TOWARDS A GROUP-THEORETICAL INTERPRETATION OF MECHANICS

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Abstract. We argue that the classical description of a symplectic manifold endowed with a Hamiltonian action of an abelian Lie group G and the corresponding quantum theory can be understood as different aspects of the unitary representation theory of G. To do so, we propose a conceptual analysis of formal tools coming from symplectic geometry (notably, Souriau's *moment map* and the *Mardsen-Weinstein symplectic reduction* formalism) and group representation theory (notably *Kirillov's orbit method*). The proposed argumentative line strongly relies on the conjecture proposed by Guillemin and Sternberg according to which "quantization commutes with (symplectic) reduction". By using the generalization of this conjecture to non-zero coadjoint orbits, we argue that phase invariance in quantum mechanics and gauge invariance have a common geometric underpinning, namely the symplectic reduction formalism. This fact points towards a gauge-theoretic interpretation of Heisenberg indeterminacy principle.

Key words: quantum mechanics; gauge theories; moment map; symplectic reduction.

1. Introduction

Group theory is pervasive in both classical and quantum mechanics, both in the framework of ordinary Hamiltonian systems and in the framework of constrained Hamiltonian systems or gauge theories (see for instance Ref.[2]). In what follows, we plead in favor of a *group-theoretical interpretation* of ordinary (non-constrained) Hamiltonian mechanics by studying the case of a symplectic manifold (M, ω) endowed with a strongly Hamiltonian symplectic action of an abelian Lie group G. The proposed approach contrasts with the C^*-algebra approach in which *classical* and *quantum* mechanics can be reobtained as the representation theories of *commutative* and *non-commutative* C^*-algebras respectively [12, 19]. In order to justify the election of a non-commutative C^*-algebra, it is common to make appeal to operationalistic arguments concerning the impossibility of joint measurements of non-commuting observables.[1] In the approach that we explore in what follows, the difference between classical and quantum mechanics is not explained by appealing to the commutativity properties of the corresponding abstract algebraic structure. Moreover, we argue that *the symplectic manifold* (M, ω) *and its quantum counterpart* $\mathcal{H}_M$ *play different roles in the representation theory of the same group* G. The restriction to the case of an abelian Lie group G, besides simplifying the discussion, allows us to stress that one of the more characteristic features of quantum mechanics, namely the existence of indeterminacy relations, does not result from the non-abelianity of the corresponding algebraic structure.[2]

In Section N°2, we use the moment map formalism in symplectic geometry to start the construction of a group-theoretical interpretation of the notions of classical *state* and classical *observable*. By using these interpretations, we argue that the *state-observable duality* can be interpreted in terms of the duality between a Lie group G and its unitary dual $\hat{G}$ (i.e., the collection of all equivalence classes of *unitary*

[1]According to Strocchi, "[...] the mathematical setting of quantum mechanics can be derived with a very strict logic solely from the C^*-algebraic structure of the observables and *the operational information of non-commutativity codified by the Heisenberg uncertainty relations*." ([19], p. 22; see also Section 2.1).

[2]The simplest example of such an abelian case is given by the (cotangent lift of) the translations generated by $G = \mathbb{R}^n$ acting on the n positions $(q_1, ..., q_n)$ of the symplectic manifold $M = \mathbb{R}^{2n}$ parameterized by $(q_1, ..., q_n; p_1, ..., p_n)$ (see Ref.[14], Th. 12.1.4).

irreducible representations–unirreps in what follows– of G). In Section N°3, we use the Guillemin-Sternberg conjecture according to which "quantization commutes with symplectic reduction" to analyze the relationship between quantization and the symplectic reduction formalism. In Section N°4, we use the generalization of this conjecture to non-zero coadjoint orbits to argue that the symplectic reduction formalism provides the common geometric underpinning of both the *gauge invariance* associated to the presence of first-class constraints and the *phase invariance* associated to the eigenstates of ordinary (non-constrained) observables. In the final section, we discuss some consequences of the proposed arguments with respect to the interpretation of mechanics.

2. Group-theoretical interpretation of the state-observable duality

In this section, we consider the duality between classical *states* and *observables* in the framework provided by the *moment map* formalism in symplectic geometry [1, 13, 14, 17, 18]. Given a symplectic manifold (M, ω), the symplectic structure ω defines a Lie algebra homomorphism between observables on M and *symplectic vector fields*:

$$\begin{array}{rcl} \mathcal{C}^{\infty}(M) & \rightarrow & Symp(M) \\ \pi : f & \rightarrow & v_f, \\ \{g, f\} & \mapsto & v_{\{g,f\}} = [v_f, v_g] \end{array}$$

defined by means of the *symplectic Hamilton equation*

$$df = i_{v_f}\omega.$$

The symplectic vector fields obtained by means of this application are called *Hamiltonian vector fields*. Let's consider now a *strongly Hamiltonian symplectic action* $\Phi : G \times M \rightarrow M$ of a Lie group G (that we shall call *phase group*) on a symplectic manifold (M, ω), i.e. an action that 1) preserves ω (i.e. such that $\Phi_g^* \omega = \omega$) and 2) has an associated infinitesimally equivariant moment map. In what follows, such a structure is called *Hamiltonian G-manifold* and denoted (M, ω, μ_M). In these cases, there exists a map (called *co-moment map*)

$$\begin{array}{rcl} \tilde{\mu}_M : \mathfrak{g} & \rightarrow & \mathcal{C}^{\infty}(M) \\ X & \mapsto & f_X \end{array} \tag{1}$$

such that the following diagram commutes

$$\mathfrak{g} \xrightarrow{\tilde{\mu}_M} \mathcal{C}^{\infty}(M) \xrightarrow{\pi} Symp(M), \qquad \lambda : \mathfrak{g} \rightarrow Symp(M)$$

where

$$\begin{array}{rcl} \lambda : \mathfrak{g} & \rightarrow & Symp(M) \\ X & \mapsto & v_X = \dfrac{d}{d\lambda}(exp(-\lambda X) \cdot m)|_{\lambda=0} \end{array} \tag{2}$$

