathbf{n}+\mathbf{1},\mathbf{k})$, and then t belongs to $E(n,k) \cap E_d(n+1,k)$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in \mathcal{F}, \varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (f)(\varphi_1, \ldots, \varphi_m)$ and

$$\#(k,t,\sigma)_{(n+1,k,d)} = P_f(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)).$$

Since $t \in E(n,k)$ we can apply assumption 2.1.10 and obtain that $n > 1$, there exists $h \in K(n-1)$: $h \sqsubseteq k$, $\varphi_1, \ldots \varphi_m \in E(n-1,h)$, $t \in E(n,h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho))$, $\#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho))$. Moreover, given $\sigma \in \Xi(k)$ and $\rho \in \Xi(h)$: $\rho \sqsubseteq \sigma$ it results $\#(k,t,\sigma) = \#(h,t,\rho)$.

Given $\sigma \in \Xi(k)$ we have $dom(h) \in \mathcal{D}$ and $dom(h) \subseteq dom(k) = dom(\sigma)$, so we can define $\rho = \sigma_{/dom(h)}$. If $h = k$ then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.11 we still get $\rho \in \Xi(h)$. Therefore

$$\#(k,t,\sigma) = \#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho),\ldots,\#(h,\varphi_m,\rho)).$$

We want to apply assumption 2.1.9. We observe that $k = (x,\vartheta)$, $h = (y,\psi) \in K(n)$, $h \in \mathcal{R}(k)$, for each $i,j \in dom(k)$ $i \neq j \rightarrow x_i \neq x_j$. Then (by lemma 2.2) for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \rightarrow \vartheta_i = \psi_j$. Moreover $\sigma = (x,s) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$, $\rho \in \mathcal{R}(\sigma)$, for each $i,j \in dom(\sigma)$ $i \neq j \rightarrow x_i \neq x_j$. Then for each

$i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \rightarrow s_i = r_j$. Since for each $\alpha = 1 \dots m$ $\varphi_\alpha \in E(n,k) \cap E(n,h)$ by 2.1.9 we obtain

$$P_f(\#(h,\varphi_1,\rho),\dots,\#(h,\varphi_m,\rho)) = P_f(\#(k,\varphi_1,\sigma),\dots,\#(k,\varphi_m,\sigma)),$$

and therefore $\#(k,t,\sigma)_{(n+1,k,d)} = \#(k,t,\sigma)$.

In this part of our verification we just need to examine the case in which t is in $\mathbf{E'(n,k)} \cap \mathbf{E'_e(n+1,k)}$, and then t belongs to $E(n,k) \cap E_e(n+1,k)$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1 \dots m$ $i \neq j \rightarrow x_i \neq x_j$,
- a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that

- $\mathcal{E}(n,k,m,x,\varphi,\phi)$,
- $t = \{\}(x_1 : \varphi_1,\dots,x_m : \varphi_m,\phi)$.

For a fixed $\sigma \in \Xi(k)$ we have

$$\#(k,t,\sigma)_{(n+1,k,e)} = \{\#(k'_m,\phi,\sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1,\varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $k'_{i+1} = k'_i + (x_{i+1},\varphi_{i+1})$.

Since $t \in E(n,k)$ we can apply assumption 2.1.10 and obtain that $n > 1$, there exists $h \in K(n-1)$ such that

- $h \sqsubseteq k$,
- for each $i = 1 \dots m$ $\varphi_i \in E(n-1)$,
- $\phi \in E(n-1)$,
- $\mathcal{E}(n-1,h,m,x,\varphi,\phi)$,
- $t \in E(n,h)$,
- for each $\sigma \in \Xi(k)$, $\rho \in \Xi(h) : \rho \sqsubseteq \sigma$ it results $\#(k,t,\sigma) = \#(h,t,\rho)$,
- for each $\rho \in \Xi(h)$ $\#(h,t,\rho) = \{\#(h'_m,\phi,\rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}$,

where $h'_1 = h + (x_1,\varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $h'_{i+1} = h'_i + (x_{i+1},\varphi_{i+1})$.

Given $\sigma \in \Xi(k)$ we have $dom(h) \in \mathcal{D}$ and $dom(h) \subseteq dom(k) = dom(\sigma)$, so we can define $\rho = \sigma_{/dom(h)}$. If $h = k$ then $\rho = \sigma \in \Xi(h)$. Otherwise by assumption 2.1.11 we still get $\rho \in \Xi(h)$. Therefore

$$\#(k,t,\sigma) = \#(h,t,\rho) = \{\#(h'_m,\phi,\rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}.$$

We clearly need to show that

$$\{\#(h'_m,\phi,\rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\} = \{\#(k'_m,\phi,\sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}.$$

To show this we need lemma 2.1.15. To apply that lemma we observe that $h = (y,\vartheta)$, $k = (z,\psi) \in K(n)$, since $\mathcal{E}(n-1,h,m,x,\varphi,\phi)$ holds then also

$\mathcal{E}(n,h,m,x,\varphi,\phi)$ holds, and $\mathcal{E}(n,k,m,x,\varphi,\phi)$ holds too. We have $h \in \mathcal{R}(k)$, so by lemma 2.2 we obtain that for each $i \in dom(k)$, $j \in dom(h)$ $z_i = y_j \to \psi_i = \vartheta_j$. Moreover $\sigma = (z,s) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$, and since $\rho \in \mathcal{R}(\sigma)$ by lemma 2.2 we also obtain that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $z_i = y_j \to s_i = r_j$. At this point we can apply lemma 2.1.15 from which it follows

$$\{\#(h'_m,\phi,\rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\} = \{\#(k'_m,\phi,\sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

and this clearly causes $\#(k,t,\sigma)_{(n+1,k,e)} = \#(k,t,\sigma)$.

$\diamond$

We now need to verify the other statement, i.e.:
for each $k \in K(n+1)$, $w_1, w_2 \in \{a,b,c,d,e\} : w_1 \neq w_2$, $t \in E'_{w_1}(n+1,k) \cap E'_{w_2}(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma)_{(n+1,k,w_1)} = \#(k,t,\sigma)_{(n+1,k,w_2)}$.

Fortunately for us, for many values of w_1, w_2 it's easy to see that $E'_{w_1}(n+1,k) \cap E'_{w_2}(n+1,k) = \emptyset$.

In fact, consider all the cases in which $w_1, w_2 \in \{b,c,d,e\}$ and $w_1 \neq w_2$. It is immediate to see that $E'_{w_1}(n+1,k) \cap E'_{w_2}(n+1,k) = \emptyset$ (actually, assumption 2.1.10 is needed to prove $E'_c(n+1,k) \cap E'_d(n+1,k) = \emptyset$).

We miss to consider the cases where $w_1 = a$ and $w_2 \in \{b,c,d,e\}$.

We can easily prove that $E'_a(n+1,k) \cap E'_b(n+1,k) = \emptyset$.

Suppose $t \in E'_a(n+1,k) \cap E'_b(n+1,k)$. This means that $t \in E_a(n+1,k)$ and $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y,\phi)$. We also have $t \in E(n,h)$, and since $t \in E_b(n+1,k)$ we have $t = y$. Since $t \in E(n,h)$ we can apply assumption 2.1.10, situations a,c,d,e can not occur, so situation b must occur. This means that $y = t \in var(h)$ and so we have reached a contradiction.

Therefore we just need to examine three cases: $t \in E'_a(n+1,k) \cap E'_c(n+1,k)$, $t \in E'_a(n+1,k) \cap E'_d(n+1,k)$, $t \in E'_a(n+1,k) \cap E'_e(n+1,k)$.

We start with the case where $t \in \mathbf{E'_a(n+1,k) \cap E'_c(n+1,k)}$, and so $t \in E_a(n+1,k) \cap E_c(n+1,k)$.

As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$ and for each $\sigma \in \Xi(k)$

$$\#(k,t,\sigma)_{(n+1,k,c)} = \#(k,\varphi,\sigma)(\#(k,\varphi_1,\sigma), \ldots, \#(k,\varphi_m,\sigma)).$$

As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y,\phi)$. We also have $t \in E(n,h)$. Given $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, and $\#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho)$.

Since $t \in E(n,h)$ we can apply assumption 2.1.10 and obtain that $n > 1$, there exists $g \in K(n-1)$: $g \sqsubseteq h$, $\varphi, \varphi_1, \ldots \varphi_m \in E(n-1,g)$, $t \in E(n,g)$, for each $\delta \in \Xi(g)$ $\#(g,\varphi,\delta)$

is a function with m arguments, $(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta))$ is a member of the domain of $\#(g,\varphi,\delta)$, $\#(g,t,\delta)=\#(g,\varphi,\delta)(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta))$.
Moreover, given $\rho\in\Xi(h)$ and $\delta\in\Xi(g)$: $\delta\sqsubseteq\rho$ it results $\#(h,t,\rho)=\#(g,t,\delta)$.

We have seen that given $\sigma\in\Xi(k)$ there exist $\rho\in\Xi(h)$, $s\in\#(h,\phi,\rho)$ such that $\sigma=\rho+(y,s)$, and $\#(k,t,\sigma)_{(n+1,k,a)}=\#(h,t,\rho)$.

We have $dom(g)\in\mathcal{D}$ and $dom(g)\subseteq dom(h)=dom(\rho)$, so we can define $\delta=\rho_{/dom(g)}$. If $g=h$ then $\delta=\rho\in\Xi(g)$. Otherwise by assumption 2.1.11 we still get $\delta\in\Xi(g)$. Therefore

$$\#(k,t,\sigma)_{(n+1,k,a)}=\#(h,t,\rho)=\#(g,t,\delta)=\#(g,\varphi,\delta)(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta)).$$

We have $g=(x,\vartheta)$, $k=(z,\psi)\in K(n)$, with $g\in\mathcal{R}(k)$. By lemma 2.2 we obtain that for each $i\in dom(k)$, $j\in dom(g)$ $z_i=x_j\to\psi_i=\vartheta_j$. Moreover $\sigma=(z,s)\in\Xi(k)$, $\delta=(x,r)\in\Xi(g)$, $\delta\in\mathcal{R}(\sigma)$ and so by lemma 2.2 we obtain that for each $i\in dom(\sigma)$, $j\in dom(\delta)$ $z_i=x_j\to s_i=r_j$. Furthermore we can see that $\varphi,\varphi_1,\ldots,\varphi_m\in E(n,k)\cap E(n,g)$. By assumption 2.1.9 we can state

$$\#(g,\varphi,\delta)(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta))=\#(k,\varphi,\sigma)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)),$$

and therefore

$$\#(k,t,\sigma)_{(n+1,k,a)}=\#(k,t,\sigma)_{(n+1,k,c)}.$$

Consider now the case where $t\in\mathbf{E'_a(n+1,k)}\cap\mathbf{E'_d(n+1,k)}$, and so $t\in E_a(n+1,k)\cap E_d(n+1,k)$.

As a consequence of $t\in E_d(n+1,k)$ there exist $f\in\mathcal{F}$, $\varphi_1,\ldots,\varphi_m$ in $E(n,k)$ such that $t=(f)(\varphi_1,\ldots,\varphi_m)$ and for each $\sigma\in\Xi(k)$

$$\#(k,t,\sigma)_{(n+1,k,d)}=P_f(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)).$$

As a consequence of $t\in E_a(n+1,k)$ we have that $k\in K(n)^+$, so there exist $h\in K(n)$, $\phi\in E_s(n,h)$, $y\in\mathcal{V}-var(h)$ such that $k=h+(y,\phi)$. We also have $t\in E(n,h)$. Given $\sigma\in\Xi(k)$ there exist $\rho\in\Xi(h)$, $s\in\#(h,\phi,\rho)$ such that $\sigma=\rho+(y,s)$, and $\#(k,t,\sigma)_{(n+1,k,a)}=\#(h,t,\rho)$.

Since $t\in E(n,h)$ we can apply assumption 2.1.10 and obtain that $n>1$, there exists $g\in K(n-1)$: $g\sqsubseteq h$, $\varphi_1,\ldots\varphi_m\in E(n-1,g)$, $t\in E(n,g)$, for each $\delta\in\Xi(g)$ $A_f(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta))$, $\#(g,t,\delta)=P_f(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta))$. Moreover, given $\rho\in\Xi(h)$ and $\delta\in\Xi(g)$: $\delta\sqsubseteq\rho$ it results $\#(h,t,\rho)=\#(g,t,\delta)$.

We have seen that given $\sigma\in\Xi(k)$ there exist $\rho\in\Xi(h)$, $s\in\#(h,\phi,\rho)$ such that $\sigma=\rho+(y,s)$, and $\#(k,t,\sigma)_{(n+1,k,a)}=\#(h,t,\rho)$.

We have $dom(g)\in\mathcal{D}$ and $dom(g)\subseteq dom(h)=dom(\rho)$, so we can define $\delta=\rho_{/dom(g)}$. If $g=h$ then $\delta=\rho\in\Xi(g)$. Otherwise by assumption 2.1.11 we still get $\delta\in\Xi(g)$. Therefore

$$\#(k,t,\sigma)_{(n+1,k,a)}=\#(h,t,\rho)=\#(g,t,\delta)=P_f(\#(g,\varphi_1,\delta),\ldots,\#(g,\varphi_m,\delta)).$$

We have $g = (x, \vartheta)$, $k = (z, \psi) \in K(n)$, with $g \in \mathcal{R}(k)$. By lemma 2.2 we obtain that for each $i \in dom(k)$, $j \in dom(g)$ $z_i = x_j \to \psi_i = \vartheta_j$. Moreover $\sigma = (z, s) \in \Xi(k)$, $\delta = (x, r) \in \Xi(g)$, $\delta \in \mathcal{R}(\sigma)$ and so by lemma 2.2 we obtain that for each $i \in dom(\sigma)$, $j \in dom(\delta)$ $z_i = x_j \to s_i = r_j$. Furthermore we can see that $\varphi_1, \dots, \varphi_m \in E(n,k) \cap E(n,g)$. By assumption 2.1.9 we can state

$$P_f(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta)) = P_f(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)),$$

and therefore

$$\#(k, t, \sigma)_{(n+1,k,a)} = \#(k, t, \sigma)_{(n+1,k,d)}.$$

Finally we examine the case where $t \in \mathbf{E'_a(n+1,k)} \cap \mathbf{E'_e(n+1,k)}$, and so $t \in E_a(n+1,k) \cap E_e(n+1,k)$.

As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that

- $\mathcal{E}(n, k, m, x, \varphi, \phi)$,
- $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$.

For a fixed $\sigma \in \Xi(k)$ we have

$$\#(k, t, \sigma)_{(n+1,k,e)} = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$.

As a consequence of $t \in E_a(n+1,k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t \in E(n,h)$. Given $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$, and $\#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho)$.

Since $t \in E(n,h)$ we can apply assumption 2.1.10 and obtain that $n > 1$, there exists $g \in K(n-1)$ such that

- $g \sqsubseteq h$,
- for each $i = 1 \dots m$ $\varphi_i \in E(n-1)$,
- $\phi \in E(n-1)$,
- $\mathcal{E}(n-1, g, m, x, \varphi, \phi)$,
- $t \in E(n,g)$,
- for each $\rho \in \Xi(h)$, $\delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h, t, \rho) = \#(g, t, \delta)$,
- for each $\delta \in \Xi(g)$ $\#(g, t, \delta) = \{\#(g'_m, \phi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}$,

where $g'_1 = g + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $g'_{i+1} = g'_i + (x_{i+1}, \varphi_{i+1})$.

We have seen that given $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$, and $\#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho)$.

We have $dom(g) \in \mathcal{D}$ and $dom(g) \subseteq dom(h) = dom(\rho)$, so we can define $\delta = \rho_{/dom(g)}$. If $g = h$ then $\delta = \rho \in \Xi(g)$. Otherwise by assumption 2.1.11 we still get $\delta \in \Xi(g)$. Therefore

$$\#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g, t, \delta) = \{\#(g'_m, \phi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}.$$

We clearly need to show that

$$\{\#(g'_m, \phi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\} = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}.$$

To show this we need lemma 2.1.15. To apply that lemma we observe that $g = (y, \vartheta)$, $k = (z, \psi) \in K(n)$, since $\mathcal{E}(n-1, g, m, x, \varphi, \phi)$ holds then also $\mathcal{E}(n, g, m, x, \varphi, \phi)$ holds, and $\mathcal{E}(n, k, m, x, \varphi, \phi)$ holds too. We have $g \in \mathcal{R}(k)$, so by lemma 2.2 we obtain that for each $i \in dom(k)$, $j \in dom(g)$ $z_i = y_j \to \psi_i = \vartheta_j$. Moreover $\sigma = (z, s) \in \Xi(k)$, $\delta = (y, r) \in \Xi(g)$, and since $\delta \in \mathcal{R}(\sigma)$ by lemma 2.2 we also obtain that for each $i \in dom(\sigma)$, $j \in dom(\delta)$ $z_i = y_j \to s_i = r_j$. At this point we can apply lemma 2.1.15 from which it follows

$$\{\#(g'_m, \phi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\} = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}.$$

and this clearly causes $\#(k, t, \sigma)_{(n+1,k,a)} = \#(k, t, \sigma)_{(n+1,k,e)}$.

$\diamond$

Let's now proceed with the verifications required to define $V_b(t)$ and $V_f(t)$. We have to verify that

- for each $k \in K(n+1)$, $w \in \{a, b, c, d, e\}$, $t \in E(n) \cap E'_w(n+1, k)$ $V_b(t) = V_b(t)_{(n+1,k,w)}$ and $V_f(t) = V_f(t)_{(n+1,k,w)}$;
- for each $k_1, k_2 \in K(n+1)$, $w_1, w_2 \in \{a, b, c, d, e\}$, $t \in E'_{w_1}(n+1, k_1) \cap E'_{w_2}(n+1, k_2)$ such that $t \notin E(n)$ we have $V_b(t)_{(n+1,k_1,w_1)} = V_b(t)_{(n+1,k_2,w_2)}$ and $V_f(t)_{(n+1,k_1,w_1)} = V_f(t)_{(n+1,k_2,w_2)}$.

Suppose $t \in \mathbf{E(n)} \cap \mathbf{E'_a(n+1, k)}$. As a consequence of $t \in E_a(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V}-var(h)$ such that $k = h + (y, \phi)$. We also have $t \in E(n, h)$, $V_b(t)_{(n+1,k,a)} = V_b(t)$ and $V_f(t)_{(n+1,k,a)} = V_f(t)$.

Suppose $t \in \mathbf{E(n)} \cap \mathbf{E'_b(n+1, k)}$. As a consequence of $t \in E_b(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V}-var(h)$ such that $k = h + (y, \phi)$. We also have $t = y$, $V_f(t)_{(n+1,k,b)} = \{y\}$, $V_b(t)_{(n+1,k,b)} = \emptyset$.
There exists $g \in K(n)$ such that $t \in E(n, g)$. By assumption 2.1.10

$$V_f(t) = \{t\} = \{y\} = V_f(t)_{(n+1,k,b)},\ V_b(t) = \emptyset = V_b(t)_{(n+1,k,b)}.$$

Suppose $t \in \mathbf{E(n)} \cap \mathbf{E'_c(n+1,k)}$. As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$,

$$V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m),$$
$$V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m).$$

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.10 we get $n > 1$, $\exists h \in K(n-1)$: $h \sqsubseteq \kappa$, $\varphi, \varphi_1, \ldots \varphi_m \in E(n-1,h)$, $t \in E(n,h)$,

$$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k,c)},$$
$$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k,c)}.$$

Suppose $t \in \mathbf{E(n)} \cap \mathbf{E'_d(n+1,k)}$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in \mathcal{F}$, $\varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (f)(\varphi_1, \ldots, \varphi_m)$

$$V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m),$$
$$V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m).$$

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.10 we get $n > 1$, $\exists h \in K(n-1)$: $h \sqsubseteq \kappa$, $\varphi_1, \ldots \varphi_m \in E(n-1,h)$, $t \in E(n,h)$,

$$V_f(t) = V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k,d)},$$
$$V_b(t) = V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k,d)}.$$

Suppose $t \in \mathbf{E(n)} \cap \mathbf{E'_e(n+1,k)}$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that

- $\mathcal{E}(n,k,m,x,\varphi,\phi)$,
- $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

If $m = 1$ we have also

$$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}),$$
$$V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi).$$

If $m > 1$ we have

$$V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup \\ \cup (V_f(\phi) - \{x_1, \ldots, x_m\}),$$
$$V_b(t)_{(n+1,k,e)} = \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi).$$

There exists $\kappa \in K(n)$ such that $t \in E(n,\kappa)$. By assumption 2.1.10 we obtain that $n > 1$, there exists $h \in K(n-1)$ such that

- $h \sqsubseteq \kappa$,
- for each $i = 1 \dots m$ $\varphi_i \in E(n-1)$,
- $\phi \in E(n-1)$,
- $\mathcal{E}(n-1, h, m, x, \varphi, \phi)$,
- $t \in E(n, h)$.

Moreover, if $m = 1$

$$V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}) = V_f(t)_{(n+1,k,e)},$$
$$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi) = V_b(t)_{(n+1,k,e)}.$$

If $m > 1$

$$\begin{aligned} V_f(t) &= V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \dots, x_{m-1}\}) \\ &\quad \cup (V_f(\phi) - \{x_1, \dots, x_m\}) = V_f(t)_{(n+1,k,e)}; \\ V_b(t) &= \{x_1, \dots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi) = V_b(t)_{(n+1,k,e)}. \end{aligned}$$

$\diamond$

We now need to verify that for each $k_1, k_2 \in K(n+1)$, $w_1, w_2 \in \{a, b, c, d, e\}$, $t \in E'_{w_1}(n+1, k_1) \cap E'_{w_2}(n+1, k_2)$ such that $t \notin E(n)$ we have

$$V_b(t)_{(n+1,k_1,w_1)} = V_b(t)_{(n+1,k_2,w_2)},\ V_f(t)_{(n+1,k_1,w_1)} = V_f(t)_{(n+1,k_2,w_2)}.$$

First of all we observe that for each $k \in K(n+1)$, $t \in E'_a(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V}-var(h)$ such that $k = h+(y, \phi)$. We also have $t \in E(n, h)$, this means that $t \in E(n)$. This implies that we just need to verify

- for each $k_1, k_2 \in K(n+1)$, $w_1, w_2 \in \{b, c, d, e\}$, $t \in E'_{w_1}(n+1, k_1) \cap E'_{w_2}(n+1, k_2)$ such that $t \notin E(n)$ we have

$$V_b(t)_{(n+1,k_1,w_1)} = V_b(t)_{(n+1,k_2,w_2)},\ V_f(t)_{(n+1,k_1,w_1)} = V_f(t)_{(n+1,k_2,w_2)}.$$

For each $k_1, k_2 \in K(n+1)$, $w_1, w_2 \in \{b, c, d, e\}$ if $w_1 \neq w_2$ then

$$E'_{w_1}(n+1, k_1) \cap E'_{w_2}(n+1, k_2) = \emptyset.$$

So we just need to verify

- for each $k_1, k_2 \in K(n+1)$, $w \in \{b, c, d, e\}$, $t \in E'_w(n+1, k_1) \cap E'_w(n+1, k_2)$ such that $t \notin E(n)$ we have

$$V_b(t)_{(n+1,k_1,w)} = V_b(t)_{(n+1,k_2,w)},\ V_f(t)_{(n+1,k_1,w)} = V_f(t)_{(n+1,k_2,w)}.$$

Suppose $t \in \mathbf{E'_b(n+1, k_1)} \cap \mathbf{E'_b(n+1, k_2)}$.
From $t \in E_b(n+1, k_1)$ we obtain that $k_1 \in K(n)^+$, so there exist $h_1 \in K(n)$, $\phi_1 \in E_s(n, h_1)$, $y_1 \in \mathcal{V} - var(h_1)$ such that $k_1 = h_1 + (y_1, \phi_1)$. We also have $t = y_1$, $V_f(t)_{(n+1,k_1,b)} = \{y_1\}$, $V_b(t)_{(n+1,k_1,b)} = \emptyset$.
From $t \in E_b(n+1, k_2)$ we obtain that $k_2 \in K(n)^+$, so there exist $h_2 \in K(n)$, $\phi_2 \in E_s(n, h_2)$, $y_2 \in \mathcal{V} - var(h_2)$ such that $k_2 = h_2 + (y_2, \phi_2)$. We also have $t = y_2$,

$V_f(t)_{(n+1,k_2,b)} = \{y_2\}$, $V_b(t)_{(n+1,k_2,b)} = \emptyset$.
Hence $V_b(t)_{(n+1,k_1,b)} = \emptyset = V_b(t)_{(n+1,k_2,b)}$, $V_f(t)_{(n+1,k_1,b)} = \{t\} = V_f(t)_{(n+1,k_2,b)}$.

Suppose $t \in \mathbf{E'_c(n+1,k_1)} \cap \mathbf{E'_c(n+1,k_2)}$.
As a consequence of $t \in E_c(n+1,k_1)$ there exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n,k_1)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$,

$$V_f(t)_{(n+1,k_1,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m),$$
$$V_b(t)_{(n+1,k_1,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m).$$

As a consequence of $t \in E_c(n+1,k_2)$ there exist $\psi, \psi_1, \ldots, \psi_p$ in $E(n,k_2)$ such that $t = (\psi)(\psi_1, \ldots, \psi_p)$,

$$V_f(t)_{(n+1,k_2,c)} = V_f(\psi) \cup V_f(\psi_1) \cup \cdots \cup V_f(\psi_p),$$
$$V_b(t)_{(n+1,k_2,c)} = V_b(\psi) \cup V_b(\psi_1) \cup \cdots \cup V_b(\psi_p).$$

So $(\varphi)(\varphi_1, \ldots, \varphi_m) = t = (\psi)(\psi_1, \ldots, \psi_p)$, it follows $p = m,\ \psi = \varphi,\ \psi_i = \varphi_i$, hence

$$V_f(t)_{(n+1,k_1,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k_2,c)},$$
$$V_b(t)_{(n+1,k_1,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k_2,c)}.$$

Suppose $t \in \mathbf{E'_d(n+1,k_1)} \cap \mathbf{E'_d(n+1,k_2)}$.
As a consequence of $t \in E_d(n+1,k_1)$ there exist $f \in \mathcal{F}$, $\varphi_1, \ldots, \varphi_m$ in $E(n,k_1)$ such that $t = (f)(\varphi_1, \ldots, \varphi_m)$

$$V_f(t)_{(n+1,k_1,d)} = V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m),$$
$$V_b(t)_{(n+1,k_1,d)} = V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m).$$

As a consequence of $t \in E_d(n+1,k_2)$ there exist $g \in \mathcal{F}$, $\psi_1, \ldots, \psi_p$ in $E(n,k_2)$ such that $t = (g)(\psi_1, \ldots, \psi_p)$

$$V_f(t)_{(n+1,k_2,d)} = V_f(\psi_1) \cup \cdots \cup V_f(\psi_p),$$
$$V_b(t)_{(n+1,k_2,d)} = V_b(\psi_1) \cup \cdots \cup V_b(\psi_p).$$

So $(f)(\varphi_1, \ldots, \varphi_m) = t = (g)(\psi_1, \ldots, \psi_p)$, if follows $g = f,\ p = m,\ \psi_i = \varphi_i$, hence

$$V_f(t)_{(n+1,k_1,d)} = V_f(\varphi_1) \cup \cdots \cup V_f(\varphi_m) = V_f(t)_{(n+1,k_2,d)},$$
$$V_b(t)_{(n+1,k_1,d)} = V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) = V_b(t)_{(n+1,k_2,d)}.$$

Suppose $t \in \mathbf{E'_e(n+1,k_1)} \cap \mathbf{E'_e(n+1,k_2)}$.
As a consequence to $t \in E_e(n+1,k_1)$ there exist

- a positive integer m,
- a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k_1)$, and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that

- $\mathcal{E}(n, k_1, m, x, \varphi, \phi)$,

- $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$.

If $m = 1$ we have also

$$V_f(t)_{(n+1,k_1,e)} = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}),$$
$$V_b(t)_{(n+1,k_1,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi).$$

If $m > 1$ we have

$$V_f(t)_{(n+1,k_1,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \dots \cup (V_f(\varphi_m) - \{x_1, \dots, x_{m-1}\}) \cup \cup (V_f(\phi) - \{x_1, \dots, x_m\}),$$

$$V_b(t)_{(n+1,k_1,e)} = \{x_1, \dots, x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi).$$

As a consequence to $t \in E_e(n+1, k_2)$ there exist

- a positive integer p,
- a function y whose domain is $\{1, \dots, p\}$ such that for each $i = 1 \dots p$ $y_i \in \mathcal{V} - var(k_2)$, and for each $i, j = 1 \dots m$ $i \neq j \to y_i \neq y_j$,
- a function ψ whose domain is $\{1, \dots, p\}$ such that for each $i = 1 \dots p$ $\psi_i \in E(n)$,
- $\vartheta \in E(n)$

such that

- $\mathcal{E}(n, k_2, p, y, \psi, \vartheta)$,
- $t = \{\}(y_1 : \psi_1, \dots, y_p : \psi_p, \vartheta)$.

If $p = 1$ we have also

$$V_f(t)_{(n+1,k_2,e)} = V_f(\psi_1) \cup (V_f(\vartheta) - \{y_1\}),$$
$$V_b(t)_{(n+1,k_2,e)} = \{y_1\} \cup V_b(\psi_1) \cup V_b(\vartheta).$$

If $p > 1$ we have

$$V_f(t)_{(n+1,k_2,e)} = V_f(\psi_1) \cup (V_f(\psi_2) - \{y_1\}) \cup \dots \cup (V_f(\psi_p) - \{y_1, \dots, y_{p-1}\}) \cup \cup (V_f(\vartheta) - \{y_1, \dots, y_p\}),$$

$$V_b(t)_{(n+1,k_2,e)} = \{y_1, \dots, y_p\} \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_p) \cup V_b(\vartheta).$$

So $\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi) = t = \{\}(y_1 : \psi_1, \dots, y_p : \psi_p, \vartheta)$, it follows $p = m$, $y = x$, $\psi = \varphi$, $\vartheta = \phi$. Hence if $m = 1$

$$V_f(t)_{(n+1,k_2,e)} = V_f(\psi_1) \cup (V_f(\vartheta) - \{y_1\}) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}) = V_f(t)_{(n+1,k_1,e)},$$
$$V_b(t)_{(n+1,k_2,e)} = \{y_1\} \cup V_b(\psi_1) \cup V_b(\vartheta) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi) = V_b(t)_{(n+1,k_1,e)}.$$

If $m > 1$

$$\begin{aligned}
V_f(t)_{(n+1,k_2,e)} &= V_f(\psi_1) \cup (V_f(\psi_2) - \{y_1\}) \cup \cdots \cup (V_f(\psi_p) - \{y_1, \ldots, y_{p-1}\}) \cup \\
&\quad \cup (V_f(\vartheta) - \{y_1, \ldots, y_p\}) \\
&= V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup \\
&\quad \cup (V_f(\phi) - \{x_1, \ldots, x_m\}) \\
&= V_f(t)_{(n+1,k_1,e)}; \\
V_b(t)_{(n+1,k_2,e)} &= \{y_1, \ldots, y_p\} \cup V_b(\psi_1) \cup \cdots \cup V_b(\psi_p) \cup V_b(\vartheta) \\
&= \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi) \\
&= V_b(t)_{(n+1,k_1,e)}.
\end{aligned}$$

$\diamond$

In the last part of our definition we need to prove that all the assumptions we have made at step n are true at step $n+1$. The order in which we'll provide these proofs is not the same in which we have listed the assumptions, but this of course is not a problem.

Proof of (assumption) 2.1.8 (at level $n+1$).

We need to prove that $K(n) \subseteq K(n+1)$, this is obvious by definition. ■

Proof of 2.1.12.

We need to prove that for each $k \in K(n)$ $E(n,k) \subseteq E(n+1,k)$.

For each $k \in K(n)$ we have $k \in K(n+1)$ and

$$\begin{aligned}
E(n+1,k) &= E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup \\
&\quad \cup E'_e(n+1,k) \\
&= E(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup \\
&\quad \cup E'_e(n+1,k).
\end{aligned}$$

■

Proof of 2.1.4.

We need to prove that for each $k \in K(n+1)$, $t \in E(n+1,k)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$;
- for each $\alpha \in \{1, \ldots, \ell(t)\}$ if $(t[\alpha] = \text{':'}) \vee (t[\alpha] = \text{','}) \vee (t[\alpha] = \text{')'})$ then $d(t,\alpha) \geqslant 1$.

We recall that

$$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k).$$

Let $\mathbf{t} \in \mathbf{E}'(\mathbf{n},\mathbf{k})$, this means that $t \in E(n,k) \subseteq E(n)$. In this case we just need to apply assumption 2.1.4.

Let $t \in \mathbf{E'_a}(\mathbf{n+1}, \mathbf{k})$. As a consequence of $t \in E_a(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t \in E(n, h)$, so we can apply assumption 2.1.4 to finish.

Let $t \in \mathbf{E'_b}(\mathbf{n+1}, \mathbf{k})$. As a consequence of $t \in E_b(n+1, k)$ we have that $k \in K(n)^+$, so there exist $h \in K(n)$, $\phi \in E_s(n, h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t = y$, so t has just one character, $t[1]$ differs from '(', ':', ',', ')' and $d(t, \ell(t)) = 0$.

Let $t \in \mathbf{E'_c}(\mathbf{n+1}, \mathbf{k})$. As a consequence of $t \in E_c(n+1, k)$ there exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n, k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = (\varphi)(\varphi_1, \varphi_2)$, if $m = 3$ then $t = (\varphi)(\varphi_1, \varphi_2, \varphi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. There are explicit occurrences of ',' only when $m > 1$. We indicate with q the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

We have $d(t, q-1) = d(t, 1 + \ell(\varphi)) = d(t, 1+1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] =$ ')' then $d(t, q) = d(t, q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.
Else $t[q-1] = \varphi[\ell(\varphi)] \notin \{\text{'('}, \text{')'}\}$, so $d(t, q) = d(t, q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

If $m > 1$ we can prove that for each $i = 1 \ldots m-1$ $d(t, q_i) = 1$.

First of all we agree that $d(t, q+2) = d(t, q) - 1 + 1 = 1$.

We have also that

$$d(t, q_1 - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q+1+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 2$ we have finished this step. Now suppose $m > 2$. Let $i = 1 \ldots m-2$ and suppose $d(t, q_i) = 1$. We'll show that $d(t, q_{i+1}) = 1$ also holds.

In fact

$$\begin{aligned} d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) &= d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = \\ &= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{\text{'('}, \text{')'}\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each $i = 1 \ldots m-1$ $d(t, q_i) = 1$.

We now want to show that $d(t, \ell(t)) = 1$.

If $m = 1$ then

$$d(t, \ell(t) - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q + 2) + d(\varphi_1, \ell(\varphi_1)) = d(t, q) + d(\varphi_1, \ell(\varphi_1)) = \\ = 1 + d(\varphi_m, \ell(\varphi_m)).$$

If $m > 1$ then

$$d(t, \ell(t) - 1) = d(t, q_{m-1} + \ell(\varphi_m)) = d(t, q_{m-1} + 1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m)).$$

If $t[\ell(t) - 1] = \varphi_m[\ell(\varphi_m)] =$ ')' then $d(t, \ell(t)) = d(t, \ell(t) - 1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$. Else $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] \notin \{$'(', ')'$\}$ so $d(t, \ell(t)) = d(t, \ell(t)-1) = 1+d(\varphi_m, \ell(\varphi_m)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '('. It's also true that $t[\ell(t)] =$ ')' and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \ldots, \ell(t)\}$ and ($t[\alpha] =$ ':' or $t[\alpha] =$ ',' or $t[\alpha] =$ ')').

If $\alpha \in \{q, q_1, \ldots, q_{m-1}, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $(\alpha > 1) \wedge (\alpha < q)$,
b. $(m = 1) \wedge (\alpha > q + 1) \wedge (\alpha < \ell(t))$,
c. $(m > 1) \wedge (\alpha > q + 1) \wedge (\alpha < q_1)$,
d. $(m > 2) \wedge (\exists i = 1 \ldots m - 2 : (\alpha > q_i) \wedge (\alpha < q_{i+1}))$,
e. $(m > 1) \wedge (\alpha > q_{m-1}) \wedge (\alpha < \ell(t))$.

In the situation a. $\varphi[\alpha - 1] = t[\alpha]$,

$$d(t, \alpha) = d(t, 1 + (\alpha - 1)) = d(t, 2) + d(\varphi, \alpha - 1) = 1 + d(\varphi, \alpha - 1) \geqslant 2.$$

In the situation b. we have

$$q + 1 < \alpha < \ell(t),$$
$$0 < \alpha - (q + 1) < \ell(t) - (q + 1),$$
$$1 \leqslant \alpha - (q + 1) \leqslant \ell(t) - q - 2 = \ell(\varphi_1),$$
$$\varphi_1[\alpha - (q + 1)] = t[\alpha],$$

$$d(t, \alpha) = d(t, q + 1 + (\alpha - (q + 1))) = d(t, q + 2) + d(\varphi_1, \alpha - (q + 1)) = \\ = 1 + d(\varphi_1, \alpha - (q + 1)) \geqslant 2.$$

In the situation c. we have

$$q + 1 < \alpha < q_1,$$
$$0 < \alpha - (q + 1) < q_1 - (q + 1),$$
$$1 \leqslant \alpha - (q + 1) \leqslant q_1 - q - 2 = \ell(\varphi_1),$$
$$\varphi_1[\alpha - (q + 1)] = t[\alpha],$$

$$d(t, \alpha) = d(t, q + 1 + (\alpha - (q + 1))) = d(t, q + 2) + d(\varphi_1, \alpha - (q + 1)) = \\ = 1 + d(\varphi_1, \alpha - (q + 1)) \geqslant 2.$$

In the situation d. we have

$$q_i < \alpha < q_{i+1},$$
$$0 < \alpha - q_i < q_{i+1} - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant q_{i+1} - q_i - 1 = \ell(\varphi_{i+1}),$$
$$\varphi_{i+1}[\alpha - q_i] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_{i+1}, \alpha - q_i) = \\ = 1 + d(\varphi_{i+1}, \alpha - q_i) \geqslant 2. \end{aligned}$$

In the situation e. we have

$$q_{m-1} < \alpha < \ell(t),$$
$$0 < \alpha - q_{m-1} < \ell(t) - q_{m-1},$$
$$1 \leqslant \alpha - q_{m-1} \leqslant \ell(t) - q_{m-1} - 1 = \ell(\varphi_m),$$
$$\varphi_m[\alpha - q_{m-1}] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t, q_{m-1} + (\alpha - q_{m-1})) = d(t, q_{m-1} + 1) + d(\varphi_m, \alpha - q_{m-1}) = \\ = 1 + d(\varphi_m, \alpha - q_{m-1}) \geqslant 2. \end{aligned}$$

Let $t \in \mathbf{E'_d(n+1,k)}$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in \mathcal{F}$, $\varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (f)(\varphi_1, \ldots, \varphi_m)$.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = (f)(\varphi_1, \varphi_2)$, if $m = 3$ then $t = (f)(\varphi_1, \varphi_2, \varphi_3)$ and so on. In this explicit representation we can see explicit occurrences of the symbols ',' and ')'. The occurrences of ')' are clearly in positions 3 and $\ell(t)$. There are explicit occurrences of ',' only when $m > 1$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of explicit occurrences of ','.

It is immediate to see that $d(t, 3) = 1$.

If $m > 1$ we can prove that for each $i = 1 \ldots m-1$ $d(t, q_i) = 1$.

We have $d(t, q_1 - 1) = d(t, 4 + \ell(\varphi_1)) = d(t, 4+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$, so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 2$ we have finished this step. Now suppose $m > 2$. Let $i = 1 \ldots m-2$ and suppose $d(t, q_i) = 1$. We'll show that $d(t, q_{i+1}) = 1$ also holds.

In fact

$$\begin{aligned} d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = \\ = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1}-1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{`(', `)'\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1}-1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each $i = 1 \dots m-1$ $d(t, q_i) = 1$.

We now want to show that $d(t, \ell(t)) = 1$.

If $m = 1$ then

$$\begin{aligned} d(t, \ell(t)-1) &= d(t, 4+\ell(\varphi_1)) = d(t, 4+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)) = \\ &= 1 + d(\varphi_m, \ell(\varphi_m)). \end{aligned}$$

If $m > 1$ then

$$d(t, \ell(t)-1) = d(t, q_{m-1}+\ell(\varphi_m)) = d(t, q_{m-1}+1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m)).$$

If $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] = `)'$ then $d(t, \ell(t)) = d(t, \ell(t)-1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$. Else $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] \notin \{`(', `)'\}$ so $d(t, \ell(t)) = d(t, \ell(t)-1) = 1 + d(\varphi_m, \ell(\varphi_m)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq `('$. It's also true that $t[\ell(t)] = `)'$ and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \dots, \ell(t)\}$ and ($t[\alpha] = `:'$ or $t[\alpha] = `,'$ or $t[\alpha] = `)'$).

If $\alpha \in \{3, q_1, \dots, q_{m-1}, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $(m = 1) \wedge (\alpha > 4) \wedge (\alpha < \ell(t))$,
b. $(m > 1) \wedge (\alpha > 4) \wedge (\alpha < q_1)$,
c. $(m > 2) \wedge (\exists i = 1 \dots m-2 : (\alpha > q_i) \wedge (\alpha < q_{i+1}))$,
d. $(m > 1) \wedge (\alpha > q_{m-1}) \wedge (\alpha < \ell(t))$.

In the situation a. we have

$$4 < \alpha < \ell(t),$$

$$0 < \alpha - 4 < \ell(t) - 4,$$

$$1 \leqslant \alpha - 4 \leqslant \ell(t) - 4 - 1 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 4] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, 4 + (\alpha - 4)) = d(t, 4+1) + d(\varphi_1, \alpha - 4) = \\ &= 1 + d(\varphi_1, \alpha - 4) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$4 < \alpha < q_1,$$

$$0 < \alpha - 4 < q_1 - 4,$$

$$1 \leqslant \alpha - 4 \leqslant q_1 - 4 - 1 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 4] = t[\alpha],$$

$$d(t,\alpha) = d(t, 4 + (\alpha - 4)) = d(t, 4+1) + d(\varphi_1, \alpha - 4) = \\ = 1 + d(\varphi_1, \alpha - 4) \geqslant 2.$$

In the situation c. we have

$$q_i < \alpha < q_{i+1},$$
$$0 < \alpha - q_i < q_{i+1} - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant q_{i+1} - q_i - 1 = \ell(\varphi_{i+1}),$$
$$\varphi_{i+1}[\alpha - q_i] = t[\alpha],$$

$$d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_{i+1}, \alpha - q_i) = \\ = 1 + d(\varphi_{i+1}, \alpha - q_i) \geqslant 2.$$

In the situation d. we have

$$q_{m-1} < \alpha < \ell(t),$$
$$0 < \alpha - q_{m-1} < \ell(t) - q_{m-1},$$
$$1 \leqslant \alpha - q_{m-1} \leqslant \ell(t) - q_{m-1} - 1 = \ell(\varphi_m),$$
$$\varphi_m[\alpha - q_{m-1}] = t[\alpha],$$

$$d(t,\alpha) = d(t, q_{m-1} + (\alpha - q_{m-1})) = d(t, q_{m-1} + 1) + d(\varphi_m, \alpha - q_{m-1}) = \\ = 1 + d(\varphi_m, \alpha - q_{m-1}) \geqslant 2.$$

Let $t \in \mathbf{E'_e(n+1,k)}$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1\ldots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

If we know m we can provide an 'explicit representation' of t. In fact if $m = 2$ then $t = \{\}(x_1 : \varphi_1, x_2 : \varphi_2, \phi)$, if $m = 3$ then $t = \{\}(x_1 : \varphi_1, x_2 : \varphi_2, x_3 : \varphi_3, \phi)$, and so on. In this explicit representation of t we can see explict occurrences of the symbols ',' and ':'. We indicate with $q_1, \ldots, q_m$ the positions of the explicit occurrences of ':' and with $r_1 \ldots r_m$ the positions of the explicit occurrences of ','. The only explicit occurrence of ')' has the position $\ell(t)$. We want to show that for each $i = 1\ldots m$ $d(t, q_i) = 1, d(t, r_i) = 1$ and that $d(t, \ell(t)) = 1$.

It is obvious that $d(t, q_1) = 1$. Moreover

$$d(t, r_1 - 1) = d(t, q_1 + (r_1 - 1 - q_1)) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = \\ = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 1$ we have shown that for each $i = 1 \dots m$ $d(t, q_i) = 1, d(t, r_i) = 1$. Now suppose $m > 1$, let $i = 1 \dots m-1$ and suppose $d(t, q_i) = 1, d(t, r_i) = 1$. We show that $d(t, q_{i+1}) = 1, d(t, r_{i+1}) = 1$.

We have $q_{i+1} = r_i + 2$ and it is immediate that $d(t, q_{i+1}) = 1$. Moreover

$$\begin{aligned} d(t, r_{i+1} - 1) &= d(t, q_{i+1} + (r_{i+1} - 1 - q_{i+1})) = d(t, q_{i+1} + \ell(\varphi_{i+1})) = \\ &= d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ‘)’ then
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$‘(’, ‘)’$\}$ so
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Furthermore

$$\begin{aligned} d(t, \ell(t) - 1) &= d(t, r_m + (\ell(t) - 1 - r_m)) = d(t, r_m + \ell(\phi)) = \\ &= d(t, r_m + 1) + d(\phi, \ell(\phi)) = 1 + d(\phi, \ell(\phi)). \end{aligned}$$

If $t[\ell(t) - 1] = \phi[\ell(\phi)] =$ ‘)’ then $d(t, \ell(t)) = d(t, \ell(t) - 1) - 1 = d(\phi, \ell(\phi)) = 1$.
Else $t[\ell(t) - 1] = \phi[\ell(\phi)] \notin \{$‘(’, ‘)’$\}$ so $d(t, \ell(t)) = d(t, \ell(t) - 1) = 1 + d(\phi, \ell(\phi)) = 1$.

Let’s now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ ‘(’. It’s also true that $t[\ell(t)] =$ ‘)’ and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \dots, \ell(t)\}$ and ($t[\alpha] =$ ‘:’ or $t[\alpha] =$ ‘,’ or $t[\alpha] =$ ‘)’).

If $\alpha \in \{q_1, \dots, q_m, r_1, \dots, r_m, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $\exists i = 1 \dots m$ such that $q_i < \alpha < r_i$,
b. $r_m < \alpha < \ell(t)$.

In the situation a. we have

$$q_i < \alpha < r_i,$$
$$0 < \alpha - q_i < r_i - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant r_i - q_i - 1 = \ell(\varphi_i),$$
$$\varphi_i[\alpha - q_i] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_i, \alpha - q_i) = \\ &= 1 + d(\varphi_i, \alpha - q_i) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$r_m < \alpha < \ell(t),$$
$$0 < \alpha - r_m < \ell(t) - r_m,$$
$$1 \leqslant \alpha - r_m \leqslant \ell(t) - r_m - 1 = \ell(\phi),$$
$$\phi[\alpha - r_m] = t[\alpha],$$

$$d(t,\alpha) = d(t, r_m + (\alpha - r_m)) = d(t, r_m + 1) + d(\phi, \alpha - r_m) = \\ = 1 + d(\phi, \alpha - r_m) \geqslant 2.$$

■

Proof of 2.1.1.

We need to prove that for each $k \in K(n+1)$ such that $k \neq \epsilon$ and for each $\sigma \in \Xi(k)$ there exist a positive integer m, a function $x\colon \{1,\dots,m\} \to \mathcal{V}$, a function $\varphi\colon \{1,\dots,m\} \to E(n+1)$, a function $s\colon \{1,\dots,m\} \to M(n+1)$ such that

- for each $i,j \in \{1\dots m\}$ $(i \neq j \to x_i \neq x_j)$,
- $k = (x,\varphi)$,
- $\sigma = (x,s)$.

We can observe the following facts.

$$E(n) = \bigcup_{k\in K(n)} E(n,k) \subseteq \bigcup_{k\in K(n)} E(n+1,k) \subseteq E(n+1),$$

for each $k \in K(n)$

$$E_s(n,k) = \{t | t \in E(n,k), \forall\sigma \in \Xi(k)\, \#(k,t,\sigma) \text{ is a set}\} \subseteq \\ \subseteq \{t | t \in E(n+1,k), \forall\sigma \in \Xi(k)\, \#(k,t,\sigma) \text{ is a set}\} = E_s(n+1,k),$$

$$M(n,k) = \bigcup_{t\in E_s(n,k)} M(k,t) \subseteq \bigcup_{t\in E_s(n+1,k)} M(k,t) = M(n+1,k),$$

$$M(n) = \bigcup_{k\in K(n)} M(n,k) \subseteq \bigcup_{k\in K(n)} M(n+1,k) \subseteq M(n+1).$$

Now let $k \in K(n+1)$ such that $k \neq \epsilon$, $\sigma \in \Xi(k)$.

If $k \in K(n)$ by our assumption there exist a positive integer m, a function $x\colon \{1,\dots,m\} \to \mathcal{V}$, a function $\varphi\colon \{1,\dots,m\} \to E(n)$, a function $s\colon \{1,\dots,m\} \to M(n)$ such that

- for each $i,j \in \{1\dots m\}$ $(i \neq j \to x_i \neq x_j)$,
- $k = (x,\varphi)$,
- $\sigma = (x,s)$.

This completes the proof in the case $k \in K(n)$.

Now suppose $k \notin K(n)$, i.e. $k \in K(n)^+$. Then there exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$.

By lemma 2.1.3 there exist $\rho \in \Xi(h), s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$.

We can observe that $\phi \in E(n) \subseteq E(n+1)$, $s \in M(h,\phi) \subseteq M(n,h) \subseteq M(n) \subseteq M(n+1)$.

If $dom(h) = \emptyset$ then $h = \epsilon, \rho = \epsilon$ and $dom(\rho) = \emptyset$. So we can define three functions x, φ, u over the domain $\{1\}$ by setting $x(1) = y,\ \varphi(1) = \phi,\ u(1) = s$. It clearly results $k = (x, \varphi),\ \sigma = (x, u)$, and in this case the proof is finished.

We still need to examine the case where $dom(h) \neq \emptyset$ and so $h \neq \epsilon$. We can apply our assumtpion 2.1.1 to h and ρ and determine that there exist a positive integer m, a function $x\colon \{1, \dots, m\} \to \mathcal{V}$, a function $\varphi\colon \{1, \dots, m\} \to E(n)$, a function $u\colon \{1, \dots, m\} \to M(n)$ such that

- for each $i, j \in \{1 \dots m\}$ $(i \neq j \to x_i \neq x_j)$,
- $h = (x, \varphi)$,
- $\rho = (x, u)$.

We define three new functions x', φ', u' over the domain $\{1, \dots, m+1\}$ as follows: for each $\alpha = 1 \dots m$ $x'(\alpha) = x(\alpha),\ \varphi'(\alpha) = \varphi(\alpha),\ u'(\alpha) = u(\alpha),\ x'(m+1) = y,\ \varphi'(m+1) = \phi,\ u'(m+1) = s$.

Since $k = h + (y, \phi)$ we have $k = (x', \varphi')$, and since $\sigma = \rho + (y, s)$ we have $\sigma = (x', u')$. We also observe that $y \notin var(h)$ so for each $i = 1 \dots m$ $y \neq x_i$. This completes the proof. ∎

Proof of 2.1.2.

We need to prove that for each $k \in K(n+1)$

$$\begin{aligned}&(k = \epsilon)\\ &\vee (\exists g \in K(n), z \in \mathcal{V} - var(g), \psi \in E_s(n, g) :\\ &\qquad k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\, \sigma \in \Xi(g), s \in \#(g, \psi, \sigma)\}).\end{aligned}$$

If $k \in K(n)$ we can apply assumption 2.1.2 and get

$$\begin{aligned}&(k = \epsilon)\\ &\vee ((n > 1) \wedge \exists g \in K(n-1), z \in \mathcal{V} - var(g), \psi \in E_s(n-1, g) :\\ &\qquad k = g + (z, \psi) \wedge \Xi(k) = \{\sigma + (z, s) |\, \sigma \in \Xi(g), s \in \#(g, \psi, \sigma)\}).\end{aligned}$$

In the case $k \neq \epsilon$ we have $g \in K(n),\ \psi \in E_s(n, g)$. Therefore when $k \in K(n)$ our result is verified.

Now suppose $k \notin K(n)$, i.e. $k \in K(n)^+$. There exist $h \in K(n),\ \phi \in E_s(n, h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. By lemma 2.1.3 we have also

$$\Xi(k) = \{\sigma + (y, s) |\ \sigma \in \Xi(h),\ s \in \#(h, \phi, \sigma)\}\,,$$

and this completes the proof. ∎

Proof of 2.1.11.

We need to prove that for each $k \in K(n+1),\ h \in \mathcal{R}(k)$: $h \neq k$ we have $h \in K(n)$ and for each $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)}$ then $\rho \in \Xi(h)$.

If $k \in K(n)$ since $k \neq \epsilon$ (and therefore $n > 1$) we can exploit assumption 2.1.11 and say that $h \in K(n-1) \subseteq K(n)$ and for each $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)}$ then $\rho \in \Xi(h)$.

Now suppose $k \notin K(n)$, i.e. $k \in K(n)^+$. There exist $g \in K(n)$, $\phi \in E_s(n, g)$, $y \in (\mathcal{V} - var(g))$ such that $k = g + (y, \phi)$. By lemma 2.1.3 we have also

$$\Xi(k) = \{\delta + (y, s) |\ \delta \in \Xi(g),\ s \in \#(g, \phi, \delta)\} .$$

By lemma 2.3 $h \in \mathcal{R}(g)$ and we can distinguish two cases: $h = g$ and $h \neq g$.

If $h = g$ then $h \in K(n)$. Let $\sigma \in \Xi(k)$ and we define $\rho = \sigma_{/dom(h)}$. There exist $\delta \in \Xi(g)$, $s \in \#(g, \phi, \delta)$ such that $\sigma = \delta + (y, s)$. We have

$$\delta = \sigma_{/dom(\delta)} = \sigma_{/dom(g)} = \sigma_{/dom(h)} = \rho,$$

so $\rho \in \Xi(h)$.

If $h \neq g$ then we can apply assumption 2.1.11 to g and h and obtain that $h \in K(n-1)$, for each $\delta \in \Xi(g)$ if we define $\rho = \delta_{/dom(h)}$ then $\rho \in \Xi(h)$. So $h \in K(n)$. Moreover, let $\sigma \in \Xi(k)$ and define $\rho = \sigma_{/dom(h)}$. There exist $\delta \in \Xi(g)$, $s \in \#(g, \phi, \delta)$ such that $\sigma = \delta + (y, s)$, so (with the assumptions that $\sigma = (x_\sigma, \varphi_\sigma)$, $\delta = (x_\delta, \varphi_\delta)$)

$$\begin{aligned} \rho = \sigma_{/dom(h)} &= ((x_\sigma)_{/dom(h)}, (\varphi_\sigma)_{/dom(h)}) = \\ &= (((x_\sigma)_{/dom(\delta)})_{/dom(h)}, ((\varphi_\sigma)_{/dom(\delta)})_{/dom(h)}) = \\ &= ((x_\delta)_{/dom(h)}, (\varphi_\delta)_{/dom(h)}) = \delta_{/dom(h)} \in \Xi(h). \end{aligned}$$

■

Proof of 2.1.10.

