A Quantum Mechanical Measurement Leading to Simultaneous Spin-Up and Spin-Down State of a Single Electron

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Abstract

In is shown that when two observers carry out a simultaneous measurement on a pair of spin-½ particles in a “singlet” state a possibility exists for an outcome that lacks physical meaning. More specifically, despite the fact that the two commuting operators formally possess simultaneous eigenvectors it is not possible for physical reasons these eigenvectors to exist simultaneously. It is pointed out that the possibility for such non-physical outcome is observed only in the case of “singlet” state which puts into question the physical meaning of such state, and following from it exotic notions such as “non-locality”. This hints that probably QM in its present form should be reconsidered with regard to its limitations.

Let us begin with some well-known background. As is known, the spin operator (the observable representing the spin of the particle) for the first particle in the compound space of the two two-state particles $\mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$\vec{S}_1 = \frac{\hbar}{2}\left(\vec{a} \cdot \vec{\sigma}\right) \otimes \mathbf{1}$$

while the same operator for the second particle is

$$\vec{S}_2 = \mathbf{1} \otimes \frac{\hbar}{2}\left(\vec{a} \cdot \vec{\sigma}\right)$$

where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, $\vec{a}$ is a unit vector in some general direction and

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ is the vector of the Pauli matrices. In other words, to make an operator act on this 4-dimensional space we have to form a direct product of the identity matrix with the vector of the Pauli matrices (the vector of Pauli matrices is the observable representing the spin of the particle in the two-dimensional space).

Now, in a usual manner, we will make three simplifications. First we will drop the coefficient $\frac{\hbar}{2}$ in the expressions of the operators – this will not affect our final result. Second we will observe the vector of the Pauli matrices directed only in z-direction, i.e. we will work only with $\overrightarrow{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Third, we will carry out the measurement not along just any vector, but along the z-direction. Thus, we may now rewrite the spin-operators in view of this new notation and show them explicitly in a matrix form

$$\vec{S}_1 = \overrightarrow{\sigma_3} \otimes \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\vec{S}_2 = \mathbf{1} \otimes \overrightarrow{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Obviously, the unit vector along the z-axis is dotted into the above matrices.

The basis states for the direct product of the single electrons, comprising a pair we are to observe are <u>four, namely</u>:

$$|++\rangle \qquad |+-\rangle$$

$$|-+\rangle \qquad |--\rangle$$

We will observe one special state in which our two spin-½ particles can supposedly exists, namely the so-called "singlet state", which at the conditions of our discussion is written as:

$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|+-\rangle \quad - \quad |-+\rangle\big)$$

Now, if we are to carry out an experiment to determine the spin of the first particle we have to act on this $|\psi\rangle$ with the operator $\vec{S}_1$.

It can be shown that if a measurement is to be performed there is a 50% chance of obtaining eigenvalue $S_1^z \;=\; 1$. There is also a 50% chance of obtaining $S_1^z \;=\; -1$. Suppose now that as a result of the measurement the eigenvalue $S_1^z \;=\; 1$ was indeed obtained (the probability of $\left|-\frac{1}{\sqrt{2}}\right|^2 \;=\; 50\%$ was realized). Then the subsequent state of the system will be $|+-\rangle$, i.e. the 1st electron will be in a spin-up state while the state of the 2nd electron will be spin-down.

The opposite will be the case if as a result of the measurement the obtained eigenvalue was $S_1^z \;=\; -1$. In such a case the subsequent state of the system will be $|-+\rangle$, i.e. the 1st electron will be in a spin-down state while the state of the 2nd electron will be spin-up.

Similar will be the case with the operator $\vec{S}_2$ and its eigenvalues and eigenvectors.

So far this is a well-known state of affairs. Now, however, we want to explore whether the two operators, namely $\vec{S}_1$ and $\vec{S}_2$ can have simultaneously the same set of eigenvectors. As is known, a criterion for two operators to have simultaneous eigenfunctions its their commutator to be equal to zero. In other words, we have to explore whether the following equality is true:

$$\left[\vec{S}_1,\vec{S}_2\right] = \vec{S}_1\vec{S}_2 - \vec{S}_2\vec{S}_1 = 0$$

This can be checked immediately:

$$\vec{S}_1\vec{S}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\vec{S}_2\vec{S}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As can be seen above two products are identical which implies that indeed the commutator in question is zero. This, however, means that the operators $\vec{S}_1$ and $\vec{S}_2$ have simultaneous eigenvectors.