is the natural map between abstract Lie algebra elements and the so-called *fundamental vector fields* on M. The commutativity of this diagram means the observable f_X obtained by means of (1) and the fundamental vector field v_X obtained by means of (2) satisfy the symplectic Hamilton equation $df_X = i_{v_X}\omega$. The dual *moment map* μ_M is given by

$$\mu_M : M \rightarrow \mathfrak{g}^*$$

such that

$$\langle \mu_M(m), X \rangle = f_X(m) = \tilde{\mu}_M(X)(m). \tag{3}$$

If X_i is an element in the basis of $\mathfrak{g}$, the observable f_{X_i} evaluated in m gives the i-component of $\mu_M(m)$. In what follows, $\mathcal{C}^\infty_{\mathfrak{g}}(M) \doteq \tilde{\mu}_M(\mathfrak{g})$ will be called *moment observable algebra* and the observables in $\mathcal{C}^\infty_{\mathfrak{g}}(M)$ will be called *moment observables*. Thanks to the existence of μ_M, we can characterize M as a *symplectic realization* of the Poisson manifold $\mathfrak{g}^*$ (endowed with the so-called *Lie-Poisson structure*). This correspondence between M and $\mathfrak{g}^*$ implies that each "classical state" $m \in M$ defines a $\mathbb{R}$-valued linear functional on $\mathfrak{g}$

$$\begin{aligned} \check{m} : \mathfrak{g} &\to \mathbb{R} \\ X &\mapsto \langle \mu_M(m), X \rangle. \end{aligned} \tag{4}$$

In turn, the co-moment map $\tilde{\mu}_M$ can be interpreted as a sort of *Gelfand transform* in the sense that it permits to realize the abstract algebra $\mathfrak{g}$ as an observable algebra $\mathcal{C}^\infty_{\mathfrak{g}}(M) \doteq \tilde{\mu}_M(\mathfrak{g})$ on the space M of $\mathbb{R}$-valued functionals on $\mathfrak{g}$. Indeed, expression (3) can be recast in terms of the usual *state-observable duality*:

$$f_X(m) = \check{m}(X).$$

The infinitesimal equivariance of the moment map means that μ_M infinitesimally intertwines the G-action on M and the G-coadjoint action on $\mathfrak{g}$ (i.e. that $(T_m\mu_M)(v_X(m)) = -ad^*_X(\mu_M(m))$ for all $X \in \mathfrak{g}$, where $T_m\mu_M : T_mM \to T\mathfrak{g}^* \simeq \mathfrak{g}^*$ and ad^* denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$). In turn, this means that the co-moment map $\tilde{\mu}$ is an homomorphism of Lie algebras, i.e. that $\tilde{\mu}_M([X,Y]) = \{\tilde{\mu}_M(X), \tilde{\mu}_M(Y)\}$ [14, 17].

We shall now argue that the duality between a physical spectrum (M, ω, μ_M) of $\mathfrak{g}$ and its moment observable algebra $\mathcal{C}^\infty_{\mathfrak{g}}(M)$ can be understood in the framework of the unitary representation theory of $\mathfrak{g}$. To do so, let's consider the role played by $\mathfrak{g}^*$ in the theory of unitary representations of $\mathfrak{g}$. If G is abelian, each *integral* linear map $\xi : \mathfrak{g} \to \mathbb{R}$ in $\mathfrak{g}^*$ defines a 1-dimensional unirrep ρ^G_ξ of G given by[3]

$$\begin{aligned} \rho^G_\xi : G &\to U(1) \\ e^X &\mapsto e^{2\pi i \langle \xi, X\rangle}, \qquad X \in \mathfrak{g}. \end{aligned} \tag{5}$$

The abelianity of G guarantees that ρ^G_ξ is indeed a representation.[4] Now, the moment map μ_M allows us to transfer this relationship between the universal functionals in $\mathfrak{g}^*$ and the unirreps of G to the points m such that $\mu_M(m)$ is integral. Indeed, we can associate the 1-dimensional unirrep $\rho^G_{\mu(m)}$ of G to each integral $m \in M$:

$$m \rightsquigarrow \rho^G_{\mu_M(m)} : e^X \mapsto e^{2\pi i \langle \mu_M(m), X\rangle}.$$

On the other hand, the co-moment map $\tilde{\mu}_M : \mathfrak{g} \to \mathcal{C}^\infty(M)$ associates to each abstract Lie algebra element X a moment observable f_X on M. The moment observable f_X encodes all possible representations of X as concrete unitary operators acting on $\mathbb{C}$. Indeed, the numerical value

$$f_X(m) = \langle \mu_M(m), X \rangle$$

obtained by evaluating f_X on m defines the concrete unitary operator

$$e^{2\pi i f_X(m)} \tag{6}$$

that represents the abstract Lie algebra element $X \in \mathfrak{g}$ in the unirrep $\rho_{\mu_M(m)}$. In this way, the duality between the so-called "classical states" in M and the moment observables in $\mathcal{C}^\infty_{\mathfrak{g}}(M)$ can be reinterpreted in the following group-theoretical terms. Whereas each element $m \in M$ defines a *single* 1-dimensional unirrep $\rho_{\mu_M(m)}$ of *all* the abstract elements in $\mathfrak{g}$, the moment observable f_X encodes *all* possible unirreps of a *single* Lie algebra element $X \in \mathfrak{g}$. *By evaluating f_X on m we obtain the concrete unitary operator (6) that represents X in the unirrep $\rho_{\mu_M(m)}$*. In this way, the duality between the "classical states" in

[3]En element ξ in $\mathfrak{g}^*$ is *integral* if it belongs to the *weight lattice* of G, i.e. if $\langle \xi, X\rangle = 2\pi i\mathbb{Z}$ for every $X \in ker(exp : \mathfrak{g} \to G)$.