We need to prove that for each $k \in K(n+1)$, $t \in E(n+1,k)$ one and only one of these five alternative situations is verified:

a.

$$t \in \mathcal{C},\ \forall \sigma \in \Xi(k)\ \#(k,t,\sigma) = \#(t),\ V_f(t) = \emptyset,\ V_b(t) = \emptyset.$$

b.

if we set $k = (x,\varphi)$ then $\exists i \in dom(k) : (t = x_i,\ \forall \sigma = (x,s) \in \Xi(k)\ \#(k,t,\sigma) = s_i)$, $V_f(t) = \{t\},\ V_b(t) = \emptyset$.

c.

$$\begin{aligned}
&\exists h \in K(n) : h \sqsubseteq k, \exists m \text{ positive integer } , \varphi, \varphi_1, \dots \varphi_m \in E(n,h) :\\
&\quad t = (\varphi)(\varphi_1, \dots, \varphi_m),\ t \in E(n+1,h),\\
&\quad \forall \rho \in \Xi(h)\ (\ \#(h,\varphi,\rho) \text{ is a function with } m \text{ arguments},\\
&\qquad (\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho)) \text{ is a member of the domain of } \#(h,\varphi,\rho),\\
&\qquad \#(h,t,\rho) = \#(h,\varphi,\rho)(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho))\ ),\\
&\quad V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),\\
&\quad V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m),\\
&\quad \forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma \text{ it results } \#(k,t,\sigma) = \#(h,t,\rho).
\end{aligned}$$

d.

$$\begin{aligned}
&\exists h \in K(n) : h \sqsubseteq k, \exists f \in \mathcal{F},\ m \text{ positive integer } , \varphi_1, \dots \varphi_m \in E(n,h) :\\
&\quad t = (f)(\varphi_1, \dots, \varphi_m),\ t \in E(n+1,h),\\
&\quad \forall \rho \in \Xi(h)\ (\ A_f(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho)),\\
&\qquad \#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho), \dots, \#(h,\varphi_m,\rho))\ ),\\
&\quad V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),\\
&\quad V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m),\\
&\quad \forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma \text{ it results } \#(k,t,\sigma) = \#(h,t,\rho).
\end{aligned}$$

e.

there exist

$h \in K(n) : h \sqsubseteq k$,

a positive integer m,

a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$

$x_i \in \mathcal{V} - var(h)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$,

a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$

$\varphi_i \in E(n)$,

$\phi \in E(n)$

such that

$\mathcal{E}(n, h, m, x, \varphi, \phi)$,

$t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$, $t \in E(n+1, h)$,

for each $\rho \in \Xi(h)$ $\#(h, t, \rho) = \{\#(h'_m, \phi, \rho'_m) | \ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}$

(where $h'_1 = h + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$

$h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$),

if $m = 1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\})$, $V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,

if $m > 1$

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup$

$\cup\, (V_f(\phi) - \{x_1, \ldots, x_m\})$,

$V_b(t) = \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi)$,

$\forall \sigma \in \Xi(k), \rho \in \Xi(h) : \rho \sqsubseteq \sigma$ it results $\#(k, t, \sigma) = \#(h, t, \rho)$.

We recall that

$$E(n+1, k) = E'(n, k) \cup E'_a(n+1, k) \cup E'_b(n+1, k) \cup E'_c(n+1, k) \cup E'_d(n+1, k) \cup E'_e(n+1, k).$$

So we need to prove that

- for each $t \in E(n, k)$ one of the five alternative situations is verified;
- for each $w \in \{a, b, c, d, e\}$ and $t \in E'_w(n+1, k)$ one of the five alternative situations is verified.

Suppose $t \in \mathbf{E}'(\mathbf{n}, \mathbf{k})$, this means that $t \in E(n, k)$ and $k \in K(n)$. This case is easily solved, in fact we apply assumption 2.1.10 and obtain that one of the five situations holds at level n, but this means the situation is also verified at level $n+1$.

Let $t \in \mathbf{E}'_{\mathbf{a}}(\mathbf{n+1}, \mathbf{k})$, this means that $t \in E_a(n+1, k)$ and $k \in K(n)^+$. There exist $h \in K(n)$, $\phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. We have $t \in E(n, h)$, so we can apply assumption 2.1.10 to h and t. Assumption 2.1.10 says that one of five alternative situations (referred to h, n) is true; we need to show that the corresponding situation, referred to $k, n+1$ is also true.

Let's consider the situation in which

$$t \in \mathcal{C},\ \forall \rho \in \Xi(h)\ \#(h,t,\rho) = \#(t),\ V_f(t) = \emptyset,\ V_b(t) = \emptyset.$$

In this case for each $\sigma = \rho + (y,s) \in \Xi(k)$

$$\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho) = \#(t).$$

So one of the five alternative situations at level $n+1$ is satisfied.

Consider the situation where $n > 1$, if we set $h = (x,\varphi)$ then $\exists i \in dom(h)$: $(t = x_i,\ \forall \rho = (x,u) \in \Xi(h)\ \#(h,t,\rho) = u_i)$, $V_f(t) = \{t\},\ V_b(t) = \emptyset$.

If we set $k = (x',\varphi')$ then, since $k = h + (y,\phi)$, we have $i \in dom(k)$, $x'_i = x_i = t$. Moreover given $\sigma = (x',u') \in \Xi(k)$ there exist $\rho \in \Xi(h),\ s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, therefore $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho) = u_i = u'_i$.

Consider the situation where

$n > 1$,
$\exists g \in K(n-1) : g \sqsubseteq h, \exists m$ positive integer $, \varphi, \varphi_1, \dots \varphi_m \in E(n-1,g)$:
$t = (\varphi)(\varphi_1,\dots,\varphi_m),\ t \in E(n,g)$,
$\forall \delta \in \Xi(g)$ ($\#(g,\varphi,\delta)$ is a function with m arguments,
$(\#(g,\varphi_1,\delta),\dots,\#(g,\varphi_m,\delta))$ is a member of the domain of $\#(g,\varphi,\delta)$,
$\#(g,t,\delta) = \#(g,\varphi,\delta)(\#(g,\varphi_1,\delta),\dots,\#(g,\varphi_m,\delta))$),
$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
$\forall \rho \in \Xi(h), \delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h,t,\rho) = \#(g,t,\delta)$.

We have

$g \in K(n) : g \sqsubseteq k, \exists m$ positive integer $, \varphi, \varphi_1, \dots \varphi_m \in E(n,g)$:
$t = (\varphi)(\varphi_1,\dots,\varphi_m),\ t \in E(n+1,g)$,
$\forall \delta \in \Xi(g)$ ($\#(g,\varphi,\delta)$ is a function with m arguments,
$(\#(g,\varphi_1,\delta),\dots,\#(g,\varphi_m,\delta))$ is a member of the domain of $\#(g,\varphi,\delta)$,
$\#(g,t,\delta) = \#(g,\varphi,\delta)(\#(g,\varphi_1,\delta),\dots,\#(g,\varphi_m,\delta))$),
$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$.

Moreover, given $\sigma = \rho + (y,s) \in \Xi(k),\ \delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$ by lemma 2.3 we have $\delta \sqsubseteq \rho$ and so it results $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = \#(h,t,\rho) = \#(g,t,\delta)$.

Consider the situation where

$n > 1,$

$\exists g \in K(n-1) : g \sqsubseteq h, \exists f \in \mathcal{F},\ m$ positive integer $, \varphi_1, \dots \varphi_m \in E(n-1, g) :$

$t = (f)(\varphi_1, \dots, \varphi_m),\ t \in E(n, g),$

$\forall \delta \in \Xi(g)\ (\ A_f(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta)),$

$\#(g, t, \delta) = P_f(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta))\ ),$

$V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),$

$V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m),$

$\forall \rho \in \Xi(h), \delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h, t, \rho) = \#(g, t, \delta).$

We have

$g \in K(n) : g \sqsubseteq k, \exists f \in \mathcal{F},\ m$ positive integer $, \varphi_1, \dots \varphi_m \in E(n, g) :$

$t = (f)(\varphi_1, \dots, \varphi_m),\ t \in E(n+1, g),$

$\forall \delta \in \Xi(g)\ (\ A_f(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta)),$

$\#(g, t, \delta) = P_f(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta))\ ),$

$V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),$

$V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m).$

Moreover, given $\sigma = \rho + (y, s) \in \Xi(k),\ \delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$ by lemma 2.3 we have $\delta \sqsubseteq \rho$ and so it results $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g, t, \delta).$

Consider the situation where

$n > 1$,

there exist

$g \in K(n-1) : g \sqsubseteq h$,

a positive integer m,

a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$

$x_i \in \mathcal{V} - var(g),$ and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$,

a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$

$\varphi_i \in E(n-1)$,

$\phi \in E(n-1)$

such that

$\mathcal{E}(n-1, g, m, x, \varphi, \phi)$,

$t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi),\ t \in E(n, g)$,

for each $\delta \in \Xi(g)$ $\#(g, t, \delta) = \{\#(g'_m, \phi, \delta'_m) | \ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}$

(where $g'_1 = g + (x_1, \varphi_1),$ and if $m > 1$ for each $i = 1 \ldots m-1$

$g'_{i+1} = g'_i + (x_{i+1}, \varphi_{i+1})$),

if $m = 1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}),\ V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,

if $m > 1$

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup$

$\cup (V_f(\phi) - \{x_1, \ldots, x_m\})$,

$V_b(t) = \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi)$,

$\forall \rho \in \Xi(h), \delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h, t, \rho) = \#(g, t, \delta)$.

We have

$g \in K(n) : g \sqsubseteq k$,
a positive integer m,
a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$
$x_i \in \mathcal{V} - var(g)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$,
a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$
$\varphi_i \in E(n)$,
$\phi \in E(n)$

such that

$\mathcal{E}(n, g, m, x, \varphi, \phi)$,
$t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$, $t \in E(n+1, g)$,
for each $\delta \in \Xi(g)$ $\#(g, t, \delta) = \{\#(g'_m, \phi, \delta'_m) | \ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}$
(where $g'_1 = g + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$
$g'_{i+1} = g'_i + (x_{i+1}, \varphi_{i+1})$),
if $m = 1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\})$, $V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,
if $m > 1$
$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \cdots \cup (V_f(\varphi_m) - \{x_1, \ldots, x_{m-1}\}) \cup$
$\cup\, (V_f(\phi) - \{x_1, \ldots, x_m\})$,
$V_b(t) = \{x_1, \ldots, x_m\} \cup V_b(\varphi_1) \cup \cdots \cup V_b(\varphi_m) \cup V_b(\phi)$.

Moreover, given $\sigma = \rho + (y, s) \in \Xi(k)$, $\delta \in \Xi(g)$: $\delta \sqsubseteq \sigma$ by lemma 2.3 we have $\delta \sqsubseteq \rho$ and so it results $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1,k,a)} = \#(h, t, \rho) = \#(g, t, \delta)$.

Let $t \in \mathbf{E'_b(n+1, k)}$, this means that $t \in E_b(n+1, k)$ and $k \in K(n)^+$. There exist $h \in K(n)$, $\phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. Let $k = (x', \varphi')$, there exists a positive integer m such that $dom(k) = \{1, \ldots, m\}$, so $t = y = x'_m$. Let $\sigma = (x', u') \in \Xi(k)$, there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$. We have

$$\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1,k,b)} = s = u'_m.$$

Moreover

$$V_f(t) = V_f(t)_{(n+1,k,b)} = \{y\} = \{t\}; \ V_b(t) = V_b(t)_{(n+1,k,b)} = \emptyset.$$

Let $t \in \mathbf{E'_c(n+1, k)}$, this means that $t \in E_c(n+1, k)$ and $k \in K(n)$. As a consequence of $t \in E_c(n+1,k)$:

$\exists m$ positive integer $,\varphi,\varphi_1,\dots\varphi_m \in E(n,k)$:

$t = (\varphi)(\varphi_1,\dots,\varphi_m),\ t \in E(n+1,k),$

$\forall \rho \in \Xi(k)$ ($\#(k,\varphi,\rho)$ is a function with m arguments,

$(\#(k,\varphi_1,\rho),\dots,\#(k,\varphi_m,\rho))$ is a member of the domain of $\#(k,\varphi,\rho)$,

$\#(k,t,\rho) = \#(k,t,\rho)_{(n+1,k,c)} = \#(k,\varphi,\rho)(\#(k,\varphi_1,\rho),\dots,\#(k,\varphi_m,\rho))$),

$V_f(t) = V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),$

$V_b(t) = V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m),$

$\forall \sigma \in \Xi(k), \rho \in \Xi(k) : \rho \sqsubseteq \sigma$ it results $\rho = \sigma$ and obviously $\#(k,t,\sigma) = \#(k,t,\rho)$.

Let $t \in \mathbf{E'_d(n+1,k)}$, this means that $t \in E_d(n+1,k)$ and $k \in K(n)$. As a consequence of $t \in E_d(n+1,k)$:

$\exists f \in \mathcal{F},\ m$ positive integer $,\varphi_1,\dots\varphi_m \in E(n,k)$:

$t = (f)(\varphi_1,\dots,\varphi_m),\ t \in E(n+1,k),$

$\forall \rho \in \Xi(k)$ ($A_f(\#(k,\varphi_1,\rho),\dots,\#(k,\varphi_m,\rho)),$

$\#(k,t,\rho) = \#(k,t,\rho)_{(n+1,k,d)} = P_f(\#(k,\varphi_1,\rho),\dots,\#(k,\varphi_m,\rho))$),

$V_f(t) = V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m),$

$V_b(t) = V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m),$

$\forall \sigma \in \Xi(k), \rho \in \Xi(k) : \rho \sqsubseteq \sigma$ it results $\rho = \sigma$ and obviously $\#(k,t,\sigma) = \#(k,t,\rho)$.

Let $t \in \mathbf{E'_e(n+1,k)}$, this means that $t \in E_e(n+1,k)$ and $k \in K(n)$. As a consequence of $t \in E_e(n+1,k)$:

there exist

a positive integer m,

a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$

$x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1\dots m$ $i \neq j \to x_i \neq x_j$,

a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$

$\varphi_i \in E(n)$,

$\phi \in E(n)$

such that

$\mathcal{E}(n,k,m,x,\varphi,\phi)$,

$t = \{\}(x_1 : \varphi_1,\dots,x_m : \varphi_m,\phi)$, $t \in E(n+1,k)$,

for each $\rho \in \Xi(k)$

$\#(k,t,\rho) = \#(k,t,\rho)_{(n+1,k,e)} = \{\#(k'_m,\phi,\rho'_m) |\ \rho'_m \in \Xi(k'_m), \rho \sqsubseteq \rho'_m\}$

(where $k'_1 = k + (x_1,\varphi_1)$, and if $m > 1$ for each $i = 1\dots m-1$

$k'_{i+1} = k'_i + (x_{i+1},\varphi_{i+1})$),

if $m = 1$

$V_f(t) = V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\})$,

$V_b(t) = V_b(t)_{(n+1,k,e)} = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,

if $m > 1$

$V_f(t) = V_f(t)_{(n+1,k,e)} = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup$

$\cup \dots \cup (V_f(\varphi_m) - \{x_1,\dots,x_{m-1}\}) \cup (V_f(\phi) - \{x_1,\dots,x_m\})$,

$V_b(t) = V_b(t)_{(n+1,k,e)} = \{x_1,\dots,x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi)$,

$\forall \sigma \in \Xi(k), \rho \in \Xi(k) : \rho \sqsubseteq \sigma$ it results $\rho = \sigma$ and obviously $\#(k,t,\sigma) = \#(k,t,\rho)$.

■

Proof of 2.1.9.

Let $k = (x,\vartheta)$, $h = (y,\chi) \in K(n)$ such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \vartheta_i = \chi_j$. Let $t \in E(n+1,k) \cap E(n+1,h)$. Let $\sigma = (x,s) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to s_i = r_j$. We need to show that $\#(k,t,\sigma) = \#(h,t,\rho)$.

We have proved that assumption 2.1.10 is true at level $n+1$, so

- $t \in E(n+1,k)$ implies that one of five alternative situations is verified,
- $t \in E(n+1,h)$ implies that one of five alternative situations is verified.

Suppose situation a. is the true situation caused by $t \in E(n+1,k)$. We have

$$t \in \mathcal{C},\ \#(k,t,\sigma) = \#(t).$$

This causes situation a. is also the true situation due to $t \in E(n+1,h)$, so

$$\#(h,t,\rho) = \#(t) = \#(k,t,\sigma).$$

The same kind of reasoning applies for the other situations. We now analyze the case where situation b. is the true situation caused by $t \in E(n+1,k)$. We have

$$\exists i \in dom(k) : (t = x_i,\ \#(k,t,\sigma) = s_i)$$
$$\exists j \in dom(h) : (t = y_j,\ \#(h,t,\rho) = r_j)$$

Since $x_i = t = y_j$ we have $\#(k,t,\sigma) = s_i = r_j = \#(h,t,\rho)$.

We turn to examine the case where situation c. is the true situation caused by $t \in E(n+1,k)$. We have

$\exists \kappa \in K(n) : \kappa \sqsubseteq k, \exists m$ positive integer $, \varphi, \varphi_1, \dots \varphi_m \in E(n,\kappa)$:
$t = (\varphi)(\varphi_1, \dots, \varphi_m),\ t \in E(n+1,\kappa)$,
$\forall \eta \in \Xi(\kappa)$ ($\#(\kappa,\varphi,\eta)$ is a function with m arguments,
$(\#(\kappa,\varphi_1,\eta), \dots, \#(\kappa,\varphi_m,\eta))$ is a member of the domain of $\#(\kappa,\varphi,\eta)$,
$\#(\kappa,t,\eta) = \#(\kappa,\varphi,\eta)(\#(\kappa,\varphi_1,\eta), \dots, \#(\kappa,\varphi_m,\eta))$),
$V_f(t) = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m)$,
$V_b(t) = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m)$,
$\forall \eta \in \Xi(\kappa) : \eta \sqsubseteq \sigma$ it results $\#(k,t,\sigma) = \#(\kappa,t,\eta)$.

$\exists g \in K(n) : g \sqsubseteq h, \exists p$ positive integer $, \psi, \psi_1, \dots \psi_p \in E(n,g)$:
$t = (\psi)(\psi_1, \dots, \psi_p),\ t \in E(n+1,g)$,
$\forall \delta \in \Xi(g)$ ($\#(g,\psi,\delta)$ is a function with p arguments,
$(\#(g,\psi_1,\delta), \dots, \#(g,\psi_p,\delta))$ is a member of the domain of $\#(g,\psi,\delta)$,
$\#(g,t,\delta) = \#(g,\psi,\delta)(\#(g,\psi_1,\delta), \dots, \#(g,\psi_p,\delta))$),
$V_f(t) = V_f(\psi) \cup V_f(\psi_1) \cup \dots \cup V_f(\psi_p)$,
$V_b(t) = V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_p)$,
$\forall \delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h,t,\rho) = \#(g,t,\delta)$.

Of course $p = m,\ \psi = \varphi,\ \forall i = 1 \dots m\ \psi_i = \varphi_i$.

Let $\eta = \sigma_{/dom(\kappa)}$ and $\delta = \rho_{/dom(g)}$. By assumption 2.1.11 we get $\eta \in \Xi(\kappa)$ and $\delta \in \Xi(g)$. Therefore

$$\#(k,t,\sigma) = \#(\kappa,t,\eta) = \#(\kappa,\varphi,\eta)(\#(\kappa,\varphi_1,\eta), \dots, \#(\kappa,\varphi_m,\eta)),$$
$$\#(h,t,\rho) = \#(g,t,\delta) = \#(g,\psi,\delta)(\#(g,\psi_1,\delta), \dots, \#(g,\psi_p,\delta)).$$

We have $\kappa = (x_{/dom(\kappa)}, \vartheta_{/dom(\kappa)})$, $g = (y_{/dom(g)}, \chi_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $\vartheta_i = \chi_j$, $(\vartheta_{/dom(\kappa)})_i = (\chi_{/dom(g)})_j$.

We have also $\eta = (x_{/dom(\kappa)}, s_{/dom(\kappa)})$, $\delta = (y_{/dom(g)}, r_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $s_i = r_j$, $(s_{/dom(\kappa)})_i = (r_{/dom(g)})_j$.

Moreover $\varphi, \varphi_1, \dots \varphi_m \in E(n, \kappa) \cap E(n, g)$. By assumption 2.1.9 we get

$$\begin{aligned}\#(k, t, \sigma) &= \#(\kappa, \varphi, \eta)(\#(\kappa, \varphi_1, \eta), \dots, \#(\kappa, \varphi_m, \eta)) \\ &= \#(g, \varphi, \delta)(\#(g, \varphi_1, \delta), \dots, \#(g, \varphi_m, \delta)) = \#(h, t, \rho).\end{aligned}$$

Next we examine the case where situation d. is the true situation caused by $t \in E(n+1, k)$. We have

$$\begin{aligned}&\exists \kappa \in K(n) : \kappa \sqsubseteq k, \exists f \in \mathcal{F},\ m \text{ positive integer }, \varphi_1, \dots \varphi_m \in E(n, \kappa) : \\ &\quad t = (f)(\varphi_1, \dots, \varphi_m),\ t \in E(n+1, \kappa), \\ &\quad \forall \eta \in \Xi(\kappa)\ (\ A_f(\#(\kappa, \varphi_1, \eta), \dots, \#(\kappa, \varphi_m, \eta)), \\ &\qquad\quad \#(\kappa, t, \eta) = P_f(\#(\kappa, \varphi_1, \eta), \dots, \#(\kappa, \varphi_m, \eta))\ ), \\ &\quad V_f(t) = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m), \\ &\quad V_b(t) = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m), \\ &\quad \forall \eta \in \Xi(\kappa) : \eta \sqsubseteq \sigma \text{ it results } \#(k, t, \sigma) = \#(\kappa, t, \eta).\end{aligned}$$

$$\begin{aligned}&\exists g \in K(n) : g \sqsubseteq h, \exists \theta \in \mathcal{F},\ p \text{ positive integer }, \psi_1, \dots \psi_p \in E(n, g) : \\ &\quad t = (\theta)(\psi_1, \dots, \psi_p),\ t \in E(n+1, g), \\ &\quad \forall \delta \in \Xi(g)\ (\ A_\theta(\#(g, \psi_1, \delta), \dots, \#(g, \psi_p, \delta)), \\ &\qquad\quad \#(g, t, \delta) = P_\theta(\#(g, \psi_1, \delta), \dots, \#(g, \psi_p, \delta))\ ), \\ &\quad V_f(t) = V_f(\psi_1) \cup \dots \cup V_f(\psi_p), \\ &\quad V_b(t) = V_b(\psi_1) \cup \dots \cup V_b(\psi_p), \\ &\quad \forall \delta \in \Xi(g) : \delta \sqsubseteq \rho \text{ it results } \#(h, t, \rho) = \#(g, t, \delta).\end{aligned}$$

Of course $p = m$, $\theta = f$, $\forall i = 1 \dots m\ \psi_i = \varphi_i$.

Let $\eta = \sigma_{/dom(\kappa)}$ and $\delta = \rho_{/dom(g)}$. By assumption 2.1.11 we get $\eta \in \Xi(\kappa)$ and $\delta \in \Xi(g)$. Therefore

$$\#(k, t, \sigma) = \#(\kappa, t, \eta) = P_f(\#(\kappa, \varphi_1, \eta), \dots, \#(\kappa, \varphi_m, \eta)),$$

$$\#(h, t, \rho) = \#(g, t, \delta) = P_\theta(\#(g, \psi_1, \delta), \dots, \#(g, \psi_p, \delta)).$$

We have $\kappa = (x_{/dom(\kappa)}, \vartheta_{/dom(\kappa)})$, $g = (y_{/dom(g)}, \chi_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $\vartheta_i = \chi_j$, $(\vartheta_{/dom(\kappa)})_i = (\chi_{/dom(g)})_j$.

We have also $\eta = (x_{/dom(\kappa)}, s_{/dom(\kappa)})$, $\delta = (y_{/dom(g)}, r_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $s_i = r_j$, $(s_{/dom(\kappa)})_i = (r_{/dom(g)})_j$.

Moreover $\varphi_1, \dots \varphi_m \in E(n,\kappa) \cap E(n,g)$. By assumption 2.1.9 we get

$$\begin{aligned} \#(k,t,\sigma) &= P_f(\#(\kappa,\varphi_1,\eta),\dots,\#(\kappa,\varphi_m,\eta)) \\ &= P_f(\#(g,\varphi_1,\delta),\dots,\#(g,\varphi_m,\delta)) = \#(h,t,\rho). \end{aligned}$$

We still need to examine the case where situation e. is the true situation caused by $t \in E(n+1,k)$. We have the following.

There exist

$\kappa \in K(n) : \kappa \sqsubseteq k$,

a positive integer m,

a function z whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$

$z_i \in \mathcal{V} - var(\kappa)$, and for each $i,j = 1 \dots m$ $i \neq j \to z_i \neq z_j$,

a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$

$\varphi_i \in E(n)$,

$\phi \in E(n)$

such that

$\mathcal{E}(n,\kappa,m,z,\varphi,\phi)$,

$t = \{\}(z_1 : \varphi_1,\dots,z_m : \varphi_m,\phi)$, $t \in E(n+1,\kappa)$,

for each $\eta \in \Xi(\kappa)$ $\#(\kappa,t,\eta) = \{\#(\kappa'_m,\phi,\eta'_m) | \ \eta'_m \in \Xi(\kappa'_m), \eta \sqsubseteq \eta'_m\}$

(where $\kappa'_1 = \kappa + (z_1,\varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$

$\kappa'_{i+1} = \kappa'_i + (z_{i+1},\varphi_{i+1})$),

if $m = 1$ $V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{z_1\})$, $V_b(t) = \{z_1\} \cup V_b(\varphi_1) \cup V_b(\phi)$,

if $m > 1$

$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{z_1\}) \cup \dots \cup (V_f(\varphi_m) - \{z_1,\dots,z_{m-1}\}) \cup$

$\cup (V_f(\phi) - \{z_1,\dots,z_m\})$,

$V_b(t) = \{z_1,\dots,z_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi)$,

$\forall \eta \in \Xi(\kappa) : \eta \sqsubseteq \sigma$ it results $\#(k,t,\sigma) = \#(\kappa,t,\eta)$.

There exist

$g \in K(n) : g \sqsubseteq h$,

a positive integer p,

a function u whose domain is $\{1, \ldots, p\}$ such that for each $i = 1 \ldots p$
$u_i \in \mathcal{V} - var(g)$, and for each $i, j = 1 \ldots p$ $i \neq j \to u_i \neq u_j$,

a function ψ whose domain is $\{1, \ldots, p\}$ such that for each $i = 1 \ldots p$
$\psi_i \in E(n)$,

$\xi \in E(n)$

such that

$\mathcal{E}(n, g, p, u, \psi, \xi)$,

$t = \{\}(u_1 : \psi_1, \ldots, u_p : \psi_p, \xi)$, $t \in E(n+1, g)$,

for each $\delta \in \Xi(g)$ $\#(g, t, \delta) = \{\#(g'_m, \xi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}$
(where $g'_1 = g + (u_1, \psi_1)$, and if $p > 1$ for each $i = 1 \ldots p - 1$
$g'_{i+1} = g'_i + (u_{i+1}, \psi_{i+1})$),

if $p = 1$ $V_f(t) = V_f(\psi_1) \cup (V_f(\xi) - \{u_1\})$, $V_b(t) = \{u_1\} \cup V_b(\psi_1) \cup V_b(\xi)$,

if $p > 1$

$V_f(t) = V_f(\psi_1) \cup (V_f(\psi_2) - \{u_1\}) \cup \cdots \cup (V_f(\psi_p) - \{u_1, \ldots, u_{m-1}\}) \cup$
$\cup (V_f(\xi) - \{u_1, \ldots, u_m\})$,

$V_b(t) = \{u_1, \ldots, u_m\} \cup V_b(\psi_1) \cup \cdots \cup V_b(\psi_m) \cup V_b(\xi)$,

$\forall \delta \in \Xi(g) : \delta \sqsubseteq \rho$ it results $\#(h, t, \rho) = \#(g, t, \delta)$.

Clearly $p = m$, $u = z$, $\psi = \varphi$, $\xi = \phi$.

Let $\eta = \sigma_{/dom(\kappa)}$ and $\delta = \rho_{/dom(g)}$. By assumption 2.1.11 we get $\eta \in \Xi(\kappa)$ and $\delta \in \Xi(g)$. Therefore

$$\#(k, t, \sigma) = \#(\kappa, t, \eta) = \{\#(\kappa'_m, \phi, \eta'_m) |\ \eta'_m \in \Xi(\kappa'_m), \eta \sqsubseteq \eta'_m\},$$

$$\#(h, t, \rho) = \#(g, t, \delta) = \{\#(g'_m, \xi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\}.$$

We have $\kappa = (x_{/dom(\kappa)}, \vartheta_{/dom(\kappa)})$, $g = (y_{/dom(g)}, \chi_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $\vartheta_i = \chi_j$, $(\vartheta_{/dom(\kappa)})_i = (\chi_{/dom(g)})_j$.

We have also $\eta = (x_{/dom(\kappa)}, s_{/dom(\kappa)})$, $\delta = (y_{/dom(g)}, r_{/dom(g)})$. For each $i \in dom(\kappa)$, $j \in dom(g)$ if $(x_{/dom(\kappa)})_i = (y_{/dom(g)})_j$ then $x_i = y_j$, $s_i = r_j$, $(s_{/dom(\kappa)})_i = (r_{/dom(g)})_j$.

By lemma 2.1.15 we get

$$\{\#(g'_m, \xi, \delta'_m) |\ \delta'_m \in \Xi(g'_m), \delta \sqsubseteq \delta'_m\} = \{\#(\kappa'_m, \phi, \eta'_m) |\ \eta'_m \in \Xi(\kappa'_m), \eta \sqsubseteq \eta'_m\},$$

and thereferore $\#(h, t, \rho) = \#(k, t, \sigma)$. ∎

2.2. Consequences of the definition process. We have finished with definition 2.7. We now prove a result that is closely related to the definition.

LEMMA 2.8. *For each positive integer n, $k \in K(n)$ and $t \in E(n,k)$*

$$V_f(t) \subseteq var(k) \wedge V_b(t) \subseteq \mathcal{V} - var(k).$$

Proof.

We use induction on n.

As for the initial step, we observe that for each $t \in E(1,\epsilon) = \mathcal{C}$

$$V_f(t) = \emptyset \subseteq \emptyset = var(\epsilon);\ V_b(t) = \emptyset \subseteq \mathcal{V} = \mathcal{V} - var(\epsilon)\ .$$

We now perform the inductive step. Let $k \in K(n+1)$ and $t \in E(n+1,k)$. We have seen that

$$E(n+1,k) = E'(n,k) \cup E'_a(n+1,k) \cup E'_b(n+1,k) \cup E'_c(n+1,k) \cup E'_d(n+1,k) \cup E'_e(n+1,k).$$

If $t \in E'(n,k)$ then $k \in K(n)$, $t \in E(n,k)$ and by the inductive hypothesis our statement holds.

Let $t \in \mathbf{E'_a(n+1,k)}$, this means that $t \in E_a(n+1,k)$ and $k \in K(n)^+$. There exist $h \in K(n)$, $\phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$. We have also $t \in E(n,h)$, $y \notin V_b(t)$.

Therefore $V_f(t) \subseteq var(h) \subseteq var(k)$.

We have also $V_b(t) \subseteq \mathcal{V} - var(h)$, and $V_b(t) \notin var(h) \cup \{y\} = var(k)$, so

$$V_b(t) \subseteq \mathcal{V} - var(k)\ .$$

Let $t \in \mathbf{E'_b(n+1,k)}$, this means that $t \in E_b(n+1,k)$ and $k \in K(n)^+$. There exist $h \in K(n)$, $\phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$. Moreover $t = y$,

$$V_f(t) = V_f(t)_{(n+1,k,b)} = \{y\} \subseteq var(k)\ .$$
$$V_b(t) = V_b(t)_{(n+1,k,b)} = \emptyset \subseteq \mathcal{V} - var(k)\ .$$

Let $t \in \mathbf{E'_c(n+1,k)}$. As a consequence of $t \in E_c(n+1,k)$ there exist $\varphi, \varphi_1, \dots, \varphi_m$ in $E(n,k)$ such that $t = (\varphi)(\varphi_1, \dots, \varphi_m)$, and

$$V_f(t) = V_f(t)_{(n+1,k,c)} = V_f(\varphi) \cup V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) \subseteq var(k)\ ,$$
$$V_b(t) = V_b(t)_{(n+1,k,c)} = V_b(\varphi) \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \subseteq \mathcal{V} - var(k)\ .$$

Let $t \in \mathbf{E'_d(n+1,k)}$. As a consequence of $t \in E_d(n+1,k)$ there exist $f \in \mathcal{F}$, $\varphi_1, \dots, \varphi_m$ in $E(n,k)$ such that $t = (f)(\varphi_1, \dots, \varphi_m)$, and

$$V_f(t) = V_f(t)_{(n+1,k,d)} = V_f(\varphi_1) \cup \dots \cup V_f(\varphi_m) \subseteq var(k)\ ,$$
$$V_b(t) = V_b(t)_{(n+1,k,d)} = V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \subseteq \mathcal{V} - var(k)\ .$$

Let $t \in \mathbf{E'_e(n+1,k)}$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1\dots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$ and $\mathcal{E}(n,k,m,x,\varphi,\phi)$.

Moreover if $m = 1$ we have

$$V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}),$$
$$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi).$$

If $m > 1$ we have

$$V_f(t) = V_f(\varphi_1) \cup (V_f(\varphi_2) - \{x_1\}) \cup \dots \cup (V_f(\varphi_m) - \{x_1,\dots,x_{m-1}\}) \cup \\ \cup (V_f(\phi) - \{x_1,\dots,x_m\}),$$
$$V_b(t) = \{x_1,\dots,x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi).$$

Let's consider the case where $m = 1$.

By the inductive hypothesis $V_f(\varphi_1) \subseteq var(k)$ and $V_f(\phi) \subseteq var(k'_1) = var(k) \cup \{x_1\}$. It follows that

$$V_f(t) = V_f(\varphi_1) \cup (V_f(\phi) - \{x_1\}) \subseteq var(k) \ .$$

Moreover, the inductive hypothesis lets us state $V_b(\varphi_1) \subseteq \mathcal{V} - var(k)$, $V_b(\phi) \subseteq \mathcal{V} - var(k'_1) = \mathcal{V} - (var(k) \cup \{x_1\}) \subseteq \mathcal{V} - var(k)$. Therefore

$$V_b(t) = \{x_1\} \cup V_b(\varphi_1) \cup V_b(\phi) \subseteq \mathcal{V} - var(k) \ .$$

We now turn to examine the case where $m > 1$.

By the inductive hypothesis $V_f(\varphi_1) \subseteq var(k)$ and for each $i = 1\dots m-1$

$$V_f(\varphi_{i+1}) \subseteq var(k'_i) = var(k) \cup \{x_1,\dots,x_i\} \text{ , so}$$
$$V_f(\varphi_{i+1}) - \{x_1,\dots,x_i\} \subseteq var(k) \ .$$

Moreover

$$V_f(\phi) \subseteq var(k'_m) = var(k) \cup \{x_1,\dots,x_m\} \text{ , so}$$
$$V_f(\phi) - \{x_1,\dots,x_m\} \subseteq var(k) \ .$$

It follows $V_f(t) \subseteq var(k)$.

The inductive hypotesis also implies $V_b(\varphi_1) \subseteq \mathcal{V} - var(k)$ and for each $i = 1\dots m-1$

$$V_b(\varphi_{i+1}) \subseteq \mathcal{V} - var(k'_i) \subseteq \mathcal{V} - var(k) \ .$$

Moreover $V_b(\phi) \subseteq \mathcal{V} - var(k'_m) \subseteq \mathcal{V} - var(k)$.

Therefore

$$V_b(t) = \{x_1,\dots,x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi) \subseteq \mathcal{V} - var(k) \ .$$

∎

This result ensures that $V_b(t)$ and $V_f(t)$ are always disjoint, so a variable cannot have both bound and free occurrences in the same expression.

3. Introduction to the deductive methodology

In this chapter we will cover some fundamental principles that underlie our inferences. An important target will be achieved with the proof of theorem 3.6, which is a simple but significant step to set up our deductive methodology.

Some preliminary definitions.

Let $K = \bigcup_{n \geqslant 1} K(n)$.

For each $k \in K$ let

$$E(k) = \bigcup_{n \geqslant 1 : k \in K(n)} E(n,k) \ ,$$

$$E_s(k) = \{t | t \in E(k), \forall \sigma \in \Xi(k) \ \#(k,t,\sigma) \text{ is a set } \} \ .$$

Let $E = \bigcup_{k \in K} E(k)$; E is the set of all expressions in our language.

One expression $t \in E(k)$ is a 'sentence with respect to k' when for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is true or $\#(k,t,\sigma)$ is false.

We define $S(k) = \{t | t \in E(k), t \text{ is a sentence with respect to } k\}$.

At the beginning of chapter 2 we have introduced the logical connectives. In our deductions, expressions will make an extensive use of the logical connectives, so we assume that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of chapter 2.

For each $t \in E(\epsilon)$ we define $\#(t) = \#(\epsilon, t, \epsilon)$.

On the way to theorem 3.6 we need some further preliminary work, beginning with the following lemma.

LEMMA 3.1. *Let* $h \in K, \ \phi \in E_s(h), \ y \in (\mathcal{V} - var(h)), \ k = h + (y, \phi)$. *We have* $k \in K$, *and if* $\vartheta \in S(k)$ *then*

- $\{\}(y : \phi, \vartheta) \in E(h)$;
- $(\forall)(\{\}(y : \phi, \vartheta)) \in S(h), \ (\exists)(\{\}(y : \phi, \vartheta)) \in S(h)$;
- $\forall \ \rho \in \Xi(h) \ \#(h, (\forall)(\{\}(y : \phi, \vartheta)), \rho) = P_\forall(\{\#(k, \vartheta, \sigma) | \ \sigma \in \Xi(k), \ \rho \sqsubseteq \sigma\})$;
- $\forall \ \rho \in \Xi(h) \ \#(h, (\exists)(\{\}(y : \phi, \vartheta)), \rho) = P_\exists(\{\#(k, \vartheta, \sigma) | \ \sigma \in \Xi(k), \ \rho \sqsubseteq \sigma\})$.

Proof.

Since $\phi \in E_s(h)$ there is a positive integer n such that $\phi \in E_s(n, h), \ h \in K(n)$. This implies that $k \in K(n)^+ \subseteq K(n+1) \subseteq K$.

Let $\vartheta \in S(k)$. There is a positive integer m such that $\vartheta \in E(m,k)$. We define $p = \max\{n+1, m\}$, then we have

- $h \in K(p)$
- $y \in (\mathcal{V} - var(h))$
- $\phi \in E_s(p,h)$
- $k \in K(p),\ \vartheta \in E(p,k)$

This implies that $\{\}(y:\phi,\vartheta) \in E_e(p+1,h) \subseteq E(p+1,h) \subseteq E(h)$.

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}\#(h,\{\}(y:\phi,\vartheta),\rho) &= \#(h,\{\}(y:\phi,\vartheta),\rho)_{/(p+1,h,e)} = \\ &= \{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\ .\end{aligned}$$

We want to show that $(\forall)(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. To obtain this we just need to show that for each $\rho \in \Xi(h)$ $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds.
Now $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ is equal to

$\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set and for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ u is true or u is false.

Clearly $\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set, furthermore for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ there is $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $u = \#(k,\vartheta,\sigma)$. Since $\vartheta \in S(k)$ u is true or u is false. So $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds.

We have proved that $(\forall)(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. Similarly we can show that $(\exists)(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. In fact to show this we just need to prove that for each $\rho \in \Xi(h)$ $A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds, and this is proved since

$$A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))\ .$$

For each $\rho \in \Xi(h)$

$$\begin{aligned}\#(h,(\forall)(\{\}(y:\phi,\vartheta)),\rho) &= \#(h,(\forall)(\{\}(y:\phi,\vartheta)),\rho)_{/(p+2,h,d)} = \\ &= P_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

$$\begin{aligned}\#(h,(\exists)(\{\}(y:\phi,\vartheta)),\rho) &= \#(h,(\exists)(\{\}(y:\phi,\vartheta)),\rho)_{/(p+2,h,d)} = \\ &= P_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\exists(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

Finally, as we have seen, for each $\rho \in \Xi(h)$

$$\#(h,(\forall)(\{\}(y:\phi,\vartheta)),\rho) = P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ ,$$

and $P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ is clearly true or false.

Hence $(\forall)(\{\}(y:\phi,\vartheta)) \in S(h)$. Similarly we obtain that $(\exists)(\{\}(y:\phi,\vartheta)) \in S(h)$. ∎

Definition 3.2. Let $x \in \mathcal{V}$, $\varphi \in E$. We define

$$H[x:\varphi] = \varphi \in E_s(\epsilon) \ .$$

If the condition $H[x:\varphi]$ holds then we define $k[x:\varphi] = \epsilon+(x,\varphi)$. Clearly $k[x:\varphi] \in K$ and $var(k[x:\varphi]) = \{x\}$.

Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_{m+1} \in E$. We can assume to have defined $H[x_1:\varphi_1, \dots, x_m:\varphi_m]$ and if this holds to have defined also $k[x_1:\varphi_1, \dots, x_m:\varphi_m] \in K$, such that

$$var(k[x_1:\varphi_1, \dots, x_m:\varphi_m]) = \{x_1, \dots, x_m\} \ .$$

We define

$$H[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}] = H[x_1:\varphi_1, \dots, x_m:\varphi_m] \\ \wedge \varphi_{m+1} \in E_s(k[x_1:\varphi_1, \dots, x_m:\varphi_m]) \ .$$

If $H[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}]$ then we define

$$k[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}] = k[x_1:\varphi_1, \dots, x_m:\varphi_m] + (x_{m+1}, \varphi_{m+1}) \ .$$

Clearly $k[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}] \in K$ and

$$var(k[x_1:\varphi_1, \dots, x_{m+1}:\varphi_{m+1}]) = \{x_1, \dots, x_{m+1}\} \ .$$

Remark 3.3. Let m be a positive integer. Let $x_1, \dots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_m \in E$ and assume $H[x_1:\varphi_1, \dots, x_m:\varphi_m]$. In these assumptions we can easily see that for each $i = 1 \dots m$ $H[x_1:\varphi_1, \dots, x_i:\varphi_i]$ holds and so $k[x_1:\varphi_1, \dots, x_i:\varphi_i]$ is defined, $k[x_1:\varphi_1, \dots, x_i:\varphi_i] \in K$, $var(k[x_1:\varphi_1, \dots, x_i:\varphi_i]) = \{x_1, \dots, x_i\}$.

In fact this is clearly true for $i = m$. Given $i = 2 \dots m$, if we suppose this is true for i, then we have $H[x_1:\varphi_1, \dots, x_{i-1}:\varphi_{i-1}]$, and so the remaining facts also hold.

In these assumptions we can define $k_0 = \epsilon$ and for each $i = 1 \dots m$ $k_i = k[x_1:\varphi_1, \dots, x_i:\varphi_i]$. We have $k_0 \in K$, $var(k_0) = \emptyset$, for each $i = 1 \dots m$ $k_i \in K$, $var(k_i) = \{x_1, \dots, x_i\}$. Hereafter we'll often use this kind of simplified notation.

We can also easily see that for each $i = 1 \dots m$ $\varphi_i \in E_s(k_{i-1})$ and $k_i = k_{i-1} + (x_i, \varphi_i)$, and $dom(k_i) = \{1, \dots, i\}$.

Definition 3.4. Let m be a positive integer. Let $x_1, \dots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_m \in E$ and assume $H[x_1:\varphi_1, \dots, x_m:\varphi_m]$. Let φ be a member of $S(k[x_1:\varphi_1, \dots, x_m:\varphi_m])$. Define

$$\gamma[x_m:\varphi_m, \varphi] = (\forall)(\{\}(x_m:\varphi_m, \varphi)) \ .$$

By lemma 3.1 we have $\gamma[x_m:\varphi_m, \varphi] \in S(k_{m-1})$.

If $m > 1$ for each $i = 2 \dots m$ suppose we have defined $\gamma[x_i:\varphi_i, \dots, x_m:\varphi_m, \varphi]$ as a member of $S(k_{i-1})$ and define

$$\gamma[x_{i-1}:\varphi_{i-1}, \dots, x_m:\varphi_m, \varphi] = (\forall)(\{\}(x_{i-1}:\varphi_{i-1}, \gamma[x_i:\varphi_i, \dots, x_m:\varphi_m, \varphi])) \ .$$

By lemma 3.1 $\gamma[x_{i-1}:\varphi_{i-1},\dots,x_m:\varphi_m,\varphi]\in S(k_{i-2})$.

Lemma 3.5. *Let m be a positive integer. Let $x_1,\dots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1,\dots,\varphi_m\in E$ and assume $H[x_1:\varphi_1,\dots,x_m:\varphi_m]$. Let $\varphi\in S(k[x_1:\varphi_1,\dots,x_m:\varphi_m])$, $m>1$, $j=2\dots m$.*

We have $\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]\in S(k_{j-1})$. We can show that for each $i=1\dots j-1$

$$\gamma[x_i:\varphi_i,\dots,x_m:\varphi_m,\varphi]=\gamma[x_i:\varphi_i,\dots,x_{j-1}:\varphi_{j-1},\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]]\ .$$

Proof.

We show this by induction on i. First we prove the property for $i=j-1$.

$$\gamma[x_{j-1}:\varphi_{j-1},\dots,x_m:\varphi_m,\varphi]=(\forall)(\{\}(x_{j-1}:\varphi_{j-1},\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]))=$$
$$=\gamma[x_{j-1}:\varphi_{j-1},\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]]\ .$$

Now we assume $j-1\geqslant 2$ and $2\leqslant i\leqslant j-1$. We assume the property is true for i and want to show it holds also for $i-1$. We have

$$\gamma[x_{i-1}:\varphi_{i-1},\dots,x_m:\varphi_m,\varphi]=(\forall)(\{\}(x_{i-1}:\varphi_{i-1},\gamma[x_i:\varphi_i,\dots,x_m:\varphi_m,\varphi]))=$$
$$=(\forall)(\{\}(x_{i-1}:\varphi_{i-1},\gamma[x_i:\varphi_i,\dots,x_{j-1}:\varphi_{j-1},\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]]))=$$
$$=\gamma[x_{i-1}:\varphi_{i-1},\dots,x_{j-1}:\varphi_{j-1},\gamma[x_j:\varphi_j,\dots,x_m:\varphi_m,\varphi]]\ .$$

■

Theorem 3.6. *Let m be a positive integer. Let $x_1,\dots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1,\dots,\varphi_m\in E$ and assume $H[x_1:\varphi_1,\dots,x_m:\varphi_m]$. Let $\varphi\in S(k[x_1:\varphi_1,\dots,x_m:\varphi_m])$. Then*

$$\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\varphi])\leftrightarrow$$
$$\leftrightarrow P_\forall(\{\#(k[x_1:\varphi_1,\dots,x_m:\varphi_m],\varphi,\sigma)|\ \sigma\in\Xi(k[x_1:\varphi_1,\dots,x_m:\varphi_m])\})$$

Proof.

We'll use the symbols $k_0,\dots,k_m$ with the meaning specified in remark 3.3, so what we need to show is:

$$\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\varphi])\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m)\})\ .$$

To this end we need to show that for each $i=m\dots 1$ and for each $\rho\in\Xi(k_{i-1})$

$$\#(k_{i-1},\gamma[x_i:\varphi_i,\dots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

We prove this by induction on i, starting with the case where $i=m$. Here we need to show that for each $\rho\in\Xi(k_{m-1})$

$$\#(k_{m-1},\gamma[x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

Actually

$$\#(k_{m-1},\gamma[x_m:\varphi_m,\varphi],\rho)=\#(k_{m-1},(\forall)(\{\}(x_m:\varphi_m,\varphi)),\rho)=$$
$$=P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

Now suppose $m > 1$, let $i = 2 \ldots m$ and suppose the property holds for i, we show it also holds for $i-1$. We need to prove that for each $\rho \in \Xi(k_{i-2})$

$$\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

We have

$$\begin{aligned}
\#(k_{i-2}, &\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi], \rho) = \\
&= \#(k_{i-2}, (\forall)(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi])), \rho) = \\
&= P_\forall(\{\#(k_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \delta) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\}) \leftrightarrow \\
&\leftrightarrow P_\forall(\{P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\}) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\})\ .
\end{aligned}$$

So it comes to showing that

$$\begin{aligned}
P_\forall(\{P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\}) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\}) \leftrightarrow \\
\leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .
\end{aligned}$$

Suppose $P_\forall(\{P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\}) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\})$.

This means that for each $\delta \in \Xi(k_{i-1})$ such that $\rho \sqsubseteq \delta$ and for each $\sigma \in \Xi(k_m) :\ \delta \sqsubseteq \sigma$ $\#(k_m, \varphi, \sigma)$ holds.

Let $\sigma \in \Xi(k_m) : \rho \sqsubseteq \sigma$, we need to prove $\#(k_m, \varphi, \sigma)$.

We define $\delta = \sigma_{/dom(k_{i-1})}$. By assumption 2.1.11 $\delta \in \Xi(k_{i-1})$. Moreover $\delta, \rho \in \mathcal{R}(\sigma)$ and $dom(\rho) = dom(k_{i-2}) \subseteq dom(k_{i-1}) = dom(\delta)$. By lemma 2.5 we obtain $\rho \sqsubseteq \delta$. Therefore $\#(k_m, \varphi, \sigma)$ holds.

Conversely suppose $P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})$, so that for each $\sigma \in \Xi(k_m) : \rho \sqsubseteq \sigma$ $\#(k_m, \varphi, \sigma)$ is true. Let $\delta \in \Xi(k_{i-1})$ be such that $\rho \sqsubseteq \delta$ and let $\sigma \in \Xi(k_m)$ be such that $\delta \sqsubseteq \sigma$. Since $\sigma \in \Xi(k_m)$ and $\rho \sqsubseteq \sigma$ we have $\#(k_m, \varphi, \sigma)$.

This completes the proof that for each $\rho \in \Xi(k_{i-2})$

$$\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

We have also finished the proof that for each $i = m \ldots 1$ and for each $\rho \in \Xi(k_{i-1})$

$$\#(k_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

It follows that for each $\rho \in \Xi(k_0)$

$$\#(k_0, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

and clearly this can be rewritten

$$\#(\epsilon, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \epsilon) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \epsilon \sqsubseteq \sigma\})\ ,$$
$$\#(\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi]) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m)\})\ .$$

■

We'll soon apply theorem 3.6 to show its importance. First we need to prove lemma 3.7, which is in some way similar to 3.1 but involves the other logical connectives.

Lemma 3.7. *Let $h \in K$, $\varphi_1, \varphi_2 \in S(h)$. Then*

- $(\wedge)(\varphi_1, \varphi_2), (\vee)(\varphi_1, \varphi_2), (\rightarrow)(\varphi_1, \varphi_2), (\leftrightarrow)(\varphi_1, \varphi_2), (\neg)(\varphi_1) \in S(h)$;
- *for each* $\rho \in \Xi(h)$ $\#(h, (\wedge)(\varphi_1, \varphi_2), \rho) = P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
- *for each* $\rho \in \Xi(h)$ $\#(h, (\vee)(\varphi_1, \varphi_2), \rho) = P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
- *for each* $\rho \in \Xi(h)$ $\#(h, (\rightarrow)(\varphi_1, \varphi_2), \rho) = P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
- *for each* $\rho \in \Xi(h)$ $\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho) = P_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;
- *for each* $\rho \in \Xi(h)$ $\#(h, (\neg)(\varphi_1), \rho) = P_\neg(\#(h, \varphi_1, \rho))$.

Proof.

For each $\rho \in \Xi(h)$ $\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false; $\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false.

We recall that for each $\rho \in \Xi(h)$ $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all defined as
($\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false) and ($\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false).

Therefore $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all true.

And for each $\rho \in \Xi(h)$ $A_\neg(\#(h, \varphi_1, \rho))$ is true.

There exists a positive integer n such that $\varphi_1, \varphi_2 \in E(n, h)$, so

$$(\wedge)(\varphi_1, \varphi_2), (\vee)(\varphi_1, \varphi_2), (\rightarrow)(\varphi_1, \varphi_2), (\leftrightarrow)(\varphi_1, \varphi_2), (\neg)(\varphi_1) \in E(h) .$$

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}
\#(h, (\wedge)(\varphi_1, \varphi_2), \rho) &= P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho));\\
\#(h, (\vee)(\varphi_1, \varphi_2), \rho) &= P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho));\\
\#(h, (\rightarrow)(\varphi_1, \varphi_2), \rho) &= P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho));\\
\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho) &= P_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho));\\
\#(h, (\neg)(\varphi_1), \rho) &= P_\neg(\#(h, \varphi_1, \rho)) .
\end{aligned}$$

so

$$\begin{aligned}
&\#(h, (\wedge)(\varphi_1, \varphi_2), \rho) \text{ is true or false;}\\
&\#(h, (\vee)(\varphi_1, \varphi_2), \rho) \text{ is true or false;}\\
&\#(h, (\rightarrow)(\varphi_1, \varphi_2), \rho) \text{ is true or false;}\\
&\#(h, (\leftrightarrow)(\varphi_1, \varphi_2), \rho) \text{ is true or false;}\\
&\#(h, (\neg)(\varphi_1), \rho) \text{ is true or false .}
\end{aligned}$$

Therefore we get

$$(\wedge)(\varphi_1, \varphi_2), (\vee)(\varphi_1, \varphi_2), (\rightarrow)(\varphi_1, \varphi_2), (\leftrightarrow)(\varphi_1, \varphi_2), (\neg)(\varphi_1) \in S(h) .$$

■

The following lemma 3.8 is an example of how theorem 3.6 is applied.

Lemma 3.8. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi_1, \psi_2 \in S(k)$.*

Under these assumptions we have $(\to)(\varphi, \psi_1), (\to)(\varphi, \psi_2), (\to)(\varphi, (\wedge)(\psi_1, \psi_2)) \in S(k)$.

Moreover, if

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_1)]),\ \#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_2)])$$

then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\wedge)(\psi_1, \psi_2))]) \ .$$

Proof.

We need to show

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\wedge)(\psi_1, \psi_2))]) \ ,$$

that is

$$\begin{aligned}
&P_\forall(\{\#(k, (\to)(\varphi, (\wedge)(\psi_1, \psi_2)), \sigma) |\ \sigma \in \Xi(k)\}) \ , \\
&P_\forall(\{P_\to(\#(k, \varphi, \sigma), \#(k, (\wedge)(\psi_1, \psi_2), \sigma)) |\ \sigma \in \Xi(k)\}) \ , \\
&P_\forall(\{P_\to(\#(k, \varphi, \sigma), P_\wedge(\#(k, \psi_1, \sigma), \#(k, \psi_2, \sigma))) |\ \sigma \in \Xi(k)\}) \ . \qquad (3.0.1)
\end{aligned}$$

But we have

$$\begin{aligned}
&\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_1)]) \ , \\
&P_\forall(\{\#(k, (\to)(\varphi, \psi_1), \sigma) |\ \sigma \in \Xi(k)\}) \ , \\
&P_\forall(\{P_\to(\#(k, \varphi, \sigma), \#(k, \psi_1, \sigma)) |\ \sigma \in \Xi(k)\}) \ .
\end{aligned}$$

And we have

$$\begin{aligned}
&\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_2)]) \ , \\
&P_\forall(\{\#(k, (\to)(\varphi, \psi_2), \sigma) |\ \sigma \in \Xi(k)\}) \ , \\
&P_\forall(\{P_\to(\#(k, \varphi, \sigma), \#(k, \psi_2, \sigma)) |\ \sigma \in \Xi(k)\}) \ .
\end{aligned}$$

So for each $\sigma \in \Xi(k)$ if $\#(k, \varphi, \sigma)$ holds true then both $\#(k, \psi_1, \sigma)$ and $\#(k, \psi_2, \sigma)$ hold. This implies 3.0.1 holds true in turn. ■

According to this lemma, if in our reasoning we have derived the sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_1)]$ and $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, \psi_2)]$, then we can derive $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\wedge)(\psi_1, \psi_2))]$. This is a first example of how our deductive methodology will work.

We terminate the chapter with other useful lemmas.

Lemma 3.9. *Let $c \in \mathcal{C}$. For each positive integer n and $k \in K(n)$ we have*

- *$c \in E(n, k)$;*
- *for each $\sigma \in \Xi(k)$ $\#(k, c, \sigma) = \#(c)$.*

Proof.

The proof is by induction on n.

For $n = 1$ we have $k = \epsilon$ so $c \in E(1, \epsilon) = E(n, k)$ and for each $\sigma \in \Xi(k)$ $\sigma = \epsilon$, so $\#(k, c, \sigma) = \#(\epsilon, c, \epsilon) = \#(c)$.

Let n be a positive integer and $k \in K(n+1) = K(n) \cup K(n)^+$.

If $k \in K(n)$ then $c \in E(n, k) \subseteq E(n+1, k)$ and for each $\sigma \in \Xi(k)$ $\#(k, c, \sigma) = \#(c)$.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. We have $c \in E(n, h)$ and for each $\rho \in \Xi(h)$ $\#(h, c, \rho) = \#(c)$.
It follows that $c \in E(n+1, k)$ and for each $\sigma = \rho + (y, s) \in \Xi(k)$

$$\#(k, c, \sigma) = \#(h, c, \rho) = \#(c) \ .$$

■

LEMMA 3.10. *Let $k \in K$, m a positive integer, $\varphi, \varphi_1, \ldots, \varphi_m \in E(k)$. Suppose for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is a function with m arguments and $(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ is a member of its domain. Then*

- $(\varphi)(\varphi_1, \ldots, \varphi_m) \in E(k)$;
- *for each* $\sigma \in \Xi(k)$
  $\#(k, (\varphi)(\varphi_1, \ldots, \varphi_m), \sigma) = \#(k, \varphi, \sigma)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$;
- $V_b((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots V_b(\varphi_m)$;
- $V_f((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots V_f(\varphi_m)$.