However, if that is the case then the simultaneous application of $\vec{S}_1$ and $\vec{S}_2$ on the state $|\psi\rangle$ may lead to a situation whereby the application of $\vec{S}_1$ will bring about the eigenvector $|+-\rangle$ while the application of $\vec{S}_2$ will bring about the eigenvector $|-+\rangle$. This is an algebraically viable situation since the operators $\vec{S}_1$ and $\vec{S}_2$ share these two eigenvectors.

However, this outcome of the measurement will bring about a confusion from a physical point of view as to what exactly the subsequent state of the electrons is. For instance, the subsequent state of the 1st electron after the measurement done with $\vec{S}_1$ will be spin-up while the subsequent state of this same 1st electron but as a result of measurement with $\vec{S}_2$ has to be spin-down. Therefore, according to this argument, in 50% of the cases after the measurement the 1st (and the 2nd) electron will be at once both in spin-up and spin-down state. This outcome of our thought experiment, however, is contrary to the laboratory experimental observations – so far an electron existing simultaneously in spin-up and spin-down state has not been observed experimentally.

One may even argue further that in allowing for the very possibility for both 1st and 2nd particle to be spin-up in a collapsed state, i.e. for the possibility the subsequent state to be $|++\rangle$ (resp. $|--\rangle$) we in fact allow for the possibility for a state the probability of whose establishment is 0%. In other words, the two observers cannot make it so that through their measurements the final state to be $|++\rangle$ (resp. $|--\rangle$). This is an argument which rejects the mere possibility for simultaneous measurement carried out by two observers – because at such measurement a situation may be encountered whereby an improbable final state will be established such as $|++\rangle$ (resp. $|--\rangle$). However, there is nothing to forbid, as far as a physical experiment is concerned, two experimenters from carrying out simultaneous measurements on this system. In other words, if an observer carries out an independent but simultaneous measurements on the 2nd particle, according to the above example, he/she may find out that the probability of getting spin-up or spin-down may not turn out to be 50%. This will be due to the fact that unsuspectedly someone else is carrying out simultaneous measurements on the 1st particle (as mentioned, there is nothing to forbid two independent observers from carrying out measurements on each one particle simultaneously). But not getting 50% probability goes contrary to the result, established on the basis of QM principles, that the probability must be 50%.

Discussion

Here we will repeat some of the above and in the process will add some more conclusions. For instance, we will note that we discuss a non-degenerate case and will again emphasize the following – the fact that the operators S_1 and S_2 commute is very important to realize because it is the pivotal point of the whole argument. Thus, suppose that we have convinced ourselves that S_1 and S_2 indeed commute (as we actually already did). In such a case we do not doubt that these two operators have simultaneous eigenvectors – this is exactly what the algebra requires. Therefore, nothing prevents us from accepting that in a case when operator S_1 has the eigenvector $\left|+-\right\rangle$ operator S_2 has simultaneously the eigenvector $\left|-+\right\rangle$. Note that even before we spoke about the commutator we had established that $\left|+-\right\rangle$ and $\left|-+\right\rangle$ are eigenvectors of S_1 and S_2, however, now, after we convinced ourselves that the commutator $\left[\vec{S}_1,\vec{S}_2\right] = \vec{S}_1\vec{S}_2 - \vec{S}_2\vec{S}_1 = 0$, we already know something more, namely, we now know that these two vectors can be the eigenvectors of the observed operators simultaneously.

Let us imagine further that there are two observers who carry out simultaneously and independently measurements of the spin of the particles. The first observer uses for this purpose operator S_1 to measure the spin of particle #1. The second observer uses the operator S_2 to measure the spin of particle #2.

As it was shown above, each one of the observers has a 50% chance to prepare the system after the measurement in a subsequent state $|+-\rangle$ as well as a 50% chance to prepare the system after the measurement in a subsequent state $|-+\rangle$.

In other words, one possibility is the first observer to have a 50% chance to prepare the system as a result of the measurement in the state $|+-\rangle$ while the second observer to have the chance to prepare the system as a result of the measurement in the state $|-+\rangle$.