[4]This can be trivially proved by means of the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Z$ with $Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + ...$

M and the moment observables on M can be understood as a duality between the unirreps of G and the elements of G, i.e. as a duality between the unitary dual $\hat{G}$ and the group G. Whereas M is a symplectic realization of the unitary representation theory of G conveyed by $\mathfrak{g}^*$, the moment observable algebra $\mathcal{C}^{\infty}_{\mathfrak{g}}(M)$ is a realization of $\mathfrak{g}$ itself as observables on M.

For the general case of a non-abelian Lie group G, we cannot simply associate an unirrep of G with each universal integral functional $\xi \in \mathfrak{g}^*$. However Kirillov's orbit method establishes a correspondence between unirreps of a nilpotent Lie group G and the (integral) orbits of the coadjoint action Ad^* of G on $\mathfrak{g}^*$ (coadjoint orbits) [10]:

$$\hat{G} \simeq int(\mathfrak{g}^*/G).$$

It can be shown that the coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ are endowed with a canonical G-invariant symplectic structure $\omega_{\mathcal{O}}$ [10]. The Kirillov's correspondence depends on the fact that each coadjoint orbit $\mathcal{O}$ is a homogeneous Hamiltonian G-manifold whose (geometric) quantization yields a Hilbert space that carries an unirrep of G. In this way, given a Lie group G we can define a Poisson family $\mathfrak{g}^*$ of Hamiltonian G-manifolds $(\mathcal{O}, \omega_{\mathcal{O}}, \mu_{\mathcal{O}})$ such that the quantization of the integral ones yields the unirreps of G. It is worth stressing that symplectic manifolds (i.e. "classical systems") canonically appear in the unitary representation theory of a Lie group G. In the case of an abelian Lie group, the coadjoint action is trivial. Hence, the coadjoint orbits are simply the elements of $\mathfrak{g}^*$. The (trivial) geometric quantization of an (integral) coadjoint orbit $\xi \in \mathfrak{g}^*$ yields a 1-dimensional Hilbert space $\mathcal{H}_{\xi}$ containing a $U(1)$-class of normalized vectors. The unitary G-action on these vectors is implemented by means of the phase factors defined in (5). If we assume that quantum states are define modulo overall phase factors, the resulting quantum theory is composed of a unique quantum state that we shall denote $|\xi\rangle$. In what follows, we shall say that the state $|\xi\rangle$ is (G, ξ)-*phase invariant*. The *concrete* group (G, ξ) (i.e. the phase group G acting in the unirrep labeled by ξ) will be called *group of automorphisms* of the state. The group of automorphisms (G, ξ), while leaving the class $|\xi\rangle$ invariant, acts non-trivially on the corresponding $U(1)$-orbit, that is on what we shall call the *internal structure* of the state.

In this way, the points m in $\mu_M^{-1}(\xi) \subset M$ (for ξ integral) are associated to the unirrep ρ_{ξ}^G of G. This representation acts by means of (G, ξ)-phase factors on the internal structure of the unique quantum state $|\xi\rangle$ obtained by quantizing ξ. In this way, the "classical states" $m \in \mu_M^{-1}(\xi)$ and the quantum state $|\xi\rangle$ play different roles in the unitary representation theory of the same group G: whereas the "classical states" $m \in \mu_M^{-1}(\xi)$ define the unirrep ρ_{ξ}^G of G, the quantum state $|\xi\rangle$ belongs to the representation space in which this unirrep acts. According to this argumentative line, we could say that we misinterpret the notion of *phase space* if we interpret as a space of states. According to the proposed arguments, a Hamiltonian G-manifold (M, ω_M, μ_M) must be understood as a symplectic realization of the Poisson manifold $\mathfrak{g}^*$ that parameterizes the unirreps of G acting in the truly representation spaces, namely the Hilbert spaces obtained by quantizing the integral elements in $\mathfrak{g}^*$.

3. On Quantization and Symplectic Reduction

Thanks to the moment map, the (integral) "classical states" in M can be put in (a non-bijective) correspondence with the elements on $\mathfrak{g}^*$. In turn, the latter define 1-dimensional unirreps of G. These representations were obtained by quantizing the (trivial) integral coadjoint orbits $\xi \in \mathfrak{g}^*$. Now, what is the relationship between the Hilbert spaces $\mathcal{H}_{\xi}$ obtained by quantizing the coadjoint orbits ξ in $\mathfrak{g}^*$ and the Hilbert space $\mathcal{H}_M$ obtained by directly quantizing M? Since G acts simplectically on M, we could expect the quantization of M to yield a Hilbert space $\mathcal{H}_M$ endowed with a unitary representation of G. Now, the quantization of the coadjoint orbits $\mathcal{O} \in int(\mathfrak{g}^*/G)$ yields the unirreps $\mathcal{H}_{\mathcal{O}}$ of G (in what follows, we denote the coadjoint orbits by $\mathcal{O}$ when the corresponding argument or formula is valid for any coadjoint orbit besides the one-point orbits). Then, we should be able to decompose $\mathcal{H}_M$ into a sum of unirreps $\mathcal{H}_{\mathcal{O}}$, that is into a direct sum of the form

$$\mathcal{H}_M = \bigoplus_{\mathcal{O}\in int(\mathfrak{g}^*/G)} m(\mathcal{O}, M)\mathcal{H}_{\mathcal{O}}.$$

Here, $m(\mathcal{O}, M) \doteq Hom_G(\mathcal{H}_0, \mathcal{H}_M)$ (being $Hom_G(\mathcal{H}_0, \mathcal{H}_M)$ the space of all G-equivariant linear maps between $\mathcal{H}_0$ and $\mathcal{H}_M$) is the multiplicity with which the unirrep $\mathcal{H}_{\mathcal{O}}$ occurs in $\mathcal{H}_M$. Therefore, we must address the following questions: 1) what are the unirreps $\mathcal{H}_{\mathcal{O}}$ of G that occur in $\mathcal{H}_M$? and 2) what are theirs multiplicities $m(\mathcal{O}, M)$?