Proof.

There exists a positive integer n such that $\varphi, \varphi_1, \ldots, \varphi_m \in E(n, k)$. This implies that $(\varphi)(\varphi_1, \ldots, \varphi_m) \in E(n+1, k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, (\varphi)(\varphi_1, \ldots, \varphi_m), \sigma) = \#(k, \varphi, \sigma)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)) \ .$$

Clearly the following also hold:

- $V_b((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_b(\varphi) \cup V_b(\varphi_1) \cup \ldots V_b(\varphi_m)$;
- $V_f((\varphi)(\varphi_1, \ldots, \varphi_m)) = V_f(\varphi) \cup V_f(\varphi_1) \cup \ldots V_f(\varphi_m)$.

■

LEMMA 3.11. *Let $k \in K$, $f \in \mathcal{F}$, m a positive integer, $\varphi_1, \ldots, \varphi_m \in E(k)$. Suppose for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ is true. Then*

- $(f)(\varphi_1, \ldots, \varphi_m) \in E(k)$;
- *for each* $\sigma \in \Xi(k)$ $\#(k, (f)(\varphi_1, \ldots, \varphi_m), \sigma) = P_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$;
- $V_b((f)(\varphi_1, \ldots, \varphi_m)) = V_b(\varphi_1) \cup \ldots V_b(\varphi_m)$;
- $V_f((f)(\varphi_1, \ldots, \varphi_m)) = V_f(\varphi_1) \cup \ldots V_f(\varphi_m)$.

Proof.

There exists a positive integer n such that $\varphi_1, \ldots, \varphi_m \in E(n,k)$. This implies that $(f)(\varphi_1, \ldots, \varphi_m) \in E(n+1,k)$ and for each $\sigma \in \Xi(k)$

$$\#(k,(f)(\varphi_1,\ldots,\varphi_m),\sigma) = P_f(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)) \ .$$

Clearly the following also hold:

- $V_b((f)(\varphi_1,\ldots,\varphi_m)) = V_b(\varphi_1) \cup \ldots V_b(\varphi_m)$;
- $V_f((f)(\varphi_1,\ldots,\varphi_m)) = V_f(\varphi_1) \cup \ldots V_f(\varphi_m)$.

■

Lemma 3.12. *Suppose the equality predicate symbol = we defined at the beginning of chapter 2 belongs to $\mathcal{F}$. Suppose $k \in K$, $\varphi_1, \varphi_2 \in E(k)$. Then $(=)(\varphi_1,\varphi_2) \in S(k)$.*

Proof.

For each $\sigma \in \Xi(k)$ $A_=(\#(k,\varphi_1,\sigma),\#(k,\varphi_2,\sigma))$ is true, so $(=)(\varphi_1,\varphi_2) \in E(k)$.

Moreover for each $\sigma \in \Xi(k)$

$$\begin{aligned} \#(k,(=)(\varphi_1,\varphi_2),\sigma) &= P_=(\#(k,\varphi_1,\sigma),\#(k,\varphi_2,\sigma)) = \\ &= \#(k,\varphi_1,\sigma) \text{ is equal to } \#(k,\varphi_2,\sigma), \end{aligned}$$

so $\#(k,(=)(\varphi_1,\varphi_2),\sigma)$ is true or false.

Therefore $(=)(\varphi_1,\varphi_2) \in S(k)$. ■

Lemma 3.13. *Suppose the membership predicate symbol $\in$ we defined at the beginning of chapter 2 belongs to $\mathcal{F}$. Suppose $k \in K$, $t, \varphi \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is a set. Then $(\in)(t,\varphi) \in S(k)$.*

Proof.

For each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is a set, so $A_\in(\#(k,t,\sigma),\#(k,\varphi,\sigma))$ holds. Therefore, by lemma 3.11, $(\in)(t,\varphi) \in E(k)$.

Using lemma 3.11 we also obtain that for each $\sigma \in \Xi(k)$

$$\#(k,(\in)(t,\varphi),\sigma) = P_\in(\#(k,t,\sigma),\#(k,\varphi,\sigma)) = \#(k,t,\sigma) \text{ belongs to } \#(k,\varphi,\sigma).$$

So $\#(k,(\in)(t,\varphi),\sigma)$ is true or false and $(\in)(t,\varphi) \in S(k)$. ■

Lemma 3.14. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$, assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$, define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$.*
Let $i = 0 \ldots m-1$ and let $\psi \in E(k_i)$ such that for each $j = i+1 \ldots m$ $x_j \notin V_b(\psi)$.
Then $\psi \in E(k)$ and for each $\sigma \in \Xi(k)$ there exists $\rho \in \Xi(k_i)$ such that $\rho \sqsubseteq \sigma$ and $\#(k,\psi,\sigma) = \#(k_i,\psi,\rho)$.

Proof.

We prove by induction on j that for each $j = i \dots m$ $\psi \in E(k_j)$ and for each $\sigma \in \Xi(k_j)$ there exists $\rho \in \Xi(k_i)$ such that $\rho \sqsubseteq \sigma$ and $\#(k_j, \psi, \sigma) = \#(k_i, \psi, \rho)$.

The initial step of the proof is obvious, so let $j = i \dots m-1$ and assume $\psi \in E(k_j)$ and for each $\sigma \in \Xi(k_j)$ there exists $\rho \in \Xi(k_i)$ such that $\rho \sqsubseteq \sigma$ and $\#(k_j, \psi, \sigma) = \#(k_i, \psi, \rho)$.

We have $\varphi_{j+1} \in E_s(k_j)$ and $k_{j+1} = k_j + (x_{j+1}, \varphi_{j+1})$, $x_{j+1} \notin V_b(\psi)$ so we can apply lemma 4.12 and obtain that $\psi \in E(k_{j+1})$ and for each $\sigma = \eta + (x_{i+1}, s) \in \Xi(k_{j+1})$ $\#(k_{j+1}, \psi, \sigma) = \#(k_j, \psi, \eta)$. Since $\eta \in \Xi(k_j)$ there exists $\rho \in \Xi(k_i)$ such that $\rho \sqsubseteq \eta \sqsubseteq \sigma$ and $\#(k_{j+1}, \psi, \sigma) = \#(k_j, \psi, \eta) = \#(k_i, \psi, \rho)$. ∎

4. Substitution

First-order logic features the notion of 'substitution' (see e.g. Enderton's book [2]). Under appropriate assumptions, we can apply substitution to a formula φ and obtain a new formula φ^x_t, by replacing the free occurrences of the variable x by the term t. In our approach we'll define a similar notion, with the difference that for us t is a generic expression.

We begin with some preliminary definitions and results, then substitution will be defined through the complex definition process 4.16.

DEFINITION 4.1. Let n be a positive integer, $n > 1$. Let $k \in K(n)$, $k \neq \epsilon$. Let p be a positive integer, $x_1, \ldots, x_p \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_p \in E$.
We define $k_0 = \epsilon$ and for each $i = 1 \ldots p$ $k_i = k_{i-1} + (x_i, \varphi_i)$.

We indicate with $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ the condition in which

$$k = k_p \text{ and for each } i = 1 \ldots p \ \ k_{i-1} \in K(n-1),\ \varphi_i \in E_s(n-1, k_{i-1}) .$$

LEMMA 4.2. *Let n be a positive integer, $n > 1$. Let $k \in K(n)$, $k \neq \epsilon$. Let p be a positive integer, $x_1, \ldots, x_p \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_p \in E$.*
Suppose $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ holds.
Then for each $i = 1 \ldots p$ $dom(k_i) = \{1, \ldots, i\}$ and if we define $k_i = (u_i, \phi_i)$ then for each $j = 1 \ldots i$ $(u_i)_j = x_j$ and $(\phi_i)_j = \varphi_j$.

Proof.

We have $k_1 = \epsilon + (x_1, \varphi_1)$, so $dom(k_1) = \{1\}$, and $(u_1)_1 = x_1$, $(\phi_1)_1 = \varphi_1$.

Given $i = 1 \ldots p-1$ we assume $dom(k_i) = \{1, \ldots, i\}$ and for each $j = 1 \ldots i$ $(u_i)_j = x_j$ and $(\phi_i)_j = \varphi_j$. We have $k_{i+1} = k_i + (x_{i+1}, \varphi_{i+1})$, so $dom(k_{i+1}) = \{1, \ldots, i+1\}$. Moreover, for each $j = 1 \ldots i$ $(u_{i+1})_j = (u_i)_j = x_j$ and $(\phi_{i+1})_j = (\phi_i)_j = \varphi_j$. Finally $(u_{i+1})_{i+1} = x_{i+1}$, $(\phi_{i+1})_{i+1} = \varphi_{i+1}$. ■

LEMMA 4.3. *Let n be a positive integer, $n > 1$. Let $k = (u, \phi) \in K(n)$, $k \neq \epsilon$. Let p be a positive integer, $x_1, \ldots, x_p \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_p \in E$.*
Suppose $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ holds.
Let also q be a positive integer, $y_1, \ldots, y_q \in \mathcal{V}$, with $y_i \neq y_j$ for $i \neq j$. Let $\psi_1, \ldots, \psi_q \in E$.
Suppose $\mathcal{K}(n; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$ holds.
Then $q = p$, for each $i = 1 \ldots p$ $y_i = x_i$, $\psi_i = \varphi_i$.

Proof.

We have $\{1 \ldots p\} = dom(k) = \{1 \ldots q\}$ and therefore $q = p$. Moreover for each $i = 1 \ldots p$ $y_i = u_i = x_i$, $\psi_i = \phi_i = \varphi_i$. ■

Lemma 4.4. *Let n be a positive integer, $n > 1$. Let $k \in K(n)$, $k \neq \epsilon$. Let p be a positive integer with $p < n$, $x_1, \ldots, x_p \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_p \in E$.*
We define $k_0 = \epsilon$ and for each $i = 1 \ldots p$ $k_i = k_{i-1} + (x_i, \varphi_i)$.
Suppose $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ holds.

Then for each $i = 1 \ldots p$, $\sigma_i \in \Xi(k_i)$ there exist $\rho \in \Xi(k_{i-1})$, $s \in \#(k_{i-1}, \varphi_i, \rho)$ such that $\sigma_i = \rho + (x_i, s)$.

Proof.

We have $k_i = k_{i-1} + (x_i, \varphi_i)$, $k_{i-1} \in K(n-1)$, $\varphi_i \in E_s(n-1, k_{i-1})$, $x_i \in \mathcal{V} - var(k_{i-1})$. Therefore $k_i \in K(n-1)^+$, and there exist $\rho \in \Xi(k_{i-1})$, $s \in \#(k_{i-1}, \varphi_i, \rho)$ such that $\sigma_i = \rho + (x_i, s)$. ■

Lemma 4.5. *Let n be a positive integer, $n > 1$. Let $k \in K(n)$, $k \neq \epsilon$. Let p be a positive integer with $p < n$, $x_1, \ldots, x_p \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_p \in E$.*
We define $k_0 = \epsilon$ and for each $i = 1 \ldots p$ $k_i = k_{i-1} + (x_i, \varphi_i)$.
Suppose $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ holds.

Let $\sigma \in \Xi(k)$, for each $i = 0 \ldots p$ define $\sigma_i = \sigma_{/dom(k_i)}$. Then for each $i = 1 \ldots p$ there exists $s_i \in \#(k_{i-1}, \varphi_i, \sigma_{i-1})$ such that $\sigma_i = \sigma_{i-1} + (x_i, s_i)$.

Proof.

By lemma 2.1.11 we obtain that $\sigma_i \in \Xi(k_i)$. Then there exist $\rho \in \Xi(k_{i-1})$, $s_i \in \#(k_{i-1}, \varphi_i, \rho)$ such that $\sigma_i = \rho + (x_i, s_i)$.

Since $\rho \in \mathcal{R}(\sigma_i)$ we have

$$\rho = (\sigma_i)_{/dom(k_{i-1})} = (\sigma_{/dom(k_i)})_{/dom(k_{i-1})} = \sigma_{/dom(k_{i-1})} = \sigma_{i-1} \ .$$

■

Lemma 4.6. *For each positive integer n and $k \in K(n)$ we have*

$k = \epsilon$ or
($n > 1$ and there exist

- *a positive integer p such that $p < n$,*
- *$x_1, \ldots, x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$,*
- *$\varphi_1, \ldots, \varphi_p \in E$*

such that $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$).

Proof.

We prove this by induction on n. The initial step is clearly satisfied because if $k \in K(1)$ then $k = \epsilon$.

Then suppose the statement holds for n and let's see it holds also for $n + 1$.

So let $k \in K(n+1)$ and $k \neq \epsilon$. By assumption 2.1.2

$$\exists g \in K(n),\ z \in \mathcal{V} - var(g),\ \psi \in E_s(n,g) : \\ k = g + (z,\psi) \wedge \Xi(k) = \{\sigma + (z,s) |\ \sigma \in \Xi(g),\ s \in \#(g,\psi,\sigma)\}\ .$$

By the inductive hypothesis

$g = \epsilon$ or
($n > 1$ and there exist

- a positive integer p such that $p < n$,
- $x_1, \ldots, x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$,
- $\varphi_1, \ldots, \varphi_p \in E$

such that $\mathcal{K}(n; g; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$).

We first consider the case where $g = \epsilon$.

Here we define $p = 1 < n+1$, $x_1 = z \in \mathcal{V}$, $\varphi_1 = \psi \in E$, and we want to show that $\mathcal{K}(n+1; k; x_1 : \varphi_1)$.

We assume $k_0 = \epsilon$ and $k_1 = \epsilon + (x_1, \varphi_1)$.

This implies $k_1 = k$, $k_0 \in K(n)$, $\varphi_1 \in E_s(n, k_0)$, so $\mathcal{K}(n+1; k; x_1 : \varphi_1)$ holds.

We now turn to the case where $g \neq \epsilon$ and so

($n > 1$ and there exist

- a positive integer p such that $p < n$,
- $x_1, \ldots, x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$,
- $\varphi_1, \ldots, \varphi_p \in E$

such that $\mathcal{K}(n; g; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$).

In this case $p+1$ is a positive integer and $p+1 < n+1$. We define $x_{p+1} = z \in \mathcal{V}$, $\varphi_{p+1} = \psi \in E$ and need to show that $\mathcal{K}(n+1; k; x_1 : \varphi_1, \ldots, x_{p+1} : \varphi_{p+1})$.

We define $k_0 = \epsilon$ and for each $i = 1 \ldots p+1$ $k_i = k_{i-1} + (x_i, \varphi_i)$.

Since $\mathcal{K}(n; g; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ we have that $g = k_p$ and for each $i = 1 \ldots p$ $k_{i-1} \in K(n-1) \subseteq K(n)$, $\varphi_i \in E_s(n-1, k_{i-1}) \subseteq E_s(n, k_{i-1})$.

To complete our proof we just need to show that $k = k_{p+1}$, $k_p \in K(n)$ and $\varphi_{p+1} \in E_s(n, k_p)$.

We have $k = g + (z, \psi) = k_p + (x_{p+1}, \varphi_{p+1}) = k_{p+1}$; $k_p = g \in K(n)$;
$\varphi_{p+1} = \psi \in E_s(n, k_p)$. ∎

Lemma 4.7. *Let n be a positive integer such that $n \geqslant 2$, let $k \in K(n)$ such that $k \neq \epsilon$. There exist a positive integer p such that $p < n$, $x_1, \ldots, x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_p \in E$ such that $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.*

Let $h \in K(n-1)$: $h \sqsubseteq k$, $h \neq \epsilon$. Then there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Moreover let m be a positive integer, $\vartheta \in E(n-1)$,
y a function whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
ψ a function whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n-1)$.

Finally suppose that $\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$, and define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.

Then for each $j = 1 \ldots m$ $\mathcal{K}(n-1; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$.

Proof.

By lemma 4.6 we have

($n - 1 > 1$ and there exist

- a positive integer q such that $q < n - 1$,
- $y_1, \ldots, y_q \in \mathcal{V}$ such that $y_i \neq y_j$ for $i \neq j$,
- $\psi_1, \ldots, \psi_q \in E$

such that $\mathcal{K}(n-1; h; y_1 : \psi_1, \ldots, y_q : \psi_q)$).

Let $k = (u, \phi)$. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$ and $h = k_{/C} = (u_{/C}, \phi_{/C})$.

By lemma 4.2 we have $dom(k) = \{1, \ldots, p\}$ and for each $i = 1 \ldots p$ $u_i = x_i$ and $\phi_i = \varphi_i$.

And by the same lemma $C = dom(h) = \{1, \ldots, q\}$ and for each $i = 1 \ldots q$ $u_i = (u_{/C})_i = y_i$ and $\phi_i = (\phi_{/C})_i = \psi_i$.

Therefore $q \leqslant p$ and for each $i = 1 \ldots q$ $y_i = x_i$ and $\psi_i = \varphi_i$.

We now turn to the second part of the lemma. We first need to prove the truth of $\mathcal{K}(n; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$.

To simplify we define for each $\alpha = 1 \ldots q$ $u_\alpha = x_\alpha$, $\xi_\alpha = \varphi_\alpha$ and for each $\alpha = q+1 \ldots q+j$ $u_\alpha = y_{\alpha-q}$, $\xi_\alpha = \psi_{\alpha-q}$.

We define $\kappa_0 = \epsilon$, for each $\alpha = 1 \ldots q+j$ $\kappa_\alpha = \kappa_{\alpha-1} + (u_\alpha, \xi_\alpha)$.

We have $h = \kappa_q$, $h'_1 = h + (y_1, \psi_1) = \kappa_q + (u_{q+1}, \xi_{q+1}) = \kappa_{q+1}$.

if $j > 1$ then for each $\beta = 1 \ldots j-1$

$$h'_{\beta+1} = h'_\beta + (y_{\beta+1}, \psi_{\beta+1}) = \kappa_{q+\beta} + (u_{q+\beta+1}, \xi_{q+\beta+1}) = \kappa_{q+\beta+1} \ .$$

It follows $h'_j = \kappa_{q+j}$.

Given $\alpha = 1 \ldots q+j$ we need to prove $\kappa_{\alpha-1} \in K(n-1)$ and $\xi_\alpha \in E_s(n-1, \kappa_{\alpha-1})$.

For each $\alpha = 1 \dots q$ $\kappa_{\alpha-1} \in K(n-2) \subseteq K(n-1)$, $\xi_\alpha = \varphi_\alpha \in E_s(n-2, \kappa_{\alpha-1})$.

We further have $\kappa_q = h \in K(n-1)$, $\xi_{q+1} = \psi_1 \in E_s(n-1, \kappa_q)$.

If $j > 1$ then for each $\alpha = q+2, \dots, q+j$

$\kappa_{\alpha-1} = h'_{\alpha-1-q} \in K(n-1);\ \xi_\alpha = \psi_{\alpha-q} \in E_s(n-1, h'_{\alpha-q-1}) = E_s(n-1, \kappa_{\alpha-1})$.

With this we have proved $\mathcal{K}(n; h'_j; x_1 : \varphi_1, \dots, x_q : \varphi_q, y_1 : \psi_1, \dots, y_j : \psi_j)$.

Now by lemma 4.6, since $h'_j \in K(n-1),\ h \neq \epsilon$, there exist

- a positive integer r such that $r < n-1$,
- $v_1, \dots, v_r \in \mathcal{V}$ such that $v_\alpha \neq v_\beta$ for $\alpha \neq \beta$,
- $\phi_1, \dots, \phi_p \in E$

such that $\mathcal{K}(n-1; h'_j; v_1 : \phi_1, \dots, v_r : \phi_r)$.

Clearly this implies $\mathcal{K}(n; h'_j; v_1 : \phi_1, \dots, v_r : \phi_r)$. By lemma 4.3 we derive that $r = q+j$, for each $\alpha = 1 \dots q$ $v_\alpha = x_\alpha$, $\phi_\alpha = \varphi_\alpha$, for each $\alpha = q+1 \dots q+j$ $v_\alpha = y_{\alpha-q}$, $\phi_\alpha = \psi_{\alpha-q}$. So we obtain that $\mathcal{K}(n-1; h'_j; x_1 : \varphi_1, \dots, x_q : \varphi_q, y_1 : \psi_1, \dots, y_j : \psi_j)$. ■

Lemma 4.8. *Let n be a positive integer such that $n \geqslant 2$, let $k \in K(n)$ such that $k \neq \epsilon$. There exist a positive integer p such that $p < n$, $x_1, \dots, x_p \in \mathcal{V}$ such that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\varphi_1, \dots, \varphi_p \in E$ such that $\mathcal{K}(n; k; x_1 : \varphi_1, \dots, x_p : \varphi_p)$.*

Let $i = 1 \dots p$, $h \in K(n)$ be such that $k_i \sqsubseteq h$.
There exist a positive integer q such that $q < n$, $y_1, \dots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \dots, \psi_q \in E$ such that $\mathcal{K}(n; h; y_1 : \psi_1, \dots, y_q : \psi_q)$.

Then $i \leqslant q$ and for each $j = 1 \dots i$ $y_j = x_j$, $\psi_j = \varphi_j$.

Proof.

Clearly $\mathcal{K}(n; k_i; x_1 : \varphi_1, \dots, x_i : \varphi_i)$. By lemma 4.2 $dom(k_i) = \{1, \dots, i\}$, and if we define $k_i = (u, \phi)$ then for each $j = 1 \dots i$ $u_j = x_j$, $\phi_j = \varphi_j$.

By lemma 4.2 $dom(h) = \{1, \dots, q\}$, and if we define $h = (v, \vartheta)$ then for each $j = 1 \dots q$ $v_j = y_j$, $\vartheta_j = \psi_j$.

Since $k_i \sqsubseteq h$ there exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \dots, q\}$, $k_i = h_{/C}$.

We have $\{1, \dots, i\} = dom(k_i) = C \subseteq \{1, \dots, q\}$ so $i \leqslant q$.

Moreover, $(u, \phi) = k_i = h_{/C} = (v_{/C}, \vartheta_{/C})$, so $u = v_{/C}$, $\phi = \vartheta_{/C}$, and for each $j = 1 \dots i$, $y_j = v_j = u_j = x_j$ and $\psi_j = \vartheta_j = \phi_j = \varphi_j$. ■

Lemma 4.9. *Let p be a positive integer and $\rho = (u, r)$ be a state-like pair whose domain is $\{1, \dots, p\}$. We define $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$ $\sigma_j = \sigma_{j-1} + (u_j, r_j)$. Then it results $\sigma_p = \rho$.*

Proof.

For each $j = 1 \dots p$ we define $\sigma_j = (v_j, \vartheta_j)$ and we prove $dom(\sigma_j) = \{1, \dots, j\}$ and for each $i = 1 \dots j$ $(v_j)_i = u_i$, $(\vartheta_j)_i = r_i$.

We have $(v_1, \vartheta_1) = \sigma_1 = \epsilon + (u_1, r_1)$, therefore $dom(\sigma_1) = \{1\}$ and $(v_1)_1 = u_1$, $(\vartheta_1)_1 = r_1$.

Let $j = 1 \dots p-1$, suppose $dom(\sigma_j) = \{1, \dots, j\}$ and for each $i = 1 \dots j$ $(v_j)_i = u_i$, $(\vartheta_j)_i = r_i$.
Then $(v_{j+1}, \vartheta_{j+1}) = \sigma_{j+1} = \sigma_j + (u_{j+1}, r_{j+1})$, so $dom(\sigma_{j+1}) = \{1, \dots, j+1\}$, for each $i = 1 \dots j$ $(v_{j+1})_i = (v_j)_i = u_i$, $(\vartheta_{j+1})_i = (\vartheta_j)_i = r_i$.
To finish, we have also $(v_{j+1})_{j+1} = u_{j+1}$ and $(\vartheta_{j+1})_{j+1} = r_{j+1}$.

Clearly we have proved $dom(\sigma_p) = \{1 \dots p\} = dom(\rho)$, and for each $i = 1 \dots p$ $(v_p)_i = u_i$, $(\vartheta_p)_i = r_i$, so $\sigma_p = (v_p, \vartheta_p) = (u, r) = \rho$. ■

LEMMA 4.10. *Let p be a positive integer and $\rho = (u, r)$ be a state-like pair whose domain is $\{1, \dots, p\}$. Let m be a positive integer and q be a non-negative integer. Let $\delta = (v, c)$ be another state-like pair whose domain is $\{1, \dots, q+m\}$.*
We define $\rho'_1 = \rho + (v_{q+1}, c_{q+1})$ and if $m > 1$ for each $j = 1 \dots m-1$

$$\rho'_{j+1} = \rho'_j + (v_{q+j+1}, c_{q+j+1}) \ .$$

In these assumptions for each $j = 1 \dots m$ if we set $\rho'_j = (u', r')$ then $dom(\rho'_j) = \{1, \dots, p+j\}$ and for each $\alpha = 1 \dots p+j$

- *if $\alpha \leqslant p$ then $u'_\alpha = u_\alpha$, $r'_\alpha = r_\alpha$;*
- *if $\alpha > p$ then $u'_\alpha = v_{q+\alpha-p}$, $r'_\alpha = c_{q+\alpha-p}$.*

Proof.

If we set $\rho'_1 = (u', r')$ then $dom(\rho'_1) = \{1, \dots, p+1\}$ and for each $\alpha = 1 \dots p+1$

- if $\alpha \leqslant p$ then $u'_\alpha = u_\alpha$, $r'_\alpha = r_\alpha$;
- if $\alpha > p$ then $u'_\alpha = u'_{p+1} = v_{q+1} = v_{q+\alpha-p}$, $r'_\alpha = r'_{p+1} = c_{q+1} = c_{q+\alpha-p}$.

Suppose $m > 1$, $j = 1 \dots m-1$. We define $\rho'_j = (u', r')$ and assume $dom(\rho'_j) = \{1, \dots, p+j\}$ and for each $\alpha = 1 \dots p+j$

- if $\alpha \leqslant p$ then $u'_\alpha = u_\alpha$, $r'_\alpha = r_\alpha$;
- if $\alpha > p$ then $u'_\alpha = v_{q+\alpha-p}$, $r'_\alpha = c_{q+\alpha-p}$.

We then define $\rho'_{j+1} = (u'', r'')$. Clearly $dom(\rho'_{j+1}) = \{1, \dots, p+j+1\}$ and for each $\alpha = 1 \dots p+j+1$

- if $\alpha \leqslant p$ then $u''_\alpha = u'_\alpha = u_\alpha$, $r''_\alpha = r'_\alpha = r_\alpha$;
- if $p+1 \leqslant \alpha \leqslant p+j$ then $u''_\alpha = u'_\alpha = v_{q+\alpha-p}$, $r''_\alpha = r'_\alpha = c_{q+\alpha-p}$.
- if $\alpha = p+j+1$ then $u''_\alpha = v_{q+j+1} = v_{q+\alpha-p}$, $r''_\alpha = c_{q+j+1} = c_{q+\alpha-p}$.

■

Lemma 4.11. *Let $h \in K$, $y \in \mathcal{V} - var(h)$, $\phi \in E_s(h)$, $k = h + (y, \phi)$. Then $k \in K$ and*

- *for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$.*
- *for each $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ $\rho + (y, s) \in \Xi(k)$.*

Proof.

There exists a positive integer n such that $\phi \in E_s(n, h)$. Clearly also $h \in K(n)$. So $k \in K(n)^+$ and

$$\Xi(k) = \{\rho + (y, s) |\, \rho \in \Xi(h), s \in \#(h, \phi, \rho)\}\,.$$

∎

Lemma 4.12. *Let $h \in K$, $y \in \mathcal{V} - var(h)$, $\varphi \in E_s(h)$, $k = h + (y, \varphi)$. Let $\psi \in E(h)$ such that $y \notin V_b(\psi)$. Then $\psi \in E(k)$ and for each $\sigma = \rho + (y, s) \in \Xi(k)$ $\#(k, \psi, \sigma) = \#(h, \psi, \rho)$.*

Proof.

There exists a positive integer n such that $\varphi \in E_s(n, h)$, $\psi \in E(n, h)$. Of course $h \in K(n)$, so $k \in K(n)^+$ and $\psi \in E_a(n + 1, k) \subseteq E(k)$.

For each $\sigma = \rho + (y, s) \in \Xi(k)$ $\#(k, \psi, \sigma) = \#(h, \psi, \rho)$. ∎

Lemma 4.13. *Let $h \in K$, $y \in \mathcal{V} - var(h)$, $\varphi \in E_s(h)$, $k = h + (y, \varphi)$. Then $y \in E(k)$ and for each $\sigma = \rho + (y, s) \in \Xi(k)$ $\#(k, y, \sigma) = s$.*

Proof.

There exists a positive integer n such that $\varphi \in E_s(n, h)$. Of course $h \in K(n)$, so $k \in K(n)^+$.

It follows that $y \in E_b(n + 1, k) \subseteq E(k)$. Moreover, for each $\sigma = \rho + (y, s) \in \Xi(k)$ $\#(k, y, \sigma) = s$. ∎

Definition 4.14. Let $k \in K$, m a positive integer, x a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \rightarrow x_i \neq x_j$, φ a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E$, and finally let $\phi \in E$. We write

$$\mathcal{E}(k, m, x, \varphi, \phi)$$

to indicate the following condition (where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m - 1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$):

- $\varphi_1 \in E_s(k)$;
- if $m > 1$ then for each $i = 1 \dots m - 1$ $k'_i \in K \wedge \varphi_{i+1} \in E_s(k'_i)$;
- $k'_m \in K \wedge \phi \in E(k'_m)$.

Lemma 4.15. *Suppose*

- *$k \in K$;*

- *m is a positive integer;*
- *x is a function whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1\dots m$ $i \neq j \rightarrow x_i \neq x_j$;*
- *φ is a function whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $\varphi_i \in E$;*
- *$\phi \in E$;*
- *$k'_1 = k + (x_1, \varphi_1)$, if $m > 1$ for each $i = 1\dots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$;*
- *$\mathcal{E}(k, m, x, \varphi, \phi)$.*

Define $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$. Then

- *$t \in E(k)$;*
- *for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}$;*
- *$V_b(t) = \{x_1,\dots,x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi)$.*

Proof.

We have

- $\varphi_1 \in E_s(k)$;
- if $m > 1$ then for each $i = 1\dots m-1$ $k'_i \in K \wedge \varphi_{i+1} \in E_s(k'_i)$;
- $k'_m \in K \wedge \phi \in E(k'_m)$.

There exist a positive integer n_1 such that $\varphi_1 \in E_s(n_1, k)$. If $m > 1$ then for each $i = 1\dots m-1$ there exists a positive integer n_{i+1} such that $\varphi_{i+1} \in E_s(n_{i+1}, k'_i)$. There exists a positive integer n_{m+1} such that $\phi \in E(n_{m+1}, k'_m)$.

We define $n = \max\{n_1,\dots,n_{m+1}\}$, then we have the following:

- $\varphi_1 \in E_s(n, k)$;
- $k \in K(n)$;
- if $m > 1$ then for each $i = 1\dots m-1$ $k'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, k'_i)$;
- $k'_m \in K(n) \wedge \phi \in E(n, k'_m)$.

This implies $\mathcal{E}(n, k, m, x, \varphi, \phi)$ and consequently

- $t \in E(n+1, k)$;
- for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}$;
- $V_b(t) = \{x_1,\dots,x_m\} \cup V_b(\varphi_1) \cup \dots \cup V_b(\varphi_m) \cup V_b(\phi)$.

■

We are ready to start the big definition process in which we define substitution. This is an inductive definition process, so be aware that at step n we may find that the notion of $k\{x_i/t\}$ or $\varphi_k\{x_i/t\}$ we are about to define has already been defined in a former step. Within the definition there are internal tasks in which we verify some expected condition. We'll use the symbol $\diamond$ to mark the end of each of those tasks.

DEFINITION 4.16. Let n be a positive integer such that $n \geqslant 2$, let $k \in K(n)$ such that $k \neq \epsilon$. There exist a positive integer p such that $p < n$, $x_1,\dots,x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_p \in E$ such that $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$. Clearly p, $x_1, \ldots, x_p$, $\varphi_1, \ldots, \varphi_p$ are univocally determined.

Given $i = 1 \ldots p$, $t \in E(k_{i-1})$ such that

- for each $\rho_{i-1} \in \Xi(k_{i-1})$ $\#(k_{i-1}, t, \rho_{i-1}) \in \#(k_{i-1}, \varphi_i, \rho_{i-1})$,
- for each $j = 1 \ldots p$: $j \neq i$ $x_j \notin V_b(t)$,
- for each $j = i+1 \ldots p$ $V_b(t) \cap V_b(\varphi_j) = \emptyset$;

what we want to do is the following.

- If $i = p$: if $k\{x_i/t\}$ has been defined in a step before the current step n we'll verify it is $k\{x_i/t\} = k_{p-1}$, otherwise we'll explicitly define $k\{x_i/t\} = k_{p-1}$.
- If $i < p$ we want to verify the following
  - $k_{p-1}\{x_i/t\}$ is defined and belongs to K;
  - $x_p \in \mathcal{V} - var(k_{p-1}\{x_i/t\})$;
  - $(\varphi_p)_{k_{p-1}}\{x_i/t\}$ is defined and belongs to $E_s(k_{p-1}\{x_i/t\})$.

  Then if $k\{x_i/t\}$ has been defined in a step before the current step n we'll verify it is

  - $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p, (\varphi_p)_{k_{p-1}}\{x_i/t\})$.

  Otherwise we'll explicitly define

  - $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p, (\varphi_p)_{k_{p-1}}\{x_i/t\})$.
- In both cases $i = p$ and $i < p$ we'll verify
  - $dom(k\{x_i/t\}) = \{1, \ldots, p-1\}$;
  - $k\{x_i/t\} \in K$;
  - $var(k\{x_i/t\}) = var(k) - \{x_i\}$;
  - $k_{i-1} \sqsubseteq k\{x_i/t\}$;
  - if we define $k\{x_i/t\} = (u, \phi)$ then for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$;
  - for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we have that for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$;
  - for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$
    * if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
    * if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
    * if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

    then $\sigma_p \in \Xi(k)$.
- For each $\varphi \in E(n, k)$ with $V_b(t) \cap V_b(\varphi) = \emptyset$
  - We'll define $\varphi_k\{x_i/t\}$.
  - We'll show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.
  - We'll prove that for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$

  * if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
  * if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
  * if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

  then $\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.
- We'll prove that $V_b(\varphi_k\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.
- We'll show that one of the following five conditions holds
  * $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.
  * $\varphi \in var(k)$, $\varphi = x_i \to \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi$.
  * $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, ψ, $\psi_1, \ldots, \psi_m \in E(n-1, h)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n, h)$, for each $\rho \in \Xi(h)$ $\#(h, \psi, \rho)$ is a function with m arguments, $(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$ is a member of the domain of $\#(h, \psi, \rho)$.

    If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

    If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have $V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results

    $$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) .$$

    Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.
  * $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n-1, h)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n, h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$.

    If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

    If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have $V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, we can define $(\psi_j)_h\{x_i/t\}$, and it results

    $$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) .$$

    Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.
  * $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, $\vartheta \in E(n-1)$, a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$;
    a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n-1)$;

    such that

    $\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$,
    $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n, h)$.

    If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \dots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.

We have $\psi_1 \in E(n-1, h)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;

for each $j = 1 \dots m-1$ $h'_j \in K(n-1)$ and by 4.7
$\mathcal{K}(n-1; h'_j; x_1 : \varphi_1, \dots, x_q : \varphi_q, y_1 : \psi_1, \dots, y_j : \psi_j)$,
for each $\alpha = 1 \dots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \dots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n-1, h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;

$h'_m \in K(n-1)$ and by 4.7
$\mathcal{K}(n-1; h'_m; x_1 : \varphi_1, \dots, x_q : \varphi_q, y_1 : \psi_1, \dots, y_m : \psi_m)$,
for each $\alpha = 1 \dots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \dots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n-1, h'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{h'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \dots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

- We'll prove the following. Given $h \in K(n)$ such that $k_i \sqsubseteq h$ we know there exist a positive integer q such that $q < n$, $y_1, \dots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \dots, \psi_q \in E$ such that $\mathcal{K}(n; h; y_1 : \psi_1, \dots, y_q : \psi_q)$.
By lemma 4.8 we know that $i \leqslant q$ and for each $j = 1 \dots i$ $y_j = x_j$, $\psi_j = \varphi_j$.
If $i < q$ then assume for each $j = i+1 \dots q$ $y_j \notin V_b(t)$, $V_b(t) \cap V_b(\psi_j) = \emptyset$.
Also assume $\varphi \in E(n, h)$.
Then $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.
- We'll prove the following. If there exists $h \in K(n)$ such that $\varphi \in E(n, h)$, $x_i \notin var(h)$ then $\varphi_k\{x_i/t\} = \varphi$.

Our definition process uses induction on $n \geqslant 2$, therefore in the initial step we have $n = 2$. If $k \in K(2)$ and $k \neq \epsilon$ then there exist $x_1 \in \mathcal{V}$, $\varphi_1 \in E$ such that $\mathcal{K}(2; k; x_1 : \varphi_1)$. This implies $k = \epsilon + (x_1, \varphi_1)$ and $\varphi_1 \in E_s(1, \epsilon)$.

Let $t \in E(\epsilon)$ be such that $\#(t) \in \#(\varphi_1)$. Clearly for each $\rho \in \Xi(\epsilon)$

$$\#(\epsilon, t, \rho) = \#(\epsilon, t, \epsilon) = \#(t) \in \#(\varphi_1) = \#(\epsilon, \varphi_1, \epsilon) = \#(\epsilon, \varphi_1, \rho) \ .$$

We define $k\{x_1/t\} = \epsilon$. Clearly $dom(\epsilon) = \emptyset$, $\epsilon \in K$,

$$var(\epsilon) = \emptyset = \{x_1\} - \{x_1\} = var(k) - \{x_1\} \ .$$

It also results $k_{i-1} = \epsilon \sqsubseteq \epsilon = k\{x_i/t\}$.

Suppose we define $\sigma_0 = \epsilon$ and $\sigma_1 = \sigma_0 + (x_1, \#(t)) = \epsilon + (x_1, \#(t))$. We have $\epsilon \in K(1)$, $\varphi_1 \in E_s(1, \epsilon)$, $x_1 \in (\mathcal{V} - var(\epsilon))$, so $k = \epsilon + (x_1, \varphi_1) \in K(1)^+$. This implies

$$\Xi(k) = \{\epsilon + (x_1, s) |\ s \in \#(\epsilon, \varphi_1, \epsilon)\} \ .$$

Now since $\#(t) \in \#(\epsilon, \varphi_1, \epsilon)$ we have $\sigma_1 \in \Xi(k)$.

Let $\varphi \in E(2, k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$. Of course

$$E(2, k) = E'(1, k) \cup E'_a(2, k) \cup E'_b(2, k) \cup E'_c(2, k) \cup E'_d(2, k) \cup E'_e(2, k).$$

Suppose $\varphi \in \mathbf{E'(1, k)}$, so $\varphi \in E(1, k)$ and $k \in K(1)$, $k = \epsilon$. This is against our assumption that $k \neq \epsilon$, so we must exclude the case where $\varphi \in E'(1, k)$.

Now suppose $\varphi \in \mathbf{E'_a(2, k)}$. This means $\varphi \in E_a(2, k)$, $k \in K(1)^+$. We have seen that $k = \epsilon + (x_1, \varphi_1)$, where $\epsilon \in K(1)$, $\varphi_1 \in E_s(1, \epsilon)$, $x_1 \in (\mathcal{V} - var(\epsilon))$. It follows that $\varphi \in E(1, \epsilon)$.

We define $\varphi_k\{x_1/t\} = \varphi \in E(\epsilon) = E(k\{x_1/t\})$.

Let $\rho \in \Xi(k\{x_1/t\})$ and define $\sigma_0 = \epsilon$ and $\sigma_1 = \sigma_0 + (x_1, \#(t)) = \epsilon + (x_1, \#(t))$. We have seen that $\sigma_1 \in \Xi(k)$. Since $\varphi \in E_a(2, k)$ we have

$$\#(k, \varphi, \sigma_1) = \#(\epsilon, \varphi, \epsilon) = \#(k\{x_1/t\}, \varphi_k\{x_1/t\}, \rho) \ .$$

Of course $V_b(\varphi_k\{x_1/t\}) = V_b(\varphi) \subseteq V_b(\varphi) \cup V_b(t)$.

The condition $\varphi \in \mathcal{C} \wedge \varphi_k\{x_1/t\} = \varphi$ is clearly satisifed.

Suppose $h \in K(2)$ such that $k_1 \sqsubseteq h$. We know there exist $y_1 \in \mathcal{V}$, $\psi_1 \in E$ such that $\mathcal{K}(2; h; y_1 : \psi_1)$. We know that $y_1 = x_1$ and $\psi_1 = \varphi_1$, therefore $h = h_1 = k_1 = k$, and clearly $\varphi_k\{x_1/t\} = \varphi_h\{x_1/t\}$.

Finally suppose there exists $h \in K(2)$ such that $\varphi \in E(2, h)$, $x_1 \notin var(h)$. We have $\varphi_k\{x_1/t\} = \varphi$ and this holds independently from the assumption, in fact this is the definition of $\varphi_k\{x_1/t\}$.

Let's examine the case where $\varphi \in \mathbf{E'_b(2, k)}$. This means $\varphi \in E_b(2, k)$, $k \in K(1)^+$. We have seen that $k = \epsilon + (x_1, \varphi_1)$, where $\epsilon \in K(1)$, $\varphi_1 \in E_s(1, \epsilon)$, $x_1 \in (\mathcal{V} - var(\epsilon))$. It results $E_b(2, k) = \{x_1\}$, so $\varphi = x_1$.

We define $\varphi_k\{x_1/t\} = t \in E(\epsilon) = E(k\{x_1/t\})$.

Let $\rho \in \Xi(k\{x_1/t\})$ and define $\sigma_0 = \epsilon$ and $\sigma_1 = \sigma_0 + (x_1, \#(t)) = \epsilon + (x_1, \#(t))$. We have seen that $\sigma_1 \in \Xi(k)$. We have

$$\#(k, \varphi, \sigma_1) = \#(k, x_1, \sigma_1) = \#(t) = \#(\epsilon, t, \epsilon) = \#(k\{x_1/t\}, \varphi_k\{x_1/t\}, \rho) \ .$$

Of course $V_b(\varphi_k\{x_1/t\}) = V_b(t) \subseteq V_b(\varphi) \cup V_b(t)$.

The following conditions hold: $\varphi \in var(k)$, $\varphi = x_1$, $\varphi_k\{x_1/t\} = t$. So the following condition is satisfied:

$$\varphi \in var(k),\ \varphi = x_1 \rightarrow \varphi_k\{x_1/t\} = t,\ \varphi \neq x_1 \rightarrow \varphi_k\{x_1/t\} = \varphi \ .$$

Suppose $h \in K(2)$ such that $k_1 \sqsubseteq h$. We know there exist $y_1 \in \mathcal{V}$, $\psi_1 \in E$ such that $\mathcal{K}(2; h; y_1 : \psi_1)$. We know that $y_1 = x_1$ and $\psi_1 = \varphi_1$, therefore $h = h_1 = k_1 = k$, and clearly $\varphi_k\{x_1/t\} = \varphi_h\{x_1/t\}$.

Finally suppose there exists $h \in K(2)$ such that $\varphi \in E(2, h)$, $x_1 \notin var(h)$. In this case, by lemma 2.8 $V_f(\varphi) \subseteq var(h)$. But since $\varphi \in E_b(2, k)$ we have also $V_f(\varphi) = \{x_1\}$. It comes out that $x_1 \in var(h)$, against our assumption. So there doesn't exist $h \in K(2)$ such that $\varphi \in E(2, h)$, $x_1 \notin var(h)$.

Now assume $\varphi \in \mathbf{E}'_{\mathbf{c}}(\mathbf{2}, \mathbf{k})$. This implies $\varphi \in E_c(2, k) \neq \emptyset$, so $k \in K(1)$, $k = \epsilon$. This is against our assumption that $k \neq \epsilon$, so we must exclude the case where $\varphi \in E'_c(2, k)$. The same way we have to exclude the cases where $\varphi \in E'_d(2, k)$ and $\varphi \in E'_e(2, k)$.

We've seen the only two 'real' cases are $\varphi \in E'_a(2, k)$, $\varphi \in E'_b(2, k)$, and the definition of $\varphi_k\{x_1/t\}$ depends on which case is verified. Clearly $E'_a(2, k)$ and $E'_b(2, k)$ are disjoint sets, so the definition we have set out is correct.

This wraps up the initial step of our definition process. To deal with the inductive step let $n \geqslant 2$, suppose we have given our definitions and verified the results at step n, and let's go on with step $n + 1$.

Let $k \in K(n + 1)$ such that $k \neq \epsilon$. Let p be a positive integer such that $p < n + 1$, $x_1, \ldots, x_p \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_p \in E$ such that $\mathcal{K}(n + 1; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.

Let $i = 1 \ldots p$, $t \in E(k_{i-1})$ such that

- for each $\rho_{i-1} \in \Xi(k_{i-1})$ $\#(k_{i-1}, t, \rho_{i-1}) \in \#(k_{i-1}, \varphi_i, \rho_{i-1})$,
- for each $j = 1 \ldots p$: $j \neq i$ $x_j \notin V_b(t)$,
- for each $j = i + 1 \ldots p$ $V_b(t) \cap V_b(\varphi_j) = \emptyset$.

Consider the case where $\mathbf{i} = \mathbf{p}$.

If $\mathbf{k} \in \mathbf{K}(\mathbf{n})$ there exist a positive integer q such that $q < n$, $y_1, \ldots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \ldots, \psi_q \in E$ such that $\mathcal{K}(n; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$. Clearly $\mathcal{K}(n + 1; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$ also holds, so by lemma 4.3 $q = p$, for each $\alpha = 1 \ldots p$ $y_\alpha = x_\alpha$ and $\psi_\alpha = \varphi_\alpha$, $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.

For this reason, by the inductive hypothesis, $k\{x_i/t\}$ is already defined and we have $k\{x_i/t\} = k_{p-1}$. We have also

- $dom(k\{x_i/t\}) = \{1, \ldots, p - 1\}$;
- $k\{x_i/t\} \in K$;
- $var(k\{x_i/t\}) = var(k) - \{x_i\}$;
- $k_{i-1} \sqsubseteq k\{x_i/t\}$;
- if we define $k\{x_i/t\} = (u, \phi)$ then for each $j = 1 \ldots i - 1$ $u_j = x_j$, for each $j = i \ldots p - 1$ $u_j = x_{j+1}$;
- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we have that for each $j = 1 \ldots i - 1$ $u_j = x_j$, for each $j = i \ldots p - 1$ $u_j = x_{j+1}$;

- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$
  - if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
  - if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
  - if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

  then $\sigma_p \in \Xi(k)$.

If on the contrary $\mathbf{k} \notin \mathbf{K}(\mathbf{n})$ then we define $k\{x_i/t\} = k_{p-1}$.

If $p > 1$ then we have $k_{p-1} \in K(n)$ and $k_{p-1} \neq \epsilon$. Using lemma 4.2 we obtain

- $dom(k_{p-1}) = \{1, \ldots, p-1\}$;
- if we define $k_{p-1} = (u_{p-1}, \phi_{p-1})$ then for each $j = 1 \ldots p-1$ $(u_{p-1})_j = x_j$, $(\phi_{p-1})_j = \varphi_j$.

Therefore the following hold

- $dom(k\{x_i/t\}) = \{1, \ldots, p-1\}$;
- $k\{x_i/t\} \in K$;
- $var(k\{x_i/t\}) = var(k) - \{x_i\}$;
- $k_{i-1} = k_{p-1} \sqsubseteq k_{p-1} = k\{x_i/t\}$;
- if we define $k\{x_i/t\} = (u, \phi)$ then for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$;
- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we have that for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$.

Moreover let $\rho = (u, r) \in \Xi(k\{x_i/t\})$, we define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$

- if $j < p$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- $\sigma_p = \sigma_{p-1} + (x_p, \#(k_{p-1}, t, \rho))$.

We need to show that $\sigma_p \in \Xi(k)$.

We have that $dom(\rho) = \{1, \ldots, p-1\}$ so by lemma 4.9

$$\sigma_{p-1} = \rho,\ \sigma_p = \rho + (x_p, \#(k_{p-1}, t, \rho))\ .$$

It also results $k = k_{p-1} + (x_p, \varphi_p)$, where $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$, $\rho \in \Xi(k_{p-1})$, $\#(k_{p-1}, t, \rho) \in \#(k_{p-1}, \varphi_p, \rho)$. Therefore we can confirm that $\sigma_p \in \Xi(k)$.

If $p = 1$ then $k\{x_i/t\} = k_{p-1} = \epsilon$.

We have

- $dom(k\{x_i/t\}) = \emptyset = \{1, \ldots, p-1\}$;
- $k\{x_i/t\} = \epsilon \in K$;
- $var(k\{x_i/t\}) = \emptyset = var(k) - \{x_i\}$;
- if we define $k\{x_i/t\} = (u, \phi)$ then for each $j = 1 \ldots 0$ $u_j = x_j$, for each $j = 1 \ldots 0$ $u_j = x_{j+1}$;

- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we have that $\rho = \epsilon$ and for each $j = 1 \ldots 0$ $u_j = x_j$, for each $j = 1 \ldots 0$ $u_j = x_{j+1}$;

Moreover let $\rho = (u, r) \in \Xi(k\{x_i/t\})$ (this implies $\rho = \epsilon$) and define $\sigma_0 = \epsilon$ and $\sigma_1 = \sigma_0 + (x_1, \#(\epsilon, t, \rho)) = \epsilon + (x_1, \#(\epsilon, t, \epsilon))$. We need to verify that $\sigma_1 \in \Xi(k)$.

We have $k = k_1 = \epsilon + (x_1, \varphi_1)$, $\epsilon \in K(n)$, $\varphi_1 \in E_s(n, \epsilon)$, $x_1 \in \mathcal{V} - var(\epsilon)$, $\epsilon \in \Xi(\epsilon)$, $\#(\epsilon, t, \epsilon) \in \#(\epsilon, \varphi_1, \epsilon)$. Therefore we can confirm that $\sigma_1 \in \Xi(k)$.

We now turn to examine the case where $\mathbf{i} < \mathbf{p}$.

If $\mathbf{k} \in \mathbf{K}(\mathbf{n})$ there exist a positive integer q such that $q < n$, $y_1, \ldots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \ldots, \psi_q \in E$ such that $\mathcal{K}(n; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$. Clearly $\mathcal{K}(n+1; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$ also holds, so by lemma 4.3 $q = p$, for each $\alpha = 1 \ldots p$ $y_\alpha = x_\alpha$ and $\psi_\alpha = \varphi_\alpha$, $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.

By the inductive hypothesis

- $k_{p-1}\{x_i/t\}$ is defined and belongs to K;
- $x_p \in \mathcal{V} - var(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k_{p-1}}\{x_i/t\}$ is defined and belongs to $E_s(k_{p-1}\{x_i/t\})$.

Moreover, $k\{x_i/t\}$ is already defined and

- $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p, (\varphi_p)_{k_{p-1}}\{x_i/t\})$.

The inductive hypothesis also ensures that

- $dom(k\{x_i/t\}) = \{1, \ldots, p-1\}$;
- $k\{x_i/t\} \in K$;
- $var(k\{x_i/t\}) = var(k) - \{x_i\}$;
- $k_{i-1} \sqsubseteq k\{x_i/t\}$;
- if we define $k\{x_i/t\} = (u, \phi)$ then for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$;
- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we have that for each $j = 1 \ldots i-1$ $u_j = x_j$, for each $j = i \ldots p-1$ $u_j = x_{j+1}$;
- for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$ if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$
  - if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
  - if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
  - if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

  then $\sigma_p \in \Xi(k)$.

If on the contrary $\mathbf{k} \notin \mathbf{K}(\mathbf{n})$ then we consider that $k_{p-1} \in K(n)$ and $k_{p-1} \neq \epsilon$. Therefore there exist a positive integer $q < n$, $y_1, \ldots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \ldots, \psi_q \in E$ such that $\mathcal{K}(n; k_{p-1}; y_1 : \psi_1, \ldots, y_q : \psi_q)$.
We recall that $\mathcal{K}(n+1; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$ also holds.

If we define $k_{p-1} = (u_{p-1}, \phi_{p-1})$ then lemma 4.2 tells us that

- $\{1, \ldots, q\} = dom(k_{p-1}) = \{1, \ldots, p-1\}$ and therefore $q = p-1$;
- for each $j = 1 \ldots p-1$ $x_j = (u_{p-1})_j = y_j$, $\varphi_j = (\phi_{p-1})_j = \psi_j$;
- as a consequence of the former results, $\mathcal{K}(n; k_{p-1}; x_1 : \varphi_1, \ldots, x_{p-1} : \varphi_{p-1})$.

By the inductive hypothesis $k_{p-1}\{x_i/t\}$ is defined, it belongs to K and $var(k_{p-1}\{x_i/t\}) = var(k_{p-1}) - \{x_i\}$. Therefore $x_p \in \mathcal{V} - var(k_{p-1}\{x_i/t\})$.

We also consider that $\varphi_p \in E_s(n, k_{p-1})$ and $V_b(t) \cap V_b(\varphi_p) = \emptyset$, so $(\varphi_p)_{k_{p-1}}\{x_i/t\}$ is also defined and belongs to $E(k_{p-1}\{x_i/t\})$.

We want to show that $(\varphi_p)_{k_{p-1}}\{x_i/t\} \in E_s(k_{p-1}\{x_i/t\})$, so we still need to prove that for each $\rho \in \Xi(k_{p-1}\{x_i/t\})$ $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho)$ is a set.

Let $\rho = (u, r) \in \Xi(k_{p-1}\{x_i/t\})$ and we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p-1$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$.

Then $\#(k_{p-1}, \varphi_p, \sigma_{p-1}) = \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho)$.

Since $\varphi_p \in E_s(n, k_{p-1})$ we have that $\#(k_{p-1}, \varphi_p, \sigma_{p-1})$ is a set and $\#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho)$ is a set too.

So we can define $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p, (\varphi_p)_{k_{p-1}}\{x_i/t\})$, and $k\{x_i/t\} \in K$.

By the inductive hypothesis $dom(k_{p-1}\{x_i/t\}) = \{1, \ldots, p-2\}$, so $dom(k\{x_i/t\}) = \{1, \ldots, p-1\}$. Moreover

$$\begin{aligned} var(k\{x_i/t\}) &= var(k_{p-1}\{x_i/t\}) \cup \{x_p\} = (var(k_{p-1}) - \{x_i\}) \cup \{x_p\} \\ &= (var(k_{p-1}) \cup \{x_p\}) - \{x_i\} = var(k) - \{x_i\} \ . \end{aligned}$$

Also, clearly, $k_{i-1} \sqsubseteq k_{p-1}\{x_i/t\} \sqsubseteq k\{x_i/t\}$.

We now define $k\{x_i/t\} = (v, \vartheta)$, $k_{p-1}\{x_i/t\} = (u, \phi)$. By the inductive hypothesis we have that for each $j = 1 \ldots i-1$ $u_j = x_j$ and for each $j = i \ldots p-2$ $u_j = x_{j+1}$.
Furthermore for each $j = 1 \ldots p-2$ $v_j = u_j$, $v_{p-1} = x_p$.
So we derive that for each $j = 1 \ldots i-1$ $v_j = u_j = x_j$; for each $j = i \ldots p-2$ $v_j = u_j = x_{j+1}$, and it follows that for each $j = i \ldots p-1$ $v_j = x_{j+1}$.

Let $\rho = (w, s) \in \Xi(k\{x_i/t\})$. We have $w = v$, so for each $j = 1 \ldots i-1$ $w_j = v_j = x_j$, for each $j = i \ldots p-1$ $w_j = v_j = x_{j+1}$.

Let $\rho = (w, s) \in \Xi(k\{x_i/t\})$, we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (w_j, s_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (w_{j-1}, s_{j-1})$;

We want to show that $\sigma_p \in \Xi(k)$.

Clearly there exist $\rho_{p-1} \in \Xi(k_{p-1}\{x_i/t\})$, $c \in \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho_{p-1})$ such that $\rho = \rho_{p-1} + (x_p, c)$.