Now notice, since the observers carry out their experiments simultaneously none of them is in a position to prepare the system in a certain definite state, suitable for a predictable experimental result to be obtained by the other observer. The results of the experiments of both observers will be purely random.

Let us imagine now that out of the several possible results the two observers happen to realize exactly the result supposed a minute ago, i.e. the first observer turns out to have prepared the system after the measurement in a state $|+-\rangle$ while the second observer, it turns out, happens to have prepared the system after the measurement in the state $|-+\rangle$. As already mentioned several times, nothing so far forbids us to suppose such a possibility: $|+-\rangle$ and $|-+\rangle$ are eigenvectors both of S_1 and of S_2, S_1 and S_2 commute, therefore the eigenvectors in question can be observed simultaneously, the observers carry out their experiments on the "virgin", if I may say so, function of state $|\psi\rangle$ (i.e. there is no reason to suppose that the mentioned 50-50 chance is not valid any more) etc. Therefore, so far everything is in accordance with the formal requirements of QM and according to the rules of QM nothing forbids us to obtain the mentioned result. However, if it turns out that there are problems with the physical interpretation of the above-obtained result, although obtained in full compliance with the QM rules, then these problems would put under question the QM theory of measurement itself – a theory which leads to a non-physical result is incorrect.

Unfortunately, even a superficial glance at the result we obtained above – a system existing after the measurement both in state $\left|+-\right\rangle$ and in state $\left|-+\right\rangle$ – shows that there are indeed problems. Thus, as already mentioned, there is a chance as a result of the QM experiment the spin of particle #1 to be both spin-up and spin-down (same applies to particle #2). Such a result, however, obtained on the basis of QM is not in accordance with the well-established experimental facts in the laboratory – a state of the particle (e.g. an electron) at which this particle has both spin-up and spin-down has never been observed experimentally. Conclusion – QM experiment conducted in full concordance with its rules has led to an outcome that does not have physical meaning. It is not uncommon in science when physical inconsistency of even such a seemingly insignificant level may lead to reconsidering the rules of a theory, even a theory of the magnitude of QM.

Here is a brief recount of the above. We determine that two operators are commuting. The fact that these operators are commuting means that they have simultaneous eigenvectors even if these eigenvectors are orthogonal. Thus, algebraically, we can choose any combination of these eigenvectors and still the eigenvectors comprising the couple can be simultaneous. Surprisingly, however, it turns out that if we want this outcome to have physical meaning, the discussed simultaneity is not always acceptable. When the two operators act on one particular state-vector ("singlet" state) two of the common eigenvectors, for purely physical reasons, cannot be simultaneous. It turns out that for physical reasons, specifically, orthogonal eigenvectors cannot be simultaneous despite the fact that algebraically there is no requirement that the two eigenvectors of these two commuting operators should not be orthogonal for these vectors to be simultaneous eigenvectors.

Thus, notice the contradiction. As seen, according to the commutativity of S_1 and S_2, algebraically, there is certainly a 50% chance to observe simultaneously the vectors $\left|+-\right\rangle$ and $\left|-+\right\rangle$. However, according to the paradoxicality of such an outcome from a physical point of view the simultaneous obtainment of these vectors is impossible. Algebraically the simultaneity of two eigenvectors of two commuting operators is undeniable. The impossibility for these two eigenvectors to exist simultaneously is only due to the fact that they emerge from a construct ("singlet" state) which is proposed to be a viable physical entity. As seen, however, such a proposal breaks down under the conditions discussed above of a simultaneous measurement.

If the above considerations are acceptable and we have to exclude the "singlet" states as viable physical entities then a concept such as "non-locality", which is a direct result of improper considering the "singlet" state as physically viable, also has to be abandoned since its physical meaning will be questionable as well.

In other words it is not that "non-locality" is some kind of property of QM which we have to take for granted and which gives rise to phenomena worth discussing. Quite the opposite – the notion of "non-locality" is a consequence of the standard formal acceptance in QM that even strange entities such as "singlet" state, as long as they are products of the Hilbert space, are physically realizable. As we saw above, from physical point of view the reality of such a state is in fact problematic.