Since the unirrep $\mathcal{H}_{\mathcal{O}}$ is supported by the coadjoint orbit $\mathcal{O}$, Kirillov conjectured that $\mathcal{H}_{\mathcal{O}}$ occurs in $\mathcal{H}_M$ if $\mu^{-1}(\mathcal{O}) \neq 0$ (see Ref.[8], Th. 6.3). According to this idea, we have to consider the "realization" $\mu^{-1}(\mathcal{O}) \subset M$ of the universal coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. Let's begin by considering the simplest coadjoint orbit, namely $0 \in \mathfrak{g}^*$. If 0 is a regular value of the moment map, $\mu_M^{-1}(0)$ is a submanifold of M. The equivariance of the G-action implies that G preserves the surface $\mu_M^{-1}(0)$, i.e. that the zero value of the moment observables is conserved by the G-action. This means that the fundamental vector fields $v_X = \lambda(X)$ (defined in (2)) that infinitesimally realize the G-action on M are tangent to $\mu_M^{-1}(0)$. Now, the 2-form $i_0^*\omega$ obtained by restricting ω to the surface $\mu_M^{-1}(0)$ is not necessarily non-degenerate (where $i_0 : \mu_M^{-1}(0) \hookrightarrow M$ is the inclusion). Indeed, $(i_0^*\omega)(v, v_X) = 0$ for all $v \in T(\mu_M^{-1}(0))$ [8]. Therefore, the symplectic complement $(T\mu^{-1}(\xi))^{\perp}$ of $T\mu^{-1}(\xi)$ is included in $T\mu^{-1}(\xi)$, which means that $\mu^{-1}(\xi)$ is a co-isotropic submanifold of M. This means that the tangent spaces to the G-orbits define null directions for the restricted 2-form $i_0^*\omega$ on $T(\mu_M^{-1}(0))$. Therefore, $i_0^*\omega$ does not define a symplectic structure on $\mu_M^{-1}(0)$. Now, we have a chance to reobtain a symplectic manifold if we eliminate these null-directions, i.e. if we take the quotient of $\mu_M^{-1}(0)$ by the group action. This is the content of the so-called *Mardsen-Weinstein symplectic reduction* procedure [12, 15]. If 0 is a regular value of the moment map μ_M and G acts freely and properly on $\mu_M^{-1}(0)$, then it can be shown that the 0-*symplectic quotient* $M_0 \doteq \mu^{-1}(0)/G$ (also known as *reduced phase space* in the physics literature) is a manifold endowed with a unique symplectic structure ω_0 satisfying

$$\pi_0^*\omega_0 = i_0^*\omega,$$

where

$$\begin{array}{ccccc} \mu_M^{-1}(0) & \overset{i_0}{\hookrightarrow} & M & \xrightarrow{\mu_M} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi_0} & & & & \\ M_0 \doteq \mu_M^{-1}(0)/G & & & & \end{array} \tag{7}$$

This procedure can be generalized to any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ by means of the so-called *shifting trick* (see Ref.[8], p. 531). To do so, let's consider the Hamiltonian G-manifold $\mathcal{O}^- \doteq (\mathcal{O}, -\omega_{\mathcal{O}})$ dual to $\mathcal{O}$ endowed with the moment map $\mu_{\mathcal{O}^-}(\xi) = -\xi$. The product $M \times \mathcal{O}^-$ endowed with the symplectic form $\pi_{\mathcal{O}^-}^*\omega_{\mathcal{O}^-} + \pi_M^*\omega_M$ (where $\pi_{\mathcal{O}^-}$ and π_M are the projections on $\mathcal{O}^-$ and M respectively) is also a Hamiltonian G-manifold. The corresponding moment map $\Psi : M \times \mathcal{O}^- \to \mathfrak{g}^*$ is given by $\Psi(m, \vartheta) = \mu_M(m) - \vartheta$. The preimage of $0 \in \mathfrak{g}^*$ in $M \times \mathcal{O}^-$ is given by $\Psi^{-1}(0) = \{(m, \xi)/\mu_M(m) = \xi\} \subset M \times \mathcal{O}^-$. This set is in bijection with the set $\mu_M^{-1}(\mathcal{O}) \subset M$. Now, an element in $\Psi^{-1}(0)/G$ is an equivalence class $[(\mu_M^{-1}(\xi), \xi)]_G$. We can (partially) "fix the gauge" by choosing the representative(s) of this class defined by an element ξ_0 in $\mathcal{O}$. However, the fact that ξ_0 might have a non-trivial isotropy group G_ξ implies that this partial gauge fixing does not yield a single representative of the G-equivalence class $[(\mu_M^{-1}(\xi), \xi)]_G$, but rather the G_ξ-class $[(\mu_M^{-1}(\xi_0), \xi_0)]_{G_\xi}$. The set of these classes is in bijection with $\mu_M^{-1}(\xi_0)/G_\xi$.[5] All

[5] The possible obstruction to the freeness of the G-action on $\Psi^{-1}(0) \simeq \mu^{-1}(\mathcal{O})$ is given by the isotropy group G_ξ of the elements $\xi \in \mathcal{O}$. Hence, the group G acts freely on $\Psi^{-1}(0) \simeq \mu^{-1}(\mathcal{O})$ if G_ξ acts freely on $\mu^{-1}(\xi)$.

in all, we have obtained three equivalent expressions of the $\mathcal{O}$*-symplectic quotient* $M_{\mathcal{O}}$, namely

$$\begin{aligned} M_{\mathcal{O}} &\doteq \Psi^{-1}(0)/G \\ &\simeq \mu_M^{-1}(\mathcal{O})/G \\ &\simeq \mu_M^{-1}(\xi)/G_{\xi}. \end{aligned}$$

In what follows, the procedure of passing from M to $M_{\mathcal{O}}$ will be called $\mathcal{O}$*-symplectic reduction*. We could say that the correct notion of quotient in the symplectic category is not the set-theoretic quotient M/G but rather the space of G-orbits in the surfaces $\mu_M^{-1}(\mathcal{O})$. In fact, the set-theoretic quotient M/G is a Poisson manifold whose symplectic leaves are the symplectic quotients $M_{\mathcal{O}}$ (see Ref.[12], Th. 1.5.5, p. 326). This means that M can be decomposed in G-fibrations $\mu_M^{-1}(\mathcal{O}) \to M_{\mathcal{O}}$ over the $\mathcal{O}$-symplectic quotients $M_{\mathcal{O}}$.