We now define $\rho_{p-1} = (w', s')$. It results

$$dom(\rho_{p-1}) = dom(k_{p-1}\{x_i/t\}) = \{1, \dots p-2\} \ .$$

For each $j = 1 \dots p-2$ we have $w_j = w'_j,\ s_j = s'_j$. Therefore for each $j = 1 \dots p-1$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (w'_j, s'_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (w'_{j-1}, s'_{j-1})$;

Clearly

$$\begin{aligned}\rho_{i-1} = \rho_{/dom(k_{i-1})} &= (w_{/dom(k_{i-1})}, s_{/dom(k_{i-1})}) = \\ &= ((w_{/\{1,\dots p-2\}})_{/dom(k_{i-1})}, (s_{/\{1,\dots p-2\}})_{/dom(k_{i-1})}) = \\ &= ((w')_{/dom(k_{i-1})}, (s')_{/dom(k_{i-1})}) = (\rho_{p-1})_{/dom(k_{i-1})} \ .\end{aligned}$$

We can apply the inductive hypothesis and obtain that $\sigma_{p-1} \in \Xi(k_{p-1})$, and

$$\#(k_{p-1}, \varphi_p, \sigma_{p-1}) = \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho_{p-1}) \ .$$

To show that $\sigma_p \in \Xi(k)$ we consider that $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$. Therefore $k \in K(n)^+$.
Moreover $\sigma_p = \sigma_{p-1} + (w_{p-1}, s_{p-1}) = \sigma_{p-1} + (x_p, c)$, and since $\sigma_{p-1} \in \Xi(k_{p-1})$, $c \in \#(k_{p-1}, \varphi_p, \sigma_{p-1})$ we have that $\sigma_p \in \Xi(k)$.

In the next step of our definition, for each $\varphi \in E(n+1, k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$

- We'll define $\varphi_k\{x_i/t\}$.
- We'll show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.
- We'll prove that for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$
  - if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
  - if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
  - if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

  then $\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.
- We'll prove that $V_b(\varphi_k\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

Remember that

$$E(n+1, k) = E'(n, k) \cup E'_a(n+1, k) \cup E'_b(n+1, k) \cup E'_c(n+1, k) \cup E'_d(n+1, k) \cup E'_e(n+1, k).$$

The definition of $\varphi_k\{x_i/t\}$ depends on the set to which φ belongs to, actually φ may belong to more than one of these sets, but this problem will be addressed later when we'll show that the definitions match each other.

Suppose $\varphi \in \mathbf{E}'(\mathbf{n},\mathbf{k})$. This means $\varphi \in E(n,k)$, $k \in K(n)$. In this case, by the inductive hypothesis, $\varphi_k\{x_i/t\}$ has already been already defined at step n and has all the properties we require at this stage of our definition.

Now suppose $\varphi \in \mathbf{E}'_{\mathbf{a}}(\mathbf{n+1},\mathbf{k})$. This implies $\varphi \in E_a(n+1,k)$, $k \in K(n)^+$.

We have $k = k_p = k_{p-1}+(x_p,\varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n,k_{p-1})$, $x_p \in \mathcal{V}-var(k_{p-1})$. Therefore $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$.

If $\mathbf{i}=\mathbf{p}$ then we define $\varphi_k\{x_i/t\} = \varphi \in E(k_{p-1}) = E(k\{x_i/t\})$.

Let $\rho=(u,r) \in \Xi(k\{x_i/t\})$, we define $\sigma_0 = \epsilon$ and for each $j = 1\ldots p$

- if $j < p$ then $\sigma_j = \sigma_{j-1} + (u_j,r_j)$,
- $\sigma_p = \sigma_{p-1} + (x_p, \#(k_{p-1},t,\rho))$.

We have already seen that $\sigma_p \in \Xi(k)$.
We need to show that $\#(k,\varphi,\sigma_p) = \#(k\{x_i/t\},\varphi_k\{x_i/t\},\rho)$.

If $p=1$ then $\rho \in \Xi(\epsilon)$ so $\rho = \epsilon = \sigma_{p-1}$.

If $p>1$ we have that $dom(\rho) = \{1,\ldots,p-1\}$ so by lemma 4.9 $\sigma_{p-1} = \rho$

In both cases $\sigma_p = \rho + (x_p,\#(k_{p-1},t,\rho))$.
We have $\rho \in \Xi(k_{p-1})$, $\#(k_{p-1},t,\rho) \in \#(k_{p-1},\varphi_p,\rho)$. Therefore

$$\#(k,\varphi,\sigma_p) = \#(k_{p-1},\varphi,\rho) = \#(k\{x_i/t\},\varphi_k\{x_i/t\},\rho)\ .$$

Moreover $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi) \subseteq V_b(\varphi) \cup V_b(t)$.

If $\mathbf{i}<\mathbf{p}$ we consider that $k_{p-1} \in K(n)$ and $k_{p-1} \neq \epsilon$. Therefore there exist a positive integer $q<n$, $y_1,\ldots,y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1,\ldots,\psi_q \in E$ such that $\mathcal{K}(n;k_{p-1};y_1:\psi_1,\ldots,y_q:\psi_q)$.
We recall that $\mathcal{K}(n+1;k;x_1:\varphi_1,\ldots,x_p:\varphi_p)$ also holds.

If we define $k_{p-1} = (u_{p-1},\phi_{p-1})$ then lemma 4.2 tells us that

- $\{1,\ldots,q\} = dom(k_{p-1}) = \{1,\ldots,p-1\}$ and therefore $q=p-1$;
- for each $j = 1\ldots p-1$ $x_j = (u_{p-1})_j = y_j$, $\varphi_j = (\phi_{p-1})_j = \psi_j$;
- as a consequence of the former results, $\mathcal{K}(n;k_{p-1};x_1:\varphi_1,\ldots,x_{p-1}:\varphi_{p-1})$.

Clearly $k_{p-1}\{x_i/t\}$ is defined; since $\varphi \in E(n,k_{p-1})$ $\varphi_{k_{p-1}}\{x_i/t\}$ is defined too, and it belongs to $E(k_{p-1}\{x_i/t\})$.

So we can define $\varphi_k\{x_i/t\} = \varphi_{k_{p-1}}\{x_i/t\} \in E(k_{p-1}\{x_i/t\})$.

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. We consider that

- $k_{p-1}\{x_i/t\} \in K$;
- $x_p \in \mathcal{V} - var(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k_{p-1}}\{x_i/t\} \in E_s(k_{p-1}\{x_i/t\})$.
- $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p,(\varphi_p)_{k_{p-1}}\{x_i/t\})$.

Moreover we can show that $x_p \notin V_b(\varphi_{k_{p-1}}\{x_i/t\})$. In fact, by the inductive hypothesis, $V_b(\varphi_{k_{p-1}}\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$. We know that $V_b(\varphi) \subseteq \mathcal{V} - var(k)$, so $x_p \notin V_b(\varphi)$. We also know that $x_p \notin V_b(t)$, hence $x_p \notin V_b(\varphi_{k_{p-1}}\{x_i/t\})$.

Using lemma 4.12 we obtain that $\varphi_k\{x_i/t\} = \varphi_{k_{p-1}}\{x_i/t\} \in E(k\{x_i/t\})$.

Let $\rho = (w, s) \in \Xi(k\{x_i/t\})$, we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (w_j, s_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (w_{j-1}, s_{j-1})$;

We have proved that $\sigma_p \in \Xi(k)$ and we need to show that

$$\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) \ .$$

Clearly there exist $\rho_{p-1} \in \Xi(k_{p-1}\{x_i/t\})$, $c \in \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho_{p-1})$ such that $\rho = \rho_{p-1} + (x_p, c)$.

We have $\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k\{x_i/t\}, \varphi_{k_{p-1}}\{x_i/t\}, \rho)$.

Since $\varphi_{k_{p-1}}\{x_i/t\} \in E(k_{p-1}\{x_i/t\})$ and $x_p \notin V_b(\varphi_{k_{p-1}}\{x_i/t\})$, by lemma 4.12, we obtain $\#(k\{x_i/t\}, \varphi_{k_{p-1}}\{x_i/t\}, \rho) = \#(k_{p-1}\{x_i/t\}, \varphi_{k_{p-1}}\{x_i/t\}, \rho_{p-1})$, and therefore

$$\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k_{p-1}\{x_i/t\}, \varphi_{k_{p-1}}\{x_i/t\}, \rho_{p-1}) \ .$$

We now define $\rho_{p-1} = (w', s')$. It results

$$dom(\rho_{p-1}) = dom(k_{p-1}\{x_i/t\}) = \{1, \dots p-2\} \ .$$

For each $j = 1 \dots p-2$ we have $w_j = w'_j,\ s_j = s'_j$. Therefore for each $j = 1 \dots p-1$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (w'_j, s'_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (w'_{j-1}, s'_{j-1})$;

Clearly

$$\begin{aligned}
\rho_{i-1} = \rho_{/dom(k_{i-1})} &= (w_{/dom(k_{i-1})}, s_{/dom(k_{i-1})}) = \\
&= ((w_{/\{1,\dots p-2\}})_{/dom(k_{i-1})}, (s_{/\{1,\dots p-2\}})_{/dom(k_{i-1})}) = \\
&= ((w')_{/dom(k_{i-1})}, (s')_{/dom(k_{i-1})}) = (\rho_{p-1})_{/dom(k_{i-1})} \ .
\end{aligned}$$

We can apply the inductive hypothesis and obtain that $\sigma_{p-1} \in \Xi(k_{p-1})$, and

$$\#(k_{p-1}, \varphi, \sigma_{p-1}) = \#(k_{p-1}\{x_i/t\}, \varphi_{k_{p-1}}\{x_i/t\}, \rho_{p-1}) \ .$$

So far we have proved that

$$\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k_{p-1}, \varphi, \sigma_{p-1}) \ .$$

To complete our proof we need a further step, consisting in proving that

$$\#(k, \varphi, \sigma_p) = \#(k_{p-1}, \varphi, \sigma_{p-1}) \ .$$

Here we consider that $k_{p-1} \in K$, $x_p \in \mathcal{V} - var(k_{p-1})$, $\varphi_p \in E_s(k_{p-1})$, $k = k_{p-1} + (x_p, \varphi_p)$, $\varphi \in E(k_{p-1})$, $x_p \notin V_b(\varphi)$, $\sigma_p = \sigma_{p-1} + (w_{p-1}, s_{p-1}) \in \Xi(k)$.

By lemma 4.11 there exist $\delta \in \Xi(k_{p-1})$, $d \in \#(k_{p-1}, \varphi_p, \delta)$ such that $\sigma_p = \delta + (x_p, d)$. Clearly $\delta = \sigma_{p-1}$, $x_p = w_{p-1}$, $d = s_{p-1}$.

By lemma 4.12 we have $\#(k, \varphi, \sigma_p) = \#(k_{p-1}, \varphi, \sigma_{p-1})$.

Finally it results $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi_{k_{p-1}}\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

Now suppose $\varphi \in \mathbf{E'_b(n+1,k)}$. This implies $\varphi \in E_b(n+1, k)$, $k \in K(n)^+$.

We have $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$. Therefore $\varphi = x_p$.

If $\mathbf{i = p}$ we define $\varphi_k\{x_i/t\} = t \in E(k_{p-1}) = E(k\{x_i/t\})$.

Let $\rho = (u, r) \in \Xi(k\{x_i/t\})$, we define $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$

- if $j < p$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- $\sigma_p = \sigma_{p-1} + (x_p, \#(k_{p-1}, t, \rho))$.

We have already seen that $\sigma_p \in \Xi(k)$.
We need to show that $\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Clearly $\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k_{p-1}, t, \rho)$, so what we need to show is

$$\#(k, x_p, \sigma_p) = \#(k_{p-1}, t, \rho) \ .$$

There exist $\delta \in \Xi(k_{p-1})$, $s \in \#(k_{p-1}, \varphi_p, \delta)$ such that $\sigma_p = \delta + (x_p, s)$. By lemma 4.13 it results $\#(k, x_p, \sigma_p) = s$.

Since $\sigma_p = \sigma_{p-1} + (x_p, \#(k_{p-1}, t, \rho))$ we have $\delta = \sigma_{p-1}$ and $s = \#(k_{p-1}, t, \rho)$. Therefore $\#(k, x_p, \sigma_p) = \#(k_{p-1}, t, \rho)$.

Moreover $V_b(\varphi_k\{x_i/t\}) = V_b(t) \subseteq V_b(\varphi) \cup V_b(t)$.

If $\mathbf{i < p}$ we define $\varphi_k\{x_i/t\} = \varphi = x_p$.

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. We consider that

- $k_{p-1}\{x_i/t\} \in K$;
- $x_p \in \mathcal{V} - var(k_{p-1}\{x_i/t\})$;
- $(\varphi_p)_{k_{p-1}}\{x_i/t\} \in E_s(k_{p-1}\{x_i/t\})$.
- $k\{x_i/t\} = k_{p-1}\{x_i/t\} + (x_p, (\varphi_p)_{k_{p-1}}\{x_i/t\})$.

By lemma 4.13 we have $\varphi_k\{x_i/t\} = x_p \in E(k\{x_i/t\})$.

Let $\rho = (w, s) \in \Xi(k\{x_i/t\})$, we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (w_j, s_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (w_{j-1}, s_{j-1})$;

We have proved that $\sigma_p \in \Xi(k)$ and we need to show that

$$\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) \ .$$

By lemma 4.11 there exist $\delta \in \Xi(k_{p-1})$, $d \in \#(k_{p-1}, \varphi_p, \delta)$ such that $\sigma_p = \delta + (x_p, d)$. Clearly $\delta = \sigma_{p-1}$, $x_p = w_{p-1}$, $d = s_{p-1}$.

Using lemma 4.13 we obtain

$$\#(k, \varphi, \sigma_p) = \#(k, x_p, \sigma_p) = s_{p-1} \ .$$

Clearly there exist $\rho_{p-1} \in \Xi(k_{p-1}\{x_i/t\})$, $c \in \#(k_{p-1}\{x_i/t\}, (\varphi_p)_{k_{p-1}}\{x_i/t\}, \rho_{p-1})$ such that $\rho = \rho_{p-1} + (x_p, c)$. And clearly $c = s_{p-1}$.

Using lemma 4.13 we obtain

$$\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) = \#(k\{x_i/t\}, x_p, \rho) = s_{p-1} \ .$$

So we can derive $\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

To finish with the current case we see that $V_b(\varphi_k\{x_i/t\}) = V_b(\varphi) \subseteq V_b(\varphi) \cup V_b(t)$.

We turn to the case where $\varphi \in \mathbf{E'_c(n+1, k)}$. This implies $\varphi \in E_c(n+1, k)$, $k \in K(n)$.

There exist a positive integer m and $\psi, \psi_1, \ldots, \psi_m \in E(n, k)$ such that

- $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$;
- for each $\sigma \in \Xi(k)$ $\#(k, \psi, \sigma)$ is a function with m arguments and $(\#(k, \psi_1, \sigma), \ldots, \#(k, \psi_m, \sigma))$ is a member of its domain.

Since $k \in K(n)$ there exist a positive integer q such that $q < n$, $y_1, \ldots, y_q \in \mathcal{V}$ such that $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, $\psi_1, \ldots, \psi_q \in E$ such that $\mathcal{K}(n; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$. Clearly $\mathcal{K}(n+1; k; y_1 : \psi_1, \ldots, y_q : \psi_q)$ also holds, so by lemma 4.3 $q = p$, for each $\alpha = 1 \ldots p$ $y_\alpha = x_\alpha$ and $\psi_\alpha = \varphi_\alpha$, $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.

We have $V_b(\varphi) = V_b(\psi) \cup V_b(\psi_1) \cup \cdots \cup V_b(\psi_m)$ and since $V_b(t) \cap V_b(\varphi) = \emptyset$ we have

- $V_b(t) \cap V_b(\psi) = \emptyset$;
- for each $j = 1 \ldots m$ $V_b(t) \cap V_b(\psi_j) = \emptyset$.

By the inductive hypothesis $\psi_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$ and for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$. So we can define

$$\varphi_k\{x_i/t\} = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) \ .$$

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. To show this we use lemma 3.10.

Let $\rho = (u, r) \in \Xi(k\{x_i/t\})$, we just need to show that $\#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$ is a function with m arguments and
$(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \ldots, \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho))$ is a member of its domain.

We define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$, for each $j = 1 \ldots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,

- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$.

By the inductive hypothesis $\sigma_p \in \Xi(k)$, $\#(k, \psi, \sigma_p) = \#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$, for each $j = 1 \dots m$ $\#(k, \psi_j, \sigma_p) = \#(k\{x_i/t\}, (\psi_j)_k\{x_i/t\}, \rho)$.

Therefore $\#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho) = \#(k, \psi, \sigma_p)$ is a function with m arguments and $(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \dots, \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho))$ is equal to $(\#(k, \psi_1, \sigma_p), \dots, \#(k, \psi_m, \sigma_p))$ and so is a member of the domain of $\#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)$.

We have proved that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Moreover for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

then $\sigma_p \in \Xi(k)$ and

$$\begin{aligned}
&\#(k, \varphi, \sigma_p) = \#(k, (\psi)(\psi_1, \dots, \psi_m), \sigma_p) = \#(k, \psi, \sigma_p)(\#(k, \psi_1, \sigma_p), \dots, \#(k, \psi_m, \sigma_p)) = \\
&= \#(k\{x_i/t\}, \psi_k\{x_i/t\}, \rho)(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \dots, \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho)) = \\
&= \#(k\{x_i/t\}, (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_k\{x_i/t\}), \rho) = \\
&= \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) \ .
\end{aligned}$$

Finally

$$\begin{aligned}
V_b(\varphi_k\{x_i/t\}) &= V_b((\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_k\{x_i/t\})) = \\
&= V_b((\psi_k\{x_i/t\})) \cup V_b((\psi_1)_k\{x_i/t\}) \cup \dots \cup V_b((\psi_m)_k\{x_i/t\}) \subseteq \\
&\subseteq V_b(\psi) \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m) \cup V_b(t) = V_b(\varphi) \cup V_b(t) \ .
\end{aligned}$$

We examine the case where $\varphi \in \mathbf{E'_d(n+1,k)}$. This implies $\varphi \in E_d(n+1, k)$, $k \in K(n)$.

There exist $f \in \mathcal{F}$, a positive integer m and $\psi_1, \dots, \psi_m \in E(n, k)$ such that

- $\varphi = (f)(\psi_1, \dots, \psi_m)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \psi_1, \sigma), \dots, \#(k, \psi_m, \sigma))$ holds true.

Since $k \in K(n)$ we have $\mathcal{K}(n; k; x_1 : \varphi_1, \dots, x_p : \varphi_p)$.

It results $V_b(\varphi) = V_b(\psi_1) \cup \dots \cup V_b(\psi_m)$ and since $V_b(t) \cap V_b(\varphi) = \emptyset$ we have

- for each $j = 1 \dots m$ $V_b(t) \cap V_b(\psi_j) = \emptyset$.

By the inductive hypothesis for each $j = 1 \dots m$ $(\psi_j)_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$. So we can define

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_k\{x_i/t\}, \dots, (\psi_m)_k\{x_i/t\}) \ .$$

We need to show that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$. To show this we use lemma 3.11.

We have $k\{x_i/t\} \in K$, $f \in \mathcal{F}$, for each $j = 1 \dots m$ $(\psi_j)_k\{x_i/t\} \in E(k\{x_i/t\})$.

Given $\rho = (u, r) \in \Xi(k\{x_i/t\})$ we need to show that $A_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \ldots, \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho))$ holds true.

We define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$, for each $j = 1 \ldots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$.

By the inductive hypothesis $\sigma_p \in \Xi(k)$ and for each $j = 1 \ldots m$ $\#(k, \psi_j, \sigma_p) = \#(k\{x_i/t\}, (\psi_j)_k\{x_i/t\}, \rho)$.

We have seen $A_f(\#(k, \psi_1, \sigma_p), \ldots, \#(k, \psi_j, \sigma_p))$ holds true, so $A_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \ldots, \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho))$ also holds.

Consequently, we have proved that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

Moreover for each $\rho = (u, r) \in \Xi(k\{x_i/t\})$, if we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$ and define $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$;

then $\sigma_p \in \Xi(k)$ and

$$\begin{aligned}\#(k, \varphi, \sigma_p) &= \#(k, (f)(\psi_1, \ldots, \psi_m), \sigma_p) = P_f(\#(k, \psi_1, \sigma_p), \ldots, \#(k, \psi_m, \sigma_p)) = \\ &= P_f(\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho), \ldots, \#(k\{x_i/t\}, (\psi_m)_k\{x_i/t\}, \rho)) = \\ &= \#(k\{x_i/t\}, (f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}), \rho) = \\ &= \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)\ .\end{aligned}$$

Furthermore

$$\begin{aligned}V_b(\varphi_k\{x_i/t\}) &= V_b((f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\})) = \\ &= V_b((\psi_1)_k\{x_i/t\}) \cup \cdots \cup V_b((\psi_m)_k\{x_i/t\}) \subseteq \\ &\subseteq V_b(\psi_1) \cup \cdots \cup V_b(\psi_m) \cup V_b(t) = V_b(\varphi) \cup V_b(t)\ .\end{aligned}$$

Finally let's consider the case where $\varphi \in \mathbf{E'_e(n+1,k)}$. This implies $\varphi \in E_e(n+1, k)$, $k \in K(n)$. There exist

- a positive integer m,
- a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(k)$, and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$,
- a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$,
- $\phi \in E(n)$

such that $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \phi)$ and $\mathcal{E}(n, k, m, y, \psi, \phi)$.

Let $k'_1 = k + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $k'_{j+1} = k'_j + (y_{j+1}, \psi_{j+1})$. We intend to define $\varphi_k\{x_i/t\}$ as follows.

If $m = 1$ $\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \phi_{k'_1}\{x_i/t\})$;

if $m > 1$

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_k\{x_i/t\}, y_2 : (\psi_2)_{k'_1}\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \phi_{k'_m}\{x_i/t\}) .$$

To accept this definition we need to verify it relies on well defined concepts. In other words we have to verify that $(\psi_1)_k\{x_i/t\}$ is defined, if $m > 1$ then for each $j = 2 \ldots m$ $(\psi_j)_{k'_{j-1}}\{x_i/t\}$ is defined, and finally that $\phi_{k'_m}\{x_i/t\}$ is defined.

Since $k \in K(n)$ we have $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$.

It results $\psi_1 \in E(n, k)$ and since $V_b(\psi_1) \subseteq V_b(\varphi)$ we have $V_b(\psi_1) \cap V_b(t) = \emptyset$. This ensures $(\psi_1)_k\{x_i/t\}$ is defined and belongs to $E(k\{x_i/t\})$.

Suppose $m > 1$ and let $j = 2 \ldots m$, we want to verify that $(\psi_j)_{k'_{j-1}}\{x_i/t\}$ is defined. We have $k'_{j-1} \in K(n)$ and $\mathcal{K}(n; k'_{j-1}; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_{j-1} : \psi_{j-1})$ follows by lemma 4.7. For each $\alpha = 1 \ldots j-1$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \subseteq V_b(\varphi)$ so $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$. We have $\psi_j \in E(n, k'_{j-1})$ and also $V_b(\psi_j) \subseteq V_b(\varphi)$ so $V_b(\psi_j) \cap V_b(t) = \emptyset$. Therefore $(\psi_j)_{k'_{j-1}}\{x_i/t\}$ is defined and belongs to $E(k'_{j-1}\{x_i/t\})$.

To verify that $\phi_{k'_m}\{x_i/t\}$ is defined we consider that $k'_m \in K(n)$ and by lemma 4.7 $\mathcal{K}(n; k'_m; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_m : \psi_m)$. For each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \subseteq V_b(\varphi)$ so $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$. We have $\phi \in E(n, k'_m)$ and also $V_b(\phi) \subseteq V_b(\varphi)$ so $V_b(\phi) \cap V_b(t) = \emptyset$. Therefore $\phi_{k'_m}\{x_i/t\}$ is defined and belongs to $E(k'_m\{x_i/t\})$.

At this point we accept the proposed definition of $\varphi_k\{x_i/t\}$, but we also need to prove that $\varphi_k\{x_i/t\} \in E(k\{x_i/t\})$.

We define $h = k\{x_i/t\}$; a function ϑ whose domain is $\{1, \ldots, m\}$ such that $\vartheta_1 = (\psi_1)_k\{x_i/t\}$, if $m > 1$ for each $j = 2 \ldots m$ $\vartheta_j = (\psi_j)_{k'_{j-1}}\{x_i/t\}$; $\theta = \phi_{k'_m}\{x_i/t\}$. With these definitions clearly

$$\varphi_k\{x_i/t\} = \{\}(y_1 : \vartheta_1, \ldots, y_m : \vartheta_m, \theta) .$$

We should be able to apply lemma 4.15. We have $var(h) = var(k) - x_i \subseteq var(k)$ and so $\mathcal{V} - var(k) \subseteq \mathcal{V} - var(h)$. Moreover

- $h \in K$;
- m is a positive integer;
- y is a function whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$, and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
- ϑ is a function whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\vartheta_j \in E$;
- $\theta \in E$.

We then define $h'_1 = h + (y_1, \vartheta_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \vartheta_{j+1})$. Clearly we need to prove $\mathcal{E}(h, m, y, \vartheta, \theta)$.

We first verify that for each $j = 1 \ldots m$ $h'_j = k'_j\{x_i/t\}$.

We have $\mathcal{K}(n; k'_1; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1)$ so $k'_1 = (k'_1)_p + (y_1, \psi_1)$, but $k'_1 = k + (y_1, \psi_1)$ also holds so $k = (k'_1)_p$. Consequently

$$k'_1\{x_i/t\} = (k'_1)_p\{x_i/t\} + (y_1, (\psi_1)_{(k'_1)_p}\{x_i/t\}) = k\{x_i/t\} + (y_1, (\psi_1)_k\{x_i/t\}) = h'_1 \ .$$

Now suppose $m > 1$ and let $j = 1 \ldots m-1$.
We have $\mathcal{K}(n; k'_{j+1}; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_{j+1} : \psi_{j+1})$,
so $k'_{j+1} = (k'_{j+1})_{p+j} + (y_{j+1}, \psi_{j+1})$, but $k'_{j+1} = k'_j + (y_{j+1}, \psi_{j+1})$ also holds so $(k'_{j+1})_{p+j} = k'_j$. It follows

$$\begin{aligned} k'_{j+1}\{x_i/t\} &= (k'_{j+1})_{p+j}\{x_i/t\} + (y_{j+1}, (\psi_{j+1})_{(k'_{j+1})_{p+j}}\{x_i/t\}) \\ &= k'_j\{x_i/t\} + (y_{j+1}, (\psi_{j+1})_{k'_j}\{x_i/t\}) = h'_j + (y_{j+1}, \vartheta_{j+1}) = h'_{j+1} \ . \end{aligned}$$

Proving $\mathcal{E}(h, m, y, \vartheta, \theta)$ means showing that

- $\vartheta_1 \in E_s(h)$;
- if $m > 1$ then for each $j = 1 \ldots m-1$ $h'_j \in K \wedge \vartheta_{j+1} \in E_s(h'_j)$;
- $h'_m \in K \wedge \theta \in E(h'_m)$.

We begin by proving that $\vartheta_1 \in E_s(h)$, in other words $(\psi_1)_k\{x_i/t\} \in E_s(k\{x_i/t\})$. Let $\rho = (u, r) \in \Xi(k\{x_i/t\})$, we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$, $\sigma_0 = \epsilon$ and for each $j = 1 \ldots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$.

Then $\sigma_p \in \Xi(k)$ and $\#(k, \psi_1, \sigma_p) = \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$. Since $\mathcal{E}(n, k, m, y, \psi, \phi)$ it results $\psi_1 \in E_s(n, k)$ so $\#(k, \psi_1, \sigma_p)$ is a set and so is $\#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$.

Suppose $m > 1$ and $j = 1 \ldots m-1$, we need to verify that $h'_j \in K$ and $\vartheta_{j+1} \in E_s(h'_j)$. In other words we need to verify $k'_j\{x_i/t\} \in K$ and $(\psi_{j+1})_{k'_j}\{x_i/t\} \in E_s(k'_j\{x_i/t\})$. Clearly $k'_j\{x_i/t\}$ belongs to K, and we have verified that $(\psi_{j+1})_{k'_j}\{x_i/t\} \in E(k'_j\{x_i/t\})$.

Let $\rho = (u, r) \in \Xi(k'_j\{x_i/t\})$, we'd like to verify that $\#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho)$ is a set. We define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$, $\sigma_0 = \epsilon$ and for each $\alpha = 1 \ldots p+j$

- if $\alpha < i$ then $\sigma_\alpha = \sigma_{\alpha-1} + (u_\alpha, r_\alpha)$,
- if $\alpha = i$ then $\sigma_\alpha = \sigma_{\alpha-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $\alpha > i$ then $\sigma_\alpha = \sigma_{\alpha-1} + (u_{\alpha-1}, r_{\alpha-1})$.

Then $\sigma_{p+j} \in \Xi(k'_j)$ and $\#(k'_j, \psi_{j+1}, \sigma_{p+j}) = \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho)$.
Since $\mathcal{E}(n, k, m, y, \psi, \phi)$ it results $\psi_{j+1} \in E_s(n, k'_j)$ so $\#(k'_j, \psi_{j+1}, \sigma_{p+j})$ is a set and so is $\#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho)$.

Finally we need to verify that $h'_m \in K \wedge \theta \in E(h'_m)$. In other words we need to verify $k'_m\{x_i/t\} \in K$ and $\phi_{k'_m}\{x_i/t\} \in E(k'_m\{x_i/t\})$. This has been proved above.

At this point $\mathcal{E}(h, m, y, \vartheta, \theta)$ is proved so by lemma 4.15 we obtain

$$\varphi_k\{x_i/t\} = \{\}(y_1 : \vartheta_1, \ldots, y_m : \vartheta_m, \theta) \in E(h) = E(k\{x_i/t\}) \ .$$

By lemma 4.15 we also obtain

$$V_b(\varphi_k\{x_i/t\}) = \{y_1, \ldots, y_m\} \cup V_b(\vartheta_1) \cup \cdots \cup V_b(\vartheta_m) \cup V_b(\theta) \ . \tag{4.0.1}$$

This will be used later.

Another point we have to verify is the following. Let $\rho = (u, r) \in \Xi(k\{x_i/t\})$, we define $\rho_{i-1} = \rho_{/dom(k_{i-1})}$, $\sigma_0 = \epsilon$ and for each $j = 1 \dots p$

- if $j < i$ then $\sigma_j = \sigma_{j-1} + (u_j, r_j)$,
- if $j = i$ then $\sigma_j = \sigma_{j-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$,
- if $j > i$ then $\sigma_j = \sigma_{j-1} + (u_{j-1}, r_{j-1})$.

It has been shown that $\sigma_p \in \Xi(k)$, we need to prove $\#(k, \varphi, \sigma_p) = \#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho)$.

Of course we have $\#(k, \varphi, \sigma_p) = \{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma_p \sqsubseteq \sigma'_m\}$, and using lemma 4.15 we derive

$$\begin{aligned}\#(k\{x_i/t\}, \varphi_k\{x_i/t\}, \rho) &= \#(h, \varphi_k\{x_i/t\}, \rho) = \{\#(h'_m, \theta, \rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\} = \\ &= \{\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m)|\ \rho'_m \in \Xi(k'_m\{x_i/t\}), \rho \sqsubseteq \rho'_m\}\ .\end{aligned}$$

So what we need to show is

$$\{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma_p \sqsubseteq \sigma'_m\} = \{\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m)|\ \rho'_m \in \Xi(k'_m\{x_i/t\}), \rho \sqsubseteq \rho'_m\}\ .$$

Suppose $w \in \{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma_p \sqsubseteq \sigma'_m\}$.

There exists $\sigma'_m = (v, c) \in \Xi(k'_m)$ such that $\sigma_p \sqsubseteq \sigma'_m$ and $w = \#(k'_m, \phi, \sigma'_m)$. Clearly $dom(\sigma'_m) = dom(k'_m) = \{1, \dots, p + m\}$.

We define $\rho'_1 = \rho + (v_{p+1}, c_{p+1})$ and if $m > 1$ for each $j = 1 \dots m - 1$ $\rho'_{j+1} = \rho'_j + (v_{p+j+1}, c_{p+j+1})$. Our goal is to show that

$$\rho'_m \in \Xi(k'_m\{x_i/t\}) \text{ and } \#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) = \#(k'_m, \phi, \sigma'_m)\ .$$

First of all we define $\delta_0 = \epsilon$ and for each $j = 1 \dots p + m$ $\delta_j = \delta_{j-1} + (v_j, c_j)$. By lemma 4.9 we derive that $\delta_{p+m} = \sigma'_m$. Therefore $\sigma_p \sqsubseteq \delta_{p+m}$.

There exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \dots p + m\}$, $\sigma_p = (\delta_{p+m})_{/C}$.

We have $C = dom(\sigma_p) = \{1, \dots, p\}$, and therefore

$$\sigma_p = (\delta_{p+m})_{/\{1,\dots,p\}} = (v_{/\{1,\dots,p\}}, c_{/\{1,\dots,p\}}) = \delta_p\ .$$

We define $k'_0 = k$. We use backward induction on j to show that for each $j = m \dots 1$

$$\delta_{p+j} \in \Xi(k'_j) \text{ and } \delta_{p+j-1} \in \Xi(k'_{j-1}),\ v_{p+j} = y_j,\ c_{p+j} \in \#(k'_{j-1}, \psi_j, \delta_{p+j-1})\ .$$

Clearly $\delta_{p+m} = \sigma'_m \in \Xi(k'_m)$.
We have $k'_m = k'_{m-1} + (y_m, \psi_m)$, $k'_{m-1} \in K(n)$, $y_m \in \mathcal{V} - var(k'_{m-1})$, $\psi_m \in E_s(n, k'_{m-1})$. By lemma 4.11 this implies there exist $\eta \in \Xi(k'_{m-1})$, $s \in \#(k'_{m-1}, \psi_m, \eta)$ such that

$$\delta_{p+m-1} + (v_{p+m}, c_{p+m}) = \delta_{p+m} = \eta + (y_m, s)\ .$$

By lemma 2.1 we obtain

$$\delta_{p+m-1} = \eta \in \Xi(k'_{m-1}),\ v_{p+m} = y_m,\ c_{p+m} = s \in \#(k'_{m-1}, \psi_m, \delta_{p+m-1})\ .$$

If $m > 1$ we need an inductive step. Let $j = m \dots 2$ and assume

$$\delta_{p+j} \in \Xi(k'_j) \text{ and } \delta_{p+j-1} \in \Xi(k'_{j-1}),\ v_{p+j} = y_j,\ c_{p+j} \in \#(k'_{j-1}, \psi_j, \delta_{p+j-1}) \ .$$

We need to prove

$$\delta_{p+j-2} \in \Xi(k'_{j-2}),\ v_{p+j-1} = y_{j-1},\ c_{p+j-1} \in \#(k'_{j-2}, \psi_{j-1}, \delta_{p+j-2}) \ .$$

We have $k'_{j-1} = k'_{j-2} + (y_{j-1}, \psi_{j-1})$, $k'_{j-2} \in K(n)$, $y_{j-1} \in \mathcal{V} - var(k'_{j-2})$, $\psi_{j-1} \in E_s(n, k'_{j-2})$.
By lemma 4.11 this implies there exist $\eta \in \Xi(k'_{j-2})$, $s \in \#(k'_{j-2}, \psi_{j-1}, \eta)$ such that

$$\delta_{p+j-2} + (v_{p+j-1}, c_{p+j-1}) = \delta_{p+j-1} = \eta + (y_{j-1}, s) \ .$$

By lemma 2.1 we obtain

$$\delta_{p+j-2} = \eta \in \Xi(k'_{j-2}),\ v_{p+j-1} = y_{j-1},\ c_{p+j-1} = s \in \#(k'_{j-2}, \psi_{j-1}, \delta_{p+j-2}) \ .$$

To show that $\rho'_m \in \Xi(k'_m\{x_i/t\})$ we show by induction on j that for each $j = 1 \dots m$ $\rho'_j \in \Xi(h'_j)$.

We begin by showing that $\rho'_1 \in \Xi(h'_1)$ using lemma 4.11. We have $h'_1 = h + (y_1, \vartheta_1)$ and $h \in K$, $y_1 \in \mathcal{V} - var(h)$, $\vartheta_1 \in E_s(h)$. We have $\rho \in \Xi(h)$ and $\rho'_1 = \rho + (y_1, c_{p+1})$.
To show $\rho'_1 \in \Xi(h'_1)$ we just need to show that $c_{p+1} \in \#(h, \vartheta_1, \rho)$.

In other words we have to prove $c_{p+1} \in \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$.

But we have proved that $c_{p+1} \in \#(k, \psi_1, \delta_p) = \#(k, \psi_1, \sigma_p)$, and since we have been able to define $(\psi_1)_k\{x_i/t\}$ we can assume $\#(k, \psi_1, \sigma_p) = \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$.

So $c_{p+1} \in \#(h, \vartheta_1, \rho)$ and $\rho'_1 \in \Xi(h'_1)$ are proved.

Suppose $m > 1$ and let $j = 1 \dots m-1$. We assume $\rho'_j \in \Xi(h'_j)$ and try to show $\rho'_{j+1} \in \Xi(h'_{j+1})$, using lemma 4.11. We have $h'_{j+1} = h'_j + (y_{j+1}, \vartheta_{j+1})$ and $h'_j \in K$, $y_{j+1} \in \mathcal{V} - var(h'_j)$, $\vartheta_{j+1} \in E_s(h'_j)$.
We have also $\rho'_{j+1} = \rho'_j + (v_{p+j+1}, c_{p+j+1}) = \rho'_j + (y_{j+1}, c_{p+j+1})$.
To show $\rho'_{j+1} \in \Xi(h'_{j+1})$ we just need to show that $c_{p+j+1} \in \#(h'_j, \vartheta_{j+1}, \rho'_j)$.

In other words we have to prove $c_{p+j+1} \in \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho'_j)$.

We have proved that $c_{p+j+1} \in \#(k'_j, \psi_{j+1}, \delta_{p+j})$.

We define two functions z, q over $\{1, \dots, p\}$ as follows: for each $\alpha = 1 \dots p$

- if $\alpha < i$ then $z_\alpha = u_\alpha$, $q_\alpha = r_\alpha$;
- if $\alpha = i$ then $z_\alpha = x_i$, $q_\alpha = \#(k_{i-1}, t, \rho_{i-1})$;
- if $\alpha > i$ then $z_\alpha = u_{\alpha-1}$, $q_\alpha = r_{\alpha-1}$.

Clearly we have $\sigma_0 = \epsilon$ and for each $\alpha = 1 \dots p$ $\sigma_\alpha = \sigma_{\alpha-1} + (z_\alpha, q_\alpha)$. Therefore by lemma 4.9 we have $\sigma_p = (z, q)$.

Since $(v_{/\{1,\dots,p\}}, c_{/\{1,\dots,p\}}) = \delta_p = \sigma_p = (z, q)$ we have that for each $\alpha = 1 \dots p$ $v_\alpha = z_\alpha$ and $c_\alpha = q_\alpha$.

Moreover we consider that $\rho = (u, r)$ is a state-like pair whose domain is $\{1, \ldots, p-1\}$ and $\sigma'_m = (v, c)$ is a state-like pair whose domain is $\{1, \ldots, p+m\}$.
We have defined $\rho'_1 = \rho + (v_{p+1}, c_{p+1})$ and if $m > 1$ for each $\beta = 1 \ldots m-1$
$\rho'_{\beta+1} = \rho'_\beta + (v_{p+\beta+1}, c_{p+\beta+1})$.
We define $\rho'_j = (u', r')$. Using lemma 4.10 we can derive that $dom(\rho'_j) = \{1, \ldots, p-1+j\}$ and for each $\alpha = 1 \ldots p-1+j$

- if $\alpha \leqslant p-1$ then $u'_\alpha = u_\alpha$, $r'_\alpha = r_\alpha$;
- if $\alpha > p-1$ then $u'_\alpha = v_{\alpha+1}$, $r'_\alpha = c_{\alpha+1}$.

For each $\alpha = 1 \ldots p+j$

- if $\alpha < i$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u_\alpha, r_\alpha) = \delta_{\alpha-1} + (u'_\alpha, r'_\alpha)$;
- if $\alpha = i$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$;
- if $i < \alpha \leqslant p$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u_{\alpha-1}, r_{\alpha-1}) = \delta_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$;
- if $\alpha > p$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$.

Also consider that $\rho_{i-1} = \rho_{/dom(k_{i-1})} = ((\rho'_j)_{/dom(\rho)})_{/dom(k_{i-1})} = (\rho'_j)_{/dom(k_{i-1})}$.

Since we have been able to define $k'_j\{x_i/t\}$ and $(\psi_{j+1})_{k'_j}\{x_i/t\}$, we must have

$$\#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho'_j) = \#(k'_j, \psi_{j+1}, \delta_{p+j}) \ .$$

At this point we have proved $c_{p+j+1} \in \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \rho'_j)$, and so also $\rho'_{j+1} \in \Xi(h'_{j+1})$ is proved. This also completes the proof of $\rho'_m \in \Xi(k'_m\{x_i/t\})$.

We still need to prove $\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$.

We consider that $\rho = (u, r)$ is a state-like pair whose domain is $\{1, \ldots, p-1\}$ and $\sigma'_m = (v, c)$ is a state-like pair whose domain is $\{1, \ldots, p+m\}$.
We have defined $\rho'_1 = \rho + (v_{p+1}, c_{p+1})$ and if $m > 1$ for each $\beta = 1 \ldots m-1$
$\rho'_{\beta+1} = \rho'_\beta + (v_{p+\beta+1}, c_{p+\beta+1})$.
We define $\rho'_m = (u', r')$. Using lemma 4.10 we can derive that
$dom(\rho'_m) = \{1, \ldots, p-1+m\}$ and for each $\alpha = 1 \ldots p-1+m$

- if $\alpha \leqslant p-1$ then $u'_\alpha = u_\alpha$, $r'_\alpha = r_\alpha$;
- if $\alpha > p-1$ then $u'_\alpha = v_{\alpha+1}$, $r'_\alpha = c_{\alpha+1}$.

For each $\alpha = 1 \ldots p+m$

- if $\alpha < i$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u_\alpha, r_\alpha) = \delta_{\alpha-1} + (u'_\alpha, r'_\alpha)$;
- if $\alpha = i$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$;
- if $i < \alpha \leqslant p$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u_{\alpha-1}, r_{\alpha-1}) = \delta_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$;
- if $\alpha > p$ then $\delta_\alpha = \delta_{\alpha-1} + (v_\alpha, c_\alpha) = \delta_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$.

Also consider that $\rho_{i-1} = \rho_{/dom(k_{i-1})} = ((\rho'_m)_{/dom(\rho)})_{/dom(k_{i-1})} = (\rho'_m)_{/dom(k_{i-1})}$.

Since we have been able to define $k'_m\{x_i/t\}$ and $\phi_{k'_m}\{x_i/t\}$, we must have

$$\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) = \#(k'_m, \phi, \delta_{p+m}) = w \ .$$

So we conclude that $w \in \{\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) |\ \rho'_m \in \Xi(k'_m\{x_i/t\}), \rho \sqsubseteq \rho'_m\}$.

For the converse implication we assume

$$w \in \{\#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) \mid \rho'_m \in \Xi(k'_m\{x_i/t\}), \rho \sqsubseteq \rho'_m\} \ ,$$

and try to show that $w \in \{\#(k'_m, \phi, \sigma'_m) \mid \sigma'_m \in \Xi(k'_m), \sigma_p \sqsubseteq \sigma'_m\}$.

There exists $\rho'_m = (u', r') \in \Xi(k'_m\{x_i/t\})$ such that $\rho \sqsubseteq \rho'_m$ and

$$w = \#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) \ .$$

Clearly $dom(\rho'_m) = dom(k'_m\{x_i/t\}) = \{1, \ldots, p+m-1\}$.

We define $\sigma'_1 = \sigma_p + (u'_p, r'_p)$ and if $m > 1$ for each $j = 1 \ldots m-1$ $\sigma'_{j+1} = \sigma'_j + (u'_{p+j}, r'_{p+j})$. Our goal is to show that

$$\sigma'_m \in \Xi(k'_m) \text{ and } \#(k'_m, \phi, \sigma'_m) = \#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) \ .$$

First of all we define $\eta_0 = \epsilon$ and for each $j = 1 \ldots p+m-1$ $\eta_j = \eta_{j-1} + (u'_j, r'_j)$. By lemma 4.9 we derive that $\eta_{p+m-1} = \rho'_m$. Therefore $\rho \sqsubseteq \eta_{p+m-1}$.

There exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \ldots p+m-1\}$, $\rho = (\eta_{p+m-1})_{/C}$.

We have $C = dom(\rho) = \{1, \ldots, p-1\}$, and therefore

$$\rho = (\eta_{p+m-1})_{/\{1,\ldots,p-1\}} = (u'_{/\{1,\ldots,p-1\}}, r'_{/\{1,\ldots,p-1\}}) = \eta_{p-1} \ .$$

We define $h'_0 = h$. We use backward induction on j to show that for each $j = m \ldots 1$

$$\eta_{p+j-1} \in \Xi(h'_j),\ \eta_{p+j-2} \in \Xi(h'_{j-1}), u'_{p+j-1} = y_j,\ r'_{p+j-1} \in \#(h'_{j-1}, \vartheta_j, \eta_{p+j-2}) \ .$$

Clearly $\eta_{p+m-1} = \rho'_m \in \Xi(h'_m)$.
We have $h'_m = h'_{m-1} + (y_m, \vartheta_m)$, $h'_{m-1} \in K$, $y_m \in \mathcal{V} - var(h'_{m-1})$, $\vartheta_m \in E_s(h'_{m-1})$.
By lemma 4.11 this implies there exist $\delta \in \Xi(h'_{m-1})$, $s \in \#(h'_{m-1}, \vartheta_m, \delta)$ such that

$$\eta_{p+m-2} + (u'_{p+m-1}, r'_{p+m-1}) = \eta_{p+m-1} = \delta + (y_m, s) \ .$$

By lemma 2.1 we obtain

$$\eta_{p+m-2} = \delta \in \Xi(h'_{m-1}),\ u'_{p+m-1} = y_m,\ r'_{p+m-1} \in \#(h'_{m-1}, \vartheta_m, \eta_{p+m-2}) \ .$$

If $m > 1$ we need an inductive step. Let $j = m \ldots 2$ and assume

$$\eta_{p+j-1} \in \Xi(h'_j),\ \eta_{p+j-2} \in \Xi(h'_{j-1}), u'_{p+j-1} = y_j,\ r'_{p+j-1} \in \#(h'_{j-1}, \vartheta_j, \eta_{p+j-2}) \ .$$

We need to prove

$$\eta_{p+j-3} \in \Xi(h'_{j-2}),\ u'_{p+j-2} = y_{j-1},\ r'_{p+j-2} \in \#(h'_{j-2}, \vartheta_{j-1}, \eta_{p+j-3}) \ .$$

We have $h'_{j-1} = h'_{j-2} + (y_{j-1}, \vartheta_{j-1})$, $h'_{j-2} \in K$, $y_{j-1} \in \mathcal{V} - var(h'_{j-2})$, $\vartheta_{j-1} \in E_s(h'_{j-2})$.
By lemma 4.11 this implies there exist $\delta \in \Xi(h'_{j-2})$, $s \in \#(h'_{j-2}, \vartheta_{j-1}, \delta)$ such that

$$\eta_{p+j-3} + (u'_{p+j-2}, r'_{p+j-2}) = \eta_{p+j-2} = \delta + (y_{j-1}, s) \ .$$

By lemma 2.1 we obtain

$$\eta_{p+j-3} = \delta \in \Xi(h'_{j-2}),\ u'_{p+j-2} = y_{j-1},\ r'_{p+j-2} \in \#(h'_{j-2}, \vartheta_{j-1}, \eta_{p+j-3}) \ .$$

To show that $\sigma'_m \in \Xi(k'_m)$ we show by induction on j that for each $j = 1 \dots m$ $\sigma'_j \in \Xi(k'_j)$.

We begin by showing that $\sigma'_1 \in \Xi(k'_1)$ using lemma 4.11. We have $k'_1 = k + (y_1, \psi_1)$ and $k \in K$, $y_1 \in \mathcal{V} - var(k)$, $\psi_1 \in E_s(k)$.
Moreover $\sigma_p \in \Xi(k)$ and $\sigma'_1 = \sigma_p + (u'_p, r'_p) = \sigma_p + (y_1, r'_p)$, so to prove $\sigma'_1 \in \Xi(k'_1)$ we just need to show that $r'_p \in \#(k, \psi_1, \sigma_p)$.

But we have proved $r'_p \in \#(h, \vartheta_1, \eta_{p-1}) = \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$. Since we have been able to define $(\psi_1)_k\{x_i/t\}$ we can assume $\#(k, \psi_1, \sigma_p) = \#(k\{x_i/t\}, (\psi_1)_k\{x_i/t\}, \rho)$.

Therefore $r'_p \in \#(k, \psi_1, \sigma_p)$ and $\sigma'_1 \in \Xi(k'_1)$ are proved.

Suppose $m > 1$ and let $j = 1 \dots m-1$. We assume $\sigma'_j \in \Xi(k'_j)$ and try to show $\sigma'_{j+1} \in \Xi(k'_{j+1})$, using lemma 4.11. We have $k'_{j+1} = k'_j + (y_{j+1}, \psi_{j+1})$, where $k'_j \in K$, $y_{j+1} \in \mathcal{V} - var(k'_j)$, $\psi_{j+1} \in E_s(k'_j)$.
Moreover $\sigma'_{j+1} = \sigma'_j + (u'_{p+j}, r'_{p+j}) = \sigma'_j + (y_{j+1}, r'_{p+j})$ and $\sigma'_j \in \Xi(k'_j)$, so to show that $\sigma'_{j+1} \in \Xi(k'_{j+1})$ we just need to prove $r'_{p+j} \in \#(k'_j, \psi_{j+1}, \sigma'_j)$.

We have proved $r'_{p+j} \in \#(h'_j, \vartheta_{j+1}, \eta_{p+j-1})$, that we can rewrite

$$r'_{p+j} \in \#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \eta_{p+j-1}) \ .$$

We'll try to exploit this. For each $\alpha = 0 \dots p$ we define $\sigma''_\alpha = \sigma_\alpha$, and for each $\alpha = p+1 \dots p+m$ let $\sigma''_\alpha = \sigma'_{\alpha-p}$.

Let u'', r'' be functions over $\{1, \dots, p+j-1\}$ such that for each $\alpha = 1 \dots p+j-1$ $u''_\alpha = u'_\alpha$ and $r''_\alpha = r'_\alpha$.
We have $\eta_0 = \epsilon$ and for each $\alpha = 1 \dots p+j-1$ $\eta_\alpha = \eta_{\alpha-1} + (u'_\alpha, r'_\alpha) = \eta_{\alpha-1} + (u''_\alpha, r''_\alpha)$.
Therefore, by lemma 4.9, $\eta_{p+j-1} = (u'', r'')$.

For each $\alpha = 1 \dots p-1$ we have $u_\alpha = u'_\alpha = u''_\alpha$, $r_\alpha = r'_\alpha = r''_\alpha$.

We have $\sigma''_0 = \epsilon$ and for each $\alpha = 1 \dots p+j$

- if $\alpha < i$ then $\sigma''_\alpha = \sigma_\alpha = \sigma_{\alpha-1} + (u_\alpha, r_\alpha) = \sigma''_{\alpha-1} + (u''_\alpha, r''_\alpha)$;
- if $\alpha = i$ then $\sigma''_\alpha = \sigma_\alpha = \sigma''_{\alpha-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$;
- if $i < \alpha \leqslant p$ then $\sigma''_\alpha = \sigma_\alpha = \sigma_{\alpha-1} + (u_{\alpha-1}, r_{\alpha-1}) = \sigma''_{\alpha-1} + (u''_{\alpha-1}, r''_{\alpha-1})$;
- if $\alpha = p+1$ then $\sigma''_\alpha = \sigma'_1 = \sigma_p + (u'_p, r'_p) = \sigma''_p + (u''_p, r''_p) = \sigma''_{\alpha-1} + (u''_{\alpha-1}, r''_{\alpha-1})$;
- if $\alpha > p+1$ then $\sigma''_\alpha = \sigma'_{\alpha-p} = \sigma'_{\alpha-p-1} + (u'_{\alpha-1}, r'_{\alpha-1}) = \sigma''_{\alpha-1} + (u''_{\alpha-1}, r''_{\alpha-1})$.

Also consider that $\rho = \eta_{p-1} \sqsubseteq \eta_{p+j-1}$, so $\rho = (\eta_{p+j-1})_{/dom(\rho)}$ and

$$\rho_{i-1} = \rho_{/dom(k_{i-1})} = ((\eta_{p+j-1})_{/dom(\rho)})_{/dom(k_{i-1})} = (\eta_{p+j-1})_{/dom(k_{i-1})} \ .$$

Since we have been able to define $k'_j\{x_i/t\}$ and $(\psi_{j+1})_{k'_j}\{x_i/t\}$, we must have

$$\#(k'_j\{x_i/t\}, (\psi_{j+1})_{k'_j}\{x_i/t\}, \eta_{p+j-1}) = \#(k'_j, \psi_{j+1}, \sigma''_{p+j}) \ .$$

It follows that $r'_{p+j} \in \#(k'_j, \psi_{j+1}, \sigma''_{p+j}) = \#(k'_j, \psi_{j+1}, \sigma'_j)$. So we have proved $\sigma'_{j+1} \in \Xi(k'_{j+1})$ and the proof of $\sigma'_m \in \Xi(k'_m)$ is finished.

We still need to show that $\#(k'_m, \phi, \sigma'_m) = \#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m)$.

For each $\alpha = 1 \dots p-1$ we have $u_\alpha = u'_\alpha$, $r_\alpha = r'_\alpha$.

We have $\sigma''_0 = \epsilon$ and for each $\alpha = 1 \dots p+m$

- if $\alpha < i$ then $\sigma''_\alpha = \sigma_\alpha = \sigma_{\alpha-1} + (u_\alpha, r_\alpha) = \sigma''_{\alpha-1} + (u'_\alpha, r'_\alpha)$;
- if $\alpha = i$ then $\sigma''_\alpha = \sigma_\alpha = \sigma''_{\alpha-1} + (x_i, \#(k_{i-1}, t, \rho_{i-1}))$;
- if $i < \alpha \leqslant p$ then $\sigma''_\alpha = \sigma_\alpha = \sigma_{\alpha-1} + (u_{\alpha-1}, r_{\alpha-1}) = \sigma''_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$;
- if $\alpha = p+1$ then $\sigma''_\alpha = \sigma'_1 = \sigma_p + (u'_p, r'_p) = \sigma''_p + (u'_p, r'_p) = \sigma''_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$;
- if $\alpha > p+1$ then $\sigma''_\alpha = \sigma'_{\alpha-p} = \sigma'_{\alpha-p-1} + (u'_{\alpha-1}, r'_{\alpha-1}) = \sigma''_{\alpha-1} + (u'_{\alpha-1}, r'_{\alpha-1})$.

Also consider that $\rho \sqsubseteq \rho'_m$, so $\rho = (\rho'_m)_{/dom(\rho)}$ and

$$\rho_{i-1} = \rho_{/dom(k_{i-1})} = ((\rho'_m)_{/dom(\rho)})_{/dom(k_{i-1})} = (\rho'_m)_{/dom(k_{i-1})} \ .$$

Since we have been able to define $k'_m\{x_i/t\}$ and $\phi_{k'_m}\{x_i/t\}$, we must have

$$\#(k'_m, \phi, \sigma'_m) = \#(k'_m, \phi, \sigma''_{p+m}) = \#(k'_m\{x_i/t\}, \phi_{k'_m}\{x_i/t\}, \rho'_m) = w \ .$$

Therefore we conclude that $w \in \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma_p \sqsubseteq \sigma'_m\}$.

To finish with the case $\varphi \in E'_e(n+1, k)$ we need to show $V_b(\varphi_k\{x_i/t\}) \subseteq V_b(\varphi) \cup V_b(t)$.