The number of points in the $\mathcal{O}$-sympletic quotient $M_{\mathcal{O}}$ tells us how many times the orbit $\mathcal{O}$ "occurs" in M. Following the argumentative line of Kirillov's conjecture (i.e. that the unirrep $\mathcal{H}_{\mathcal{O}}$ occurs in $\mathcal{H}_M$ if $\mathcal{O}$ is in the image of μ_M), we could conjecture that the multiplicity of $\mathcal{H}_{\mathcal{O}}$ in $\mathcal{H}_M$ somehow depends on this number. We shall now show that this heuristic conjecture can be made precise by reconsidering what we understand by the notion of *point* in a symplectic framework. According to Guillemin and Sternberg's rephrasing of Weinstein' "*symplectic creed*" [21], "*the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds [...] it makes sense to regard the Lagrangian submanifolds as being its true "points"*." [8]. This notion of point can be formalized by using the idea coming from category theory according to which the morphisms $X \to Y$ between two objects in a category define the X-points of Y. In the present case, we must use what we call the G*-symplectic "category"* $Symp(G)$ [21]. The objects of $Symp(G)$ are Hamiltonian G-manifolds (M, ω_M, μ_M) and the morphisms between two objects (M_2, ω_2, μ_2) and (M_1, ω_1, μ_1) are the Lagrangian submanifolds of the symplectic manifold $(M_1 \times M_2^-, \pi_1^*\omega_1 - \pi_2^*\omega_2, \Psi \doteq \mu_1 - \mu_2)$ (where $\pi_{1,2} : M_1 \times M_2 \to M_{1,2}$ are the projections) contained in $\Psi^{-1}(0)$ (these morphisms in $Symp(G)$ are also called *canonical relations* from M_2 to M_1). In particular, the morphisms between the zero-dimensional symplectic manifold $\xi \in \mathfrak{g}^*$ and (M, ω_M, μ_M) are given by the Lagrangian submanifolds of $M \times \xi^- \simeq M$ contained in $\mu^{-1}(\xi)$. In categorical terms, the Lagrangian submanifolds of M in $\mu^{-1}(\xi)$ define the ξ-*points of* M. Coming back to the general case, $M_{\mathcal{O}}$ is a Hamiltonian G-manifold with trivial moment map $\mu_{M_{\mathcal{O}}}(m) = 0$ for all $m \in M_{\mathcal{O}}$. Indeed, G acts trivially on $M_{\mathcal{O}}$ given that $M_{\mathcal{O}}$ was defined by quotiening out the G-action. Then, the 0-points of $M_{\mathcal{O}}$ are simply its Lagrangian submanifolds. Now, the Lagrangian submanifolds of $M \times \mathcal{O}^-$ contained in the coisotropic submanifold $\Psi^{-1}(0)$ are G-invariant (Ref.[23], Prop.3.1). Therefore, they are in bijective correspondence with the Lagrangian submanifolds of $M_{\mathcal{O}} \doteq \Psi^{-1}(0)/G$ [7, 8, 22].[6] In other terms, the 0-points of $M_{\mathcal{O}}$ are in bijective correspondence with the (necessarily G-invariant) $\mathcal{O}$-points of M, that is with the G-equivariant morphisms $\mathcal{O} \to M$ ([7], Th. 3.1.1). Briefly, we have found the following bijection:

$$M_{\mathcal{O}} \simeq Hom_G(\mathcal{O}, M), \tag{8}$$

where $M_{\mathcal{O}}$ must be understood here as the set of its 0-points. According to this expression, the $\mathcal{O}$-symplectic quotient $M_{\mathcal{O}}$ can be called the *classical intertwiner space* between $\mathcal{O}$ and M [22]. From a conceptual viewpoint, it is worth stressing that the symplectic points of a Hamiltonian G-manifold, far from being simple geometric entities, have an internal structure. The important point in the present context is that the variable acted upon by G belongs to the internal structure of the $\mathcal{O}$-points of M.

[6]Let's consider for instance the symplectic manifold $M = (q_1, q_2; p_1, p_2)$. If the value of p_1 is fixed to p_1^0, then the p_1^0-points of M are given by the G-invariant Lagrangian submanifolds of $\mu_M^{-1}(p_1^0) = (q_1, q_2; p_1^0, p_2)$. These G-invariant Lagrangian submanifolds are $\mathcal{L}_{p_2}(q_1, q_2)$ and $\mathcal{L}_{q_2}(q_1, p_2)$. In both cases, the coordinate q_1 becomes an internal coordinate of the p_1^0-points of M. Now, these G-invariant Lagrangian submanifolds are in bijective correspondence with the Lagrangian submanifolds of $M_{p_1^0} \doteq \mu_M^{-1}(p_1^0)/G = \{q_2, p_2\}$. These Lagrangian submanifolds are $\mathcal{L}_{p_2}(q_2)$ and $\mathcal{L}_{q_2}(p_2)$.