Using (4.0.1) on page 110 and the inductive hypothesis we obtain

$$\begin{aligned}
&V_b(\varphi_k\{x_i/t\}) = \{y_1, \dots, y_m\} \cup V_b(\vartheta_1) \cup \dots \cup V_b(\vartheta_m) \cup V_b(\theta) = \\
&= \{y_1, \dots, y_m\} \cup V_b((\psi_1)_k\{x_i/t\}) \cup \dots \cup V_b((\psi_m)_{k'_{m-1}}\{x_i/t\}) \cup V_b(\phi_{k'_m}\{x_i/t\}) \subseteq \\
&\subseteq \{y_1, \dots, y_m\} \cup (V_b(\psi_1) \cup V_b(t)) \cup \dots \cup (V_b(\psi_m) \cup V_b(t)) \cup (V_b(\phi) \cup V_b(t)) = \\
&= \{y_1, \dots, y_m\} \cup V_b(\psi_1) \cup \dots \cup V_b(\psi_m) \cup V_b(\phi) \cup V_b(t) = \\
&= V_b(\varphi) \cup V_b(t) \ .
\end{aligned}$$

$\diamond$

We have defined $\varphi_k\{x_i/t\}$ for each $\varphi \in E(n+1, k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$. Recall that

$$E(n+1, k) = E'(n, k) \cup E'_a(n+1, k) \cup E'_b(n+1, k) \cup E'_c(n+1, k) \cup E'_d(n+1, k) \cup E'_e(n+1, k).$$

and recall that the definition of $\varphi_k\{x_i/t\}$ depends on the set to which φ belongs to. Actually φ may belong to more than one of these sets. We need to check that, in every case in which φ belongs to two of the six sets, the two definitions of $\varphi_k\{x_i/t\}$ match each other.

We split the task in two steps. The first step requires to verify that

- for each $w \in \{a, b, c, d, e\}$ if $\varphi \in E'(n, k) \cap E'_w(n+1, k)$ then $(\varphi_k\{x_i/t\})_w = \varphi_k\{x_i/t\}$.

The second step requires to verify that

- for each $w_1, w_2 \in \{a, b, c, d, e\}$ if $w_1 \neq w_2$, $\varphi \in E'_{w_1}(n+1, k) \cap E'_{w_2}(n+1, k)$ then $(\varphi_k\{x_i/t\})_{w_1} = (\varphi_k\{x_i/t\})_{w_2}$.

We begin with the first step and examine the case where $\varphi \in \mathbf{E}'(\mathbf{n},\mathbf{k}) \cap \mathbf{E}'_{\mathbf{a}}(\mathbf{n}+\mathbf{1},\mathbf{k})$.

Of course $\varphi \in E(n,k) \cap E_a(n+1,k)$ and $k \in K(n)^+$.

We have $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$. Therefore $\varphi \in E(n, k_{p-1})$, $x_p \notin V_b(\varphi)$.

Consider the case where $i = p$. Here we have $(\varphi_k\{x_i/t\})_a = \varphi$.

We also see that $k_{p-1} \in K(n)$, $\varphi \in E(n, k_{p-1})$, $x_i \notin var(k_{p-1})$. At the beginning of our definition we declared the intention to show the truth of some properties. Clearly we will show these properties are true at step $n+1$, and we can assume their truth at step n. One of those properties tells us that in this case $\varphi_k\{x_i/t\} = \varphi$. So

$$(\varphi_k\{x_i/t\})_a = \varphi = \varphi_k\{x_i/t\} \ .$$

We now examine the case where $i < p$. Here we defined $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\}$.

It also holds true that $k_{p-1} \in K(n)$, $k_i \sqsubseteq k_{p-1}$, $\mathcal{K}(n; k_{p-1}; x_1 : \varphi_1, \ldots, x_{p-1} : \varphi_{p-1})$, $\varphi \in E(n, k_{p-1})$. Another declared property tells us that $\varphi_k\{x_i/t\} = \varphi_{k_{p-1}}\{x_i/t\}$. So

$$(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = \varphi_k\{x_i/t\} \ .$$

Let's turn to examine the case where $\varphi \in \mathbf{E}'(\mathbf{n},\mathbf{k}) \cap \mathbf{E}'_{\mathbf{b}}(\mathbf{n}+\mathbf{1},\mathbf{k})$.

Of course $\varphi \in E(n,k) \cap E_b(n+1,k)$ and $k \in K(n)^+$.

We have $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$. Therefore $\varphi = x_p$.

Since $\varphi \in E(n,k)$ the following condition holds:

$$\varphi \in var(k),\ \varphi = x_i \to \varphi_k\{x_i/t\} = t,\ \varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi \ .$$

Consider the case where $i = p$. Here we defined $(\varphi_k\{x_i/t\})_b = t$ and since $\varphi = x_p = x_i$ we have $\varphi_k\{x_i/t\} = t = (\varphi_k\{x_i/t\})_b$.

Turn to the case where $i < p$. Here we defined $(\varphi_k\{x_i/t\})_b = \varphi$ and since $\varphi = x_p \neq x_i$ we have $\varphi_k\{x_i/t\} = \varphi = (\varphi_k\{x_i/t\})_b$.

Let's examine the case where $\varphi \in \mathbf{E}'(\mathbf{n},\mathbf{k}) \cap \mathbf{E}'_{\mathbf{c}}(\mathbf{n}+\mathbf{1},\mathbf{k})$.

Of course $\varphi \in E(n,k) \cap E_c(n+1,k)$.

Since $\varphi \in E(n,k)$ the following condition holds:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n-1,h)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments, $(\#(h,\psi_1,\rho), \ldots, \#(h,\psi_m,\rho))$ is a member of the domain of $\#(h,\psi,\rho)$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_c(n+1, k)$ the following condition holds:

there exist a positive integer r and $\vartheta, \vartheta_1, \ldots, \vartheta_r \in E(n, k)$ such that

- $\varphi = (\vartheta)(\vartheta_1, \ldots, \vartheta_r)$;
- for each $\sigma \in \Xi(k)$ $\#(k, \vartheta, \sigma)$ is a function with r arguments and $(\#(k, \vartheta_1, \sigma), \ldots, \#(k, \vartheta_r, \sigma))$ is a member of its domain;
- $(\varphi_k\{x_i/t\})_c = (\vartheta_k\{x_i/t\})((\vartheta_1)_k\{x_i/t\}, \ldots, (\vartheta_r)_k\{x_i/t\})$.

This implies that $r = m$, $\vartheta = \psi$ and for each $j = 1 \ldots m$ $\vartheta_j = \psi_j$. Therefore

$$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) \ .$$

Suppose $h \neq \epsilon$ and $i \leqslant q$, in this case we have

$$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

We have $k_i \sqsubseteq h$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $\psi, \psi_1, \ldots, \psi_m \in E(n, h)$. We can apply one of our declared properties and obtain that $\psi_k\{x_i/t\} = \psi_h\{x_i/t\}$, for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\} = (\psi_j)_h\{x_i/t\}$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_c &= (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ &= (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) = \varphi_k\{x_i/t\} \ .\end{aligned}$$

Consider instead the case where $h = \epsilon$ or $h \neq \epsilon \wedge i > q$. In this case $x_i \notin var(h)$, and $\psi, \psi_1, \ldots, \psi_m \in E(n, h)$, so by one of our declared properties

$$\begin{aligned}(\varphi_k\{x_i/t\})_c &= (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ &= (\psi)(\psi_1, \ldots, \psi_m) = \varphi = \varphi_k\{x_i/t\} \ .\end{aligned}$$

Let's examine the case where $\varphi \in \mathbf{E}'(\mathbf{n}, \mathbf{k}) \cap \mathbf{E}'_{\mathbf{d}}(\mathbf{n} + \mathbf{1}, \mathbf{k})$.

Of course $\varphi \in E(n, k) \cap E_d(n+1, k)$.

Since $\varphi \in E(n, k)$ the following condition holds:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n-1, h)$, $f \in \mathcal{F}$, such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n, h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p$ such that $q < n - 1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_d(n+1,k)$ there exist $f \in \mathcal{F}$, a positive integer r and $\vartheta_1,\ldots,\vartheta_r \in E(n,k)$ such that

- $\varphi = (f)(\vartheta_1,\ldots,\vartheta_r)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k,\vartheta_1,\sigma),\ldots,\#(k,\vartheta_r,\sigma))$ holds true;
- $(\varphi_k\{x_i/t\})_d = (f)((\vartheta_1)_k\{x_i/t\},\ldots,(\vartheta_r)_k\{x_i/t\})$.

This implies that $r = m$ and for each $j = 1\ldots m$ $\vartheta_j = \psi_j$. Therefore

$$(\varphi_k\{x_i/t\})_d = (f)((\psi_1)_k\{x_i/t\},\ldots,(\psi_m)_k\{x_i/t\}) \ .$$

Suppose $h \neq \epsilon$ and $i \leqslant q$, in this case we have

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\},\ldots,(\psi_m)_h\{x_i/t\}) \ .$$

We have $k_i \sqsubseteq h$, $\mathcal{K}(n;h;x_1:\varphi_1,\ldots,x_q:\varphi_q)$, $\psi_1,\ldots,\psi_m \in E(n,h)$. We can apply one of our declared properties and obtain that for each $j = 1\ldots m$ $(\psi_j)_k\{x_i/t\} = (\psi_j)_h\{x_i/t\}$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_d &= (f)((\psi_1)_k\{x_i/t\},\ldots,(\psi_m)_k\{x_i/t\}) = \\ &= (f)((\psi_1)_h\{x_i/t\},\ldots,(\psi_m)_h\{x_i/t\}) = \varphi_k\{x_i/t\} \ .\end{aligned}$$

Consider instead the case where $h = \epsilon$ or $h \neq \epsilon \wedge i > q$. In this case $x_i \notin var(h)$, and $\psi_1,\ldots,\psi_m \in E(n,h)$, so by one of our declared properties

$$\begin{aligned}(\varphi_k\{x_i/t\})_d &= (f)((\psi_1)_k\{x_i/t\},\ldots,(\psi_m)_k\{x_i/t\}) = \\ &= (f)(\psi_1,\ldots,\psi_m) = \varphi = \varphi_k\{x_i/t\} \ .\end{aligned}$$

Let's examine the case where $\varphi \in \mathbf{E'(n,k)} \cap \mathbf{E'_e(n+1,k)}$.

Of course $\varphi \in E(n,k) \cap E_e(n+1,k)$.

Since $\varphi \in E(n,k)$ the following condition holds:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, $\vartheta \in E(n-1)$,
a function y whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha,\beta = 1\ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $\psi_j \in E(n-1)$;

such that

$\mathcal{E}(n-1,h,m,y,\psi,\vartheta)$,
$\varphi = \{\}(y_1:\psi_1,\ldots,y_m:\psi_m,\vartheta),\ \varphi \in E(n,h)$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p$ such that $q < n-1$,
$\mathcal{K}(n-1;h;x_1:\varphi_1,\ldots,x_q:\varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h+(y_1,\psi_1)$, and if $m > 1$ for each $j = 1\ldots m-1$ $h'_{j+1} = h'_j+(y_{j+1},\psi_{j+1})$.

We have $\psi_1 \in E(n-1,h)$, $V_b(t) \cap V_b(\psi_1) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;

for each $j = 1\ldots m-1$ $h'_j \in K(n-1)$ and
$\mathcal{K}(n-1;h'_j;x_1:\varphi_1,\ldots,x_q:\varphi_q,y_1:\psi_1,\ldots,y_j:\psi_j)$,
$\psi_{j+1} \in E(n-1,h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) = \emptyset$, $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;

$h'_m \in K(n-1)$ and
$\mathcal{K}(n-1; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
$\vartheta \in E(n-1, h'_m)$, $V_b(t) \cap V_b(\vartheta) = \emptyset$, $\vartheta_{h'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E_e(n+1, k)$ there exist

- a positive integer r,
- a function z whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $z_j \in \mathcal{V} - var(k)$, and for each $\alpha, \beta = 1 \ldots r$ $\alpha \neq \beta \rightarrow z_\alpha \neq z_\beta$,
- a function ϕ whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $\phi_j \in E(n)$,
- $\theta \in E(n)$

such that $\varphi = \{\}(z_1 : \phi_1, \ldots, z_r : \phi_r, \theta)$, $\mathcal{E}(n, k, r, z, \phi, \theta)$ and

$$(\varphi_k\{x_i/t\})_e = \{\}(z_1 : (\phi_1)_k\{x_i/t\}, \ldots, z_r : (\phi_r)_{k'_{r-1}}\{x_i/t\}, \theta_{k'_r}\{x_i/t\}) \ ,$$

where $k'_1 = k + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $k'_{j+1} = k'_j + (y_{j+1}, \psi_{j+1})$.

Clearly $r = m$, $z = y$, $\phi = \psi$, $\theta = \vartheta$, therefore

$$(\varphi_k\{x_i/t\})_e = \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \vartheta_{k'_m}\{x_i/t\}) \ .$$

Suppose $h \neq \epsilon$ and $i \leqslant q$, we have

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \ .$$

We recall that $k \in K(n)$, $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$, $\psi_1 \in E(n, k)$ and $V_b(\psi_1) \cap V_b(t) = \emptyset$. This ensures $(\psi_1)_k\{x_i/t\}$ is defined, and we have $h \in K(n)$, $k_i \sqsubseteq h$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $\psi_1 \in E(n, h)$. By one of our declared properties we obtain that $(\psi_1)_k\{x_i/t\} = (\psi_1)_h\{x_i/t\}$.

If $m > 1$ let $j = 1 \ldots m-1$, we want to show that $(\psi_{j+1})_{k'_j}\{x_i/t\} = (\psi_{j+1})_{h'_j}\{x_i/t\}$. We recall that $k'_j \in K(n)$ and $\mathcal{K}(n; k'_j; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_j : \psi_j)$, for each $\alpha = i+1 \ldots p$ $x_\alpha \notin V_b(t)$ and $V_b(\varphi_\alpha) \cap V_b(t) = \emptyset$, for each $\alpha = 1 \ldots j$ $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$, $\psi_{j+1} \in E(n, k'_j)$, $V_b(\psi_{j+1}) \cap V_b(t) = \emptyset$. As a result of these conditions we were able to define $(\psi_{j+1})_{k'_j}\{x_i/t\}$.

We have also $h'_j \in K(n)$, $\mathcal{K}(n; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$, $k_i \sqsubseteq h \sqsubseteq h'_j$, $\psi_{j+1} \in E(n, h'_j)$. By one of our declared properties we obtain that

$$(\psi_{j+1})_{k'_j}\{x_i/t\} = (\psi_{j+1})_{h'_j}\{x_i/t\} \ .$$

We also want to show that $\vartheta_{k'_m}\{x_i/t\}) = \vartheta_{h'_m}\{x_i/t\})$.
We recall that $k'_m \in K(n)$ and $\mathcal{K}(n; k'_m; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_m : \psi_m)$, for each $\alpha = i+1 \ldots p$ $x_\alpha \notin V_b(t)$ and $V_b(\varphi_\alpha) \cap V_b(t) = \emptyset$, for each $\alpha = 1 \ldots m$ $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$. We have $\vartheta \in E(n, k'_m)$ and also $V_b(\vartheta) \cap V_b(t) = \emptyset$. As a result of these conditions we were able to define $\vartheta_{k'_m}\{x_i/t\}$.

We have also $h'_m \in K(n)$, $\mathcal{K}(n; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$, $k_i \sqsubseteq h \sqsubseteq h'_m$, $\vartheta \in E(n, h'_m)$. By one of our declared properties we obtain that

$$\vartheta_{k'_m}\{x_i/t\} = \vartheta_{h'_m}\{x_i/t\} \ .$$

Hence

$$\begin{aligned}\varphi_k\{x_i/t\} &= \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) = \\ &= \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \vartheta_{k'_m}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e \ .\end{aligned}$$

We now consider the alternative case $h = \epsilon$ or $h \neq \epsilon \wedge i > q$. In this case

$$\varphi_k\{x_i/t\} = \varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta) \ .$$

We could define $(\psi_1)_k\{x_i/t\}$, and we have $h \in K(n)$, $\psi_1 \in E(n, h)$ (follows by $\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$), $x_i \notin var(h)$. By one declared property $(\psi_1)_k\{x_i/t\} = \psi_1$.

If $m > 1$ suppose $j = 1 \ldots m-1$. We want to show that $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$. Recall we were able to define $(\psi_{j+1})_{k'_j}\{x_i/t\}$. Recall that $\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$ holds, so $h'_j \in K(n)$, $\psi_{j+1} \in E(n, h'_j)$. Moreover, for each $\alpha = 1 \ldots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_j)$. By one declared property $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$.

We also need to show $\vartheta_{k'_m}\{x_i/t\} = \vartheta$. Recall we were able to define $(\vartheta)_{k'_m}\{x_i/t\}$. Recall that $\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$ holds, so $h'_m \in K(n)$, $\vartheta \in E(n, h'_m)$. Moreover, for each $\alpha = 1 \ldots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_m)$. By one declared property $\vartheta_{k'_m}\{x_i/t\} = \vartheta$.

Therefore

$$\begin{aligned}\varphi_k\{x_i/t\} &= \varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta) = \\ &= \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \vartheta_{k'_m}\{x_i/t\}) = (\varphi_k\{x_i/t\})_e \ .\end{aligned}$$

$\diamond$

We now turn to the second step of our task. This requires to verify that

- for each $w_1, w_2 \in \{a, b, c, d, e\}$ if $w_1 \neq w_2$, $\varphi \in E'_{w_1}(n+1, k) \cap E'_{w_2}(n+1, k)$ then $(\varphi_k\{x_i/t\})_{w_1} = (\varphi_k\{x_i/t\})_{w_2}$.

Within definition 2.7 we have seen that for many values of w_1, w_2 it results $E'_{w_1}(n+1, k) \cap E'_{w_2}(n+1, k) = \emptyset$.

In fact, we have seen that in all the cases in which $w_1, w_2 \in \{b, c, d, e\}$ and $w_1 \neq w_2$ $E'_{w_1}(n+1, k) \cap E'_{w_2}(n+1, k) = \emptyset$.
Moreover, we have proved that $E'_a(n+1, k) \cap E'_b(n+1, k) = \emptyset$.

Therefore we just need to examine three cases: $\varphi \in E'_a(n+1, k) \cap E'_c(n+1, k)$, $\varphi \in E'_a(n+1, k) \cap E'_d(n+1, k)$, $\varphi \in E'_a(n+1, k) \cap E'_e(n+1, k)$.

We start with the case where $\varphi \in \mathbf{E'_a(n+1, k) \cap E'_c(n+1, k)}$. Clearly φ belongs to $E_a(n+1, k) \cap E_c(n+1, k)$.

Since $\varphi \in E_c(n+1,k)$ the following condition holds: there exist a positive integer m and $\psi, \psi_1, \ldots, \psi_m \in E(n,k)$ such that

- $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$;
- $(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\})$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n,k_{p-1})$, $x_p \notin V_b(\varphi)$. We distinguish the case where $i = p$ from the case where $i < p$.

If $i = p$ then $(\varphi_k\{x_i/t\})_a = \varphi$. Given that $\varphi \in E(n,k_{p-1})$ we can use assumption 2.1.10 to obtain that $n > 1$ and there exist $h \in K(n-1)$ such that $h \sqsubseteq k_{p-1}$ and a positive integer r, $\vartheta, \vartheta_1, \ldots \vartheta_r \in E(n-1,h)$ such that $\varphi = (\vartheta)(\vartheta_1, \ldots, \vartheta_r)$.

Clearly $m = r$, $\psi = \vartheta \in E(n-1,h)$ and for each $j = 1 \ldots m$ $\psi_j = \vartheta_j \in E(n-1,h)$. Therefore $\psi, \psi_1, \ldots, \psi_m \in E(n-1,h)$.

We can apply one of our declared properties. In fact $h \in K(n)$ and $x_i \notin var(h)$, so $\psi_k\{x_i/t\} = \psi$ and for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_c &= (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ &= (\psi)(\psi_1, \ldots, \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a \ .\end{aligned}$$

Now suppose $i < p$. Here $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\}$. Since $\varphi \in E(n,k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, a positive integer r,
ϑ, $\vartheta_1, \ldots, \vartheta_r \in E(n-1,h)$ such that $\varphi = (\vartheta)(\vartheta_1, \ldots, \vartheta_r)$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p-1$ such that $q < n-1$,
$\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, we can define $\vartheta_h\{x_i/t\}$, and similarly we can define $(\vartheta_j)_h\{x_i/t\}$, and it results

$$\varphi_{k_{p-1}}\{x_i/t\} = (\vartheta_h\{x_i/t\})((\vartheta_1)_h\{x_i/t\}, \ldots, (\vartheta_r)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Clearly $m = r$, $\psi = \vartheta \in E(n-1,h)$ and for each $j = 1 \ldots m$ $\psi_j = \vartheta_j \in E(n-1,h)$. Therefore $\psi, \psi_1, \ldots, \psi_m \in E(n-1,h)$.

Moreover, if $x_i \in var(h)$ it results

$$(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ ;$$

otherwise $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Suppose $x_i \in var(h)$. It follows that $k_i \sqsubseteq h$. Since $\psi, \psi_1, \ldots, \psi_m \in E(n-1,h)$ we can apply one of our declared inductive assumptions and get $\psi_k\{x_i/t\} = \psi_h\{x_i/t\}$ and for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\} = (\psi_j)_h\{x_i/t\}$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_a &= \varphi_{k_{p-1}}\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) = \\ &= (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = (\varphi_k\{x_i/t\})_c \ .\end{aligned}$$

Suppose instead $x_i \notin var(h)$. Since $\psi, \psi_1, \ldots, \psi_m \in E(n-1, h)$ by one of our declared inductive assumptions we obtain $\psi_k\{x_i/t\} = \psi$ and for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$$(\varphi_k\{x_i/t\})_c = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ = (\psi)(\psi_1, \ldots, \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a \ .$$

We now examine the case where $\varphi \in \mathbf{E'_a(n+1,k)} \cap \mathbf{E'_d(n+1,k)}$. Clearly φ belongs to $E_a(n+1,k) \cap E_d(n+1,k)$.

Since $\varphi \in E_d(n+1,k)$ the following condition holds: there exist $f \in \mathcal{F}$, a positive integer m and $\psi_1, \ldots, \psi_m \in E(n,k)$ such that

- $\varphi = (f)(\psi_1, \ldots, \psi_m)$;
- $(\varphi_k\{x_i/t\})_d = (f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\})$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n, k_{p-1})$, $x_p \notin V_b(\varphi)$. We distinguish the case where $i = p$ from the case where $i < p$.

If $i = p$ then $(\varphi_k\{x_i/t\})_a = \varphi$. Given that $\varphi \in E(n, k_{p-1})$ we can use assumption 2.1.10 to obtain that $n > 1$ and there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, $g \in \mathcal{F}$, a positive integer r, $\vartheta_1, \ldots \vartheta_r \in E(n-1, h)$: $\varphi = (g)(\vartheta_1, \ldots, \vartheta_r)$.

Clearly $f = g$, $m = r$ and for each $j = 1 \ldots m$ $\psi_j = \vartheta_j \in E(n-1, h)$. Therefore $\psi_1, \ldots, \psi_m \in E(n-1, h)$.

We can apply one of our declared properties. In fact $h \in K(n)$ and $x_i \notin var(h)$, so for each $j = 1 \ldots m$ $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$$(\varphi_k\{x_i/t\})_d = (f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ = (f)(\psi_1, \ldots, \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a \ .$$

Now suppose $i < p$. Here $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\}$. Since $\varphi \in E(n, k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, $g \in \mathcal{F}$, a positive integer r, $\vartheta_1, \ldots, \vartheta_r \in E(n-1, h)$ such that $\varphi = (g)(\vartheta_1, \ldots, \vartheta_r)$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p-1$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, we can define $(\vartheta_j)_h\{x_i/t\}$, and it results

$$\varphi_{k_{p-1}}\{x_i/t\} = (g)((\vartheta_1)_h\{x_i/t\}, \ldots, (\vartheta_r)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Clearly $f = g$, $m = r$ and for each $j = 1 \ldots m$ $\psi_j = \vartheta_j \in E(n-1, h)$. Therefore $\psi_1, \ldots, \psi_m \in E(n-1, h)$.

Moreover, if $x_i \in var(h)$ it results

$$(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ ;$$

otherwise $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Suppose $x_i \in var(h)$. It follows that $k_i \sqsubseteq h$. Since $\psi_1, \ldots, \psi_m \in E(n-1, h)$ we can apply one of our declared inductive assumptions and get, for each $j = 1 \ldots m$, $(\psi_j)_k\{x_i/t\} = (\psi_j)_h\{x_i/t\}$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_a &= \varphi_{k_{p-1}}\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) = \\ &= (f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = (\varphi_k\{x_i/t\})_d \ .\end{aligned}$$

Suppose instead $x_i \notin var(h)$. Since $\psi_1, \ldots, \psi_m \in E(n-1, h)$ by one of our declared inductive assumptions we obtain, for each $j = 1 \ldots m$, $(\psi_j)_k\{x_i/t\} = \psi_j$. Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_d &= (f)((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) = \\ &= (f)(\psi_1, \ldots, \psi_m) = \varphi = (\varphi_k\{x_i/t\})_a \ .\end{aligned}$$

Finally we turn to the case where $\varphi \in \mathbf{E'_a(n+1,k)} \cap \mathbf{E'_e(n+1,k)}$. Clearly φ belongs to $E_a(n+1,k) \cap E_e(n+1,k)$.

Since $\varphi \in E_e(n+1,k)$ the following condition holds: there exist

- a positive integer m,
- a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(k)$, and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$,
- a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$,
- $\phi \in E(n)$

such that $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \phi)$, $\mathcal{E}(n, k, m, y, \psi, \phi)$ and

$$(\varphi_k\{x_i/t\})_e = \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \phi_{k'_m}\{x_i/t\}) \ .$$

where $k'_1 = k + (y_1, \psi_1)$ and if $m > 1$ for each $j = 1 \ldots m-1$ $k'_{j+1} = k'_j + (y_{j+1}, \psi_{j+1})$.

Since $\varphi \in E_a(n+1,k)$ we have $\varphi \in E(n, k_{p-1})$, $x_p \notin V_b(\varphi)$. We distinguish the case where $i = p$ from the case where $i < p$.

If $i = p$ then $(\varphi_k\{x_i/t\})_a = \varphi$. Given that $\varphi \in E(n, k_{p-1})$ we can use assumption 2.1.10 to obtain that

$n > 1$ and there exist

- $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$,
- a positive integer r,
- a function z whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $z_j \in \mathcal{V} - var(h)$, and for each $\alpha, \beta = 1 \ldots r$ $\alpha \neq \beta \to z_\alpha \neq z_\beta$,
- a function ϑ whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $\vartheta_j \in E(n-1)$,
- $\theta \in E(n-1)$

such that $\varphi = \{\}(z_1 : \vartheta_1, \ldots, z_r : \vartheta_r, \theta)$ and $\mathcal{E}(n-1, h, r, z, \vartheta, \theta)$.

Clearly $m = r$, $y = z$, $\psi = \vartheta$, $\phi = \theta$.

We define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.

We could define $(\psi_1)_k\{x_i/t\}$, and we have $h \in K(n)$, $\psi_1 \in E(n,h)$ (follows by $\mathcal{E}(n-1,h,m,y,\psi,\phi)$), $x_i \notin var(h)$. By one declared property $(\psi_1)_k\{x_i/t\} = \psi_1$.

If $m > 1$ suppose $j = 1 \ldots m-1$. We want to show that $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$. Recall we were able to define $(\psi_{j+1})_{k'_j}\{x_i/t\}$. Recall that $\mathcal{E}(n-1,h,m,y,\psi,\phi)$ holds, so $h'_j \in K(n)$, $\psi_{j+1} \in E(n,h'_j)$. Moreover, for each $\alpha = 1 \ldots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_j)$. By one declared property $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$.

We also need to show $\phi_{k'_m}\{x_i/t\} = \phi$. Recall we were able to define $(\phi)_{k'_m}\{x_i/t\}$. Recall that $\mathcal{E}(n-1,h,m,y,\psi,\phi)$ holds, so $h'_m \in K(n)$, $\phi \in E(n,h'_m)$. Moreover, for each $\alpha = 1 \ldots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_m)$. By one declared property $\phi_{k'_m}\{x_i/t\} = \phi$.

Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_e &= \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \phi_{k'_m}\{x_i/t\}) = \\ &= \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \phi) = \varphi = (\varphi_k\{x_i/t\})_a \ .\end{aligned}$$

Now suppose $i < p$. Here $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\}$. Since $\varphi \in E(n, k_{p-1})$ we can apply one of our inductive assumptions and obtain the following:

$n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, a positive integer r, $\theta \in E(n-1)$, a function z whose domain is $\{1,\ldots,r\}$ such that for each $j = 1 \ldots r$ $z_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots r$ $\alpha \neq \beta \to z_\alpha \neq z_\beta$;
a function ϑ whose domain is $\{1,\ldots,r\}$ such that for each $j = 1 \ldots r$ $\vartheta_j \in E(n-1)$;

such that

$\mathcal{E}(n-1,h,r,z,\vartheta,\theta)$,
$\varphi = \{\}(z_1 : \vartheta_1, \ldots, z_r : \vartheta_r, \theta)$, $\varphi \in E(n,h)$.

If $h \neq \epsilon$ there exists a positive integer $q \leqslant p-1$ such that $q < n-1$,
$\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h + (z_1, \vartheta_1)$, and if $r > 1$ for each $j = 1 \ldots r-1$ $h'_{j+1} = h'_j + (z_{j+1}, \vartheta_{j+1})$.

We have $\vartheta_1 \in E(n-1,h)$, $V_b(t) \cap V_b(\vartheta_1) = \emptyset$, therefore $(\vartheta_1)_h\{x_i/t\}$ is defined;

for each $j = 1 \ldots r-1$ $h'_j \in K(n-1)$ and
$\mathcal{K}(n-1; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, z_1 : \vartheta_1, \ldots, z_j : \vartheta_j)$,
for each $\alpha = 1 \ldots j$ $z_\alpha \notin V_b(t)$ and $V_b(t) \cap V_b(\vartheta_\alpha) = \emptyset$, $\vartheta_{j+1} \in E(n-1,h'_j)$,
$V_b(t) \cap V_b(\vartheta_{j+1}) = \emptyset$, therefore $(\vartheta_{j+1})_{h'_j}\{x_i/t\}$ is defined;

$h'_r \in K(n-1)$ and $\mathcal{K}(n-1; h'_r; x_1 : \varphi_1, \ldots, x_q : \varphi_q, z_1 : \vartheta_1, \ldots, z_r : \vartheta_r)$,
for each $\alpha = 1 \ldots r$ $z_\alpha \notin V_b(t)$ and $V_b(t) \cap V_b(\vartheta_\alpha) = \emptyset$,
$\theta \in E(n-1,h'_r)$, $V_b(t) \cap V_b(\theta) = \emptyset$, therefore $\theta_{h'_r}\{x_i/t\}$ is defined;

it results

$$\varphi_{k_{p-1}}\{x_i/t\} = \{\}(z_1 : (\vartheta_1)_h\{x_i/t\}, \ldots, z_r : (\vartheta_r)_{h'_{r-1}}\{x_i/t\}, \theta_{h'_r}\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Clearly $m = r$, $y = z$, $\psi = \vartheta$ and $\phi = \theta$.

Therefore, if $x_i \in var(h)$ it results

$$\begin{aligned}(\varphi_k\{x_i/t\})_a &= \varphi_{k_{p-1}}\{x_i/t\} = \\ &= \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \phi_{h'_m}\{x_i/t\}) \ ;\end{aligned}$$

otherwise $(\varphi_k\{x_i/t\})_a = \varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

Suppose $x_i \in var(h)$.

We recall that $k \in K(n)$, $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$, $\psi_1 \in E(n,k)$ and $V_b(\psi_1) \cap V_b(t) = \emptyset$. This ensures $(\psi_1)_k\{x_i/t\}$ is defined, and we have $h \in K(n)$, $k_i \sqsubseteq h$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $\psi_1 \in E(n,h)$. By one of our declared properties we obtain that $(\psi_1)_k\{x_i/t\} = (\psi_1)_h\{x_i/t\}$.

If $m > 1$ let $j = 1 \ldots m-1$, we want to show that $(\psi_{j+1})_{k'_j}\{x_i/t\} = (\psi_{j+1})_{h'_j}\{x_i/t\}$. We recall that $k'_j \in K(n)$ and $\mathcal{K}(n; k'_j; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_j : \psi_j)$, for each $\alpha = i+1 \ldots p$ $x_\alpha \notin V_b(t)$ and $V_b(\varphi_\alpha) \cap V_b(t) = \emptyset$, for each $\alpha = 1 \ldots j$ $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$, $\psi_{j+1} \in E(n, k'_j)$, $V_b(\psi_{j+1}) \cap V_b(t) = \emptyset$. As a result of these conditions we were able to define $(\psi_{j+1})_{k'_j}\{x_i/t\}$.

We have also $h'_j \in K(n)$, $\mathcal{K}(n; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$, $k_i \sqsubseteq h \sqsubseteq h'_j$, $\psi_{j+1} \in E(n, h'_j)$. By one of our declared properties we obtain that

$$(\psi_{j+1})_{k'_j}\{x_i/t\} = (\psi_{j+1})_{h'_j}\{x_i/t\} \ .$$

We also want to show that $\phi_{k'_m}\{x_i/t\}) = \phi_{h'_m}\{x_i/t\})$.
We recall that $k'_m \in K(n)$ and $\mathcal{K}(n; k'_m; x_1 : \varphi_1, \ldots, x_p : \varphi_p, y_1 : \psi_1, \ldots, y_m : \psi_m)$, for each $\alpha = i+1 \ldots p$ $x_\alpha \notin V_b(t)$ and $V_b(\varphi_\alpha) \cap V_b(t) = \emptyset$, for each $\alpha = 1 \ldots m$ $y_\alpha \notin V_b(t)$ and $V_b(\psi_\alpha) \cap V_b(t) = \emptyset$. We have $\phi \in E(n, k'_m)$ and also $V_b(\phi) \cap V_b(t) = \emptyset$. As a result of these conditions we were able to define $\phi_{k'_m}\{x_i/t\}$.

We have also $h'_m \in K(n)$, $\mathcal{K}(n; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$, $k_i \sqsubseteq h \sqsubseteq h'_m$, $\phi \in E(n, h'_m)$. By one of our declared properties we obtain that

$$\phi_{k'_m}\{x_i/t\} = \phi_{h'_m}\{x_i/t\} \ .$$

Hence

$$\begin{aligned}(\varphi_k\{x_i/t\})_a &= \varphi_{k_{p-1}}\{x_i/t\} = \\ &= \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \phi_{h'_m}\{x_i/t\}) = \\ &= \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \ldots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \phi_{k'_m}\{x_i/t\}) = \\ &= (\varphi_k\{x_i/t\})_e \ .\end{aligned}$$

Now let $x_i \notin var(h)$. In this case $(\varphi_k\{x_i/t\})_a = \varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \phi)$.

As seen above, we could define $(\psi_1)_k\{x_i/t\}$, and we have $h \in K(n)$, $\psi_1 \in E(n,h)$, $x_i \notin var(h)$. By one declared property $(\psi_1)_k\{x_i/t\} = \psi_1$.

If $m > 1$ suppose $j = 1 \ldots m-1$. We want to show that $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$. As seen above, we were able to define $(\psi_{j+1})_{k'_j}\{x_i/t\}$. We have also $h'_j \in K(n)$,

$\psi_{j+1} \in E(n, h'_j)$. Moreover, for each $\alpha = 1 \dots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_j)$. By one declared property $(\psi_{j+1})_{k'_j}\{x_i/t\} = \psi_{j+1}$.

We also need to show $\phi_{k'_m}\{x_i/t\} = \phi$. As seen above, we were able to define $(\phi)_{k'_m}\{x_i/t\}$. We have also $h'_m \in K(n)$, $\phi \in E(n, h'_m)$. Moreover, for each $\alpha = 1 \dots m$ since $y_\alpha \in \mathcal{V} - var(k)$ it also results $y_\alpha \neq x_i$, so $x_i \notin var(h'_m)$. By one declared property $\phi_{k'_m}\{x_i/t\} = \phi$.

Therefore

$$\begin{aligned}(\varphi_k\{x_i/t\})_a &= \{\}(y_1 : \psi_1, \dots, y_m : \psi_m, \phi) = \\ &= \{\}(y_1 : (\psi_1)_k\{x_i/t\}, \dots, y_m : (\psi_m)_{k'_{m-1}}\{x_i/t\}, \phi_{k'_m}\{x_i/t\}) = \\ &= (\varphi_k\{x_i/t\})_e \ .\end{aligned}$$

◇

At this point we have completed the proof that $\varphi_k\{x_i/t\}$ is defined unambiguosly. Our definition process requires now to verify that (for $\varphi \in E(n+1, k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$) one of the following five conditions holds:

a1. $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.

a2. $\varphi \in var(k)$, $\varphi = x_i \rightarrow \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \rightarrow \varphi_k\{x_i/t\} = \varphi$.

a3. there exist $h \in K(n)$: $h \sqsubseteq k$, a positive integer m, ψ, $\psi_1, \dots, \psi_m \in E(n, h)$ such that $\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1, h)$, for each $\rho \in \Xi(h)$ $\#(h, \psi, \rho)$ is a function with m arguments, $(\#(h, \psi_1, \rho), \dots, \#(h, \psi_m, \rho))$ is a member of the domain of $\#(h, \psi, \rho)$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \dots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

$$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \dots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a4. there exist $h \in K(n)$: $h \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \dots, \psi_m \in E(n, h)$ such that $\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1, h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), \dots, \#(h, \psi_m, \rho))$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \dots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

$$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a5. there exist $h \in K(n)$: $h \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$;
such that
$\mathcal{E}(n, h, m, y, \psi, \vartheta)$, $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n+1, h)$.
If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.
Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.
We have $\psi_1 \in E(n, h)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;
for each $j = 1 \ldots m-1$ $h'_j \in K(n)$ and by 4.7
$\mathcal{K}(n; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$,
for each $\alpha = 1 \ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n, h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;
$h'_m \in K(n)$ and by 4.7
$\mathcal{K}(n; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
for each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n, h'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{h'_m}\{x_i/t\}$ is defined;
it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

In this case too we need to remember that

$$E(n+1, k) = E'(n, k) \cup E'_a(n+1, k) \cup E'_b(n+1, k) \cup E'_c(n+1, k) \cup E'_d(n+1, k) \cup E'_e(n+1, k).$$

Suppose $\varphi \in \mathbf{E}'(\mathbf{n}, \mathbf{k})$. By the inductive hypothesis one of the following five conditions holds:

b1. $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.

b2. $\varphi \in var(k)$, $\varphi = x_i \to \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi$.

b3. $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, ψ, $\psi_1, \ldots, \psi_m \in E(n-1,h)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments, $(\#(h,\psi_1,\rho), \ldots, \#(h,\psi_m,\rho))$ is a member of the domain of $\#(h,\psi,\rho)$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

$$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$

we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

b4. $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n-1,h)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h,\psi_1,\rho), \ldots, \#(h,\psi_m,\rho))$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

$$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$

we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

b5. $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k$, a positive integer m, $\vartheta \in E(n-1)$,
a function y whose domain is $\{1, \ldots, m\}$ such that
for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1, \ldots, m\}$ such that
for each $j = 1 \ldots m$ $\psi_j \in E(n-1)$;

such that

$\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$,
$\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n,h)$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.

We have $\psi_1 \in E(n-1,h)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;

for each $j = 1 \ldots m-1$ $h'_j \in K(n-1)$ and by 4.7
$\mathcal{K}(n-1; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$,
for each $\alpha = 1 \ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n-1, h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;

$h'_m \in K(n-1)$ and by 4.7
$\mathcal{K}(n-1; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
for each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n-1, h'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{h'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \,.$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Clearly if b1. holds then a1. holds too, if b2. holds then a2. holds too. If b3. holds then a3. holds too, if b4. holds then a4. holds too. Finally if b5. holds then a5. holds too.

We turn to the case where $\varphi \in \mathbf{E'_a(n+1,k)}$. This implies $\varphi \in E_a(n+1,k)$, $k \in K(n)^+$.

We have $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V}-var(k_{p-1})$. Therefore $\varphi \in E(n, k_{p-1})$, $x_p \notin V_b(\varphi)$. We have to distinguish the case where $i < p$ from the one where $i = p$.

First we suppose $i < p$. In this case $1 \leqslant i \leqslant p-1$, so $k_{p-1} \neq \epsilon$. As we have seen above $\mathcal{K}(n; k_{p-1}; x_1 : \varphi_1, \ldots, x_{p-1} : \varphi_{p-1})$ holds. So we can apply the inductive hypothesis to φ and obtain that one of the following five conditions holds:

- $\varphi \in \mathcal{C}$ and $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

- $\varphi \in var(k_{p-1})$, $\varphi = x_i \to \varphi_{k_{p-1}}\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

- $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, a positive integer m, ψ, $\psi_1, \ldots, \psi_m \in E(n-1,h)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments, $(\#(h,\psi_1,\rho), \ldots, \#(h,\psi_m,\rho))$ is a member of the domain of $\#(h,\psi,\rho)$.

  If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p-1$ such that $q < n-1$, $\mathcal{K}(n-1; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

  If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

  $$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$

we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results

$$\varphi_{k_{p-1}}\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\},\ldots,(\psi_m)_h\{x_i/t\})\ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

- $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, $f \in \mathcal{F}$, a positive integer m, $\psi_1,\ldots,\psi_m \in E(n-1,h)$ such that $\varphi = (f)(\psi_1,\ldots,\psi_m)$, $\varphi \in E(n,h)$, for each $\rho \in \Xi(h)$ $A_f(\#(h,\psi_1,\rho),\ldots,\#(h,\psi_m,\rho))$.

  If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p-1$ such that $q < n-1$, $\mathcal{K}(n-1;h;x_1:\varphi_1,\ldots,x_q:\varphi_q)$.

  If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have

  $$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$

  we can define $(\psi_j)_h\{x_i/t\}$, and it results

  $$\varphi_{k_{p-1}}\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\},\ldots,(\psi_m)_h\{x_i/t\})\ .$$

  Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

- $n > 1$, there exist $h \in K(n-1)$: $h \sqsubseteq k_{p-1}$, a positive integer m, $\vartheta \in E(n-1)$, a function y whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $y_j \in \mathcal{V}-var(h)$ and for each $\alpha,\beta = 1\ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$; a function ψ whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $\psi_j \in E(n-1)$;

  such that

  $\mathcal{E}(n-1,h,m,y,\psi,\vartheta)$,
  $\varphi = \{\}(y_1:\psi_1,\ldots,y_m:\psi_m,\vartheta)$, $\varphi \in E(n,h)$.

  If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p-1$ such that $q < n-1$, $\mathcal{K}(n-1;h;x_1:\varphi_1,\ldots,x_q:\varphi_q)$.

  Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
  We define $h'_1 = h + (y_1,\psi_1)$, and if $m > 1$ for each $j = 1\ldots m-1$
  $h'_{j+1} = h'_j + (y_{j+1},\psi_{j+1})$.

  We have $\psi_1 \in E(n-1,h)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;

  for each $j = 1\ldots m-1$ $h'_j \in K(n-1)$ and by 4.7
  $\mathcal{K}(n-1;h'_j;x_1:\varphi_1,\ldots,x_q:\varphi_q,y_1:\psi_1,\ldots,y_j:\psi_j)$,
  for each $\alpha = 1\ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
  for each $\alpha = 1\ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
  $\psi_{j+1} \in E(n-1,h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
  therefore $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;

  $h'_m \in K(n-1)$ and by 4.7
  $\mathcal{K}(n-1;h'_m;x_1:\varphi_1,\ldots,x_q:\varphi_q,y_1:\psi_1,\ldots,y_m:\psi_m)$,
  for each $\alpha = 1\ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
  for each $\alpha = 1\ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,

$\vartheta \in E(n-1, h'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{h'_m}\{x_i/t\}$ is defined;
it results

$$\varphi_{k_{p-1}}\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\}) \ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_{k_{p-1}}\{x_i/t\} = \varphi$.

In this case $i < p$ we defined $\varphi_k\{x_i/t\} = \varphi_{k_{p-1}}\{x_i/t\}$, therefore one of the following five conditions holds:

- $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.

- $\varphi \in var(k)$, $\varphi = x_i \to \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi$.

- there exist $h \in K(n)$: $h \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n, h)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, h)$, for each $\rho \in \Xi(h)$ $\#(h, \psi, \rho)$ is a function with m arguments, $(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$ is a member of the domain of $\#(h, \psi, \rho)$.
  If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.
  If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have
  $$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$
  we can define $\psi_h\{x_i/t\}$, and similarly we can define $(\psi_j)_h\{x_i/t\}$, and it results
  $$\varphi_k\{x_i/t\} = (\psi_h\{x_i/t\})((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$
  Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

- there exist $h \in K(n)$: $h \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n, h)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, h)$,
  for each $\rho \in \Xi(h)$ $A_f(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$.
  If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.
  If $i \leqslant q$, or in other words $x_i \in var(h)$, since we have
  $$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset,$$
  we can define $(\psi_j)_h\{x_i/t\}$, and it results
  $$\varphi_k\{x_i/t\} = (f)((\psi_1)_h\{x_i/t\}, \ldots, (\psi_m)_h\{x_i/t\}) \ .$$
  Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

- there exist $h \in K(n)$: $h \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$; a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$;

such that

$\mathcal{E}(n, h, m, y, \psi, \vartheta)$,
$\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta),\ \varphi \in E(n+1, h)$.

If $h \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(h)$.
We define $h'_1 = h + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$
$h'_{j+1} = h'_j + (y_{j+1}, \psi_{j+1})$.
We have $\psi_1 \in E(n, h)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_h\{x_i/t\}$ is defined;

for each $j = 1 \ldots m-1$ $h'_j \in K(n)$ and by 4.7
$\mathcal{K}(n; h'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$,
for each $\alpha = 1 \ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n, h'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{h'_j}\{x_i/t\}$ is defined;

$h'_m \in K(n)$ and by 4.7
$\mathcal{K}(n; h'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
for each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n, h'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{h'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_h\{x_i/t\}, \ldots, y_m : (\psi_m)_{h'_{m-1}}\{x_i/t\}, \vartheta_{h'_m}\{x_i/t\})\ .$$

Otherwise (when $h = \epsilon$ or $h \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

We now consider the case where $i = p$, in which we defined $\varphi_k\{x_i/t\} = \varphi$.
Since $\varphi \in E(n, k_{p-1})$ we can apply assumption 2.1.10 to establish that one of the following five conditions holds:

c1. $\varphi \in \mathcal{C}$.

c2. there exists $j = 1 \ldots p-1$ such that $\varphi = x_j$.

c3.

$\exists h \in K(n-1) : h \sqsubseteq k_{p-1}, \exists m$ positive integer $, \psi, \psi_1, \ldots \psi_m \in E(n-1, h)$:
$\varphi = (\psi)(\psi_1, \ldots, \psi_m),\ \varphi \in E(n, h)$,
$\forall \rho \in \Xi(h)$ ($\#(h, \psi, \rho)$ is a function with m arguments,
$(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_m, \rho))$ is a member of the domain of $\#(h, \psi, \rho)$.

c4.

$$\exists h \in K(n-1) : h \sqsubseteq k_{p-1}, \exists f \in \mathcal{F},\ m \text{ positive integer }, \psi_1, \dots \psi_m \in E(n-1,h) :$$
$$\varphi = (f)(\psi_1, \dots, \psi_m),\ \varphi \in E(n,h),$$
$$\forall \rho \in \Xi(h)\ (\ A_f(\#(h,\psi_1,\rho), \dots, \#(h,\psi_m,\rho)).$$

c5.

there exist

$h \in K(n-1) : h \sqsubseteq k_{p-1}$,

a positive integer m,

a function y whose domain is $\{1, \dots, m\}$ such that for each $j = 1 \dots m$

$y_j \in \mathcal{V} - var(h)$, and for each $\alpha, \beta = 1 \dots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$,

a function ψ whose domain is $\{1, \dots, m\}$ such that for each $j = 1 \dots m$

$\psi_j \in E(n-1)$,

$\vartheta \in E(n-1)$

such that

$\mathcal{E}(n-1, h, m, y, \psi, \vartheta)$,

$\varphi = \{\}(y_1 : \psi_1, \dots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n,h)$.

If c1. holds then $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$ so a1. holds.

If c2. holds then $\varphi \in var(k)$, $\varphi \neq x_i$, $\varphi_k\{x_i/t\} = \varphi$ so a2. holds.

If c3. holds then there exist $h \in K(n)$: $h \sqsubseteq k_{p-1} \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \dots, \psi_m \in E(n,h)$ such that $\varphi = (\psi)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1,h)$, for each $\rho \in \Xi(h)$ $\#(h,\psi,\rho)$ is a function with m arguments, $(\#(h,\psi_1,\rho), \dots, \#(h,\psi_m,\rho))$ is a member of the domain of $\#(h,\psi,\rho)$.

Moreover if $h \neq \epsilon$ then by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \dots, x_q : \varphi_q)$. Suppose $p = i \leqslant q$, this would imply that $q = p$, so $h = k_p = k$. But $h \sqsubseteq k_{p-1}$ also holds. So $dom(h) = \{1, \dots, p\}$ and $dom(h) \subseteq dom(k_{p-1}) \subseteq \{1, \dots, p-1\}$. This is a contradiction, so we must have $i > q$.

Therefore we have $h = \epsilon \vee (h \neq \epsilon \wedge i > q)$ and $\varphi_k\{x_i/t\} = \varphi$. This implies that a3. is satisfied.

If c4. holds then there exist $h \in K(n)$: $h \sqsubseteq k_{p-1} \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \dots, \psi_m \in E(n,h)$ such that $\varphi = (f)(\psi_1, \dots, \psi_m)$, $\varphi \in E(n+1,h)$,
for each $\rho \in \Xi(h)$ $A_f(\#(h,\psi_1,\rho), \dots, \#(h,\psi_m,\rho))$.

Moreover if $h \neq \epsilon$ then by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \dots, x_q : \varphi_q)$. Suppose $p = i \leqslant q$, this would imply that $q = p$, so $h = k_p = k$. But $h \sqsubseteq k_{p-1}$ also holds. So $dom(h) = \{1, \dots, p\}$ and $dom(h) \subseteq dom(k_{p-1}) \subseteq \{1, \dots, p-1\}$. This is a contradiction, so we must have $i > q$.

Therefore we have $h = \epsilon \vee (h \neq \epsilon \wedge i > q)$ and $\varphi_k\{x_i/t\} = \varphi$. This implies that a4. is satisfied.

If c5. holds then there exist $h \in K(n)$: $h \sqsubseteq k_{p-1} \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1,\ldots,m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(h)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1,\ldots,m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$;

such that

$\mathcal{E}(n, h, m, y, \psi, \vartheta)$, $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n+1, h)$.

Moreover if $h \neq \epsilon$ then by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; h; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$. Suppose $p = i \leqslant q$, this would imply that $q = p$, so $h = k_p = k$. But $h \sqsubseteq k_{p-1}$ also holds. So $dom(h) = \{1,\ldots,p\}$ and $dom(h) \subseteq dom(k_{p-1}) \subseteq \{1,\ldots,p-1\}$. This is a contradiction, so we must have $i > q$.

Therefore we have $h = \epsilon \vee (h \neq \epsilon \wedge i > q)$ and $\varphi_k\{x_i/t\} = \varphi$. This implies that a5. is satisfied.

Let's examine the case where $\varphi \in \mathbf{E'_b(n+1,k)}$. This implies $\varphi \in E_b(n+1,k)$, $k \in K(n)^+$.

We have $k = k_p = k_{p-1} + (x_p, \varphi_p)$, $k_{p-1} \in K(n)$, $\varphi_p \in E_s(n, k_{p-1})$, $x_p \in \mathcal{V} - var(k_{p-1})$. Therefore $\varphi = x_p \in var(k)$.

If $i = p$ we have $\varphi = x_i$ and $\varphi_k\{x_i/t\} = t$.

If $i < p$ we have $\varphi \neq x_i$ and $\varphi_k\{x_i/t\} = \varphi$.

This implies that a2. is satisfied.

We now consider the case where $\varphi \in \mathbf{E'_c(n+1,k)}$. This implies $\varphi \in E_c(n+1,k)$, $k \in K(n)$.

There exist a positive integer m and $\psi, \psi_1, \ldots, \psi_m \in E(n,k)$ such that

- $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$;
- for each $\sigma \in \Xi(k)$ $\#(k, \psi, \sigma)$ is a function with m arguments and $(\#(k, \psi_1, \sigma), \ldots, \#(k, \psi_m, \sigma))$ is a member of its domain.

We have also $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_p : \varphi_p)$. We can define $\psi_k\{x_i/t\}$, $(\psi_j)_k\{x_i/t\}$ and it results

$$\varphi_k\{x_i/t\} = (\psi_k\{x_i/t\})((\psi_1)_k\{x_i/t\}, \ldots, (\psi_m)_k\{x_i/t\}) \; .$$

This implies that a3. is satisfied.

The case where $\varphi \in \mathbf{E'_d(n+1,k)}$ is similar. In fact this implies $\varphi \in E_d(n+1,k)$, $k \in K(n)$.

There exist $f \in \mathcal{F}$, a positive integer m and $\psi_1, \ldots, \psi_m \in E(n,k)$ such that

- $\varphi = (f)(\psi_1, \ldots, \psi_m)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \psi_1, \sigma), \ldots, \#(k, \psi_m, \sigma))$ holds true.

We have also $\mathcal{K}(n;k;x_1:\varphi_1,\ldots,x_p:\varphi_p)$. We can define $(\psi_j)_k\{x_i/t\}$ and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_k\{x_i/t\},\ldots,(\psi_m)_k\{x_i/t\}) \ .$$

This implies that a4. is satisfied.

Finally we examine the case where $\varphi \in \mathbf{E'_e(n+1,k)}$. This implies $\varphi \in E_e(n+1,k)$, $k \in K(n)$. There exist

- a positive integer m,
- a function y whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $y_j \in \mathcal{V} - var(k)$, and for each $\alpha,\beta = 1\ldots m$ $\alpha \neq \beta \to y_\alpha \neq y_\beta$,
- a function ψ whose domain is $\{1,\ldots,m\}$ such that for each $j = 1\ldots m$ $\psi_j \in E(n)$,
- $\phi \in E(n)$

such that $\varphi = \{\}(y_1:\psi_1,\ldots,y_m:\psi_m,\phi)$ and $\mathcal{E}(n,k,m,y,\psi,\phi)$.

We have $\mathcal{K}(n;k;x_1:\varphi_1,\ldots,x_p:\varphi_p)$.

Let $k'_1 = k+(y_1,\psi_1)$, and if $m>1$ for each $j=1\ldots m-1$ $k'_{j+1} = k'_j + (y_{j+1},\psi_{j+1})$.

In our assumptions $(\psi_1)_k\{x_i/t\}$ is defined, if $m > 1$ then for each $j = 2\ldots m$ $(\psi_j)_{k'_{j-1}}\{x_i/t\}$ is defined, and finally that $\phi_{k'_m}\{x_i/t\}$ is defined. It results

$$\varphi_k\{x_i/t\} = \{\}(y_1:(\psi_1)_k\{x_i/t\},\ldots,y_m:(\psi_m)_{k'_{m-1}}\{x_i/t\},\phi_{k'_m}\{x_i/t\}) \ .$$

This implies that a5. is satisfied.

$\diamond$

Another step has been completed. We maintain the assumption that $\varphi \in E(n+1,k)$ is such that $V_b(t)\cap V_b(\varphi) = \emptyset$. To go on with the next step we assume $h \in K(n+1)$ is such that $k_i \sqsubseteq h$.

We know there exist a positive integer u such that $u < n+1$, $w_1,\ldots,w_u \in \mathcal{V}$ such that $w_\alpha \neq w_\beta$ for $\alpha \neq \beta$, $\xi_1,\ldots,\xi_u \in E$ such that $\mathcal{K}(n+1;h;w_1:\xi_1,\ldots,w_u:\xi_u)$.
By lemma 4.8 we know that $i \leqslant u$ and for each $j = 1\ldots i$ $w_j = x_j$, $\xi_j = \varphi_j$.
If $i < u$ then we assume for each $j = i+1\ldots u$ $w_j \notin V_b(t)$, $V_b(t)\cap V_b(\xi_j) = \emptyset$. We also assume $\varphi \in E(n+1,h)$.

We need to show that $\varphi_k\{x_i/t\} = \varphi_h\{x_i/t\}$.