In a seminal work, Guillemin and Sternberg found the quantum version of (8) [8] (see also Ref.[20]). Firstly, they conjecture (and demonstrate it for the particular case of a Khäler manifold) that *quantization (Q) commutes with symplectic reduction with respect to the trivial coadjoint orbit (R_0).* This is sometimes symbolically summarized by means of the expression

$$[Q, R_0] = 0.$$

More precisely, this means that

$$\mathcal{H}_{M_0} \simeq \mathcal{H}_M^G, \tag{9}$$

where $\mathcal{H}_M^G$ is the Hilbert space containing the G-invariant quantum states on M. This means that the Hilbert space obtained by quantizing the 0-symplectic quotient M_0 defined by the trivial coadjoint orbit $0 \in \mathfrak{g}^*$ is in bijective correspondence with the Hilbert space containing the G-invariant quantum states on M. Now, expression (9) just tells us that the dimension of $\mathcal{H}_{M_0}$ coincides with the dimension of $\mathcal{H}_M^G$, i.e. with the number of copies of the trivial 1-dimensional unirrep of G defined by $0 \in \mathfrak{g}^*$. In other terms, $dim(\mathcal{H}_{M_0})$ coincides with the multiplicity of the trivial unirrep of G in $\mathcal{H}_M$. Hence, we can rephrase (9) by means of the following expression:

$$\mathcal{H}_{M_0} \simeq Hom_G(\mathcal{H}_0, \mathcal{H}_M), \tag{10}$$

where $\mathcal{H}_0$ is the Hilbert space (obtained by quantizing $0 \in \mathfrak{g}^*$) carrying the trivial unirrep of G. In Ref.[8], Guillemin and Sternberg found the straightforward generalization of this expression to the general case of a non-trivial unirrep $\mathcal{H}_\mathcal{O}$:

$$\mathcal{H}_{M_\mathcal{O}} \cong Hom_G(\mathcal{H}_\mathcal{O}, \mathcal{H}_M). \tag{11}$$

This means that the dimension of $\mathcal{H}_{M_\mathcal{O}}$ yields the number of independent G-intertwiners between $\mathcal{H}_\mathcal{O}$ and $\mathcal{H}_M$. In this way, whereas the quantization of the coadjoint orbits $\mathcal{O}$ belonging to the image of μ_M yields the unirreps $\mathcal{H}_\mathcal{O}$ appearing in the decomposition of $\mathcal{H}_M$, the quantization of the $\mathcal{O}$-symplectic quotients $M_\mathcal{O}$ yields the multiplicity of the unirreps $\mathcal{H}_\mathcal{O}$ in $\mathcal{H}_M$. It is worth noting that in the general case each state in $\mathcal{H}_{M_\mathcal{O}}$ does not define a single state in $\mathcal{H}_M$ (as it happens in the particular case given by (9)), but rather a copy of $\mathcal{H}_\mathcal{O}$ in $\mathcal{H}_M$.

Expression (11) can be considered the quantum counterpart of (8): whereas at the classical level the 0-points of $M_\mathcal{O}$ define the G-equivariant maps $\mathcal{O} \to M$, at the quantum level the states in $\mathcal{H}_{M_\mathcal{O}}$ define the G-intertwiners $\mathcal{H}_\mathcal{O} \to \mathcal{H}_M$. In this way, *the (geometric) quantization of the classical intertwiner space $M_\mathcal{O} \simeq Hom_G(\mathcal{O}, M)$ yields the quantum intertwiner space $\mathcal{H}_{M_\mathcal{O}} \simeq Hom_G(\mathcal{H}_\mathcal{O}, \mathcal{H}_M)$*. In particular, if $\mathcal{O}$ is not in the image of μ, there is neither a $\mathcal{O}$-symplectic quotient $M_\mathcal{O}$ nor a reduced Hilbert space $\mathcal{H}_{M_\mathcal{O}}$. Hence, $m(\mathcal{O}, M)$ is zero, that is the unirrep $\mathcal{H}_\mathcal{O}$ does not occur in $\mathcal{H}_M$ (Kirillov's conjecture).

4. Gauge Invariance and Phase Invariance

The importance of the result (9) has been mainly stressed in the framework of the theory of *constrained Hamiltonian systems* (or *gauge theories*) [6, 9]. Indeed, the symplectic reduction procedure that we have just described is a particular case of the reduction formalism used in gauge theories [11]. If (9) is valid, one can quantize a gauge theory and impose the (quantum operators associated to the) constraints on the resulting quantum states instead of reducing at the classical level (which is in general problematic) and quantizing the reduced theory afterwards. In other terms, the following "diagram" commutes:

$$\begin{array}{ccc} M & \xrightarrow{\text{Quantization}} & \mathcal{H}_M \\ {\scriptstyle 0\text{-symplectic reduction}}\big\downarrow & & \big\downarrow{\scriptstyle G\text{-invariant states}} \\ M_0 & \longrightarrow & \mathcal{H}_{M_0} \simeq \mathcal{H}_M^G. \end{array} \tag{12}$$

The important point in the present context is that *the G-invariance of the quantum states of a gauge theory is the quantum counterpart of the classical symplectic reduction with respect to the coadjoint orbit* 0.

We shall now argue that expression (11) generalizes this correspondence between group invariance and symplectic reduction from first-class constraints to the properties defined by the moment observables in $\mathcal{C}^{\infty}_{\mathfrak{g}}(M)$. The important difference is that the moment observables are not constrained (like the constraints) to take a unique value in $\mathfrak{g}^*$. In the case of a one-point coadjoint orbit $\xi \neq 0$, expression (11) leads to a straightforward generalization of the bijection (9) given by (see Appendix for details)

$$\mathcal{H}_{M_\xi} \simeq \mathcal{H}_M^{(G,\xi)}. \tag{13}$$

Here, $\mathcal{H}_M^{(G,\xi)}$ denotes the Hilbert space containing the (G,ξ)-phase invariant states in $\mathcal{H}_M$, i.e. the quantum states $|\xi, ...\rangle$ whose group of automorphisms is given by the 1-dimensional unirrep of G defined by ξ:

$$\rho^G_\xi : e^X \mapsto e^{2\pi i \langle \xi, X\rangle}.$$

In this way, the ξ-symplectic reduction of M does not yield the G-invariant states in $\mathcal{H}_M$, but rather the (G,ξ)-phase invariant states. The corresponding commuting "diagram" is:

$$\begin{array}{ccc} M & \xrightarrow{\text{Quantization}} & \mathcal{H}_M \\ \xi\text{-symplectic reduction} \downarrow & & \downarrow (G,\xi)\text{-phase invariant states} \\ M_\xi & \longrightarrow & \mathcal{H}_{M_\xi} \simeq \mathcal{H}_M^{(G,\xi)}. \end{array} \tag{14}$$

The important point is that the (G,ξ)-phase invariance of quantum states is the quantum counterpart of the symplectic reduction with respect to the non-zero one-point coadjoint orbit ξ. This means that *quantum phase invariance is the generalization of the G-invariance appearing in gauge theories for the ξ-symplectic reductions with $\xi \neq 0$.*

We shall now use the correspondence (13) to revisit Heisenberg indeterminacy principle. As we have said before, the ξ-points of M (or, as we shall call them, the *symplectic realizations of $\xi \in \mathfrak{g}^*$ in M*) are G-invariant. We could thus say the variable acted upon by G belongs to the internal structure of the ξ-points of M. This is just the counterpart at the level of the symplectic realization M of $\mathfrak{g}^*$ of the fact that the isotropy group of ξ in $\mathfrak{g}^*$ is G itself. Now, (13) implies that the ξ-points of M (considered as 0-points in M_ξ) "support" the (G,ξ)-phase invariant quantum states $|\xi, ...\rangle$ in $\mathcal{H}_M$. We could thus say that the quantum states supported by the ξ-points of M have G as internal phase symmetry group. Exactly as it happens with ξ and $|\xi\rangle$, *the internal G-structure of the ξ-points of M is lifted to the (G,ξ)-phase invariance of the quantum states $|\xi, ...\rangle$ in $\mathcal{H}_M$ supported by these ξ-points.* Now, the quantum state $|\xi, ...\rangle$ is (G,ξ)-phase invariant, that is its group of automorphisms is (G,ξ). This means that the variable acted upon by G, far from being an "external" variable that could be used to localize the state $|\xi, ...\rangle$, is an "internal" variable acted upon by the group of automorphisms of $|\xi, ...\rangle$. Hence, the variable acted upon by G can be transformed without modifying the state $|\xi, ...\rangle$. In more usual terms, the variable acted upon by G is completely "undetermined".[7] This indeterminacy just reflects the internal G-structure of the ξ-points in M_ξ that support the states $|\xi, ...\rangle$. We could thus propose the following group-theoretical interpretation of Heisenberg indeterminacy principle: *if the variable that fixes the unirreps of G is sharply defined, then the conjugated variable acted upon by G is completely undetermined.*

We have argued that non-constrained quantum mechanics encodes a generalization of the symplectic reduction procedure used in gauge theories in the sense that it allows reductions with respect to non-zero

[7]In gauge-theoretic terms, the observables on M that are not G-invariant do not induce well-defined *phase invariant observables* on the ξ-symplectic quotient M_ξ that supports the states $|\xi, ...\rangle$.

coadjoint orbits. This is just a consequence of the difference between *moment observables* and (first-class) *constraints*: whereas one considers all the possible values of a moment observable (like for instance the different values of the momentum p), one restricts the theory to a single value of a constraint (namely zero). In other terms, whereas in gauge theory one only considers G-invariant quantum states (i.e. states whose group of automorphisms is given by the trivial unirrep of the gauge group), in non-constrained quantum mechanics one considers (G, ξ)-phase invariant states *for all the integer values* ξ (i.e. states whose group of automorphisms can be given by the different unirreps of the phase group). Now, in quantum mechanics one can superpose the (G, ξ)-phase invariant states defined by the different unirreps ρ^G_ξ. In other terms, $\mathcal{H}_M$ contains states of the form $|\psi\rangle = \sum_\xi \psi(\xi)|\xi, ...\rangle$. The important point is that the state $|\psi\rangle$ is not G-phase invariant since the action of G changes the relative phases between the terms $|\xi, ...\rangle$ of the superposition. Therefore, *a state* $|\psi\rangle$ *not transforming in a well-defined unirrep of* G *is not* G*-phase invariant*. This means that the superposition of states transforming under different unirreps of G breaks the G-phase symmetry. In turn, the presence of states transforming under different unirreps of G is just a consequence of the fact that one does not "constraint" the theory to a single value of the corresponding moment observable. In this way, the introduction of an indeterminacy in the variable that labels the unirrep of G breaks the complete indeterminacy in the variable acted upon by G. We can thus conclude that *Heisenberg indeterminacy principle relates the indeterminacy in the value of the variable that defines a unirrep of* G *and the (inversely correlated) indeterminacy in the value of the variable acted upon by* G.

5. Conclusion

In order to plead in favor of a group-theoretical interpretation of mechanics, we have analyzed the case of a symplectic manifold (M, ω) endowed with a strongly Hamiltonian symplectic action of an abelian Lie group G. We shall now summarize the results of this analysis.

Firstly, we have argued that the classical and quantum description of a Hamiltonian G-manifold (M, ω) should not be considered as alternative descriptions of the same physical system, but rather as different structures of the unitary representation theory of G. Whereas the ξ-points of M are the symplectic realizations of the multiplicative characters $\xi \in \mathfrak{g}^*$ that define the unirreps ρ^G_ξ of G, the quantum states $|\xi, ...\rangle$ obtained by quantizing these ξ-points belong to the representation spaces wherein these unirreps act.

Secondly, we have argued that the phase-invariance of quantum states generalizes the strict invariance appearing in gauge theories to the case of (moment) observables that are not constrained to take a single value in $\mathfrak{g}^*$. Now, the quantum phase invariance can be understood as a manifestation of Heisenberg indeterminacy principle. Therefore, the fact that phase invariance and gauge invariance have the same geometric underpinning (namely, as we have argued, the symplectic reduction formalism) points towards a gauge-theoretic interpretation of Heisenberg indeterminacy principle. In gauge theories, the variable canonically conjugated to a constraint $G(q, p)$–i.e. the variable acted upon by the Hamiltonian vector field v_G–is completely "gauged out" by the gauge transformations generated by the operator defined by $G(q, p)$. Analogously, the variable canonically conjugated to a moment observable f taking a well-defined value $f = f_0$ is completely "phased out" by the phase transformations generated by the operator defined by f.[8] In group-theoretical terms, by fixing a unirrep ξ of G we obtain quantum states $|\xi, ...\rangle$ that do not depend on the variable acted upon by G. Roughly speaking, the value of such a variable is "pure phase". In this conceptual framework, Heisenberg indeterminacy principle is not a consequence of the commutativity properties of the phase group G, but rather a consequence of the *duality* between the unirreps of the phase group G and the variables acted upon by G in certain G-sets (namely, the ξ-points of M).[9]

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[8] This gauge-like interpretation of Heisenberg indeterminacy principle was proposed in Refs.[3, 4, 5].