We have just seen that one of the following five conditions holds:

a1. $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.

a2. $\varphi \in var(k)$, $\varphi = x_i \to \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi$.

a3. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$ $\#(\kappa, \psi, \rho)$ is a function with m arguments, $(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$ is a member of the domain of $\#(\kappa, \psi, \rho)$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(\kappa)$, since we have

$$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $\psi_\kappa\{x_i/t\}$, and similarly we can define $(\psi_j)_\kappa\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a4. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(\kappa)$, since we have

$$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $(\psi_j)_\kappa\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a5. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(\kappa)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$; such that

$\mathcal{E}(n, \kappa, m, y, \psi, \vartheta)$, $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n+1, \kappa)$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(\kappa)$.
We define $\kappa'_1 = \kappa + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m-1$ $\kappa'_{j+1} = \kappa'_j + (y_{j+1}, \psi_{j+1})$.
We have $\psi_1 \in E(n, \kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each $j = 1 \ldots m-1$ $\kappa'_j \in K(n)$ and by 4.7
$\mathcal{K}(n; \kappa'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$,
for each $\alpha = 1 \ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,

for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n, \kappa'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{\kappa'_j}\{x_i/t\}$ is defined;

$\kappa'_m \in K(n)$ and by 4.7
$\mathcal{K}(n; \kappa'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
for each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n, \kappa'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{\kappa'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_\kappa\{x_i/t\}, \ldots, y_m : (\psi_m)_{\kappa'_{m-1}}\{x_i/t\}, \vartheta_{\kappa'_m}\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Given that $\varphi \in E(n, h+1)$ we have also to accept that one of the following five conditions holds:

d1. $\varphi \in \mathcal{C}$ and $\varphi_h\{x_i/t\} = \varphi$.

d2. $\varphi \in var(h)$, $\varphi = x_i \rightarrow \varphi_h\{x_i/t\} = t$, $\varphi \neq x_i \rightarrow \varphi_h\{x_i/t\} = \varphi$.

d3. there exist $\eta \in K(n)$: $\eta \sqsubseteq h$, a positive integer r,
χ, $\chi_1, \ldots, \chi_r \in E(n, \eta)$ such that $\varphi = (\chi)(\chi_1, \ldots, \chi_r)$, $\varphi \in E(n+1, \eta)$, for each $\rho \in \Xi(\eta)$ $\#(\eta, \chi, \rho)$ is a function with r arguments, $(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_r, \rho))$ is a member of the domain of $\#(\eta, \chi, \rho)$.

If $\eta \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$.

If $i \leqslant q'$, or in other words $x_i \in var(\eta)$, since we have

$$V_b(t) \cap V_b(\chi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $\chi_\eta\{x_i/t\}$, and similarly we can define $(\chi_j)_\eta\{x_i/t\}$, and it results

$$\varphi_h\{x_i/t\} = (\chi_\eta\{x_i/t\})((\chi_1)_\eta\{x_i/t\}, \ldots, (\chi_r)_\eta\{x_i/t\}) \ .$$

Otherwise (when $\eta = \epsilon$ or $\eta \neq \epsilon \wedge i > q'$) $\varphi_h\{x_i/t\} = \varphi$.

d4. there exist $\eta \in K(n)$: $\eta \sqsubseteq h$, $g \in \mathcal{F}$, a positive integer r, $\chi_1, \ldots, \chi_r \in E(n, \eta)$ such that $\varphi = (g)(\chi_1, \ldots, \chi_r)$, $\varphi \in E(n+1, \eta)$,
for each $\rho \in \Xi(\eta)$ $A_g(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_r, \rho))$.

If $\eta \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$.

If $i \leqslant q'$, or in other words $x_i \in var(\eta)$, since we have

$$V_b(t) \cap V_b(\chi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $(\chi_j)_\eta\{x_i/t\}$, and it results

$$\varphi_h\{x_i/t\} = (g)((\chi_1)_\eta\{x_i/t\}, \ldots, (\chi_r)_\eta\{x_i/t\}) \ .$$

Otherwise (when $\eta = \epsilon$ or $\eta \neq \epsilon \wedge i > q'$) $\varphi_h\{x_i/t\} = \varphi$.

d5. there exist $\eta \in K(n)$: $\eta \sqsubseteq h$, a positive integer r, $\phi \in E(n)$, a function z whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $z_j \in \mathcal{V} - var(\eta)$ and for each $\alpha, \beta = 1 \ldots r$ $\alpha \neq \beta \to z_\alpha \neq z_\beta$;
a function χ whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$ $\chi_j \in E(n)$;
such that
$\mathcal{E}(n, \eta, r, z, \chi, \phi)$, $\varphi = \{\}(z_1 : \chi_1, \ldots, z_r : \chi_r, \phi)$, $\varphi \in E(n+1, \eta)$.
If $\eta \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$.
Suppose $i \leqslant q'$, or in other words $x_i \in var(\eta)$.
We define $\eta'_1 = \eta + (z_1, \chi_1)$, and if $r > 1$ for each $j = 1 \ldots r - 1$ $\eta'_{j+1} = \eta'_j + (z_{j+1}, \chi_{j+1})$.
We have $\chi_1 \in E(n, \eta)$, $V_b(t) \cap V_b(\chi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\chi_1)_\eta\{x_i/t\}$ is defined;
for each $j = 1 \ldots r - 1$ $\eta'_j \in K(n)$ and by 4.7
$\mathcal{K}(n; \eta'_j; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'}, z_1 : \chi_1, \ldots, z_j : \chi_j)$,
for each $\alpha = 1 \ldots j$ $z_\alpha \in V_b(\varphi)$ so $z_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\chi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\chi_{j+1} \in E(n, \eta'_j)$, $V_b(t) \cap V_b(\chi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\chi_{j+1})_{\eta'_j}\{x_i/t\}$ is defined;
$\eta'_r \in K(n)$ and by 4.7
$\mathcal{K}(n; \eta'_r; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'}, z_1 : \chi_1, \ldots, z_r : \chi_r)$,
for each $\alpha = 1 \ldots r$ $z_\alpha \in V_b(\varphi)$ so $z_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots r$ $V_b(t) \cap V_b(\chi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\phi \in E(n, \eta'_r)$, $V_b(t) \cap V_b(\phi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\phi_{\eta'_r}\{x_i/t\}$ is defined;
it results

$$\varphi_h\{x_i/t\} = \{\}(z_1 : (\chi_1)_\eta\{x_i/t\}, \ldots, z_r : (\chi_r)_{\eta'_{r-1}}\{x_i/t\}, \phi_{\eta'_r}\{x_i/t\}) \ .$$

Otherwise (when $\eta = \epsilon$ or $\eta \neq \epsilon \wedge i > q'$) $\varphi_h\{x_i/t\} = \varphi$.

If **a1**. occurs then d1. also occurs and $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

If **a2**. occurs then d2. also occurs and $\varphi = x_i \to \varphi_h\{x_i/t\} = t = \varphi_k\{x_i/t\}$, $\varphi \neq x_i \to \varphi_h\{x_i/t\} = \varphi = \varphi_k\{x_i/t\}$.

We now consider the case where **a3**. occurs. As a consequence d3. occurs too.

There exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$

$\#(\kappa,\psi,\rho)$ is a function with m arguments, $(\#(\kappa,\psi_1,\rho),\ldots,\#(\kappa,\psi_m,\rho))$ is a member of the domain of $\#(\kappa,\psi,\rho)$.

There exist $\eta \in K(n)$: $\eta \sqsubseteq h$, a positive integer r,
χ, $\chi_1,\ldots,\chi_r \in E(n,\eta)$ such that $\varphi = (\chi)(\chi_1,\ldots,\chi_r)$, $\varphi \in E(n+1,\eta)$, for each $\rho \in \Xi(\eta)$ $\#(\eta,\chi,\rho)$ is a function with r arguments, $(\#(\eta,\chi_1,\rho),\ldots,\#(\eta,\chi_r,\rho))$ is a member of the domain of $\#(\eta,\chi,\rho)$.

We have

$$(\psi)(\psi_1,\ldots,\psi_m) = \varphi = (\chi)(\chi_1,\ldots,\chi_r)\ ,$$

and therefore $r = m$, $\chi = \psi$ and for each $j = 1\ldots m$ $\chi_j = \psi_j$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, ψ, $\psi_1,\ldots,\psi_m \in E(n,\eta)$, $\varphi \in E(n+1,\eta)$, for each $\rho \in \Xi(\eta)$ $\#(\eta,\psi,\rho)$ is a function with m arguments, $(\#(\eta,\psi_1,\rho),\ldots,\#(\eta,\psi_m,\rho))$ is a member of the domain of $\#(\eta,\psi,\rho)$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1:\varphi_1,\ldots,x_q:\varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $\psi_\kappa\{x_i/t\}$, $(\psi_j)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\kappa_i = h_i = \eta_i \sqsubseteq \eta$.

We know that $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$. Lemma 4.8 confirms that $i \leqslant q'$ and for each $j = 1\ldots i$ $w_j = x_j$, $\xi_j = \varphi_j$.

Our assumptions also ensure that if $i < q'$ then for each $j = i+1\ldots q'$ $w_j \notin V_b(t)$, $V_b(t) \cap V_b(\xi_j) = \emptyset$. We have also seen that ψ, $\psi_1,\ldots,\psi_m \in E(n,\eta)$.

So by the inductive hypothesis we obtain

$$\psi_\kappa\{x_i/t\} = \psi_\eta\{x_i/t\},\ (\psi_j)_\kappa\{x_i/t\} = (\psi_j)_\eta\{x_i/t\}\ ,$$

and then

$$\begin{aligned}\varphi_k\{x_i/t\} &= (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\},\ldots,(\psi_m)_\kappa\{x_i/t\}) = \\ &= (\psi_\eta\{x_i/t\})((\psi_1)_\eta\{x_i/t\},\ldots,(\psi_m)_\eta\{x_i/t\}) = \varphi_h\{x_i/t\}\ .\end{aligned}$$

We now need to consider another subcase of our a3. and d3. case.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1:\varphi_1,\ldots,x_q:\varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $\psi_\kappa\{x_i/t\}$, $(\psi_j)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\psi,\psi_1,\ldots,\psi_m \in E(n,\eta)$.

If $\eta = \epsilon$ then clearly $x_i \notin var(\eta)$. Otherwise we know there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$. Clearly $i > q'$, so for each $j = 1 \ldots q'$ $w_j = x_j$ and $x_i \notin var(\eta)$ holds in this case too.

Therefore

$$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = \\ = (\psi)(\psi_1, \ldots, \psi_m) = \varphi = \varphi_h\{x_i/t\} \ .$$

We turn to consider a third subcase of our a3. and d3. case.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\eta \in K(n)$ and we can define $\psi_\eta\{x_i/t\}$, $(\psi_j)_\eta\{x_i/t\}$. Moreover $\kappa \in K(n)$, $\psi, \psi_1, \ldots, \psi_m \in E(n, \kappa)$.

If $\kappa = \epsilon$ then clearly $x_i \notin var(\kappa)$. Otherwise we know there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$. Clearly $i > q$ so $x_i \notin var(\kappa)$ still holds.

Thus we get

$$\varphi_k\{x_i/t\} = \varphi = (\psi)(\psi_1, \ldots, \psi_m) = \\ = (\psi_\eta\{x_i/t\})((\psi_1)_\eta\{x_i/t\}, \ldots, (\psi_m)_\eta\{x_i/t\}) = \varphi_h\{x_i/t\} \ .$$

There is still another subcase to consider.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Here clearly $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

Let's turn to the case where **a4**. holds, and accordingly d4. holds too.

There exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$.

There exist $\eta \in K(n)$: $\eta \sqsubseteq h$, $g \in \mathcal{F}$, a positive integer r, $\chi_1, \ldots, \chi_r \in E(n, \eta)$ such that $\varphi = (g)(\chi_1, \ldots, \chi_r)$, $\varphi \in E(n+1, \eta)$, for each $\rho \in \Xi(\eta)$ $A_g(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_r, \rho))$.

We have

$$(f)(\psi_1, \ldots, \psi_m) = \varphi = (g)(\chi_1, \ldots, \chi_r) \ ,$$

and therefore $g = f$, $r = m$, for each $j = 1 \ldots m$ $\chi_j = \psi_j$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, $\psi_1, \ldots, \psi_m \in E(n, \eta)$, $\varphi \in E(n+1, \eta)$, for each $\rho \in \Xi(\eta)$ $A_f(\#(\eta, \psi_1, \rho), \ldots, \#(\eta, \psi_m, \rho))$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_j)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\kappa_i = h_i = \eta_i \sqsubseteq \eta$.

We know that $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$. Lemma 4.8 confirms that $i \leqslant q'$ and for each $j = 1 \ldots i$ $w_j = x_j$, $\xi_j = \varphi_j$.

Our assumptions also ensure that if $i < q'$ then for each $j = i+1 \ldots q'$ $w_j \notin V_b(t)$, $V_b(t) \cap V_b(\xi_j) = \emptyset$. We have also seen that $\psi_1, \ldots, \psi_m \in E(n, \eta)$.

So by the inductive hypothesis we obtain $(\psi_j)_\kappa\{x_i/t\} = (\psi_j)_\eta\{x_i/t\}$ and then

$$\begin{aligned}\varphi_k\{x_i/t\} &= (f)((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = \\ &= (f)((\psi_1)_\eta\{x_i/t\}, \ldots, (\psi_m)_\eta\{x_i/t\}) = \varphi_h\{x_i/t\}\ .\end{aligned}$$

We now need to consider another subcase of our a4. and d4. case.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_j)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\psi_1, \ldots, \psi_m \in E(n, \eta)$.

If $\eta = \epsilon$ then clearly $x_i \notin var(\eta)$. Otherwise we know there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$. Clearly $i > q'$, so for each $j = 1 \ldots q'$ $w_j = x_j$ and $x_i \notin var(\eta)$ holds in this case too.

Therefore

$$\begin{aligned}\varphi_k\{x_i/t\} &= (f)((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = \\ &= (f)(\psi_1, \ldots, \psi_m) = \varphi = \varphi_h\{x_i/t\}\ .\end{aligned}$$

We turn to consider a third subcase of our a4. and d4. case.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\eta \in K(n)$ and we can define $(\psi_j)_\eta\{x_i/t\}$. Moreover $\kappa \in K(n)$, $\psi_1, \ldots, \psi_m \in E(n, \kappa)$.

If $\kappa = \epsilon$ then clearly $x_i \notin var(\kappa)$. Otherwise we know there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1 : \varphi_1, \dots, x_q : \varphi_q)$. Clearly $i > q$ so $x_i \notin var(\kappa)$ still holds.

Thus we get

$$\begin{aligned}\varphi_k\{x_i/t\} = \varphi = (f)(\psi_1, \dots, \psi_m) = \\ = (f)((\psi_1)_\eta\{x_i/t\}, \dots, (\psi_m)_\eta\{x_i/t\}) = \varphi_h\{x_i/t\} .\end{aligned}$$

There is still another subcase to consider.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1 : \varphi_1, \dots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1 : \xi_1, \dots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Here clearly $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

Finally we examine the case where **a5**. holds, and accordingly d5. also occurs.

There exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1, \dots, m\}$ such that for each $j = 1 \dots m$ $y_j \in \mathcal{V} - var(\kappa)$ and for each $\alpha, \beta = 1 \dots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1, \dots, m\}$ such that for each $j = 1 \dots m$ $\psi_j \in E(n)$;

such that

$\mathcal{E}(n, \kappa, m, y, \psi, \vartheta)$, $\varphi = \{\}(y_1 : \psi_1, \dots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n+1, \kappa)$.

There exist $\eta \in K(n)$: $\eta \sqsubseteq h$, a positive integer r, $\phi \in E(n)$, a function z whose domain is $\{1, \dots, r\}$ such that for each $j = 1 \dots r$ $z_j \in \mathcal{V} - var(\eta)$ and for each $\alpha, \beta = 1 \dots r$ $\alpha \neq \beta \rightarrow z_\alpha \neq z_\beta$;
a function χ whose domain is $\{1, \dots, r\}$ such that for each $j = 1 \dots r$ $\chi_j \in E(n)$;

such that

$\mathcal{E}(n, \eta, r, z, \chi, \phi)$, $\varphi = \{\}(z_1 : \chi_1, \dots, z_r : \chi_r, \phi)$, $\varphi \in E(n+1, \eta)$.

We have $\varphi \in E_e(n+1, \kappa)$ and

$$\{\}(y_1 : \psi_1, \dots, y_m : \psi_m, \vartheta) = \varphi = \{\}(z_1 : \chi_1, \dots, z_r : \chi_r, \phi) .$$

Therefore $r = m$, $z = y$, $\chi = \psi$ and $\phi = \vartheta$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, for each $j = 1 \dots m$ $y_j \in \mathcal{V} - var(\eta)$; $\mathcal{E}(n, \eta, m, y, \psi, \vartheta)$, $\varphi \in E(n+1, \eta)$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1 : \varphi_1, \dots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1 : \xi_1, \dots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_1)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\kappa_i = h_i = \eta_i \sqsubseteq \eta$.

We know that $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$. Lemma 4.8 confirms that $i \leqslant q'$ and for each $j = 1\ldots i$ $w_j = x_j$, $\xi_j = \varphi_j$.

Our assumptions also ensure that if $i < q'$ then for each $j = i+1\ldots q'$ $w_j \notin V_b(t)$, $V_b(t) \cap V_b(\xi_j) = \emptyset$. We have also seen that $\psi_1 \in E(n,\eta)$.

So by the inductive hypothesis we obtain

$$(\psi_1)_\kappa\{x_i/t\} = (\psi_1)_\eta\{x_i/t\}\ .$$

Now suppose $m > 1$ and let $j = 1\ldots m-1$. It results $\kappa'_j \in K(n)$ and we can define $(\psi_{j+1})_{\kappa'_j}\{x_i/t\}$. Moreover $\eta'_j \in K(n)$, $(\kappa'_j)_i = \kappa_i = h_i = \eta_i \sqsubseteq \eta'_j$.

We know that $\mathcal{K}(n;\eta'_j;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'},y_1:\psi_1,\ldots,y_j:\psi_j)$.
For each $\alpha = 1\ldots j$ $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$. Moreover $\psi_{j+1} \in E(n,\eta'_j)$. Therefore

$$(\psi_{j+1})_{\kappa'_j}\{x_i/t\} = (\psi_{j+1})_{\eta'_j}\{x_i/t\}\ .$$

We still need to show that $\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta_{\eta'_m}\{x_i/t\}$.

To this end we see that $\kappa'_m \in K(n)$ and we can define $\vartheta_{\kappa'_m}\{x_i/t\}$. Moreover $\eta'_m \in K(n)$, $(\kappa'_m)_i = \kappa_i = h_i = \eta_i \sqsubseteq \eta'_m$.

We know that $\mathcal{K}(n;\eta'_m;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'},y_1:\psi_1,\ldots,y_m:\psi_m)$.
For each $\alpha = 1\ldots m$ $y_\alpha \notin V_b(t)$, $V_b(t) \cap V_b(\psi_\alpha) = \emptyset$. Moreover $\vartheta \in E(n,\eta'_m)$. Therefore

$$\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta_{\eta'_m}\{x_i/t\}\ .$$

Finally we can establish

$$\begin{aligned}\varphi_k\{x_i/t\} &= \{\}(y_1:(\psi_1)_\kappa\{x_i/t\},\ldots,y_m:(\psi_m)_{\kappa'_{m-1}}\{x_i/t\},\vartheta_{\kappa'_m}\{x_i/t\}) = \\ &= \{\}(y_1:(\psi_1)_\eta\{x_i/t\},\ldots,y_m:(\psi_m)_{\eta'_{m-1}}\{x_i/t\},\vartheta_{\eta'_m}\{x_i/t\}) = \varphi_h\{x_i/t\}\ .\end{aligned}$$

We now need to consider another subcase of our a5. and d5. case.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1:\varphi_1,\ldots,x_q:\varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$, $i \leqslant q'$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_1)_\kappa\{x_i/t\}$. Moreover $\eta \in K(n)$, $\psi_1 \in E(n,\eta)$.

If $\eta = \epsilon$ then clearly $x_i \notin var(\eta)$. Otherwise we know there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1:\xi_1,\ldots,w_{q'}:\xi_{q'})$. Clearly $i > q'$, so for each $j = 1\ldots q'$ $w_j = x_j$ and $x_i \notin var(\eta)$ holds in this case too.

We obtain that $(\psi_1)_\kappa\{x_i/t\} = \psi_1$.

Now suppose $m > 1$ and let $j = 1 \dots m-1$. It results $\kappa'_j \in K(n)$ and we can define $(\psi_{j+1})_{\kappa'_j}\{x_i/t\}$. Recall that $\mathcal{E}(n,\eta,m,y,\psi,\vartheta)$ holds, so $\eta'_j \in K(n)$, $\psi_{j+1} \in E(n,\eta'_j)$. For each $\alpha = 1 \dots j$ $y_\alpha \notin var(\kappa)$ and since $x_i \in var(\kappa)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\eta'_j)$.

Therefore $(\psi_{j+1})_{\kappa'_j}\{x_i/t\} = \psi_{j+1}$.

We still need to show that $\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta$.

It results $\kappa'_m \in K(n)$ and we can define $\vartheta_{\kappa'_m}\{x_i/t\}$. Recall that $\mathcal{E}(n,\eta,m,y,\psi,\vartheta)$ holds, so $\eta'_m \in K(n)$, $\vartheta \in E(n,\eta'_m)$. For each $\alpha = 1 \dots m$ $y_\alpha \notin var(\kappa)$ and since $x_i \in var(\kappa)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\eta'_m)$.

Therefore $\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta$.

Finally we establish

$$\begin{aligned}\varphi_k\{x_i/t\} &= \{\}(y_1 : (\psi_1)_\kappa\{x_i/t\}, \dots, y_m : (\psi_m)_{\kappa'_{m-1}}\{x_i/t\}, \vartheta_{\kappa'_m}\{x_i/t\}) = \\ &= \{\}(y_1 : \psi_1, \dots, y_m : \psi_m, \vartheta) = \varphi = \varphi_h\{x_i/t\}\ .\end{aligned}$$

We turn to consider a third subcase of our a5. and d5. case.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1 : \varphi_1, \dots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta \neq \epsilon$ and there exists a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n;\eta;w_1 : \xi_1, \dots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Consider that $\eta \in K(n)$ and we can define $(\psi_1)_\eta\{x_i/t\}$. Moreover $\kappa \in K(n)$, and because of $\mathcal{E}(n,\kappa,m,y,\psi,\vartheta)$ we have $\psi_1 \in E(n,\kappa)$.

If $\kappa = \epsilon$ then clearly $x_i \notin var(\kappa)$. Otherwise we know there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n;\kappa;x_1 : \varphi_1, \dots, x_q : \varphi_q)$. Clearly $i > q$ so $x_i \notin var(\kappa)$ still holds.

We obtain that $(\psi_1)_\eta\{x_i/t\} = \psi_1$.

Now suppose $m > 1$ and let $j = 1 \dots m-1$. It results $\eta'_j \in K(n)$ and we can define $(\psi_{j+1})_{\eta'_j}\{x_i/t\}$. Recall that $\mathcal{E}(n,\kappa,m,y,\psi,\vartheta)$ holds, so $\kappa'_j \in K(n)$, $\psi_{j+1} \in E(n,\kappa'_j)$. For each $\alpha = 1 \dots j$ $y_\alpha \notin var(\eta)$ and since $x_i = w_i \in var(\eta)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\kappa'_j)$.

Therefore $(\psi_{j+1})_{\eta'_j}\{x_i/t\} = \psi_{j+1}$.

We still need to show that $\vartheta_{\eta'_m}\{x_i/t\} = \vartheta$.

It results $\eta'_m \in K(n)$ and we can define $\vartheta_{\eta'_m}\{x_i/t\}$. Recall that $\mathcal{E}(n,\kappa,m,y,\psi,\vartheta)$ holds, so $\kappa'_m \in K(n)$, $\vartheta \in E(n,\kappa'_m)$. For each $\alpha = 1 \dots m$ $y_\alpha \notin var(\eta)$ and since $x_i = w_i \in var(\eta)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\kappa'_m)$.

Therefore $\vartheta_{\eta'_m}\{x_i/t\} = \vartheta$.

Finally we establish

$$\varphi_h\{x_i/t\} = \{\}(y_1 : (\psi_1)_\eta\{x_i/t\}, \ldots, y_m : (\psi_m)_{\eta'_{m-1}}\{x_i/t\}, \vartheta_{\eta'_m}\{x_i/t\}) = \\ = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta) = \varphi = \varphi_k\{x_i/t\} \ .$$

There is still another subcase to consider.

Suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Also suppose $\eta = \epsilon$ or $\eta \neq \epsilon$ and there doesn't exist a positive integer $q' \leqslant u$ such that $q' < n$, $\mathcal{K}(n; \eta; w_1 : \xi_1, \ldots, w_{q'} : \xi_{q'})$, $i \leqslant q'$.

Here clearly $\varphi_k\{x_i/t\} = \varphi = \varphi_h\{x_i/t\}$.

$\diamond$

Our definition process requires just a final step. As in the former step, we maintain the assumption that $\varphi \in E(n+1, k)$ is such that $V_b(t) \cap V_b(\varphi) = \emptyset$. In addition we assume that $h \in K(n+1)$ is such that $\varphi \in E(n+1, h)$, $x_i \notin var(h)$. We want to prove that $\varphi_k\{x_i/t\} = \varphi$.

Because of $\varphi \in E(n+1, k)$ and $V_b(t) \cap V_b(\varphi) = \emptyset$ one of the following five conditions holds:

a1. $\varphi \in \mathcal{C}$ and $\varphi_k\{x_i/t\} = \varphi$.

a2. $\varphi \in var(k)$, $\varphi = x_i \to \varphi_k\{x_i/t\} = t$, $\varphi \neq x_i \to \varphi_k\{x_i/t\} = \varphi$.

a3. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (\psi)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$ $\#(\kappa, \psi, \rho)$ is a function with m arguments, $(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$ is a member of the domain of $\#(\kappa, \psi, \rho)$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(\kappa)$, since we have

$$V_b(t) \cap V_b(\psi) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $\psi_\kappa\{x_i/t\}$, and similarly we can define $(\psi_j)_\kappa\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a4. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \ldots, \psi_m \in E(n, \kappa)$ such that $\varphi = (f)(\psi_1, \ldots, \psi_m)$, $\varphi \in E(n+1, \kappa)$, for each $\rho \in \Xi(\kappa)$ $A_f(\#(\kappa, \psi_1, \rho), \ldots, \#(\kappa, \psi_m, \rho))$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

If $i \leqslant q$, or in other words $x_i \in var(\kappa)$, since we have

$$V_b(t) \cap V_b(\psi_j) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset \ ,$$

we can define $(\psi_j)_\kappa\{x_i/t\}$, and it results

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

a5. there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\vartheta \in E(n)$, a function y whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(\kappa)$ and for each $\alpha, \beta = 1 \ldots m$ $\alpha \neq \beta \rightarrow y_\alpha \neq y_\beta$;
a function ψ whose domain is $\{1, \ldots, m\}$ such that for each $j = 1 \ldots m$ $\psi_j \in E(n)$;

such that

$\mathcal{E}(n, \kappa, m, y, \psi, \vartheta)$, $\varphi = \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta)$, $\varphi \in E(n+1, \kappa)$.

If $\kappa \neq \epsilon$ by 4.7 we can derive there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$.

Suppose $i \leqslant q$, or in other words $x_i \in var(\kappa)$.
We define $\kappa'_1 = \kappa + (y_1, \psi_1)$, and if $m > 1$ for each $j = 1 \ldots m - 1$ $\kappa'_{j+1} = \kappa'_j + (y_{j+1}, \psi_{j+1})$.
We have $\psi_1 \in E(n, \kappa)$, $V_b(t) \cap V_b(\psi_1) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$, therefore $(\psi_1)_\kappa\{x_i/t\}$ is defined;

for each $j = 1 \ldots m - 1$ $\kappa'_j \in K(n)$ and by 4.7
$\mathcal{K}(n; \kappa'_j; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_j : \psi_j)$,
for each $\alpha = 1 \ldots j$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots j$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\psi_{j+1} \in E(n, \kappa'_j)$, $V_b(t) \cap V_b(\psi_{j+1}) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $(\psi_{j+1})_{\kappa'_j}\{x_i/t\}$ is defined;

$\kappa'_m \in K(n)$ and by 4.7
$\mathcal{K}(n; \kappa'_m; x_1 : \varphi_1, \ldots, x_q : \varphi_q, y_1 : \psi_1, \ldots, y_m : \psi_m)$,
for each $\alpha = 1 \ldots m$ $y_\alpha \in V_b(\varphi)$ so $y_\alpha \notin V_b(t)$,
for each $\alpha = 1 \ldots m$ $V_b(t) \cap V_b(\psi_\alpha) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
$\vartheta \in E(n, \kappa'_m)$, $V_b(t) \cap V_b(\vartheta) \subseteq V_b(t) \cap V_b(\varphi) = \emptyset$,
therefore $\vartheta_{\kappa'_m}\{x_i/t\}$ is defined;

it results

$$\varphi_k\{x_i/t\} = \{\}(y_1 : (\psi_1)_\kappa\{x_i/t\}, \ldots, y_m : (\psi_m)_{\kappa'_{m-1}}\{x_i/t\}, \vartheta_{\kappa'_m}\{x_i/t\}) \ .$$

Otherwise (when $\kappa = \epsilon$ or $\kappa \neq \epsilon \wedge i > q$) $\varphi_k\{x_i/t\} = \varphi$.

Since $\varphi \in E(n+1, h)$ we can apply assumption 2.1.10 to establish that one of the following five conditions holds:

e1. $\varphi \in \mathcal{C}$.

e2. $\varphi \in var(h)$.

e3.

$$\begin{aligned}
&\exists \eta \in K(n) : \eta \sqsubseteq h, \exists r \text{ positive integer }, \chi, \chi_1, \ldots \chi_r \in E(n, \eta) : \\
&\quad \varphi = (\chi)(\chi_1, \ldots, \chi_r),\ \varphi \in E(n+1, \eta), \\
&\quad \forall \rho \in \Xi(\eta)\ (\ \#(\eta, \chi, \rho) \text{ is a function with } r \text{ arguments}, \\
&\qquad (\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_r, \rho)) \text{ is a member of the domain of } \#(\eta, \chi, \rho).
\end{aligned}$$

e4.

$$\begin{aligned}
&\exists \eta \in K(n) : \eta \sqsubseteq h, \exists g \in \mathcal{F},\ r \text{ positive integer }, \chi_1, \ldots \chi_r \in E(n, \eta) : \\
&\quad \varphi = (g)(\chi_1, \ldots, \chi_r),\ \varphi \in E(n+1, \eta), \\
&\quad \forall \rho \in \Xi(\eta)\ (\ A_g(\#(\eta, \chi_1, \rho), \ldots, \#(\eta, \chi_r, \rho)).
\end{aligned}$$

e5.

there exist

$\eta \in K(n) : \eta \sqsubseteq h$,

a positive integer r,

a function z whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$

$z_j \in \mathcal{V} - var(\eta)$, and for each $\alpha, \beta = 1 \ldots r$ $\alpha \neq \beta \rightarrow z_\alpha \neq z_\beta$,

a function χ whose domain is $\{1, \ldots, r\}$ such that for each $j = 1 \ldots r$

$\chi_j \in E(n)$,

$\phi \in E(n)$

such that

$\mathcal{E}(n, \eta, r, z, \chi, \phi)$,

$\varphi = \{\}(z_1 : \chi_1, \ldots, z_r : \chi_r, \phi),\ \varphi \in E(n+1, \eta)$.

If **a1**. occurs then clearly $\varphi_k\{x_i/t\} = \varphi$.

If **a2**. occurs then e2. also holds. Since $\varphi \in var(h)$ and $x_i \notin var(h)$ we have $\varphi \neq x_i$, so $\varphi_k\{x_i/t\} = \varphi$.

If **a3**. occurs then e3. also holds. We have

$$(\psi)(\psi_1, \ldots, \psi_m) = \varphi = (\chi)(\chi_1, \ldots, \chi_r)\ ,$$

and therefore $r = m$, $\chi = \psi$ and for each $j = 1 \ldots m$ $\chi_j = \psi_j$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, ψ, $\psi_1, \ldots, \psi_m \in E(n, \eta)$, $\varphi \in E(n+1, \eta)$, for each $\rho \in \Xi(\eta)$ $\#(\eta, \psi, \rho)$ is a function with m arguments, $(\#(\eta, \psi_1, \rho), \ldots, \#(\eta, \psi_m, \rho))$ is a member of the domain of $\#(\eta, \psi, \rho)$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Consider that $\kappa \in K(n)$ and we can define $\psi_\kappa\{x_i/t\}$, $(\psi_j)_\kappa\{x_i/t\}$. Also consider that $\eta \in K(n)$, ψ, $\psi_1, \ldots, \psi_m \in E(n, \eta)$, and since $\eta \sqsubseteq h$, $x_i \notin var(\eta)$. Therefore

$$\psi_\kappa\{x_i/t\} = \psi \ , \ (\psi_j)_\kappa\{x_i/t\} = \psi_j \ .$$

It follows that

$$\varphi_k\{x_i/t\} = (\psi_\kappa\{x_i/t\})((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = (\psi)(\psi_1, \ldots, \psi_m) = \varphi \ .$$

Now suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Here it's easier, as we immediately get $\varphi_k\{x_i/t\} = \varphi$.

If **a4**. occurs then e4. also holds. We have

$$(f)(\psi_1, \ldots, \psi_m) = \varphi = (g)(\chi_1, \ldots, \chi_r) \ ,$$

and therefore $g = f$, $r = m$, for each $j = 1 \ldots m$ $\chi_j = \psi_j$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, $\psi_1, \ldots, \psi_m \in E(n, \eta)$, $\varphi \in E(n+1, \eta)$, for each $\rho \in \Xi(\eta)$ $A_f(\#(\eta, \psi_1, \rho), \ldots, \#(\eta, \psi_m, \rho))$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_j)_\kappa\{x_i/t\}$. Also consider that $\eta \in K(n)$, $\psi_1, \ldots, \psi_m \in E(n, \eta)$, and since $\eta \sqsubseteq h$, $x_i \notin var(\eta)$. Therefore $(\psi_j)_\kappa\{x_i/t\} = \psi_j$.

It follows that

$$\varphi_k\{x_i/t\} = (f)((\psi_1)_\kappa\{x_i/t\}, \ldots, (\psi_m)_\kappa\{x_i/t\}) = (f)(\psi_1, \ldots, \psi_m) = \varphi \ .$$

Now suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Here it's easier, as we immediately get $\varphi_k\{x_i/t\} = \varphi$.

If **a5**. occurs then e5. also holds. We have $\varphi \in E_e(n+1, \kappa)$ and

$$\{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta) = \varphi = \{\}(z_1 : \chi_1, \ldots, z_r : \chi_r, \phi) \ .$$

Therefore $r = m$, $z = y$, $\chi = \psi$ and $\phi = \vartheta$.

It follows that $\eta \in K(n)$, $\eta \sqsubseteq h$, for each $j = 1 \ldots m$ $y_j \in \mathcal{V} - var(\eta)$; $\mathcal{E}(n, \eta, m, y, \psi, \vartheta)$, $\varphi \in E(n+1, \eta)$.

Suppose $\kappa \neq \epsilon$ and there exists a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Consider that $\kappa \in K(n)$ and we can define $(\psi_1)_\kappa\{x_i/t\}$. Also consider that $\eta \in K(n)$, $\psi_1 \in E(n, \eta)$, and since $\eta \sqsubseteq h$, $x_i \notin var(\eta)$. Therefore $(\psi_1)_\kappa\{x_i/t\} = \psi_1$.

Now suppose $m > 1$ and let $j = 1 \ldots m-1$. It results $\kappa'_j \in K(n)$ and we can define $(\psi_{j+1})_{\kappa'_j}\{x_i/t\}$. Moreover $\eta'_j \in K(n)$, $\psi_{j+1} \in E(n, \eta'_j)$. For each $\alpha = 1 \ldots j$ $y_\alpha \notin var(\kappa)$ and since $x_i \in var(\kappa)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\eta'_j)$.

Therefore $(\psi_{j+1})_{\kappa'_j}\{x_i/t\} = \psi_{j+1}$.

We still need to show that $\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta$.

It results $\kappa'_m \in K(n)$ and we can define $\vartheta_{\kappa'_m}\{x_i/t\}$. Recall that $\mathcal{E}(n, \eta, m, y, \psi, \vartheta)$ holds, so $\eta'_m \in K(n)$, $\vartheta \in E(n, \eta'_m)$. For each $\alpha = 1 \ldots m$ $y_\alpha \notin var(\kappa)$ and since $x_i \in var(\kappa)$ we have that $y_\alpha \neq x_i$. This implies that $x_i \notin var(\eta'_m)$.

Therefore $\vartheta_{\kappa'_m}\{x_i/t\} = \vartheta$.

We conclude

$$\begin{aligned} \varphi_k\{x_i/t\} &= \{\}(y_1 : (\psi_1)_\kappa\{x_i/t\}, \ldots, y_m : (\psi_m)_{\kappa'_{m-1}}\{x_i/t\}, \vartheta_{\kappa'_m}\{x_i/t\}) = \\ &= \{\}(y_1 : \psi_1, \ldots, y_m : \psi_m, \vartheta) = \varphi\ . \end{aligned}$$

Now suppose $\kappa = \epsilon$ or $\kappa \neq \epsilon$ and there doesn't exist a positive integer $q \leqslant p$ such that $q < n$, $\mathcal{K}(n; \kappa; x_1 : \varphi_1, \ldots, x_q : \varphi_q)$, $i \leqslant q$.

Here it's easier, as we immediately get $\varphi_k\{x_i/t\} = \varphi$.

The final step of our definition process has been completed.

5. Proofs and deductive methodology

In chapter 2 we have seen that our language is identified by a 4-tuple $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$. In chapter 3 we have given some definitions which are important with respect to the deductive methodology. For instance we have defined the set $S(k)$ of sentences with respect to a context k. A sentence with respect to ϵ will simply be called a 'sentence'.

At this point we need to define what is a proof in our language. To define this we need to define the notions of axiom and rule.

An axiom is a set A such that

- $A \subseteq S(\epsilon)$
- for each $\varphi \in A$ $\#(\varphi)$ holds.

The property 'for each $\varphi \in A$ $\#(\varphi)$ holds' states that axiom A is 'sound'.

Given a positive integer n we indicate with $S(\epsilon)^n$ the set of all n-tuples $(\varphi_1, \dots, \varphi_n)$ for $\varphi_1, \dots, \varphi_n \in S(\epsilon)$. An n-ary rule is a set $R \subseteq S(\epsilon)^{n+1}$ such that

- for each $(\varphi_1, \dots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \dots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds.

The property 'for each $(\varphi_1, \dots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \dots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds' states that rule R is 'sound'.

Both in the definition of axiom and rule we have included a requirement of soundness.

A deductive system is built on top of a language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$, and is identified by a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a set of axioms in $\mathcal{L}$ and $\mathcal{R}$ is a set of rules in $\mathcal{L}$.

Given a language $\mathcal{L}$, $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$, φ, $\psi_1, \dots, \psi_m$ sentences in $\mathcal{L}$, we say that $(\psi_1, \dots, \psi_m)$ is a proof of φ in $\mathcal{D}$ if and only if

- there exists $A \in \mathcal{A}$ such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \dots m$ one of the following holds
  - there exists $A \in \mathcal{A}$ such that $\psi_j \in A$,
  - there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \dots, i_n < j$ such that $(\psi_{i_1}, \dots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

Given $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$ and φ sentence in $\mathcal{L}$ we say that φ *is derivable in* $\mathcal{D}$ and write $\vdash_{\mathcal{D}} \varphi$ if and only if there exist $\psi_1, \dots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \dots, \psi_m)$ is a proof of φ in $\mathcal{D}$.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *sound* if and only if for each φ sentence in $\mathcal{L}$ if $\vdash_{\mathcal{D}} \varphi$ then $\#(\varphi)$ holds. In the next lemma we easily prove that each of our systems is sound.

Lemma 5.1. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is sound.*

Proof.

Let φ be a sentence in $\mathcal{L}$. Suppose $\vdash_{\mathcal{D}} \varphi$. There exist $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \ldots, \psi_m)$ is a proof of φ in $\mathcal{D}$. We can show that for each $j = 1 \ldots m$ $\#(\psi_j)$ holds.

There exists $A \in \mathcal{A}$ such that $\psi_1 \in A$, so $\#(\psi_1)$ holds.

If $m > 1$ suppose $j = 2 \ldots m$.

If there exists $A \in \mathcal{A}$ such that $\psi_j \in A$ then $\#(\psi_j)$ holds.

Otherwise there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \ldots, i_n < j$ such that

$$(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R\ .$$

Since $\#(\psi_{i_1}), \ldots, \#(\psi_{i_n})$ all hold then $\#(\psi_j)$ also holds. ∎

We assume that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ are in our set $\mathcal{F}$ (this is the same assumption we made in chapter 3). We also add to $\mathcal{F}$ the membership predicate $\in$ and the equality predicate $=$ (they have both been explained at the beginning of chapter 2).

We now need to list a set of axioms and rules that can be used in every language with the aforementioned symbols within the set $\mathcal{F}$. For every axiom/rule we first prove a result which ensures the soundness of the axiom/rule and then define properly the axiom/rule itself.

In our proofs we'll frequently use the following simple result.

Lemma 5.2. *Let S be a set and q, r be functions over S such that for each $\sigma \in S$ $q(\sigma)$ and $r(\sigma)$ are true (in these assumptions q, r can be called 'predicates over S'). Then*

$$P_\forall(\{q(\sigma) |\ \sigma \in S\}) \leftrightarrow \textit{ for each } \sigma \in S\ q(\sigma),$$

$$P_\exists(\{q(\sigma) |\ \sigma \in S\}) \leftrightarrow \textit{ there exists } \sigma \in S:\ q(\sigma),$$

$$P_\forall(\{q(\sigma) |\ \sigma \in S, r(\sigma)\}) \leftrightarrow \textit{ for each } \sigma \in S \textit{ if } r(\sigma) \textit{ then } q(\sigma),$$

$$P_\exists(\{q(\sigma) |\ \sigma \in S, r(\sigma)\}) \leftrightarrow \textit{ there exists } \sigma \in S : r(\sigma) \textit{ and } q(\sigma).$$

Proof.

Let $x_1 = \{q(\sigma) |\ \sigma \in S\}$.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ $q(\sigma)$.
Let $\sigma \in S$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ $q(\sigma)$ and try to prove $P_\forall(x_1)$. Let $x \in x_1$, there exists $\sigma \in S$ such that $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S$ $q(\sigma)$. There exists x in x_1 such that (x is true). There exists $\sigma \in S$ such that $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S$ $q(\sigma)$ and try to prove $P_\exists(x_1)$. Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved.

Now, to prove the other result, let $x_1 = \{q(\sigma)|\ \sigma \in S, r(\sigma)\}$.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$. Let $\sigma \in S$ and assume $r(\sigma)$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$ and try to prove $P_\forall(x_1)$. Let $x \in x_1$, there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$. There exists x in x_1 such that x is true. So there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$ and try to prove $P_\exists(x_1)$. Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved. ■

The first rule we introduce is based on lemma 3.8. In fact that lemma allows us to create a rule $R_{3.8}$ which is the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi_1)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi_2)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, (\wedge)(\psi_1, \psi_2))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi_1, \psi_2 \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

We continue the list of our rules with another simple one.

Lemma 5.3. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $(\leftrightarrow)(\varphi, \psi), (\rightarrow)(\varphi, \psi), (\rightarrow)(\psi, \varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\leftrightarrow)(\varphi, \psi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)] \in S(\epsilon)$,

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \varphi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\leftrightarrow)(\varphi, \psi)])$ *then* $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$ *and* $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \varphi)])$.

Proof.

We have

$$
\begin{aligned}
&P_\forall(\{\#(k, (\leftrightarrow)(\varphi, \psi), \sigma)|\ \sigma \in \Xi(k)\}),\\
&P_\forall(\{P_\leftrightarrow(\#(k, \varphi, \sigma), \#(k, \psi, \sigma))|\ \sigma \in \Xi(k)\}),\\
&P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \psi, \sigma))|\ \sigma \in \Xi(k)\}),\\
&P_\forall(\{\#(k, (\rightarrow)(\varphi, \psi), \sigma)|\ \sigma \in \Xi(k)\}),\\
&\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)]).
\end{aligned}
$$

In addition

$$
\begin{aligned}
&P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\}),\\
&P_\forall(\{\#(k, (\rightarrow)(\psi, \varphi), \sigma)|\ \sigma \in \Xi(k)\}),\\
&\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \varphi)]).
\end{aligned}
$$

■

This lemma allows us to create a unary rule $R_{5.3}$ which is the union of two sets of pairs.

Let G_1 be the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\leftrightarrow)(\varphi, \psi)], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Let G_2 be the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\leftrightarrow)(\varphi, \psi)], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Then $R_{5.3}$ is the union of G_1 and G_2.

LEMMA 5.4. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $(\wedge)(\varphi,\psi), (\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right), (\rightarrow)\left((\wedge)(\varphi,\psi),\psi\right) \in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right)] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\psi\right)] \in S(\epsilon)$.

Moreover $\#\left(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right)]\right)$ *and* $\#\left(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\psi\right)]\right)$ *are both true.*

Proof.

We can rewrite $\#\left(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right)]\right)$ as follows:

$$P_\forall(\{\#(k,(\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right),\sigma)|\ \sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\rightarrow\left(\#\left(k,(\wedge)(\varphi,\psi),\sigma\right),\#\left(k,\varphi,\sigma\right)\right)\mid\sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\rightarrow\left(P_\wedge\left(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)\right),\#\left(k,\varphi,\sigma\right)\right)\mid\sigma\in\Xi(k)\}).$$

This can be expressed as

$$\text{for each } \sigma\in\Xi(k) \text{ if } \#(k,\varphi,\sigma) \text{ and } \#(k,\psi,\sigma) \text{ then } \#(k,\varphi,\sigma),$$

which is clearly true.

In the same way we can prove the truth of

$$\#\left(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\psi\right)]\right).$$

■

Lemma 5.4 permits us to create an axiom $A_{5.4}$ which is the union of two sets of sentences.

Let G_1 be the set of all sentences $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\varphi\right)]$ such that

- m is a positive integer, $x_1,\ldots,x_m\in\mathcal{V}$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1,\ldots,\varphi_m\in E$, $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$,
- $\varphi,\psi\in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$.

Let G_2 be the set of all sentences $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)\left((\wedge)(\varphi,\psi),\psi\right)]$ such that

- m is a positive integer, $x_1,\ldots,x_m\in\mathcal{V}$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1,\ldots,\varphi_m\in E$, $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$,
- $\varphi,\psi\in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$.

Then $A_{5.4}$ is the union of G_1 and G_2.

LEMMA 5.5. *Let m be a positive integer. Let $x_1,\ldots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1,\ldots,\varphi_m\in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k=k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi,\chi\in S(k)$.*

Under these assumptions we have

- $(\rightarrow)(\varphi,\psi),(\rightarrow)(\psi,\chi),(\rightarrow)(\varphi,\chi)\in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\varphi,\psi)]\in S(\epsilon)$,

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \chi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \chi)])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)])$.

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$ as follows:

$$P_\forall(\{\#(k, (\rightarrow)(\varphi, \psi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

And we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \chi)])$ as follows:

$$P_\forall(\{\#(k, (\rightarrow)(\psi, \chi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \chi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

In other words for each $\sigma \in \Xi(k)$ if $\#(k, \varphi, \sigma)$ then $\#(k, \psi, \sigma)$, and if $\#(k, \psi, \sigma)$ then $\#(k, \chi, \sigma)$. So, for each $\sigma \in \Xi(k)$, if $\#(k, \varphi, \sigma)$ then $\#(k, \chi, \sigma)$. This can be written as follows:

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \chi, \sigma)) \mid \sigma \in \Xi(k)\})$$

$$P_\forall(\{\#(k, (\rightarrow)(\varphi, \chi), \sigma) |\ \sigma \in \Xi(k)\}),$$

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)]).$$

■

Lemma 5.5 allows us to create a rule $R_{5.5}$ which is the set of all 3-tuples

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\psi, \chi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)] \end{array}\right)$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.6. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$.

Let $t \in E(h)$ *such that* $\forall \rho \in \Xi(h)\ \#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$.
Let $t' \in E(h)$ *such that* $\forall \rho \in \Xi(h)\ \#(h, t', \rho) \in \#(h, \varphi_{m+1}, \rho)$.
Let $\varphi \in S(k)$ *such that* $V_b(t) \cap V_b(\varphi) = \emptyset$, $V_b(t') \cap V_b(\varphi) = \emptyset$.

Then we can define $\varphi_k\{x_{m+1}/t\}, \varphi_k\{x_{m+1}/t'\} \in S(h)$ *and therefore*

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (=)(t, t'))] \in S(\epsilon)$
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t'\})] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})])$
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (=)(t, t'))])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t'\})])$.

Proof.

We define $k_0 = \epsilon$ and for each $i = 1 \ldots m+1$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$.

We saw in remark 3.3 that for each $i = 1 \ldots m+1$

$$\varphi_i \in E_s(k_{i-1}),\ k_i = k_{i-1} + (x_i, \varphi_i),\ dom(k_i) = \{1, \ldots, i\}\ .$$

There exists a positive integer n such that $k \in K(n)$, $\varphi \in E(n, k)$ and for each $i = 1 \ldots m+1$ $\varphi_i \in E_s(n-1, k_{i-1})$. Clearly $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] = k_{m+1}$ also holds, so we have $\mathcal{K}(n; k; x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1})$.

Moreover $h = k_m$ so $t \in E(k_m)$ is such that $\forall \rho \in \Xi(k_m)\ \#(k_m, t, \rho) \in \#(k_m, \varphi_{m+1}, \rho)$.

We have $V_b(t) \subseteq \mathcal{V} - var(k_m) = \mathcal{V} - \{x_1, \ldots, x_m\}$, so for each $j = 1 \ldots m$ $x_j \notin V_b(t)$.

Therefore we can define $\varphi_k\{x_{m+1}/t\} \in E(k\{x_{m+1}/t\}) = E(h)$, and clearly the same holds for t', so we can define $\varphi_k\{x_{m+1}/t'\} \in E(k\{x_{m+1}/t'\}) = E(h)$.

By definition 4.16 we know that for each $\rho \in \Xi(h)$ there exists $\sigma \in \Xi(k)$ such that $\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma)$. Since $\#(k, \varphi, \sigma)$ is true or false then so is $\#(h, \varphi_k\{x_{m+1}/t\}, \rho)$. Therefore $\varphi_k\{x_{m+1}/t\} \in S(h)$.

Clearly the same holds for t', so $\varphi_k\{x_{m+1}/t'\} \in S(h)$.

We can derive that $(\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\}), (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t'\}) \in S(h)$.

Furtherly, $(=)(t, t') \in S(h)$ and so $(\rightarrow)(\chi, (=)(t, t')) \in S(h)$. Therefore

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (=)(t, t'))] \in S(\epsilon)$
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t'\})] \in S(\epsilon)$.

Suppose the following both hold

a. $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})])$
b. $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (=)(t, t'))])$

We can rewrite a. like this:

$$P_\forall(\{\#(h,(\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\}),\rho)|\ \rho\in\Xi(h)\})$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),\#(h,\varphi_k\{x_{m+1}/t\},\rho))\mid\rho\in\Xi(h)\})\ .$$

And we can rewrite b. like this:

$$P_\forall(\{\#(h,(\rightarrow)(\chi,(=)(t,t')),\rho)|\ \rho\in\Xi(h)\})$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),\#(h,(=)(t,t'),\rho))\mid\rho\in\Xi(h)\})$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),P_=(\#(h,t,\rho),\#(h,t',\rho)))\mid\rho\in\Xi(h)\})\ .$$

We have to show $\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,(\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})])$, which can be rewritten:

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),\#(h,\varphi_k\{x_{m+1}/t'\},\rho))\mid\rho\in\Xi(h)\})\ .$$

In other words we need to show that for each $\rho\in\Xi(h)$

$$\text{if } \#(h,\chi,\rho) \text{ then } \#(h,\varphi_k\{x_{m+1}/t'\},\rho).$$

Let $\rho=(u,r)\in\Xi(h)$ and assume $\#(h,\chi,\rho)$.

We have $dom(\rho)=dom(h)=dom(k_m)$, so $\rho_{/dom(k_m)}=\rho$. Let's define $\sigma_0=\epsilon$,

- for each $j=1\dots m$ $\sigma_j=\sigma_{j-1}+(u_j,r_j)$,
- $\sigma_{m+1}=\sigma_m+(x_{m+1},\#(k_m,t,\rho))$.

Because of a. $\#(h,\varphi_k\{x_{m+1}/t\},\rho)$ holds, so $\#(k,\varphi,\sigma_{m+1})$ holds too.

Because of b. $\#(k_m,t',\rho)=\#(h,t',\rho)=\#(h,t,\rho)=\#(k_m,t,\rho)$.

Therefore $\#(h,\varphi_k\{x_{m+1}/t'\},\rho)$ holds too. ■

Lemma 5.6 allows us to create a rule $R_{5.6}$ which is the set of all 3-tuples

$$\begin{pmatrix}\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,(\rightarrow)(\chi,\varphi_k\{x_{m+1}/t\})],\\ \gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,(\rightarrow)(\chi,(=)(t,t'))],\\ \gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,(\rightarrow)(\chi,\varphi_k\{x_{m+1}/t'\})]\end{pmatrix}$$

such that

- m is a positive integer, $x_1,\dots,x_{m+1}\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$, $\varphi_1,\dots,\varphi_{m+1}\in E$, $H[x_1:\varphi_1,\dots,x_{m+1}:\varphi_{m+1}]$;
- if we define $k=k[x_1:\varphi_1,\dots,x_{m+1}:\varphi_{m+1}]$ and $h=k[x_1:\varphi_1,\dots,x_m:\varphi_m]$ then
  - $\chi\in S(h)$,
  - $t\in E(h)$, $\forall\rho\in\Xi(h)$ $\#(h,t,\rho)\in\#(h,\varphi_{m+1},\rho)$,
  - $t'\in E(h)$, $\forall\rho\in\Xi(h)$ $\#(h,t',\rho)\in\#(h,\varphi_{m+1},\rho)$,
  - $\varphi\in S(k)$, $V_b(t)\cap V_b(\varphi)=\emptyset$, $V_b(t')\cap V_b(\varphi)=\emptyset$.

LEMMA 5.7. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $(\to)(\psi, \varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\psi, \varphi)] \in S(\epsilon)$.

Moreover if $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi])$ then $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\psi, \varphi)])$ also holds.

Proof.

Suppose $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi])$ holds. It can be rewritten as

$$P_\forall(\{\#(k, \varphi, \sigma) |\ \sigma \in \Xi(k)\}) \ .$$

We can rewrite $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\psi, \varphi)])$ as

$$P_\forall(\{\#(k, (\to)(\psi, \varphi), \sigma) |\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\to(\#(k, \psi, \sigma), \#(k, \varphi, \sigma)) |\ \sigma \in \Xi(k)\}) \ .$$

For each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ holds, this implies that

$$P_\forall(\{P_\to(\#(k, \psi, \sigma), \#(k, \varphi, \sigma)) |\ \sigma \in \Xi(k)\})$$

holds too and this completes the proof. ∎

Lemma 5.7 allows us to create a rule $R_{5.7}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\psi, \varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.8. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi, \chi \in E(k)$, $\vartheta \in S(k)$.*

Under these assumptions we have

- $(\to)(\vartheta, (=)(\varphi, \psi)), (\to)(\vartheta, (=)(\psi, \chi)), (\to)(\vartheta, (=)(\varphi, \chi)) \in S(k)$;
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \psi))] \in S(\epsilon)$;
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\psi, \chi))] \in S(\epsilon)$;
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \chi))] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \psi))])$,

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\psi, \chi))])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \chi))])$.

Proof.

We rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \psi))])$ as

$$P_\forall(\{\#(k, (\to)(\vartheta, (=)(\varphi, \psi)), \sigma) |\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k, \vartheta, \sigma), \#(k, (=)(\varphi, \psi), \sigma)) \mid \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k, \vartheta, \sigma), P_=(\#(k, \varphi, \sigma), \#(k, \psi, \sigma))) \mid \sigma \in \Xi(k)\})\ .$$

Similarly we rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\psi, \chi))])$ as

$$P_\forall(\{P_\to(\#(k, \vartheta, \sigma), P_=(\#(k, \psi, \sigma), \#(k, \chi, \sigma))) \mid \sigma \in \Xi(k)\})\ ,$$

and we rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \chi))])$ as

$$P_\forall(\{P_\to(\#(k, \vartheta, \sigma), P_=(\#(k, \varphi, \sigma), \#(k, \chi, \sigma))) \mid \sigma \in \Xi(k)\})\ .$$

If

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \psi))])$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\psi, \chi))])$

both hold, then for each $\sigma \in \Xi(k)$ if $\#(k, \vartheta, \sigma)$ then $\#(k, \varphi, \sigma)$ is equal to $\#(k, \psi, \sigma)$, which is equal to $\#(k, \chi, \sigma)$.

This implies that $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \chi))])$ holds. ■

Lemma 5.8 allows us to create a rule $R_{5.8}$ which is the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \psi))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\psi, \chi))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\vartheta, (=)(\varphi, \chi))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in E(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, $\vartheta \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 5.9. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$.

Let $t \in E(h)$ such that $\forall \rho \in \Xi(h)\ \#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$. Let $\varphi \in S(k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$.

Then we can define $\varphi_k\{x_{m+1}/t\} \in S(h)$ and furthermore

- $(\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$,

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})])$ *then*

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))]) \ .$$

Proof.

As we have seen in lemma 5.6, in these assumptions we can prove $\varphi_k\{x_{m+1}/t\} \in S(h)$.

Since $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$, $k = h + (x_{m+1}, \varphi_{m+1})$ we can apply lemma 3.1 and obtain that $(\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$.

Therefore

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))] \in S(\epsilon)$.

Suppose $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})])$ holds, it can be rewritten as

$$P_\forall(\{\#(h, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\}), \rho)|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \varphi_k\{x_{m+1}/t\}, \rho))|\ \rho \in \Xi(h)\}) \ .$$

We need to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))])$, and this can be rewritten

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)), \rho))|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\exists\,(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}))|\ \rho \in \Xi(h)\}) \ .$$

Let $\rho \in \Xi(h)$ and suppose $\#(h, \chi, \rho)$. We need to show there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi, \sigma)$.

We have $\#(h, \varphi_k\{x_{m+1}/t\}, \rho)$, $\rho \in \Xi(h) = \Xi(k\{x_{m+1}/t\})$, $dom(\rho) = \{1, \ldots, m\}$. Let $\rho = (u, r)$ and define $\sigma_0 = \epsilon$, for each $j = 1 \ldots m$ $\sigma_j = \sigma_{j-1} + (u_j, r_j)$, $\sigma_{m+1} = \sigma_m + (x_{m+1}, \#(h, t, \rho))$.

By definition 4.16 it results $\sigma_{m+1} \in \Xi(k)$ and $\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma_{m+1})$, so $\#(k, \varphi, \sigma_{m+1})$ holds true. Clearly $\rho = \sigma_m \sqsubseteq \sigma_{m+1}$, so our proof is finished. ∎

Lemma 5.9 allows us to create a rule $R_{5.9}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\exists)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then
  - $\chi \in S(h)$,
  - $t \in E(h)$, $\forall \rho \in \Xi(h)\ \#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$,
  - $\varphi \in S(k)$, $V_b(t) \cap V_b(\varphi) = \emptyset$.

LEMMA 5.10. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.*

Under these assumptions we have

- $(\to)((\wedge)(\varphi, \psi), \chi), (\to)(\varphi, (\to)(\psi, \chi)) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)((\wedge)(\varphi, \psi), \chi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\to)(\psi, \chi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)((\wedge)(\varphi, \psi), \chi)])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\to)(\psi, \chi))])$.

Proof.

We assume $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)((\wedge)(\varphi, \psi), \chi)])$ which can be rewritten

$$P_\forall(\{\#(k, (\to)((\wedge)(\varphi, \psi), \chi), \sigma) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\to(\#(k, (\wedge)(\varphi, \psi), \sigma), \#(k, \chi, \sigma)) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\to(P_\wedge(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)), \#(k, \chi, \sigma)) |\ \sigma \in \Xi(k)\})\ .$$

Of course we now try to show $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\to)(\psi, \chi))])$ which in turn can be rewritten

$$P_\forall(\{\#(k, (\to)(\varphi, (\to)(\psi, \chi)), \sigma) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\to(\#(k, \varphi, \sigma), \#(k, (\to)(\psi, \chi), \sigma)) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\to(\#(k, \varphi, \sigma), P_\to(\#(k, \psi, \sigma), \#(k, \chi, \sigma))) |\ \sigma \in \Xi(k)\})\ .$$

Let $\sigma \in \Xi(k)$, suppose $\#(k, \varphi, \sigma)$ and $\#(k, \psi, \sigma)$, then we have $\#(k, \chi, \sigma)$ and this completes the proof. ∎

Lemma 5.10 allows us to create a rule $R_{5.10}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)((\wedge)(\varphi, \psi), \chi)], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\varphi, (\to)(\psi, \chi))])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.11. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\psi \in S(h) \cap S(k)$ and $\varphi \in S(k)$.

Then the following hold

- $(\to)(\psi, \varphi) \in S(k)$,
- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,

- $(\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi]) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi])] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]])$ *then*

$$\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi])]) \ .$$

Proof.

The two facts

- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,
- $(\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi]) \in S(h)$

clearly follow from definition 3.4.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]])$ as

$$P_\forall(\{\#(h, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)], \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{\#(h, (\forall)\left(\{\}\left(x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)\right)\right), \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\forall\left(\{\#(k, (\to)(\psi, \varphi), \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}\right) |\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\forall\left(\{P_\to(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}\right) |\ \rho \in \Xi(h)\}) \ .$$

In turn $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi])])$ can be rewritten as

$$P_\forall(\{\#(h, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi]), \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \psi, \rho), \#(h, \gamma[x_{m+1} : \varphi_{m+1}, \varphi], \rho)\right) |\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \psi, \rho), \#(h, (\forall)\left(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)\right), \rho)\right) |\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \psi, \rho), P_\forall\left(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}\right)\right) |\ \rho \in \Xi(h)\}) \ .$$

We suppose $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]])$ holds and try to show that $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi])])$ holds too.

In this view let $\rho \in \Xi(h)$ and suppose $\#(h, \psi, \rho)$, let $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$. We want to show that $\#(k, \varphi, \sigma)$ holds.

We want to apply lemma 4.12.

Remark 3.3 tells us that $k = h + (x_{m+1}, \varphi_{m+1})$, $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$.

Moreover, since $\psi \in E(k)$, $V_b(\psi) \subseteq \mathcal{V} - var(k)$ and $x_{m+1} \notin V_b(\psi)$.

Clearly there exist $\delta \in \Xi(h)$, $s \in \#(h, \varphi_{m+1}, \delta)$ such that $\sigma = \delta + (x_{m+1}, s)$.

By lemma 4.12 we obtain that $\#(k, \psi, \sigma) = \#(h, \psi, \delta)$.

We have $\rho, \delta \in \mathcal{R}(\sigma)$, so $\delta = \sigma_{/dom(\delta)}$ and $\rho = \sigma_{/dom(\rho)}$.
Since $dom(\delta) = dom(h) = dom(\rho)$ it follows that $\rho = \delta$ and $\#(k, \psi, \sigma) = \#(h, \psi, \rho)$.

Therefore $\#(k, \psi, \sigma)$ holds true.

Using our rewriting of $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]])$ we obtain that $\#(k, \varphi, \sigma)$ holds true, and this completes the proof. ∎

Lemma 5.11 allows us to create a rule $R_{5.11}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)(\psi, \gamma[x_{m+1} : \varphi_{m+1}, \varphi])] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\psi \in S(h) \cap S(k)$ and $\varphi \in S(k)$.

Lemma 5.12. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\psi \in S(k)$ and $\varphi \in S(h) \cap S(k)$.

Under these assumptions we have

- $(\to)(\psi, \varphi) \in S(k)$,
- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,
- $(\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right) \in S(h)$,
- $(\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]])$ then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)]) \ .$$

Proof.

Using remark 3.3 (and the notation in it) we can easily determine that $h = k_m \in K$, $k = k_{m+1} = k_m + (x_{m+1}, \varphi_{m+1}) = h + (x_{m+1}, \varphi_{m+1})$, $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$.

Clearly $(\to)(\psi, \varphi) \in S(k)$ also holds so

- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,
- $(\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right) \in S(h)$,
- $(\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right) \in S(h)$

and this implies

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)] \in S(\epsilon)$.

Suppose $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\rightarrow)(\psi, \varphi)]])$. It can be rewritten as

$$P_\forall(\{\#(h, \gamma[x_{m+1} : \varphi_{m+1}, (\rightarrow)(\psi, \varphi)], \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{\#(h, (\forall)\,(\{\}\,(x_{m+1} : \varphi_{m+1}, (\rightarrow)(\psi, \varphi)))\,, \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\forall\,(\{\#(k, (\rightarrow)(\psi, \varphi), \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\})\mid \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\forall\,(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\})\mid \rho \in \Xi(h)\})\ .$$

In turn $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)\,((\exists)\,(\{\}(x_{m+1} : \varphi_{m+1}, \psi))\,, \varphi)])$ can be rewritten as

$$P_\forall(\{\#(h, (\rightarrow)\,((\exists)\,(\{\}(x_{m+1} : \varphi_{m+1}, \psi))\,, \varphi)\,, \rho)|\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\rightarrow\,(\#(h, (\exists)\,(\{\}(x_{m+1} : \varphi_{m+1}, \psi))\,, \rho), \#(h, \varphi, \rho))\mid \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\rightarrow\,(P_\exists\,(\{\#(k, \psi, \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\})\,, \#(h, \varphi, \rho))\mid \rho \in \Xi(h)\})\ .$$

To prove the last statement we suppose $\rho \in \Xi(h)$ and suppose there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \psi, \rho)$. We need to prove $\#(h, \varphi, \rho)$.

By our assumption we know that $\#(k, \varphi, \sigma)$ holds.

We want to apply lemma 4.12.

Remark 3.3 tells us that $k = h + (x_{m+1}, \varphi_{m+1})$, $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$.

Moreover, since $\varphi \in E(k)$, $V_b(\varphi) \subseteq \mathcal{V} - var(k)$ and $x_{m+1} \notin V_b(\varphi)$.

Clearly there exist $\delta \in \Xi(h)$, $s \in \#(h, \varphi_{m+1}, \delta)$ such that $\sigma = \delta + (x_{m+1}, s)$.

By lemma 4.12 we obtain that $\#(k, \varphi, \sigma) = \#(h, \varphi, \delta)$.

We have $\rho, \delta \in \mathcal{R}(\sigma)$, so $\delta = \sigma_{/dom(\delta)}$ and $\rho = \sigma_{/dom(\rho)}$.
Since $dom(\delta) = dom(h) = dom(\rho)$ it follows that $\rho = \delta$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$.

So $\#(h, \varphi, \rho)$ holds true and our proof is finished. ■

Lemma 5.12 allows us to create a rule $R_{5.12}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \gamma[x_{m+1} : \varphi_{m+1}, (\rightarrow)(\psi, \varphi)]], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)\,((\exists)\,(\{\}(x_{m+1} : \varphi_{m+1}, \psi))\,, \varphi)] \end{array}\right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\psi \in S(k)$ and $\varphi \in S(h) \cap S(k)$.

The next rule is a variation of the former one.

Lemma 5.13. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $\psi \in S(k)$ and $\varphi \in S(h) \cap S(k)$.

Under these assumptions we have

- $(\to)(\psi, \varphi) \in S(k)$,
- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,
- $(\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right) \in S(h)$,
- $(\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right) \in S(h)$,
- $(\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right) \in S(h)$,
- $(\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right)])$ then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right)]) .$$

Proof.

Just as in the proof of 5.12 we can derive

- $(\to)(\psi, \varphi) \in S(k)$
- $\gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)] \in S(h)$,
- $(\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right) \in S(h)$,
- $(\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right) \in S(h)$.

It clearly follows that

- $(\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right) \in S(h)$,
- $(\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right)] \in S(\epsilon)$.

Suppose $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right)])$. It can be rewritten as

$$P_\forall(\{\#(h, (\to)\left(\chi, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)]\right), \rho) |\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \chi, \rho), \#(h, \gamma[x_{m+1} : \varphi_{m+1}, (\to)(\psi, \varphi)], \rho)\right) \mid \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \chi, \rho), P_\forall\left(\{\#(k, (\to)(\psi, \varphi), \sigma) |\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}\right)\right) \mid \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \chi, \rho), P_\forall\left(\{P_\to(\#(k, \psi, \sigma), \#(k, \varphi, \sigma)) |\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}\right)\right) \mid \rho \in \Xi(h)\}).$$

In turn $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right)])$ can be rewritten as

$$P_\forall(\{\#(h, (\to)\left(\chi, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right)\right), \rho) |\ \rho \in \Xi(h)\})$$
$$P_\forall(\{P_\to\left(\#(h, \chi, \rho), \#(h, (\to)\left((\exists)\left(\{\}(x_{m+1} : \varphi_{m+1}, \psi)\right), \varphi\right), \rho)\right) \mid \rho \in \Xi(h)\})$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho), P_\rightarrow(\#(h,(\exists)(\{\}(x_{m+1}:\varphi_{m+1},\psi)),\rho),\#(h,\varphi,\rho)))\mid \rho\in\Xi(h)\})$$
$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho), P_\rightarrow(P_\exists(\{\#(k,\psi,\sigma)\mid \sigma\in\Xi(k),\rho\sqsubseteq\sigma\}),\#(h,\varphi,\rho)))\mid \rho\in\Xi(h)\}).$$

To prove the last statement we suppose $\rho \in \Xi(h)$ and $\#(h,\chi,\rho)$. Moreover we suppose there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k,\psi,\rho)$. We need to prove $\#(h,\varphi,\rho)$.

By our assumption we know that $\#(k,\varphi,\sigma)$ holds.

We want to apply lemma 4.12.

Remark 3.3 tells us that $k = h + (x_{m+1},\varphi_{m+1})$, $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$.

Moreover, since $\varphi \in E(k)$, $V_b(\varphi) \subseteq \mathcal{V} - var(k)$ and $x_{m+1} \notin V_b(\varphi)$.

Clearly there exist $\delta \in \Xi(h)$, $s \in \#(h,\varphi_{m+1},\delta)$ such that $\sigma = \delta + (x_{m+1}, s)$.

By lemma 4.12 we obtain that $\#(k,\varphi,\sigma) = \#(h,\varphi,\delta)$.

We have $\rho,\delta \in \mathcal{R}(\sigma)$, so $\delta = \sigma_{/dom(\delta)}$ and $\rho = \sigma_{/dom(\rho)}$.
Since $dom(\delta) = dom(h) = dom(\rho)$ it follows that $\rho = \delta$ and $\#(k,\varphi,\sigma) = \#(h,\varphi,\rho)$.

So $\#(h,\varphi,\rho)$ holds true and our proof is finished. ■

Lemma 5.13 allows us to create a rule $R_{5.13}$ which is the set of all pairs

$$\begin{pmatrix} \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\chi,\gamma[x_{m+1}:\varphi_{m+1},(\rightarrow)(\psi,\varphi)])], \\ \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\chi,(\rightarrow)((\exists)(\{\}(x_{m+1}:\varphi_{m+1},\psi)),\varphi))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1,\ldots,x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_{m+1} \in E$, $H[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$;
- if we define $k = k[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ then $\chi \in S(h)$, $\psi \in S(k)$ and $\varphi \in S(h) \cap S(k)$.

The next rule recalls the rule of standard logic which is called 'modus ponens' and can be itself called 'modus ponens' .

Lemma 5.14. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi,\chi \in S(k)$.*

Under these assumptions we have

- $(\rightarrow)(\varphi,\psi), (\rightarrow)(\varphi,(\rightarrow)(\psi,\chi)), (\rightarrow)(\varphi,\chi) \in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\varphi,\psi)] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\varphi,(\rightarrow)(\psi,\chi))] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\varphi,\chi)] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\varphi,\psi)])$,

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi))])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)])$.

Proof.

We assume that

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi))])$

both hold.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)])$ as

$$P_\forall(\{\#(k, (\rightarrow)(\varphi, \psi), \sigma) | \ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)) | \ \sigma \in \Xi(k)\}) \ .$$

In turn $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi))])$ can be rewritten

$$P_\forall(\{\#(k, (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi)), \sigma) | \ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, (\rightarrow)(\psi, \chi), \sigma)) | \ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\rightarrow(\#(k, \psi, \sigma), \#(k, \chi, \sigma))) | \ \sigma \in \Xi(k)\}) \ .$$

We have to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)])$ which can be rewritten

$$P_\forall(\{\#(k, (\rightarrow)(\varphi, \chi), \sigma) | \ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \chi, \sigma)) | \ \sigma \in \Xi(k)\}) \ .$$

Let $\sigma \in \Xi(k)$ and let $\#(k, \varphi, \sigma)$. We need to prove $\#(k, \chi, \sigma)$.

We have $\#(k, \psi, \sigma)$ and so $\#(k, \chi, \sigma)$ holds too. ∎

Lemma 5.14 allows us to create a rule $R_{5.14}$ which is the set of all 3-tuples

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \psi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\varphi, \chi)] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 5.15. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds and we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$.

Let $t \in E(h)$ such that $\forall \rho \in \Xi(h)$ $\#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$.
Let $\varphi \in S(k)$ such that $V_b(t) \cap V_b(\varphi) = \emptyset$.

Then we can define $\varphi_k\{x_{m+1}/t\} \in S(h)$ *and furthermore*

- $(\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))])$ *then*

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})]) \ .$$

Proof.

As we have seen in lemma 5.6, in these assumptions we can prove $\varphi_k\{x_{m+1}/t\} \in S(h)$.

Since $\varphi_{m+1} \in E_s(h)$, $x_{m+1} \in \mathcal{V} - var(h)$, $k = h + (x_{m+1}, \varphi_{m+1})$ we can apply lemma 3.1 and obtain that $(\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$.

Therefore

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \in S(\epsilon)$.

Suppose $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))])$ holds, it can be rewritten

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)), \rho))|\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\forall\,(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k), \rho \sqsubseteq \sigma\}))|\ \rho \in \Xi(h)\}) \ .$$

We need to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})])$ it can be rewritten as

$$P_\forall(\{\#(h, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\}), \rho)|\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \varphi_k\{x_{m+1}/t\}, \rho))|\ \rho \in \Xi(h)\}) \ .$$

Let $\rho \in \Xi(h)$ and suppose $\#(h, \chi, \rho)$. We need to show $\#(h, \varphi_k\{x_{m+1}/t\}, \rho)$.

We have $\rho \in \Xi(h) = \Xi(k\{x_{m+1}/t\})$, $dom(\rho) = \{1, \ldots, m\}$. Let $\rho = (u, r)$ and define $\sigma_0 = \epsilon$, for each $j = 1 \ldots m$ $\sigma_j = \sigma_{j-1} + (u_j, r_j)$, $\sigma_{m+1} = \sigma_m + (x_{m+1}, \#(h, t, \rho))$.

By definition 4.16 it results $\sigma_{m+1} \in \Xi(k)$ and $\#(h, \varphi_k\{x_{m+1}/t\}, \rho) = \#(k, \varphi, \sigma_{m+1})$.

Clearly $\rho = \sigma_m \sqsubseteq \sigma_{m+1}$, so $\#(k, \varphi, \sigma_{m+1})$ holds and $\#(h, \varphi_k\{x_{m+1}/t\}, \rho)$ holds too. ■

Lemma 5.15 allows us to create a rule $R_{5.15}$ which is the set of all pairs

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, \varphi_k\{x_{m+1}/t\})] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then
  - $\chi \in S(h)$,

– $t \in E(h)$, $\forall \rho \in \Xi(h)$ $\#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$,
– $\varphi \in S(k)$, $V_b(t) \cap V_b(\varphi) = \emptyset$.

LEMMA 5.16. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$.*

Let $i = 1 \ldots m$ such that for each $j = i \ldots m$ $x_j \notin V_b(\varphi_i)$. Then

- $(\in)(x_i, \varphi_i) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)] \in S(\epsilon)$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)])$.

Proof.

We have $k_{i-1} \in K$, $x_i \in \mathcal{V} - var(k_{i-1})$, $\varphi_i \in E_s(k_{i-1})$, $k_i = k_{i-1} + (x_i, \varphi_i)$. By lemma 4.13 $x_i \in E(k_i)$ and for each $\rho_i = \rho_{i-1} + (x_i, s) \in \Xi(k_i)$

$$\#(k_i, x_i, \rho_i) = s \in \#(k_{i-1}, \varphi_i, \rho_{i-1}).$$

If $i = m$ then we have proved $x_i \in E(k)$.

If $i < m$, since for each $j = i + 1 \ldots m$ $x_j \notin V_b(x_i)$, we can apply lemma 3.14 and derive that $x_i \in E(k)$ and for each $\rho \in \Xi(k)$ there exists $\rho_i \in \Xi(k_i)$ such that $\rho_i \sqsubseteq \rho$ and $\#(k_i, x_i, \rho_i) = \#(k, x_i, \rho)$.

It also results $\varphi_i \in E_s(k_{i-1})$ and for each $j = i \ldots m$ $x_j \notin V_b(\varphi_i)$. Therefore, by lemma 3.14, $\varphi_i \in E(k)$ and for each $\rho \in \Xi(k)$ there exists $\rho_{i-1} \in \Xi(k_{i-1})$ such that $\rho_{i-1} \sqsubseteq \rho$ and $\#(k, \varphi_i, \rho) = \#(k_{i-1}, \varphi_i, \rho_{i-1})$ is a set.

By lemma 3.13 we derive that $(\in)(x_i, \varphi_i) \in S(k)$, and consequently

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)] \in S(\epsilon) \ .$$

Moreover we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)])$ as follows

$$P_\forall(\{\#(k, (\in)(x_i, \varphi_i), \rho) | \ \rho \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\in(\#(k, x_i, \rho), \#(k, \varphi_i, \rho)) | \ \rho \in \Xi(k)\}) \ .$$

To show this we have to prove that for each $\rho \in \Xi(k)$ $\#(k, x_i, \rho)$ belongs to $\#(k, \varphi_i, \rho)$.

We know there exists $\rho_i \in \Xi(k_i)$ such that $\rho_i \sqsubseteq \rho$ and $\#(k_i, x_i, \rho_i) = \#(k, x_i, \rho)$.

We also know there exist $\rho_{i-1} \in \Xi(k_{i-1})$, $s \in \#(k_{i-1}, \varphi_i, \delta_{i-1})$ such that $\rho_i = \rho_{i-1} + (x_i, s)$ and $\#(k_i, x_i, \rho_i) = s \in \#(k_{i-1}, \varphi_i, \rho_{i-1})$.

Furthermore there exists $\delta_{i-1} \in \Xi(k_{i-1})$ such that $\delta_{i-1} \sqsubseteq \rho$ and $\#(k, \varphi_i, \rho) = \#(k_{i-1}, \varphi_i, \delta_{i-1})$.

We have $\delta_{i-1} = \rho_{/dom(k_{i-1})} = \rho_{i-1}$, so

$$\#(k, x_i, \rho) = \#(k_i, x_i, \rho_i) \in \#(k_{i-1}, \varphi_i, \rho_{i-1}) = \#(k, \varphi_i, \rho) \ .$$

■

Lemma 5.16 permits us to create an axiom $A_{5.16}$ which is the set of all sentences $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\in)(x_i, \varphi_i)]$ such that

- m is a positive integer, $x_1, \dots, x_m \in \mathcal{V}$, $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\varphi_1, \dots, \varphi_m \in E$, $H[x_1 : \varphi_1, \dots, x_m : \varphi_m]$,
- $i = 1 \dots m$,
- for each $j = i \dots m$ $x_j \notin V_b(\varphi_i)$.

Lemma 5.17. *Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \dots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \dots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $t \in E(h)$.

Let $\varphi \in E_s(h)$ and $x_{m+1} \notin V_b(\varphi)$.

Under these assumptions

- $(\in)(x_{m+1}, \varphi) \in S(k)$,
- $(\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))] \in S(\epsilon)$,
- $(\in)(t, \varphi_{m+1}) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))] \in S(\epsilon)$,
- $(\in)(t, \varphi) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi))] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))])$ *and*
- $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))])$

then $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi))])$.

Proof.

We have $k = h + (x_{m+1}, \varphi_{m+1})$, where $h \in K$, $x_{m+1} \in \mathcal{V} - var(h)$, $\varphi_{m+1} \in E_s(h)$. Using lemma 4.13 we can derive that $x_{m+1} \in E(k)$.

Since $\varphi \in E_s(h)$ and $x_{m+1} \notin V_b(\varphi)$ we can apply lemma 4.12 and obtain that $\varphi \in E(k)$ and for each $\sigma = \rho + (x_{m+1}, s) \in \Xi(k)$ $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$ is a set.

Therefore, by lemma 3.13 we get $(\in)(x_{m+1}, \varphi) \in S(k)$.

By lemma 3.1 we obtain $(\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))) \in S(h)$.

Clearly this implies that

$$\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))] \in S(\epsilon).$$

Furthermore we have $t \in E(h)$, $\varphi_{m+1} \in E_s(h)$, so $(\in)(t, \varphi_{m+1}) \in S(h)$. It clearly follows that $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))] \in S(\epsilon)$.

We have also $\varphi \in E_s(h)$, so $(\in)(t,\varphi) \in S(h)$. It follows that

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))] \in S(\epsilon).$$

We now assume

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))])$ and
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))])$

both hold and we try to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t,\varphi))])$.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))])$$

as

$$P_\forall(\{\#\,(h, (\rightarrow)\,(\chi, (\forall)\,(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi)))), \rho) \mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow\,(\#\,(h, \chi, \rho)\,, \#\,(h, (\forall)\,(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi)))\,, \rho)) \mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow\,(\#\,(h, \chi, \rho)\,, P_\forall\,(\{\#\,(k, (\in)(x_{m+1}, \varphi), \sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow\,(\#\,(h, \chi, \rho)\,, P_\forall\,(\{P_\in\,(\#(k, x_{m+1}, \sigma), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\})\ .$$

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))])$$

as

$$P_\forall(\{\#(h, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1})), \rho)\mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, (\in)(t, \varphi_{m+1}), \rho))\mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\in(\#(h, t, \rho), \#(h, \varphi_{m+1}, \rho)))\mid \rho \in \Xi(h)\})\ .$$

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi))])$$

as

$$P_\forall(\{\#(h, (\rightarrow)(\chi, (\in)(t, \varphi)), \rho)\mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, (\in)(t, \varphi), \rho))\mid \rho \in \Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\in(\#(h, t, \rho), \#(h, \varphi, \rho)))\mid \rho \in \Xi(h)\})\ .$$

Let $\rho \in \Xi(h)$ and assume $\#(h, \chi, \rho)$. We need to show that $\#(h, t, \rho)$ belongs to $\#(h, \varphi, \rho)$.

Let $\sigma = \rho + (x_{m+1}, \#(h, t, \rho))$.

Since $k = h + (x_{m+1}, \varphi_{m+1})$ and $\#(h, t, \rho)$ belongs to $\#(h, \varphi_{m+1}, \rho)$ we have $\sigma \in \Xi(k)$.

By lemma 4.13 $\#(k, x_{m+1}, \sigma) = \#(h, t, \rho)$, so $\#(h, t, \rho) \in \#(k, \varphi, \sigma)$.

Since $\varphi \in E(h)$ and $x_{m+1} \notin V_b(\varphi)$ we can apply lemma 4.12 and obtain

$$\#(k, \varphi, \sigma) = \#(h, \varphi, \rho).$$

■

By virtue of lemma 5.17 we can create a rule $R_{5.17}$ which is the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\forall)(\{\}\,(x_{m+1} : \varphi_{m+1}, (\in)(x_{m+1}, \varphi))))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi_{m+1}))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)(\chi, (\in)(t, \varphi))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then
  - $\chi \in S(h)$,
  - $t \in E(h)$,
  - $\varphi \in E_s(h)$, $x_{m+1} \notin V_b(\varphi)$.

LEMMA 5.18. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $(\rightarrow)\,(\varphi, (\wedge)\,(\psi, (\neg)(\psi)))\,, (\neg)(\varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)\,(\varphi, (\wedge)\,(\psi, (\neg)(\psi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\neg)(\varphi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)\,(\varphi, (\wedge)\,(\psi, (\neg)(\psi)))])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\neg)(\varphi)])$.

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\rightarrow)\,(\varphi, (\wedge)\,(\psi, (\neg)(\psi)))])$ as

$$P_\forall(\{\#(k, (\rightarrow)\,(\varphi, (\wedge)\,(\psi, (\neg)(\psi)))\,, \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(k, \varphi, \sigma), \#(k, (\wedge)\,(\psi, (\neg)(\psi))\,, \sigma))\mid \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(k, \varphi, \sigma), P_\wedge\,(\#(k, \psi, \sigma), \#(k, (\neg)(\psi), \sigma)))\mid \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(k, \varphi, \sigma), P_\wedge\,(\#(k, \psi, \sigma), P_\neg(\#(k, \psi, \sigma))))\mid \sigma \in \Xi(k)\})\ .$$

This can be expressed as 'for each $\sigma \in \Xi(k)$ either $\#(k, \varphi, \sigma)$ is false or both $\#(k, \psi, \sigma)$ and ($\#(k, \psi, \sigma)$ is false) are true'.

Since $\#(k, \psi, \sigma)$ cannot be both true and false at the same time we have that 'for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is false'. This is formally expressed as

$$P_\forall(\{P_\neg(\#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{\#(k, (\neg)(\varphi), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

which we can finally rewrite as $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\neg)(\varphi)]$. ■

Lemma 5.18 allows us to create a rule $R_{5.18}$ which is the set of all pairs

$$(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\wedge)(\psi,(\neg)(\psi)))],\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)(\varphi)])$$

such that

- m is a positive integer, $x_1,\ldots,x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_m \in E$, $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$,
- $\varphi,\psi \in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$.

Lemma 5.19. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi \in S(k)$.*

Under these assumptions we have

- $(\neg)((\wedge)(\varphi,\psi)),(\to)(\varphi,(\neg)(\psi)) \in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)((\wedge)(\varphi,\psi))] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\neg)(\psi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)((\wedge)(\varphi,\psi))])$ then $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\neg)(\psi))])$.

Proof.

We can rewrite $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)((\wedge)(\varphi,\psi))])$ as

$$P_\forall(\{\#(k,(\neg)((\wedge)(\varphi,\psi)),\sigma)|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\neg(\#(k,(\wedge)(\varphi,\psi),\sigma))|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\neg(P_\wedge(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)))|\ \sigma\in\Xi(k)\})\ .$$

We can rewrite $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\neg)(\psi))])$ as

$$P_\forall(\{\#(k,(\to)(\varphi,(\neg)(\psi)),\sigma)|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,(\neg)(\psi),\sigma))|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma),P_\neg(\#(k,\psi,\sigma)))|\ \sigma\in\Xi(k)\})\ .$$

Thus if $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)((\wedge)(\varphi,\psi))])$ we have that ‘for each $\sigma\in\Xi(k)$ it is false that $\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma)$ are both true’.

In other words for each $\sigma\in\Xi(k)$ ($\#(k,\varphi,\sigma)$ is false) or ($\#(k,\psi,\sigma)$ is false).

In other words for each $\sigma\in\Xi(k)$ $P_\to(\#(k,\varphi,\sigma),P_\neg(\#(k,\psi,\sigma)))$.

The last condition clearly implies $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\neg)(\psi))])$. ∎

Lemma 5.19 allows us to create a rule $R_{5.19}$ which is the set of all pairs

$$(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\neg)((\wedge)(\varphi,\psi))],\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\to)(\varphi,(\neg)(\psi))])$$

such that

- m is a positive integer, $x_1, \dots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \dots, \varphi_m \in E$, $H[x_1 : \varphi_1, \dots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \dots, x_m : \varphi_m])$.

Lemma 5.20. *Let m be a positive integer. Let $x_1, \dots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \dots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \dots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \dots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $\varphi \in S(k)$.

Under these assumptions we have

- $(\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $(\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) \in S(h)$,
- $(\rightarrow)\,(\chi, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))] \in S(\epsilon)$,
- $(\neg)(\varphi) \in S(k)$,
- $(\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi))) \in S(h)$,
- $(\rightarrow)\,(\chi, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi)))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi))))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ then

$$\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi))))]) \ .$$

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ as

$$P_\forall(\{\#(h, (\rightarrow)\,(\chi, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))) , \rho) |\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), \#(h, (\neg)\,((\forall)(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) , \rho)) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), P_\neg\,(\#(h, (\forall)\,(\{\}\,(x_{m+1} : \varphi_{m+1}, \varphi)) , \rho))) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), P_\neg\,(P_\forall(\{\#(k, \varphi, \sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}))) \mid \rho \in \Xi(h)\}) \ .$$

We can furtherly express this as

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that $P_\forall(\{\#(k, \varphi, \sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$',

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that (for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k, \varphi, \sigma)$ holds)',

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi, \sigma)$ is false)'.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, (\rightarrow)\,(\chi, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi))))])$ as

$$P_\forall(\{\#(h, (\rightarrow)\,(\chi, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi)))) , \rho) |\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), \#(h, (\exists)(\{\}(x_{m+1} : \varphi_{m+1}, (\neg)(\varphi))), \rho)) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), P_\exists(\{\#(k, (\neg)(\varphi), \sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\,(\#(h, \chi, \rho), P_\exists(\{P_\neg(\#(k, \varphi, \sigma)) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ .$$

This can be furtherly rewritten as

'for each $\rho \in \Xi(h)$ if $\#(h,\chi,\rho)$ then $P_\exists(\{P_\neg(\#(k,\varphi,\sigma))|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ ',
'for each $\rho \in \Xi(h)$ if $\#(h,\chi,\rho)$ then (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k,\varphi,\sigma)$ is false)'.

The last condition is clearly ensured by our hypothesis. ∎

Lemma 5.20 allows us to create a rule $R_{5.20}$ which is the set of all pairs

$$\begin{pmatrix} \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\chi,(\neg)((\forall)(\{\}(x_{m+1}:\varphi_{m+1},\varphi))))], \\ \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,(\rightarrow)(\chi,(\exists)(\{\}(x_{m+1}:\varphi_{m+1},(\neg)(\varphi))))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1,\ldots,x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_{m+1} \in E$, $H[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$;
- if we define $k = k[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$ and $h = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ then $\chi \in S(h)$, $\varphi \in S(k)$.

The next lemma is just a degenerate case of rule 5.12, we probably could modify that lemma to enclose also this case, but we choose to treat it separately.

Lemma 5.21. *Let $x_1 \in \mathcal{V}$, $\varphi_1 \in E$ and assume $H[x_1:\varphi_1]$. Define $k = k[x_1:\varphi_1]$. Let $\psi \in S(k)$ and $\varphi \in S(k) \cap S(\epsilon)$. Under these assumptions we have*

- $(\rightarrow)(\psi,\varphi) \in S(k)$,
- $\gamma[x_1:\varphi_1,(\rightarrow)(\psi,\varphi)] \in S(\epsilon)$,
- $(\exists)(\{\}(x_1:\varphi_1,\psi)) \in S(\epsilon)$,
- $(\rightarrow)((\exists)(\{\}(x_1:\varphi_1,\psi)),\varphi) \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,(\rightarrow)(\psi,\varphi)])$ then $\#((\rightarrow)((\exists)(\{\}(x_1:\varphi_1,\psi)),\varphi))$.

Proof.

Suppose $\#(\gamma[x_1:\varphi_1,(\rightarrow)(\psi,\varphi)])$. By definition we have

$$\#((\forall)(\{\}(x_1:\varphi_1,(\rightarrow)(\psi,\varphi)))) \ ,$$

and then

$$P_\forall(\{\#(k,(\rightarrow)(\psi,\varphi),\sigma)|\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(\#(k,\psi,\sigma),\#(k,\varphi,\sigma))|\ \sigma \in \Xi(k)\}) \ .$$

In turn $\#((\rightarrow)((\exists)(\{\}(x_1:\varphi_1,\psi)),\varphi))$ can be rewritten as

$$P_\rightarrow(\#((\exists)(\{\}(x_1:\varphi_1,\psi))),\#(\varphi)) \ ,$$

$$P_\rightarrow(P_\exists(\{\#(k,\psi,\sigma)|\ \sigma \in \Xi(k)\}),\#(\varphi)) \ .$$

In order to prove the last statement, we suppose there exists $\sigma \in \Xi(k)$ such that $\#(k,\psi,\sigma)$. This implies $\#(k,\varphi,\sigma)$, but we need to show that $\#(\varphi)$ holds.

To this end we can consider that $\varphi \in S(\epsilon)$ and, since $V_b(\varphi) \subseteq \mathcal{V} - var(k)$, $x_1 \notin V_b(\varphi)$. So we can apply lemma 4.12. There exists $s \in \#(\varphi_1)$ such that $\sigma = \epsilon + (x_1, s)$, and by the mentioned lemma we obtain $\#(k, \varphi, \sigma) = \#(\epsilon, \varphi, \epsilon) = \#(\varphi)$. ∎

Lemma 5.21 allows us to create a rule $R_{5.21}$ which is the set of all pairs

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, (\rightarrow)(\psi, \varphi)], \\ (\rightarrow)\left((\exists)\left(\{\}(x_1 : \varphi_1, \psi)\right), \varphi\right) \end{pmatrix}$$

such that $x_1 \in \mathcal{V}$, $\varphi_1 \in E$, $H[x_1 : \varphi_1]$, $\psi \in S(k[x_1 : \varphi_1])$ and $\varphi \in S(k[x_1 : \varphi_1]) \cap S(\epsilon)$.

Lemma 5.22. *Let $\varphi, \psi, \chi \in S(\epsilon)$. We have*

- $(\rightarrow)(\varphi, (\rightarrow)(\psi, \chi)) \in S(\epsilon)$,
- $(\rightarrow)((\wedge)(\varphi, \psi), \chi) \in S(\epsilon)$.

Moreover if $\#((\rightarrow)(\varphi, (\rightarrow)(\psi, \chi)))$ then $\#((\rightarrow)((\wedge)(\varphi, \psi), \chi))$.

Proof.

Suppose $\#((\rightarrow)(\varphi, (\rightarrow)(\psi, \chi)))$ holds. It can be rewritten

$$P_\rightarrow(\#(\varphi), \#((\rightarrow)(\psi, \chi))) ,$$
$$P_\rightarrow(\#(\varphi), P_\rightarrow(\#(\psi), \#(\chi))) .$$

In turn, $\#((\rightarrow)((\wedge)(\varphi, \psi), \chi))$ can be rewritten

$$P_\rightarrow(\#((\wedge)(\varphi, \psi)), \#(\chi)) ,$$
$$P_\rightarrow(P_\wedge(\#(\varphi), \#(\psi)), \#(\chi)) .$$

Suppose $\#(\varphi)$ and $\#(\psi)$ both hold, we need to show that $\#(\chi)$ holds. This is granted by

$$P_\rightarrow(\#(\varphi), P_\rightarrow(\#(\psi), \#(\chi))) .$$

∎

Lemma 5.22 allows us to create a rule $R_{5.22}$ which is the set of all pairs

$$\begin{pmatrix} (\rightarrow)(\varphi, (\rightarrow)(\psi, \chi)), \\ (\rightarrow)((\wedge)(\varphi, \psi), \chi) \end{pmatrix}$$

such that $\varphi, \psi, \chi \in S(\epsilon)$.

6. Deduction examples

6.1. First example. For each x, y natural numbers we say that x divides y if there exists a natural number α such that $y = x\alpha$.

In our example we want to show that for each x, y, z natural numbers if x divides y and y divides z then x divides z.

Of course, we first need to build an expression in our language to express this. To build that expression we must add to our language two constant symbols:

- a constant symbol N to represent the set of natural numbers $\mathbb{N}$, so that we have $\#(N) = \mathbb{N}$;
- a constant symbol $|$ to represent the 'divides' relation, so that $\#(|)$ is a function defined on $\mathbb{N} \times \mathbb{N}$ by $\#(|)(\alpha, \beta) = \exists \eta \in \mathbb{N} : \beta = \alpha\eta$.

The set $\mathcal{F}$ of operators is the same we have assumed in chapter 5, so it must contain all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$.

The statement we wish to prove is the following:

$$\gamma[x : N, y : N, z : N, (\rightarrow)((\wedge)((|)(x, y), (|)(y, z)), (|)(x, z))] \ , \tag{Th_1}$$

where x, y, z of course are variables in our language.

First of all we need to know this is a sentence in our language and that its meaning is as expected. To this purpose we'll use the following technical lemma.

Lemma 6.1. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$.*

We have $H[x_1 : N, \ldots, x_m : N]$ and we define $k = k[x_1 : N, \ldots, x_m : N]$.

Then for each $i = 1 \ldots m$ $x_i \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, x_i, \sigma) \in \mathbb{N}$.

Moreover for each $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ if we define $\sigma_0 = \epsilon$ and for each $i = 0 \ldots m-1$ $\sigma_{i+1} = \sigma_i + (x_{i+1}, \alpha_{i+1})$ then $\sigma_m \in \Xi(k)$ and for each $i = 1 \ldots m$ $\#(k, x_i, \sigma_m) = \alpha_i$.

Proof.

We first show that $H[x_1 : N, \ldots, x_m : N]$ holds. Let $k_0 = \epsilon$.

First consider that $N \in \mathcal{C} \subseteq E(\epsilon)$ and $\#(\epsilon, N, \epsilon) = \#(N) = \mathbb{N}$ is a set. Therefore $N \in E_s(\epsilon)$, so $H[x_1 : N]$ holds. Let $k_1 = k[x_1 : N]$, clearly $var(k_1) = \{x_1\}$.

Suppose $m > 1$ and let $i = 1 \ldots m - 1$. Assume $H[x_1 : N, \ldots, x_i : N]$ and $k_i = k[x_1 : N, \ldots, x_i : N]$, $var(k_i) = \{x_1, \ldots, x_i\}$. To prove $H[x_1 : N, \ldots, x_{i+1} : N]$ we just need to prove $N \in E_s(k_i)$.

There exists a positive integer n such that $k_i \in K(n)$, so by lemma 3.9 $N \in E(n, k_i)$ and for each $\sigma \in \Xi(k_i)$ $\#(k_i, N, \sigma) = \#(N)$ is a set. So we have proved $N \in E_s(k_i)$, and it results $H[x_1 : N, \dots, x_{i+1} : N]$. We also define $k_{i+1} = k[x_1 : N, \dots, x_{i+1} : N]$, and we have $var(k_{i+1}) = \{x_1, \dots, x_{i+1}\}$.

We have proved that $H[x_1 : N, \dots, x_m : N]$ holds.

Let $i = 1 \dots m$. Clearly $k_i = k_{i-1} + (x_i, N)$. There exists a positive integer n such that $k_{i-1} \in K(n)$, and $N \in E_s(k_{i-1})$, so $N \in E_s(n, k_{i-1})$. Moreover $x_i \notin var(k_{i-1})$, so $k_i \in K(n)^+$.

It follows that $x_i \in E(n+1, k_i) \subseteq E(k_i)$, for each $\sigma = \rho + (x_i, s) \in \Xi(k_i)$

$$\#(k_i, x_i, \sigma) = s \in \#(k_{i-1}, N, \rho) = \#(N) = \mathbb{N};\ V_b(x_i) = \emptyset\ .$$

If $i < m$ then for each $j = i \dots m-1$ we can assume $x_i \in E(k_j)$ and for each $\rho \in \Xi(k_j)$ $\#(k_j, x_i, \rho) \in \mathbb{N}$.

Clearly $k_{j+1} = k_j + (x_{j+1}, N)$. There exists a positive integer n such that $x_i \in E(n, k_j)$, so $k_j \in K(n)$. We have also $N \in E_s(k_j)$, so $N \in E_s(n, k_j)$. Moreover $x_{j+1} \notin var(k_j)$, so $k_{j+1} \in K(n)^+$.

Since $x_{j+1} \notin V_b(x_i)$ we have $x_i \in E(n+1, k_{j+1}) \subseteq E(k_{j+1})$. In addition, for each $\sigma = \rho + (x_{j+1}, s) \in \Xi(k_{j+1})$ $\#(k_{j+1}, x_i, \sigma) = \#(k_j, x_i, \rho) \in \mathbb{N}$.

We have proved that $x_i \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, x_i, \sigma) \in \mathbb{N}$.

Let $\alpha_1, \dots, \alpha_m \in \mathbb{N}$, $\sigma_0 = \epsilon$ and for each $i = 0 \dots m-1$ $\sigma_{i+1} = \sigma_i + (x_{i+1}, \alpha_{i+1})$.

We have $\sigma_0 = \epsilon \in \Xi(\epsilon) = \Xi(k_0)$.

Given $i = 0 \dots m-1$ we assume $\sigma_i \in \Xi(k_i)$. We have $k_{i+1} = k_i + (x_{i+1}, N)$ and there exists a positive integer n such that $k_i \in K(n)$ and $N \in E_s(n, k_i)$. Moreover $x_{i+1} \notin var(k_i)$, so $k_{i+1} \in K(n)^+$. To prove that $\sigma_{i+1} \in \Xi(k_{i+1})$ we just need to prove that $\alpha_{i+1} \in \#(k_i, N, \sigma_i) = \#(N) = \mathbb{N}$. This is true, of course.

So we have proved that $\sigma_m \in \Xi(k)$.

Now let $i = 1 \dots m$, we want to show that $\#(k, x_i, \sigma_m) = \alpha_i$.

We begin by showing that $\#(k_i, x_i, \sigma_i) = \alpha_i$. We have $k_i = k_{i-1} + (x_i, N)$ and there exists a positive integer n such that $k_{i-1} \in K(n)$ and $N \in E_s(n, k_{i-1})$. Moreover $x_i \notin var(k_{i-1})$, so $k_i \in K(n)^+$, $x_i \in E(n+1, k_i)$. We have $\sigma_i = \sigma_{i-1} + (x_i, \alpha_i) \in \Xi(k_i)$ and $\#(k_i, x_i, \sigma_i) = \alpha_i$.

If $i < m$ then let $j = i \dots m-1$, we assume $\#(k_j, x_i, \sigma_j) = \alpha_i$ and try to show $\#(k_{j+1}, x_i, \sigma_{j+1}) = \alpha_i$. Clearly $k_{j+1} = k_j + (x_{j+1}, N)$. There exists a positive integer n such that $x_i \in E(n, k_j)$, so $k_j \in K(n)$. We have also $N \in E_s(k_j)$, so $N \in E_s(n, k_j)$. Moreover $x_{j+1} \notin var(k_j)$, so $k_{j+1} \in K(n)^+$. Since $x_{j+1} \notin V_b(x_i)$ we have $x_i \in E(n+1, k_{j+1})$.

It results $\sigma_{j+1} = \sigma_j + (x_{j+1}, \alpha_{j+1}) \in \Xi(k_{j+1})$ and

$$\#(k_{j+1}, x_i, \sigma_{j+1}) = \#(k_j, x_i, \sigma_j) = \alpha_i.$$

■

To show that expression Th_1 belongs to $S(\epsilon)$ we define $k = k[x : N, y : N, z : N]$. By 6.1 we obtain that $x, y, z \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, x, \sigma), \#(k, y, \sigma), \#(k, z, \sigma)$ are all members of $\mathbb{N}$.

Moreover, $| \in E(k)$, for each $\sigma \in \Xi(k)$ $\#(k, |, \sigma) = \#(|)$ is a function with two arguments and $(\#(k, x, \sigma), \#(k, y, \sigma)), (\#(k, y, \sigma), \#(k, z, \sigma)), (\#(k, x, \sigma), \#(k, z, \sigma))$ are members of its domain.

So, by lemma 3.10, $(|)(x, y), (|)(x, z), (|)(y, z)$ all belong to $E(k)$.

Moreover, for each $\sigma \in \Xi(k)$

$$\#(k, (|)(x, y), \sigma) = \#(|)(\#(k, x, \sigma), \#(k, y, \sigma)) = \exists \eta \in \mathbb{N} : \ (\#(k, y, \sigma) = \#(k, x, \sigma) \cdot \eta) \ ;$$

so $\#(k, (|)(x, y), \sigma)$ is true or false. Therefore $(|)(x, y) \in S(k)$. In the same way we can show that $(|)(y, z), (|)(x, z) \in S(k)$.

By lemma 3.7 we have

$$(\wedge)((|)(x, y), (|)(y, z)) \in S(k), \quad (\rightarrow)((\wedge)((|)(x, y), (|)(y, z)), (|)(x, z)) \in S(k).$$

By definition 3.4

$$\gamma[x : N, y : N, z : N, (\rightarrow)((\wedge)((|)(x, y), (|)(y, z)), (|)(x, z))] \in S(\epsilon) \ .$$

We have proved Th_1 is a sentence and we'll now show it has the correct meaning.

By theorem 3.6 $\#(\gamma[x : N, y : N, z : N, (\rightarrow)((\wedge)((|)(x, y), (|)(y, z)), (|)(x, z))])$ is equivalent to

$$P_\forall(\{\#(k, (\rightarrow)((\wedge)((|)(x, y), (|)(y, z)), (|)(x, z)), \sigma) |\ \sigma \in \Xi(k)\}) \ .$$

This condition can be rewritten in the following ways:

$$P_\forall(\{P_\rightarrow(\#(k, (\wedge)((|)(x, y), (|)(y, z)), \sigma), \#(k, (|)(x, z)), \sigma)) |\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(P_\wedge(\#(k, (|)(x, y), \sigma), \#(k, (|)(y, z), \sigma)), \#(k, (|)(x, z)), \sigma)) |\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(P_\wedge\left(\begin{array}{l} \#(|)(\#(k, x, \sigma), \#(k, y, \sigma)), \\ \#(|)(\#(k, y, \sigma), \#(k, z, \sigma)) \end{array}\right), \#(|)(\#(k, x, \sigma), \#(k, z, \sigma))) |\ \sigma \in \Xi(k)\}) \ .$$

The last statement can be rewritten as follows:

for each $\sigma \in \Xi(k)$

$$P_\rightarrow(P_\wedge\left(\begin{array}{l} \#(|)(\#(k, x, \sigma), \#(k, y, \sigma)), \\ \#(|)(\#(k, y, \sigma), \#(k, z, \sigma)) \end{array}\right), \#(|)(\#(k, x, \sigma), \#(k, z, \sigma))).$$

By lemma 6.1 we can furtherly rewrite it like this:

for each $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$

$$P_\rightarrow(P_\wedge\left(\ \#(|)(\alpha_1, \alpha_2), \#(|)(\alpha_2, \alpha_3) \ \right), \#(|)(\alpha_1, \alpha_3)).$$

Finally this can be rewritten

for each $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$ if $\#(|)(\alpha_1, \alpha_2)$ and $\#(|)(\alpha_2, \alpha_3)$ then $\#(|)(\alpha_1, \alpha_3)$.

This is the meaning of our sentence Th_1 and that meaning is exactly as expected.

Our proof of statement Th_1 will begin by trying to exploit the definition of symbol $|$. To this end we need to add another constant symbol in our language. This is the symbol $*$ that stands for the product (or multiplication) operation in the domain $\mathbb{N}$ of natural numbers. Therefore $\#(*)$ is a function defined on $\mathbb{N}\times\mathbb{N}$ and for each $\alpha,\beta\in\mathbb{N}$ $\#(*)(\alpha,\beta)$ is the product of α and β, in other words $\#(*)(\alpha,\beta)=\alpha\cdot\beta$. Given two expressions φ,ψ in our language if $(*)(\varphi,\psi)$ is also an expression in our language then it can be abbreviated as $(\varphi\psi)$ (as used in mathematics).

Lemma 6.2. *Let m be a positive integer, $x_1,\dots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$.*

We have $H[x_1:N,\dots,x_m:N]$ and we define $k=k[x_1:N,\dots,x_m:N]$.

Suppose $i,j=1\dots m, i\neq j$, suppose $c\in\mathcal{V}-var(k)$. Then

$$\gamma\,[x_1:N,\dots,x_m:N,(\leftrightarrow)\,((|)(x_i,x_j),(\exists)\,(\{\}\,(c:N,(=)(x_j,(x_ic)))))]\in S(\epsilon);$$

$$\#\,(\gamma\,[x_1:N,\dots,x_m:N,(\leftrightarrow)\,((|)(x_i,x_j),(\exists)\,(\{\}\,(c:N,(=)(x_j,(x_ic)))))])\ \textit{is true.}$$

Proof.

We have also $H[x_1:N,\dots,x_m:N,c:N]$ and we can define

$$k'=k[x_1:N,\dots,x_m:N,c:N].$$

By lemma 6.1 we obtain that $x_i,x_j,c\in E(k')$. Moreover $*\in E(k')$ also holds.

For each $\sigma'\in\Xi(k')$ $\#(k',*,\sigma')=\#(*)$ is a function with two arguments, $\#(k',x_i,\sigma')$ and $\#(k',c,\sigma')$ belong to $\mathbb{N}$, so by lemma 3.10 $(*)(x_i,c)\in E(k')$.

By lemma 3.12 we have $(=)(x_j,(x_ic))\in S(k')$.

By lemma 3.1

- $\{\}\,(c:N,(=)(x_j,(x_ic)))\in E(k)$;
- $(\exists)\,(\{\}\,(c:N,(=)(x_j,(x_ic))))\in S(k)$;
- for each $\sigma\in\Xi(k)$

$$\#(k,(\exists)\,(\{\}\,(c:N,(=)(x_j,(x_ic))))\,,\sigma)=$$
$$=P_\exists\,(\{\#(k',(=)(x_j,(x_ic)),\sigma')|\ \sigma'\in\Xi(k'),\ \sigma\sqsubseteq\sigma'\})\ .$$

Lemma 6.1 also tells us that $x_i,x_j\in E(k)$ and for each $\sigma\in E(k)$ $\#(k,x_i,\sigma)\in\mathbb{N}$, $\#(k,x_j,\sigma)\in\mathbb{N}$. Moreover $|\in E(k)$ also holds.

For each $\sigma\in\Xi(k)$ $\#(k,|,\sigma)=\#(|)$ is a function with two arguments and $(\#(k,x_i,\sigma),\#(k,x_j,\sigma))$ is a member of its domain, therefore $(|)(x_i,x_j)\in E(k)$.

Moreover for each $\sigma\in\Xi(k)$

$$\#(k,(|)(x_i,x_j),\sigma)=\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma))$$
$$=\exists\eta\in\mathbb{N}:\#(k,x_j,\sigma)=\#(k,x_i,\sigma)\cdot\eta;$$

so $\#(k,(|)(x_i,x_j),\sigma)$ is true or false and $(|)(x_i,x_j)\in S(k)$.

From there follows that

$$(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right)\right)\in S(k)\ ;$$
$$\gamma\left[x_1:N,\ldots,x_m:N,(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right)\right)\right]\in S(\epsilon)\ .$$

By theorem 3.6 we can rewrite

$$\#\left(\gamma\left[x_1:N,\ldots,x_m:N,(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right)\right)\right]\right)$$

as follows

$$P_\forall(\{\#(k,(\leftrightarrow)\left((|)(x_i,x_j),(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right)\right),\sigma)|\ \sigma\in\Xi(k)\})$$

and this can be further rewritten

$$P_\forall(\{P_\leftrightarrow(\#(k,(|)(x_i,x_j),\sigma),\#(k,(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right),\sigma))|\ \sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\leftrightarrow\begin{pmatrix}\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma)),\\ \#(k,(\exists)\left(\{\}\left(c:N,(=)(x_j,(x_ic))\right)\right),\sigma)\end{pmatrix}|\ \sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\leftrightarrow\begin{pmatrix}\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma)),\\ P_\exists\left(\{\#(k',(=)(x_j,(x_ic)),\sigma')|\ \sigma'\in\Xi(k'),\ \sigma\sqsubseteq\sigma'\}\right)\end{pmatrix}|\ \sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\leftrightarrow\begin{pmatrix}\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma)),\\ P_\exists\left(\{P_=(\#(k',x_j,\sigma'),\#(k',(x_ic),\sigma'))|\ \sigma'\in\Xi(k'),\ \sigma\sqsubseteq\sigma'\}\right)\end{pmatrix}|\ \sigma\in\Xi(k)\})$$
$$P_\forall(\{P_\leftrightarrow\begin{pmatrix}\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma)),\\ P_\exists\left(\{P_=(\#(k',x_j,\sigma'),\#(k',x_i,\sigma')\cdot\#(k',c,\sigma'))|\,\sigma'\in\Xi(k'),\sigma\sqsubseteq\sigma'\}\right)\end{pmatrix}|\,\sigma\in\Xi(k)\})\,.$$

The final statement can also be expressed (more 'textually') as follows:

for each $\sigma\in\Xi(k)$
$\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma))$ if and only if
there exists $\sigma'\in\Xi(k')$ such that $\sigma\sqsubseteq\sigma'$ and $\#(k',x_j,\sigma')=\#(k',x_i,\sigma')\cdot\#(k',c,\sigma')$.

By definition we have

$$\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma))=\exists\eta\in\mathbb{N}:(\#(k,x_j,\sigma)=\#(k,x_i,\sigma)\cdot\eta)\ .$$

Suppose $\#(|)(\#(k,x_i,\sigma),\#(k,x_j,\sigma))$ holds.
There exists $\eta\in\mathbb{N}$ such that $\#(k,x_j,\sigma)=\#(k,x_i,\sigma)\cdot\eta$.