[9] In Ref.[16], we provide a Galoisian interpretation of the duality between unirreps and the elements of a G-space.

By resuming ideas proposed by Weinstein, Guillemin and Sternberg [7, 8, 21], we have argued that in order to unveil the rational necessity of Heisenberg indeterminacy principle, we cannot simply understand a Hamiltonian G-space M as a *partes extra partes* arrangement of mutually external and structureless set-theoretic points. Rather, a Hamiltonian G-space M must be understood in terms of its ξ-points, i.e. in terms of its G-invariant Lagrangian submanifolds. The fact that these ξ-points are G-invariant implies that the variable acted upon by G, far from being an "external" variable that could be use to separate the different ξ-points, belong to the "internal" G-structure of the latter. This difference between *structureless set-theoretic points* and *structure-endowed symplectic points* explains the fact that the quantum states $|\xi, ...\rangle$ supported by the ξ-points can be defined by using only half the number of variables needed to localize a set-theoretic point in M (that is, a "classical state"). The (im)possibility of localizing a quantum state in M depends on the notion of point that we are using: whereas quantum states cannot be completely localized in the set-theoretic points of M, they can be completely localized in the symplectic ξ-points of M. Of course, considered from a set-theoretic perspective, the symplectic ξ-points of M are intrinsically delocalized.

6. Appendix. Multiplicities of non-trivial 1-dimensional unirreps

Since the conjecture (9) only concerns the value $0 \in \mathfrak{g}^*$ of the moment map, it cannot be directly applied to a non-zero element ξ. In other terms, the fact that M_ξ cannot be obtained as a symplectic reduction of M with respect to the zero value of the original moment map μ_M implies that the quantum states in $\mathcal{H}_{M_\xi}$ are not in bijection with the G-invariant states in $\mathcal{H}_M$. However, we shall now show (by following Ref.[8]) that the shifted moment map $\Psi : M \times \mathcal{O}^- \to \mathfrak{g}^*$ allows us to obtain a correspondence analogous to (9) for ξ-symplectic quotients M_ξ with $\xi \neq 0$ (namely, the correspondence (13)).

Let's consider the line bundle $L_M \boxtimes L^*_\xi \doteq \pi^*_M L_M \otimes \pi^*_{\xi^-} L^*_{\xi^-}$ defined by the following diagram

$$\begin{array}{ccccc} L_M & & L_M \boxtimes L^*_\xi & & L^*_\xi \\ \downarrow & & \downarrow & & \downarrow \\ M & \xleftarrow[\pi_M]{} & M \times \xi^- & \xrightarrow[\pi_{\xi^-}]{} & \xi^- \end{array}$$

By using that $M_\xi \simeq \Psi^{-1}(0)/G$, the Guillemin-Sternberg conjecture (9) yields the following bijection:

$$\mathcal{H}_{M_\xi} \simeq \mathcal{H}^G_{M\times\xi^-}$$

where

$$\mathcal{H}^G_{M\times\xi^-} \doteq \Gamma^G_{\mathcal{P}}(L_M \boxtimes L^*_\xi)$$

is the Hilbert space of G-invariant sections of $L_M \boxtimes L^*_\xi$ polarized with respect to a suitable polarization $\mathcal{P}$. Now, the quantum theory over ξ is composed of a unique state, namely the state $|\xi\rangle$ associated to the $U(1)$-class of normalized section of L_ξ. The phase group G acts on the elements of this equivalence class by means of phase factors of the form $e^{2\pi i\langle\xi,X\rangle}$ for $X \in \mathfrak{g}$. Let's write now an element in $L_M \boxtimes L^*_\xi$ as $s = s_M \otimes \langle\xi|_0$, where $\langle\xi|_0$ is a representative in the class $|\xi\rangle$. The G-action on $L_M \boxtimes L^*_\xi$ is given by

$$(e^X \cdot s) = \rho^G_M(e^X) \cdot s_M \otimes e^{-2\pi i\langle\xi,X\rangle}\langle\xi|_0.$$

A section s will be G-invariant iff

$$\rho^G_M(e^X) \cdot s_M \otimes e^{-2\pi i\langle\xi,X\rangle}\langle\xi|_0 = s.$$

This will be the case for the sections s^ξ_M in L_M that transform under G as

$$\rho^G_M(e^X) \cdot s^\xi_M = e^{2\pi i\langle\xi,X\rangle} s^\xi_M.$$

Hence, the G-invariant states in $L_M \boxtimes L^*_\xi$ are defined by the sections s^ξ_M in L_M that transform under G by means of phase factors of the form $e^{2\pi i\langle \xi, X\rangle}$. On the other hand, a G-invariant section $s = s^\xi_M \otimes \langle\xi|_0$ in $L_M \boxtimes L^*_\xi$ defines a morphism $T_s : L_\xi \to L_M$ given by

$$T_s(|\xi\rangle_0) = s^\xi_M \otimes \langle\xi|_0\xi\rangle_0 = s^\xi_M.$$

The fact that s^ξ_M transforms under the G-action by means of phase factors of the form $e^{2\pi i\langle \xi, X\rangle}$ guarantees that T_s intertwines the two representations. All in all, we have the following maps

$$\mathcal{H}_{M_\xi} \to \mathcal{H}^G_{M\times\xi^-} \to \mathcal{H}_M$$

given by

$$\psi \mapsto s_\psi \mapsto s^\xi_M = T_{s_\psi}(|\xi\rangle_0).$$

In this way, a state ψ in $\mathcal{H}_{M_\xi}$ defines a (G,ξ)-phase invariant state in $\mathcal{H}_M$.

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