We define $\sigma'=\sigma+(c,\eta)$. We have $\sigma'\in\Xi(k')$ and $\sigma\sqsubseteq\sigma'$.

Moreover since $x_j\in E(k)$, $V_b(x_j)=\emptyset$ and then $c\notin V_b(x_j)$ we can apply lemma 4.12 and obtain that $\#(k',x_j,\sigma')=\#(k,x_j,\sigma)$. Similarly $\#(k',x_i,\sigma')=\#(k,x_i,\sigma)$. Using lemma 4.13 we obtain $\#(k',c,\sigma')=\eta$. Therefore

$$\#(k',x_j,\sigma')=\#(k,x_j,\sigma)=\#(k,x_i,\sigma)\cdot\eta=\#(k',x_i,\sigma')\cdot\#(k',c,\sigma')\ .$$

Conversely suppose there exists $\sigma'\in\Xi(k')$ such that $\sigma\sqsubseteq\sigma'$ and

$$\#(k',x_j,\sigma')=\#(k',x_i,\sigma')\cdot\#(k',c,\sigma').$$

There exists a positive integer n such that $k \in K(n)$, $N \in E_s(n,k)$. Since $c \in \mathcal{V} - var(k)$ we have $k' = k + (c, N) \in K(n)^+$, so there exist $\rho \in \Xi(k)$, $s \in \#(k, N, \rho) = \mathbb{N}$ such that $\rho + (c, s) = \sigma'$.

Now we have to consider that $\rho \sqsubseteq \sigma'$, so $\rho = {\sigma'}_{/dom(\rho)}$, and similarly $\sigma \sqsubseteq \sigma'$, so $\sigma = {\sigma'}_{/dom(\sigma)}$. Moreover $dom(\rho) = dom(k) = dom(\sigma)$ so

$$\rho = {\sigma'}_{/dom(\rho)} = {\sigma'}_{/dom(\sigma)} = \sigma \ .$$

Since $x_j \in E(k)$, $V_b(x_j) = \emptyset$ and then $c \notin V_b(x_j)$ we can apply lemma 4.12 and obtain that $\#(k', x_j, \sigma') = \#(k, x_j, \rho) = \#(k, x_j, \sigma)$. Similarly $\#(k', x_i, \sigma') = \#(k, x_i, \sigma)$. So we have

$$\#(k, x_j, \sigma) = \#(k, x_i, \sigma) \cdot \#(k', c, \sigma').$$

By lemma 6.1 we have $\#(k', c, \sigma') \in \mathbb{N}$, so $\#(|)(\#(k, x_i, \sigma), \#(k, x_j, \sigma))$ is proved. ■

This lemma allows us to create an axiom which is the set $A_{6.2}$ of all expressions

$$\gamma\,[x_1 : N, \ldots, x_m : N, (\leftrightarrow)\,((|)(x_i, x_j), (\exists)\,(\{\}\,(c : \mathbb{N}, (=)(x_j, (x_i c)))))]$$

such that m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $i, j = 1 \ldots m$, $i \neq j$, $c \in \mathcal{V} - var(k[x_1 : N, \ldots, x_m : N])$.

Lemma 6.3. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$.*

We have $H[x_1 : N, \ldots, x_m : N]$ and we define $k = k[x_1 : N, \ldots, x_m : N]$.

Suppose i_1, i_2, i_3 are distinct members of $\{1, \ldots, m\}$. Then

$$\gamma\,[x_1 : N, \ldots, x_m : N, (=)\,((*)\,((*)(x_{i_1}, x_{i_2}), x_{i_3})\,, (*)\,(x_{i_1}, (*)(x_{i_2}, x_{i_3})))] \in S(\epsilon);$$

$$\#\,(\gamma\,[x_1 : N, \ldots, x_m : N, (=)\,((*)\,((*)(x_{i_1}, x_{i_2}), x_{i_3})\,, (*)\,(x_{i_1}, (*)(x_{i_2}, x_{i_3})))])$$ *is true.*

Proof.

By lemma 6.1 we obtain that

- for each $j = 1 \ldots 3$ $x_{i_j} \in E(k)$;
- for each $\sigma \in \Xi(k)$ $\#(k, x_{i_j}, \sigma) \in \mathbb{N}$.

Also $*$ belongs to E(k). For each $\sigma \in \Xi(k)$ $\#(k, *, \sigma) = \#(*)$ is a function with two arguments and $(\#(k, x_{i_1}, \sigma), \#(k, x_{i_2}, \sigma))$ is a member of its domain, so $(*)(x_{i_1}, x_{i_2}) \in E(k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, (*)(x_{i_1}, x_{i_2}), \sigma) = \#(*)(\#(k, x_{i_1}, \sigma), \#(k, x_{i_2}, \sigma)) = \#(k, x_{i_1}, \sigma) \cdot \#(k, x_{i_2}, \sigma) \in \mathbb{N}.$$

Clearly we have also $(*)\,((*)(x_{i_1}, x_{i_2}), x_{i_3}) \in E(k)$.

Similarly $(*)\,(x_{i_1}, (*)(x_{i_2}, x_{i_3})) \in E(k)$ so by lemma 3.12

$$(=)\,((*)\,((*)(x_{i_1}, x_{i_2}), x_{i_3})\,, (*)\,(x_{i_1}, (*)(x_{i_2}, x_{i_3}))) \in S(k)\ ,$$

and

$$\gamma\,[x_1 : N, \ldots, x_m : N, (=)\,((*)\,((*)(x_{i_1}, x_{i_2}), x_{i_3})\,, (*)\,(x_{i_1}, (*)(x_{i_2}, x_{i_3})))] \in S(\epsilon).$$

By theorem 3.6 we can rewrite

$$\#\left(\gamma\left[x_1 : N, \ldots, x_m : N, (=)\left((*)\left((*)(x_{i_1}, x_{i_2}), x_{i_3}\right), (*)\left(x_{i_1}, (*)(x_{i_2}, x_{i_3})\right)\right)\right]\right)$$

as follows

$$P_\forall(\{\#(k, (=)\left((*)\left((*)(x_{i_1}, x_{i_2}), x_{i_3}\right), (*)\left(x_{i_1}, (*)(x_{i_2}, x_{i_3})\right)\right), \sigma)|\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_= \left(\begin{array}{c} \#(k, (*)\left((*)(x_{i_1}, x_{i_2}), x_{i_3}\right), \sigma), \\ \#(k, (*)\left(x_{i_1}, (*)(x_{i_2}, x_{i_3})\right), \sigma) \end{array}\right) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_= \left(\begin{array}{c} \#(*)\left(\#(k, (*)(x_{i_1}, x_{i_2}), \sigma), \#(k, x_{i_3}, \sigma)\right), \\ \#(*)\left(\#(k, x_{i_1}, \sigma), \#(k, (*)(x_{i_2}, x_{i_3}), \sigma)\right) \end{array}\right) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_= \left(\begin{array}{c} \#(*)\left(\#(*)\left(\#(k, x_{i_1}, \sigma), \#(k, x_{i_2}, \sigma)\right), \#(k, x_{i_3}, \sigma)\right), \\ \#(*)\left(\#(k, x_{i_1}, \sigma), \#(*)\left(\#(k, x_{i_2}, \sigma), \#(k, x_{i_3}, \sigma)\right)\right) \end{array}\right) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_= \left(\begin{array}{c} (\#(k, x_{i_1}, \sigma) \cdot \#(k, x_{i_2}, \sigma)) \cdot \#(k, x_{i_3}, \sigma), \\ \#(k, x_{i_1}, \sigma) \cdot (\#(k, x_{i_2}, \sigma) \cdot \#(k, x_{i_3}, \sigma)) \end{array}\right) |\ \sigma \in \Xi(k)\})$$

The last condition is clearly true, since for each $\sigma \in \Xi(k)$ $(\#(k, x_{i_1}, \sigma) \cdot \#(k, x_{i_2}, \sigma)) \cdot \#(k, x_{i_3}, \sigma)$ and $\#(k, x_{i_1}, \sigma) \cdot (\#(k, x_{i_2}, \sigma) \cdot \#(k, x_{i_3}, \sigma))$ are the same. ■

Lemma 6.3 allows us to create an axiom which is the set $A_{6.3}$ of all expressions

$$\gamma\left[x_1 : N, \ldots, x_m : N, (=)\left((*)\left((*)(x_{i_1}, x_{i_2}), x_{i_3}\right), (*)\left(x_{i_1}, (*)(x_{i_2}, x_{i_3})\right)\right)\right]$$

such that m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, i_1, i_2, i_3 are distinct members of $\{1, \ldots, m\}$.

6.1.1. The proof. We have already defined the sets $\mathcal{C}$ and $\mathcal{F}$ in our language, as follows:

$$\mathcal{C} = \{\mathbb{N}, |, *\};$$

$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\}.$$

Moreover we define $\mathcal{V} = \{x, y, z, c, d, e\}$.

Our deductive system includes the axioms and rules we've listed in chapter 5 and in this section 6.1.

The first step in our proof of statement Th_1 uses axiom $A_{6.2}$:

$$\gamma\left[x : N, y : N, z : N, (\leftrightarrow)\left((|)(x, y), (\exists)\left(\{\}\left(c : N, (=)(y, (xc))\right)\right)\right)\right]. \tag{6.1.1}$$

Then we can use $R_{5.3}$ to derive a new statement from 6.1.1:

$$\gamma\left[x : N, y : N, z : N, (\rightarrow)\left((|)(x, y), (\exists)\left(\{\}\left(c : N, (=)(y, (xc))\right)\right)\right)\right]. \tag{6.1.2}$$

In the next step we use axiom $A_{5.4}$:

$$\gamma[x : N, y : N, z : N, (\rightarrow)\left((\wedge)((|)(x, y), (|)(y, z)), (|)(x, y)\right)]. \tag{6.1.3}$$

At this point we can apply rule $R_{5.5}$ to 6.1.3 and 6.1.2 and obtain

$$\gamma[x : N, y : N, z : N, (\rightarrow)\left((\wedge)((|)(x, y), (|)(y, z)), (\exists)\left(\{\}\left(c : N, (=)(y, (xc))\right)\right)\right)]. \tag{6.1.4}$$

In much the same way we can obtain

$$\gamma[x : N, y : N, z : N, (\to)\,((\wedge)((|)(x, y), (|)(y, z)), (\exists)\,(\{\}\,(d : N, (=)(z, (yd)))))]. \quad (6.1.5)$$

The next two statements are instances of axiom $A_{5.4}$:

$$\gamma\left[x : N, y : N, z : N, c : N, d : N, (\to)\left((\wedge)\left(\begin{array}{c}(=)(y, (xc)),\\ (=)(z, (yd))\end{array}\right), (=)(y, (xc))\right)\right], \quad (6.1.6)$$

$$\gamma\left[x : N, y : N, z : N, c : N, d : N, (\to)\left((\wedge)\left(\begin{array}{c}(=)(y, (xc)),\\ (=)(z, (yd))\end{array}\right), (=)(z, (yd))\right)\right]. \quad (6.1.7)$$

In fact if we define $h = k[x : N, y : N, z : N, c : N, d : N]$ then $x, y, z, c, d \in E(h)$ and for each $\sigma \in \Xi(h)$ $\#(h, x, \sigma), \#(h, y, \sigma), \#(h, z, \sigma), \#(h, c, \sigma), \#(h, d, \sigma) \in \mathbb{N}$.

Moreover $* \in E(h)$ and for each $\sigma \in \Xi(h)$ $\#(h, *, \sigma) = \#(*)$ is a function with two arguments and $(\#(h, x, \sigma), \#(h, c, \sigma))$ is a member of its domain. Therefore $(xc) \in E(k)$ and similarly $(yd) \in E(h)$.

By lemma 3.12 we get $(=)(y, (xc)) \in S(h)$ and $(=)(z, (yd)) \in S(h)$ and the two statements are instances of $A_{5.4}$.

To proceed with our proof, our idea is to apply rule $R_{5.6}$ to 6.1.7 and 6.1.6.

We have $x, y, z, c, d, e \in \mathcal{V}$, $N \in E$, $H[x : N, y : N, z : N, c : N, d : N, e : N]$.

We have already defined $h = k[x : N, y : N, z : N, c : N, d : N]$ and we define $k = k[x : N, y : N, z : N, c : N, d : N, e : N]$.

We want to apply rule $R_{5.6}$ with

- $(\wedge)\,((=)(y, (xc)), (=)(z, (yd)))$ in the role of χ,
- $(=)(z, (ed))$ in the role of φ,
- y in the role of t,
- (xc) in the role of t'.

It has been shown above that $(\wedge)\,((=)(y, (xc)), (=)(z, (yd))) \in S(h)$.

It's easy to see that $(=)(z, (ed)) \in S(k)$. In fact $e, d, * \in E(k)$, for each $\sigma \in \Xi(k)$ $\#(k, *, \sigma) = \#(*)$ is a function with two arguments, and $\#(k, d, \sigma), \#(k, e, \sigma) \in \mathbb{N}$. This implies that $(ed) \in E(k)$. Since $z \in E(k)$ we obtain $(=)(z, (ed)) \in S(k)$.

Clearly $y \in E(h)$ and for each $\rho \in \Xi(h)$ $\#(h, y, \rho) \in \mathbb{N} = \#(h, N, \rho)$.

Moreover $x, c, * \in E(h)$, for each $\rho \in E(h)$ $\#(h, *, \rho) = \#(*)$ is a function with two arguments and $\#(h, x, \rho), \#(h, c, \rho) \in \mathbb{N}$. This implies that $(xc) \in E(h)$ and for each $\rho \in \Xi(h)$

$$\#(h, (xc), \rho) = \#(*)(\#(h, x, \rho), \#(h, c, \rho)) = \#(h, x, \rho) \cdot \#(h, c, \rho) \in \mathbb{N} = \#(h, N, \rho).$$

We can use assumption 2.1.10 to evaluate $V_b(y)$ and $V_b((xc))$. That assumption tells us that $V_b(y) = \emptyset$ and $V_b((xc)) = V_b(*) \cup V_b(x) \cup V_b(c) = \emptyset$. Therefore, clearly,

$$V_b(y) \cap V_b((=)(z, (ed))) = \emptyset;\ V_b((xc)) \cap V_b((=)(z, (ed))) = \emptyset\ .$$

In order to calculate $(=)(z,(ed))_k\{e/y\}$ and $(=)(z,(ed))_k\{e/(xc)\}$ we can exploit definition 4.16. In one part of it we established that one of five conditions holds true and a consequent calculation of $\varphi_k\{x_i/t\}$.

By lemma 5.6 we know there exists a positive integer n' such that $\mathcal{K}(n';k;x:N,y:N,z:N,c:N,d:N,e:N)$, $(=)(z,(ed)) \in E(n',k)$ and we can define $(=)(z,(ed))_k\{e/y\}$ and $(=)(z,(ed))_k\{e/(xc)\}$.

There also exists a positive integer n'' such that $z,(ed) \in E(n'',k)$. If we set $n = max\{n',n''\}$, then clearly $z,(ed) \in E(n,k)$, so $(=)(z,(ed))$ belongs to $E_d(n+1,k)$.

We have also $\mathcal{K}(n+1;k;x:N,y:N,z:N,c:N,d:N,e:N)$ and $(=)(z,(ed)) \in E(n+1,k)$. So definition 4.16 tells us there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1,\dots,\psi_m \in E(n,\kappa)$ such that $(=)(z,(ed)) = (f)(\psi_1,\dots,\psi_m)$, $(=)(z,(ed))$ belongs to $E(n+1,\kappa)$ etc..

Clearly f is the symbol $=$ and $m=2$, $z=\psi_1 \in E(n,\kappa)$, $(ed)=\psi_2 \in E(n,\kappa)$. At this point we observe that $V_f((ed)) \subseteq var(\kappa)$.

We can use assumption 2.1.10 to evaluate $V_f((ed))$. That assumption tells us that $V_f((ed)) = V_f(*) \cup V_f(e) \cup V_f(d) = \{d,e\}$. So $e \in V_f((ed)) \subseteq var(\kappa)$.

This implies $\kappa \neq \epsilon$. Let's rename our variables as follows

$$u_1 = x, u_2 = y, u_3 = z, u_4 = c, u_5 = d, u_6 = e$$

and for each $j = 1\dots 6$ let $\vartheta_j = N$.

By 4.7 we can derive there exists a positive integer $q \leqslant 6$ such that $q < n$, $\mathcal{K}(n;\kappa;u_1:\vartheta_1,\dots,u_q:\vartheta_q)$. We have $var(\kappa) = \{u_1,\dots,u_q\}$, so if $q < 6$ then $e \notin var(\kappa)$. But $e \in var(\kappa)$ holds so $q = 6$ and $\kappa = k$. We have also

$$(=)(z,(ed))_k\{e/y\} = (=)\left(z_k\{e/y\},(ed)_k\{e/y\}\right)$$

and similarly

$$(=)(z,(ed))_k\{e/(xc)\} = (=)\left(z_k\{e/(xc)\},(ed)_k\{e/(xc)\}\right) .$$

We can see immediately that $z_k\{e/y\} = z$ and $z_k\{e/(xc)\} = z$.

In order to evaluate $(ed)_k\{e/y\}$ and $(ed)_k\{e/(xc)\}$, we know that $\mathcal{K}(n;k;x:N,y:N,z:N,c:N,d:N,e:N)$ holds, $(ed) \in E(n,k)$ and we can define both $(ed)_k\{e/y\}$ and $(ed)_k\{e/(xc)\}$.

For reasons of clarity we need to redefine n', n'' and n. We can start by saying that there exists a positive integer n' such that $\mathcal{K}(n';k;x:N,y:N,z:N,c:N,d:N,e:N)$ holds, $(ed) \in E(n',k)$ and we can define both $(ed)_k\{e/y\}$ and $(ed)_k\{e/(xc)\}$.

There also exists a positive integer n'' such that $*,e,d \in E(n'',k)$. If we set $n = max\{n',n''\}$, then clearly $*,e,d \in E(n,k)$. For each $\sigma \in \Xi(k)$ $\#(k,*,\sigma) = \#(*)$ is a function with two arguments, and $\#(k,d,\sigma), \#(k,e,\sigma) \in \mathbb{N}$. This implies that $(ed) \in E_c(n+1,k)$.

We have also $\mathcal{K}(n+1;k;x:N,y:N,z:N,c:N,d:N,e:N)$, $(ed) \in E(n+1,k)$. So definition 4.16 tells us there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, a positive integer m, $\psi, \psi_1, \ldots, \psi_m \in E(n,\kappa)$ such that $(*)(e,d) = (\psi)(\psi_1,\ldots,\psi_m)$, $(*)(e,d) \in E(n+1,\kappa)$ etc..

Clearly $* = \psi$, $m = 2$, $e = \psi_1 \in E(n,\kappa)$, $d = \psi_2 \in E(n,\kappa)$. At this point we observe that $V_f(e) \subseteq var(\kappa)$.

We can use assumption 2.1.10 to evaluate $V_f(e)$. That assumption tells us that $V_f(e) = \{e\}$. So $e \in V_f(e) \subseteq var(\kappa)$.

This implies $\kappa \neq \epsilon$. We rename our variables as above

$$u_1 = x, u_2 = y, u_3 = z, u_4 = c, u_5 = d, u_6 = e$$

and for each $j = 1 \ldots 6$ let $\vartheta_j = N$.

By 4.7 we can derive there exists a positive integer $q \leqslant 6$ such that $q < n$, $\mathcal{K}(n;\kappa;u_1:\vartheta_1,\ldots,u_q:\vartheta_q)$. We have $var(\kappa) = \{u_1,\ldots,u_q\}$, so if $q < 6$ then $e \notin var(\kappa)$. But $e \in var(\kappa)$ holds so $q = 6$ and $\kappa = k$. We have also

$$(ed)_k\{e/y\} = ((*)_k\{e/y\})((e)_k\{e/y\},(d)_k\{e/y\}) = (*)(y,d) \ ,$$

and similarly

$$(ed)_k\{e/(xc)\} = ((*)_k\{e/(xc)\})((e)_k\{e/(xc)\},(d)_k\{e/(xc)\}) = (*)((xc),d) \ .$$

Therefore

$$(=)(z,(ed))_k\{e/y\} = (=)\left(z_k\{e/y\},(ed)_k\{e/y\}\right) = (=)\left(z,(*)(y,d)\right)$$

and similarly

$$(=)(z,(ed))_k\{e/(xc)\} = (=)\left(z_k\{e/(xc)\},(ed)_k\{e/(xc)\}\right) = (=)\left(z,(*)((xc),d)\right) \ .$$

So if we apply rule $R_{5.6}$ to 6.1.7 and 6.1.6 we obtain

$$\gamma\left[x:N,y:N,z:N,c:N,d:N,(\to)\left((\wedge)\left(\begin{array}{c}(=)(y,(xc)),\\(=)(z,(yd))\end{array}\right),(=)\left(z,(*)((xc),d)\right)\right)\right]. \quad (6.1.8)$$

The next statement is an instance of axiom $A_{6.3}$:

$$\gamma\left[x:N,y:N,z:N,c:N,d:N,(=)\left((*)((xc),d),(*)(x,(cd))\right)\right]. \quad (6.1.9)$$

Using rule $R_{5.7}$ we obtain

$$\gamma\left[x:N,y:N,z:N,c:N,d:N,(\to)\left((\wedge)\left(\begin{array}{c}(=)(y,(xc)),\\(=)(z,(yd))\end{array}\right),(=)\left((*)((xc),d),(*)(x,(cd))\right)\right)\right]. \quad (6.1.10)$$

We can apply rule $R_{5.8}$ to 6.1.8 and 6.1.10 to obtain

$$\gamma\left[x:N,y:N,z:N,c:N,d:N,(\to)\left((\wedge)\left(\begin{array}{c}(=)(y,(xc)),\\(=)(z,(yd))\end{array}\right),(=)\left(z,(*)(x,(cd))\right)\right)\right]. \quad (6.1.11)$$

To proceed, our idea is now to apply rule $R_{5.9}$ to 6.1.11.

We have $x, y, z, c, d, e \in \mathcal{V}$, $N \in E$, $H[x : N, y : N, z : N, c : N, d : N, e : N]$.

We want to apply our rule with the following assumptions:

- $k[x : N, y : N, z : N, c : N, d : N, e : N]$ takes the role of k;
- $h[x : N, y : N, z : N, c : N, d : N]$ takes the role of h;
- $(\wedge)\,((=)(y, (xc)), (=)(z, (yd)))$ takes the role of χ;
- (cd) takes the role of t;
- $(=)(z, (*)(x, e))$ takes the role of φ.

It has been shown above that $(\wedge)\,((=)(y, (xc)), (=)(z, (yd))) \in S(h)$.

We observe $c, d, * \in E(h)$, for each $\rho \in E(h)$ $\#(h, *, \rho) = \#(*)$ is a function with two arguments and $\#(h, c, \rho), \#(h, d, \rho) \in \mathbb{N}$. This implies that $(cd) \in E(h)$ and for each $\rho \in \Xi(h)$

$$\#(h, (cd), \rho) = \#(*)(\#(h, c, \rho), \#(h, d, \rho)) = \#(h, c, \rho) \cdot \#(h, d, \rho) \in \mathbb{N} = \#(h, N, \rho).$$

We can add that $V_b((cd)) = V_b(*) \cup V_b(c) \cup V_b(d) = \emptyset$.

It's also easy to see that $(=)(z, (xe)) \in S(k)$. In fact $x, e, * \in E(k)$, for each $\sigma \in \Xi(k)$ $\#(k, *, \sigma) = \#(*)$ is a function with two arguments, and $\#(k, x, \sigma), \#(k, e, \sigma) \in \mathbb{N}$. This implies that $(xe) \in E(k)$. Since $z \in E(k)$ we obtain $(=)(z, (xe)) \in S(k)$.

In order to calculate $(=)(z, (xe))_k\{e/(cd)\}$ we can exploit definition 4.16. In one part of it we established that one of five conditions holds true and a consequent calculation of $\varphi_k\{x_i/t\}$.

By the proof of lemma 5.6 we know there exists a positive integer n' such that $\mathcal{K}(n'; k; x : N, y : N, z : N, c : N, d : N, e : N)$, $(=)(z, (xe)) \in E(n', k)$ and we can define $(=)(z, (xe))_k\{e/(cd)\}$ at step n' of our inductive process in definition 4.16.

There also exists a positive integer n'' such that $z, (xe) \in E(n'', k)$. If we set $n = max\{n', n''\}$, then clearly $z, (xe) \in E(n, k)$, so $(=)(z, (xe))$ belongs to $E_d(n + 1, k)$.

We have also $\mathcal{K}(n + 1; k; x : N, y : N, z : N, c : N, d : N, e : N)$ and $(=)(z, (xe)) \in E(n + 1, k)$. So $(=)(z, (xe))_k\{e/(cd)\}$ can also be defined at step $n + 1$ of our inductive process and definition 4.16 tells us there exist $\kappa \in K(n)$: $\kappa \sqsubseteq k$, $f \in \mathcal{F}$, a positive integer m, $\psi_1, \dots, \psi_m \in E(n, \kappa)$ such that $(=)(z, (xe)) = (f)(\psi_1, \dots, \psi_m)$, $(=)(z, (xe))$ belongs to $E(n + 1, \kappa)$ etc..

Clearly f is the symbol $=$ and $m = 2$, $z = \psi_1 \in E(n, \kappa)$, $(xe) = \psi_2 \in E(n, \kappa)$. At this point we observe that $V_f((xe)) \subseteq var(\kappa)$.

We can use assumption 2.1.10 to evaluate $V_f((xe))$. That assumption tells us that $V_f((xe)) = V_f(*) \cup V_f(x) \cup V_f(e) = \{e, x\}$. So $e \in V_f((xe)) \subseteq var(\kappa)$.

This implies $\kappa \neq \epsilon$. Let's rename our variables as follows

$$u_1 = x, u_2 = y, u_3 = z, u_4 = c, u_5 = d, u_6 = e$$

and for each $j = 1 \dots 6$ let $\vartheta_j = N$.

By 4.7 we can derive there exists a positive integer $q \leqslant 6$ such that $q < n$, $\mathcal{K}(n;\kappa;u_1:\vartheta_1,\ldots,u_q:\vartheta_q)$. We have $var(\kappa)=\{u_1,\ldots,u_q\}$, so if $q<6$ then $e\notin var(\kappa)$. But $e\in var(\kappa)$ holds so $q=6$ and $\kappa=k$. We have also

$$(=)(z,(xe))_k\{e/(cd)\}=(=)\left(z_k\{e/(cd)\},(xe)_k\{e/(cd)\}\right)\ .$$

We can see immediately that $z_k\{e/(cd)\}=z$.

In order to evaluate $(xe)_k\{e/(cd)\}$, we know that $\mathcal{K}(n;k;x:N,y:N,z:N,c:N,d:N,e:N)$ holds, $(xe)\in E(n,k)$ and we can define $(xe)_k\{e/(cd)\}$ at step n of our inductive defintion process.

For reasons of clarity we need to redefine n',n'' and n. We can start by saying that there exists a positive integer n' such that $\mathcal{K}(n';k;x:N,y:N,z:N,c:N,d:N,e:N)$ holds, $(xe)\in E(n',k)$ and we can define $(xe)_k\{e/(cd)\}$ at step n'.

There also exists a positive integer n'' such that $*,x,e\in E(n'',k)$. If we set $n=max\{n',n''\}$, then clearly $*,x,e\in E(n,k)$. For each $\sigma\in\Xi(k)$ $\#(k,*,\sigma)=\#(*)$ is a function with two arguments, and $\#(k,x,\sigma),\#(k,e,\sigma)\in\mathbb{N}$. This implies that $(xe)\in E_c(n+1,k)$.

We have also $\mathcal{K}(n+1;k;x:N,y:N,z:N,c:N,d:N,e:N)$, $(xe)\in E(n+1,k)$ and we can define $(xe)_k\{e/(cd)\}$ at step $n+1$. So definition 4.16 tells us there exist $\kappa\in K(n)$: $\kappa\sqsubseteq k$, a positive integer m, $\psi,\psi_1,\ldots,\psi_m\in E(n,\kappa)$ such that $(*)(x,e)=(\psi)(\psi_1,\ldots,\psi_m)$, $(*)(x,e)\in E(n+1,\kappa)$ etc..

Clearly $*=\psi$, $m=2$, $x=\psi_1\in E(n,\kappa)$, $e=\psi_2\in E(n,\kappa)$. At this point we observe that $V_f(e)\subseteq var(\kappa)$.

We can use assumption 2.1.10 to evaluate $V_f(e)$. That assumption tells us that $V_f(e)=\{e\}$. So $e\in V_f(e)\subseteq var(\kappa)$.

This implies $\kappa\neq\epsilon$. We rename our variables as above

$$u_1=x,u_2=y,u_3=z,u_4=c,u_5=d,u_6=e$$

and for each $j=1\ldots6$ let $\vartheta_j=N$.

By 4.7 we can derive there exists a positive integer $q \leqslant 6$ such that $q < n$, $\mathcal{K}(n;\kappa;u_1:\vartheta_1,\ldots,u_q:\vartheta_q)$. We have $var(\kappa)=\{u_1,\ldots,u_q\}$, so if $q<6$ then $e\notin var(\kappa)$. But $e\in var(\kappa)$ holds so $q=6$ and $\kappa=k$. We have also

$$(xe)_k\{e/(cd)\}=((*)_k\{e/(cd)\})((x)_k\{e/(cd)\},(e)_k\{e/(cd)\})=(*)(x,(cd))$$

and therefore

$$(=)(z,(xe))_k\{e/(cd)\}=(=)\left(z_k\{e/(cd)\},(xe)_k\{e/(cd)\}\right)=(=)\left(z,(*)(x,(cd))\right)\ .$$

If we go back to our proof, we see that we can derive

$$\gamma\left[x:N,y:N,z:N,c:N,d:N,(\rightarrow)\left((\wedge)\left(\begin{array}{c}(=)(y,(xc)),\\(=)(z,(yd))\end{array}\right),(\exists)\left(\{\}(e:N,(=)(z,(xe)))\right)\right)\right]. \tag{6.1.12}$$

We can use the following instance of axiom $A_{6.2}$:

$$\gamma\,[x:N, y:N, z:N, c:N, d:N, (\leftrightarrow)\,((|)(x,z), (\exists)\,(\{\}\,(e:\mathbb{N}, (=)(z,(xe)))))] \ . \quad (6.1.13)$$

Using rule $R_{5.3}$ we can derive

$$\gamma\,[x:N, y:N, z:N, c:N, d:N, (\rightarrow)\,((\exists)\,(\{\}\,(e:\mathbb{N}, (=)(z,(xe)))), (|)(x,z))] \ . \quad (6.1.14)$$

We can apply rule $R_{5.5}$ to 6.1.12 and 6.1.14 to obtain

$$\gamma\left[x:N, y:N, z:N, c:N, d:N, (\rightarrow)\left((\wedge)\left(\begin{array}{c}(=)(y,(xc)),\\(=)(z,(yd))\end{array}\right), (|)(x,z)\right)\right] . \quad (6.1.15)$$

We can now apply rule $R_{5.10}$. With the definition $h = k[x:N, y:N, z:N, c:N, d:N]$ we have $(|)(x,z), (=)(y,(xc)), (=)(z,(yd)) \in S(h)$. So by $R_{5.10}$ we obtain

$$\gamma\,[x:N, y:N, z:N, c:N, d:N, (\rightarrow)\,((=)(y,(xc)), (\rightarrow)((=)(z,(yd)), (|)(x,z)))] \, . \quad (6.1.16)$$

By lemma 3.5 this can be rewritten

$$\gamma\,[x:N, y:N, z:N, c:N, \gamma\,[d:N, (\rightarrow)\,((=)(y,(xc)), (\rightarrow)((=)(z,(yd)), (|)(x,z)))]] \, . \quad (6.1.17)$$

We can apply rule $R_{5.11}$ using $k = k[x:N, y:N, z:N, c:N, d:N]$, $h = k[x:N, y:N, z:N, c:N]$. We consider that $(\rightarrow)((=)(z,(yd)), (|)(x,z)) \in S(k)$, $(=)(y,(xc)) \in S(k)$.

Moreover, $x, c, * \in E(h)$ and for each $\sigma \in \Xi(h)$ $\#(h, *, \sigma) = \#(*)$ is a function with two arguments and $(\#(h,x,\sigma), \#(h,c,\sigma))$ is a member of its domain. Therefore $(xc) \in E(h)$. We also observe that $y \in E(h)$ and therefore $(=)(y,(xc)) \in S(h)$. Thus we derive

$$\gamma\,[x:N, y:N, z:N, c:N, (\rightarrow)\,((=)(y,(xc)), \gamma\,[d:N, (\rightarrow)((=)(z,(yd)), (|)(x,z))])] \, . \quad (6.1.18)$$

This can be rewritten

$$\gamma\,[x:N, y:N, z:N, \gamma\,[c:N, (\rightarrow)\,((=)(y,(xc)), \gamma\,[d:N, (\rightarrow)((=)(z,(yd)), (|)(x,z))])]] \, . \quad (6.1.19)$$

We intend to apply rule $R_{5.12}$ using

- $k = k[x:N, y:N, z:N, c:N]$,
- $h = k[x:N, y:N, z:N]$,
- $\psi = (=)(y,(xc)) \in S(k)$,
- $\varphi = \gamma\,[d:N, (\rightarrow)((=)(z,(yd)), (|)(x,z))] \in S(k)$.

To be able to apply that rule we need to show that

$$\gamma\,[d:N, (\rightarrow)((=)(z,(yd)), (|)(x,z))] \in S(h) \ .$$

Let $\kappa = k[x:N, y:N, z:N, d:N]$. By lemma 6.1 $x, y, z, d \in E(\kappa)$, for each $\sigma \in \Xi(\kappa)$ $\#(\kappa,x,\sigma), \#(\kappa,y,\sigma), \#(\kappa,z,\sigma), \#(\kappa,d,\sigma) \in \mathbb{N}$.

Therefore $(yd) \in E(\kappa)$, $(=)(z,(yd)) \in S(\kappa)$, $(|)(x,z) \in S(\kappa)$, $(\to)((=)(z,(yd)),(|)(x,z)) \in S(\kappa)$ and $\gamma\,[d : N, (\to)((=)(z,(yd)),(|)(x,z))] \in S(h)$.

We obtain

$$\gamma[x : N, y : N, z : N, (\to)\left((\exists)\left(\{\}(c : N, (=)(y,(xc)))\right), \gamma\,[d : N, (\to)((=)(z,(yd)),(|)(x,z))]\right)] \; . \tag{6.1.20}$$

We can apply rule $R_{5.5}$ to 6.1.4 and 6.1.20 and obtain

$$\gamma[x : N, y : N, z : N, (\to)\left((\wedge)((|)(x,y),(|)(y,z)), \gamma\,[d : N, (\to)((=)(z,(yd)),(|)(x,z))]\right)] \; . \tag{6.1.21}$$

At this point we need to apply rule $R_{5.13}$ using

- $k = k[x : N, y : N, z : N, d : N]$,
- $h = k[x : N, y : N, z : N]$,
- $\chi = (\wedge)((|)(x,y),(|)(y,z)) \in S(h)$,
- $\psi = (=)(z,(yd)) \in S(k)$,
- $\varphi = (|)(x,z) \in S(h) \cap S(k)$.

We obtain

$$\gamma[x : N, y : N, z : N, (\to)\left((\wedge)((|)(x,y),(|)(y,z)), (\to)\left((\exists)\left(\{\}(d : N, (=)(z,(yd)))\right), (|)(x,z)\right)\right)] \; . \tag{6.1.22}$$

The final step in our proof consists in applying the 'modus ponens' rule $R_{5.14}$ to 6.1.5 and 6.1.22. We get

$$\gamma[x : N, y : N, z : N, (\to)\left((\wedge)((|)(x,y),(|)(y,z)), (|)(x,z)\right)] \; . \tag{6.1.23}$$

6.2. Second example. In this other example we want to prove a form of the Bocardo syllogism. In Ferreirós' referenced paper ([3]), on paragraph 3.1, the syllogism is expressed as follows:

Some A are not B. All C are B. Therefore, some A are not C.

Suppose A, B and C represent sets, the statement we actually want to prove is the following:

If ((there exists $x \in A$ such that $x \notin B$) and (for each $y \in C$ $y \in B$)) then (there exists $z \in A$ such that $z \notin C$).

In order to formalize this, our language must be as follows

$$\mathcal{C} = \{A, B, C\},$$
$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\},$$
$$\mathcal{V} = \{x, y, z\},$$

where A, B, C are constants each representing a set.

At this point we suppose we can formalize the statement as

$$(\rightarrow)\left((\wedge)\left(\begin{array}{l}(\exists)\left(\{\}\left(x : A, (\neg)\left((\in)(x, B)\right)\right)\right), \\ (\forall)\left(\{\}\left(y : C, (\in)(y, B)\right)\right)\end{array}\right), (\exists)\left(\{\}\left(z : A, (\neg)\left((\in)(z, C)\right)\right)\right)\right). \tag{Th_2}$$

We'll soon see a proof of this statement and within the proof we'll also prove Th_2 is a sentence in our language.

First of all we need the following lemma, that can be applied to the general language of chapter 5.

Lemma 6.4. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $A_1, \ldots, A_m \in \mathcal{C}$ such that for each $i = 1 \ldots m$ $\#(A_i)$ is a set. Let $D \in \mathcal{C}$ such that $\#(D)$ is a set. We have $H[x_1 : A_1, \ldots, x_m : A_m]$. If we define $k = k[x_1 : A_1, \ldots, x_m : A_m]$ then for each $i = 1 \ldots m$ $(\in)(x_i, D) \in S(k)$.*

Proof.

We first consider that $A_1 \in E(\epsilon)$ and $\#(A_1)$ is a set, so $A_1 \in E_s(\epsilon)$ and $H[x : A_1]$. Let $k_1 = k[x : A_1]$.

If $m > 1$ then for each $i = 1 \ldots m-1$ we suppose $H[x_1 : A_1, \ldots, x_i : A_i]$ holds and we define $k_i = k[x_1 : A_1, \ldots, x_i : A_i]$.
Clearly $A_{i+1} \in E(k_i)$ and for each $\rho \in \Xi(k_i)$ $\#(k_i, A_{i+1}, \rho) = \#(A_{i+1})$ is a set.
So $A_{i+1} \in E_s(k_i)$, which implies $H[x_1 : A_1, \ldots, x_{i+1} : A_{i+1}]$ (and we can define $k_{i+1} = k[x_1 : A_1, \ldots, x_{i+1} : A_{i+1}]$).

This proves that $H[x_1 : A_1, \ldots, x_m : A_m]$ holds.

Let $i = 1 \ldots m$. We have $A_i \in E_s(k_{i-1})$ and $k_i = k_{i-1} + (x_i, A_i)$. So we can apply lemma 4.13 and obtain that $x_i \in E(k_i)$. If $i = m$ this implies $x_i \in E(k)$.

If $i < m$ we consider that for each $j = i+1 \ldots m$ $x_j \notin V_b(x_i)$. So we can apply lemma 3.14 and prove $x_i \in E(k)$.

Moreover $D \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, D, \sigma) = \#(D)$ is a set. By lemma 3.13 we have $(\in)(x_i, D) \in S(k)$. ∎

6.2.1. The proof. To provide a proof of statement Th_2 we'll make use of a deductive system which includes all the axioms and rules listed in chapter 5.

If we go back to the language we have introduced for our proof, using the former lemma we can derive $H[x : A]$ and we can define $h = k[x : A]$. Moreover $(\in)(x, B) \in S(h)$, so $(\neg)((\in)(x, B)) \in S(h)$.

We also have $H[x : A, y : C]$ and we define $k_y = k[x : A, y : C]$.
We have $(\in)(y, B) \in S(k_y)$ and by lemma 3.1 $(\forall)(\{\}(y : C, (\in)(y, B))) \in S(h)$.

Thus $(\wedge)\,((\neg)((\in)(x, B)), (\forall)(\{\}(y : C, (\in)(y, B))))$ also belongs to $S(h)$.

Moreover $H[x : A, z : A]$ and we define $k_z = k[x : A, z : A]$.
We have $(\in)(z, C) \in S(k_z)$ and by lemma 3.1 $(\forall)(\{\}(z : A, (\in)(z, C))) \in S(h)$.

The first sentence in our proof is an instance of axiom $A_{5.4}$.

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x, B)),\\ (\forall)(\{\}(y : C, (\in)(y, B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z, C)))\end{array}\right), (\wedge)\left(\begin{array}{l}(\neg)((\in)(x, B)),\\ (\forall)(\{\}(y : C, (\in)(y, B)))\end{array}\right)\right)\right]. \tag{6.2.1}$$

By $A_{5.4}$ we also obtain

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\neg)((\in)(x, B)),\\ (\forall)(\{\}(y : C, (\in)(y, B)))\end{array}\right), (\neg)((\in)(x, B))\right)\right]. \tag{6.2.2}$$

By 6.2.1, 6.2.2 and rule $R_{5.5}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x, B)),\\ (\forall)(\{\}(y : C, (\in)(y, B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z, C)))\end{array}\right), (\neg)((\in)(x, B))\right)\right]. \tag{6.2.3}$$

Another instance of $A_{5.4}$ is the following

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x, B)),\\ (\forall)(\{\}(y : C, (\in)(y, B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z, C)))\end{array}\right), (\forall)(\{\}(z : A, (\in)(z, C)))\right)\right]. \tag{6.2.4}$$

By axiom $A_{5.16}$ we obtain

$$\gamma[x : A, (\in)(x, A)]. \tag{6.2.5}$$

By 6.2.5 and rule $R_{5.7}$ we also get

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right), (\in)(x,A)\right)\right]. \tag{6.2.6}$$

Since $x \in E(h)$, $C \in E_s(h)$ etc. we can apply rule $R_{5.17}$ to 6.2.4 and 6.2.6 and obtain

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right), (\in)(x,C)\right)\right]. \tag{6.2.7}$$

By axiom $A_{5.4}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right), (\forall)(\{\}(y : C, (\in)(y,B)))\right)\right]. \tag{6.2.8}$$

By 6.2.1, 6.2.8 and rule $R_{5.5}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right), (\forall)(\{\}(y : C, (\in)(y,B)))\right)\right]. \tag{6.2.9}$$

Since $x \in E(h)$, $B \in E_s(h)$ etc. we can apply rule $R_{5.17}$ to 6.2.7 and 6.2.9 and obtain

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right), (\in)(x,B)\right)\right]. \tag{6.2.10}$$

By 6.2.10, 6.2.3 and $R_{3.8}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right), (\wedge)\left(\begin{array}{l}(\in)(x,B),\\ (\neg)((\in)(x,B))\end{array}\right)\right)\right]. \tag{6.2.11}$$

By $R_{5.18}$

$$\gamma\left[x : A, (\neg)\left((\wedge)\left(\begin{array}{l}(\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right),\\ (\forall)(\{\}(z : A, (\in)(z,C)))\end{array}\right)\right)\right]. \tag{6.2.12}$$

By $R_{5.19}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right), (\neg)\left((\forall)(\{\}(z : A, (\in)(z,C)))\right)\right)\right]. \tag{6.2.13}$$

By $R_{5.20}$

$$\gamma\left[x : A, (\rightarrow)\left((\wedge)\left(\begin{array}{l}(\neg)((\in)(x,B)),\\ (\forall)(\{\}(y : C, (\in)(y,B)))\end{array}\right), (\exists)(\{\}(z : A, (\neg)((\in)(z,C))))\right)\right]. \tag{6.2.14}$$

By $R_{5.10}$

$$\gamma\left[x:A,(\to)\left((\neg)((\in)(x,B)),(\to)\left(\begin{array}{l}(\forall)(\{\}(y:C,(\in)(y,B))),\\(\exists)(\{\}(z:A,(\neg)((\in)(z,C))))\end{array}\right)\right)\right]. \tag{6.2.15}$$

Using lemma 6.4 we obtain that $(\in)(y,B)\in S(k[y:C])$ and $(\in)(z,C)\in S(k[z:A])$.

By lemma 3.1 we obtain that $(\forall)(\{\}(y:C,(\in)(y,B)))\in S(\epsilon)$ and similarly $(\exists)(\{\}(z:A,(\neg)((\in)(z,C))))\in S(\epsilon)$.

We can apply rule $R_{5.21}$ to 6.2.15 and obtain

$$(\to)\left((\exists)\left(\{\}(x:A,(\neg)((\in)(x,B)))\right),(\to)\left(\begin{array}{l}(\forall)(\{\}(y:C,(\in)(y,B))),\\(\exists)(\{\}(z:A,(\neg)((\in)(z,C))))\end{array}\right)\right) \tag{6.2.16}$$

Finally, by $R_{5.22}$, we obtain

$$(\to)\left((\wedge)\left(\begin{array}{l}(\exists)\left(\{\}(x:A,(\neg)((\in)(x,B)))\right),\\(\forall)(\{\}(y:C,(\in)(y,B)))\end{array}\right),(\exists)(\{\}(z:A,(\neg)((\in)(z,C))))\right) \tag{6.2.17}$$

6.2.2. Additional notes. We have proved statement Th_2, this also means that Th_2 is a sentence in our language. It seems quite obvious that the statement's meaning is as expected, anyway to complete the argument we also want to prove this.

We need the following lemma, that can be applied to the general language of chapter 5.

LEMMA 6.5. *Let $u\in\mathcal{V}$, $D\in\mathcal{C}$ such that $\#(D)$ is a set. We have $H[u:D]$ and we can define $h=k[u:D]$. Then, for each $\sigma\in\Xi(h)$ $\#(h,u,\sigma)\in\#(D)$. Moreover, for each $\alpha\in\#(D)$, if we define $\sigma=\epsilon+(u,\alpha)$ then $\sigma\in\Xi(h)$ and $\#(h,u,\sigma)=\alpha$.*

Proof.

We have $D\in E(1,\epsilon)$ and $\#(\epsilon,D,\epsilon)$ is a set, so $D\in E_s(1,\epsilon)$, $h=\epsilon+(u,D)\in K(1)^+$ and

$$\Xi(h)=\{\epsilon+(u,s)|\, s\in\#(\epsilon,D,\epsilon)\}=\{\epsilon+(u,s)|\, s\in\#(D)\}\,.$$

Therefore, for each $\sigma\in\Xi(h)$ there exists $s\in\#(D)$ such that $\sigma=\epsilon+(u,s)$.

It follows by lemma 4.13 that $\#(h,u,\sigma)=s\in\#(D)$.

Now let $\alpha\in\#(D)$ and $\sigma=\epsilon+(u,\alpha)$, clearly $\sigma\in\Xi(h)$ and by 4.13 $\#(h,u,\sigma)=\alpha$. ■

We now examine the meaning of $(\exists)\left(\{\}(x:A,(\neg)((\in)(x,B)))\right)$.

We can rewrite $\#((\exists)\left(\{\}(x:A,(\neg)((\in)(x,B)))\right))$ as

$$P_\exists(\{\#(k[x:A],(\neg)((\in)(x,B)),\sigma)|\ \sigma\in\Xi(k[x:A])\})\ ,$$

$$P_\exists(\{P_\neg(\#(k[x:A],(\in)(x,B),\sigma))|\ \sigma\in\Xi(k[x:A])\})\ ,$$

$$P_\exists(\{P_\neg(P_\in(\#(k[x:A],x,\sigma),\#(B)))|\ \sigma\in\Xi(k[x:A])\})\ .$$

This can be furtherly expressed as

'there exists $\sigma \in \Xi(k[x:A])$ such that $P_\neg(P_\in(\#(k[x:A],x,\sigma),\#(B)))$',

which is the same as

'there exists $\alpha_x \in \#(A)$ such that $P_\neg(P_\in(\alpha_x,\#(B)))$',
'there exists $\alpha_x \in \#(A)$ such that α_x doesn't belong to $\#(B)$'.

Similarly we can rewrite $\#((\forall)(\{\}(y:C,(\in)(y,B))))$ as

$$P_\forall(\{\#(k[y:C],(\in)(y,B),\sigma)|\ \sigma \in \Xi(k[y:C])\})$$
$$P_\forall(\{P_\in(\#(k[y:C],y,\sigma),\#(k[y:C],B,\sigma))|\ \sigma \in \Xi(k[y:C])\})$$
$$P_\forall(\{P_\in(\#(k[y:C],y,\sigma),\#(B))|\ \sigma \in \Xi(k[y:C])\})$$

This can be furtherly expressed as

'for each $\sigma \in \Xi(k[y:C])$ $P_\in(\#(k[y:C],y,\sigma),\#(B))$',

which is the same as

'for each $\alpha_y \in \#(C)$ $P_\in(\alpha_y,\#(B))$',
'for each $\alpha_y \in \#(C)$ α_y belongs to $\#(B)$'.

Similarly we can also rewrite $\#((\exists)\,(\{\}(z:A,(\neg)((\in)(z,C)))))$ as

$$P_\exists(\{\#(k[z:A],(\neg)((\in)(z,C)),\sigma)|\ \sigma \in \Xi(k[z:A])\})\ ,$$
$$P_\exists(\{P_\neg(\#(k[z:A],(\in)(z,C),\sigma))|\ \sigma \in \Xi(k[z:A])\})\ ,$$
$$P_\exists(\{P_\neg(P_\in(\#(k[z:A],z,\sigma),\#(C)))|\ \sigma \in \Xi(k[z:A])\})\ .$$

This can be furtherly expressed as

'there exists $\sigma \in \Xi(k[z:A])$ such that $P_\neg(P_\in(\#(k[z:A],z,\sigma),\#(C)))$',

which is the same as

'there exists $\alpha_z \in \#(A)$ such that $P_\neg(P_\in(\alpha_z,\#(C)))$',
'there exists $\alpha_z \in \#(A)$ such that α_z doesn't belong to $\#(C)$'.

At this point we can rewrite

$$\#\left((\to)\left((\wedge)\left(\begin{array}{l}(\exists)\,(\{\}(x:A,(\neg)((\in)(x,B))))\,,\\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{array}\right),(\exists)(\{\}(z:A,(\neg)((\in)(z,C))))\right)\right)$$

as

$$P_\to\left(\#\left((\wedge)\left(\begin{array}{l}(\exists)\,(\{\}(x:A,(\neg)((\in)(x,B))))\,,\\ (\forall)(\{\}(y:C,(\in)(y,B)))\end{array}\right)\right),\#\,((\exists)(\{\}(z:A,(\neg)((\in)(z,C)))))\right)$$

and then

$$P_\to\left(P_\wedge\left(\begin{array}{l}\#\,((\exists)\,(\{\}(x:A,(\neg)((\in)(x,B)))))\,,\\ \#\,((\forall)(\{\}(y:C,(\in)(y,B))))\end{array}\right),\#\,((\exists)(\{\}(z:A,(\neg)((\in)(z,C)))))\right)$$

This can be furtherly expressed as:

'if (there exists $\alpha_x \in \#(A)$ such that α_x doesn't belong to $\#(B)$) and
(for each $\alpha_y \in \#(C)$ α_y belongs to $\#(B)$) then
(there exists $\alpha_z \in \#(A)$ such that α_z doesn't belong to $\#(C)$)'.

So the statement which we have proved has the expected meaning.

7. Consistency, paradoxes and further study

We have proved that a deductive system is sound, i.e. if we can derive a sentence φ in our system then $\#(\varphi)$ holds. We now discuss the consistency of a deductive system.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *consistent* if and only if for each φ sentence in $\mathcal{L}$ $(\vdash_{\mathcal{D}} \varphi)$ and $(\vdash_{\mathcal{D}} (\neg)(\varphi))$ aren't both true.

LEMMA 7.1. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is consistent.*

Proof.

Suppose there exists a sentence φ such that $\vdash_{\mathcal{D}} \varphi$ and $\vdash_{\mathcal{D}} (\neg)(\varphi)$ both hold. By the soundness property we have $\#(\varphi)$ and $\#((\neg)(\varphi))$. Clearly

$$\#((\neg)(\varphi)) = \#(\epsilon, (\neg)(\varphi), \epsilon) = P_{\neg}(\#(\varphi)) = \#(\varphi) \text{ is false.}$$

So $\#(\varphi)$ would be true and false at the same time, a plain contradiction. ■

A paradox is usually a situation in which a contradiction or inconsistency occurs, in other words a paradox arises when we can build a sentence φ such that both φ and $(\neg)(\varphi)$ can be derived. Since our system is consistent it shouldn't be possible to have true paradoxes in it, anyway it seems appropriate to discuss how our system relates with some of the most famous paradoxical arguments.

We begin with Russell's paradox. Assume we can build the set A of all those sets X such that X is not a member of X. Clearly, if $A \in A$ then $A \notin A$ and conversely if $A \notin A$ then $A \in A$. We have proved both $A \in A$ and its negation, and this is the Russel's paradox.
It seems in our system we cannot generate this paradox since building a set is permitted only if you rely on already defined sets. When trying to build set A in our language we could obtain something like this:

$$\{\}((\neg)((\in)(X, X)), X) \ .$$

However it is clear this isn't a legal expression in our language, since in our language if you want to build a context-independent expression using a variable X, then you have to assign a domain to X.

We now turn to Cantor's paradox. Often the wording of this paradox involves the theory of cardinal numbers (see e.g. Mendelson's book [4]), but here we use a simpler wording.

First of all we prove that for each set A there doesn't exist a surjective function with domain A and codomain $\mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the power set of A).

Let f be a function from A to $\mathcal{P}(A)$. Let $B = \{x \in A | x \notin f(x)\}$.
Suppose there exists $y \in A$ such that $B = f(y)$. If $y \in B$ then $y \notin f(y) = B$, and conversely if $y \notin B = f(y)$ then $y \in B$. So there isn't $y \in A$ such that $B = f(y)$ and therefore f is not surjective.

At this point, suppose there exists a set Ω such that any member of Ω is a set and any set is a member of Ω. Clearly Ω and all of its subsets belong to Ω, so we can define a function f from Ω to $\mathcal{P}(\Omega)$ such that for each $X \subseteq \Omega$ $f(X) = X$. Obviously this is a surjective function, and we have a contradiction.

The contradiction is due to having assumed the existence of Ω. In this case too in our language we cannot build an expression with such meaning. One expression like the following:

$$\{\}(set(X), X)$$

is not a valid expression in our language.

Finally we want to examine the liar paradox. Let's consider how the paradox is stated in Mendelson's book.

A man says, 'I am lying'. If he is lying, then what he says is true, so he is not lying. If he is not lying, then what he says is false, so he is lying. In any case, he is lying and he is not lying.

Mendelson classifies this paradox as a 'semantic paradox' because it makes use of concepts which need not occur within our standard mathematical language. I agree that, in his formulation, the paradox has some step which seems not mathematically rigorous.

We'll try to provide a more rigorous wording of the paradox.

Let A be a set, and let δ be the condition 'for each x in A x is false'. Suppose δ is the only member of A. In this case if δ is true then it is false; if on the contrary δ is false then it is true.

The explanation of the paradox is the following: simply δ cannot be the only item in set A. In fact, suppose A has only one element, and let's call it φ. This implies δ is equivalent to 'φ is false' so it seems acceptable that δ is not φ.

Another approach to the explanation is the following.

If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false.

Moreover if δ is false and the uniqueness condition 'for each x in A $x = \delta$' is true then δ is true, thus if δ is false then 'for each x in A $x = \delta$' is false too. By contraposition if 'for each x in A $x = \delta$' then δ is true.

Therefore if δ is the only element in A then δ is true and false at the same time. This implies δ cannot be the only item in A.

On the basis of this argument I consider the liar paradox as an apparent paradox that actually has an explanation. What is the relation between our approach to logic and the liar paradox?

Standard logic isn't very suitable to express this paradox. In fact first-order logic is not designed to construct a condition like our condition δ (= 'for each x in A x is false'), and moreover, it is clearly not designed to say 'δ belongs to set A'. These conditions aren't plainly leading to inconsistency, so it is desirable they can be expressed in a general approach to logic. And our system permits to express them. The paradox isn't ought to simply using these conditions, it is due to an assumption that is clearly false, and the so-called paradox is simply the proof of its falseness.

Of course, further investigations about our approach to logic can be performed. For instance, we can be asked about the completeness of the system. A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *complete* if and only if for each φ sentence in $\mathcal{L}$ if $\#(\varphi)$ holds then $\vdash_{\mathcal{D}} \varphi$. It was easy to prove the soundness of our system, unfortunately the topic of completeness is more difficult, and in general there is no reason to expect that completeness holds. For instance Cutland's book [1] has interesting material in this regard.

Another interesting (and not extremely easy) topic is about comparing the expressive power of our system with the one of standard logic systems.

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