A few historical-critical glances on mathematical ontology through the Hermann Weyl and Edmund Husserl works

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Abstract. From the general history of culture, with a particular attention turned towards the personal and intellectual relationships between Hermann Weyl and Edmund Husserl, it will be possible to identify certain historical-critical moments from which a philosophical reflection concerning aspects of the ontology of mathematics may be carried out. In particular, a notable epistemological relevance of group theory methods will stand out.

The term *mathematical entity* is quite often used in the everyday mathematical terminology without, nevertheless, putting that right epistemological attention which is needed whenever we use a term belonging to one of the most crucial philosophical issues, like that regarding the term *entity* which goes through the whole course of the history of philosophy. Therefore, attention should be paid to such a term and to its contextual use. In this paper, without having a full completeness' aim, we would like to devote a bit of our epistemological reflection to try to contextually clarify some of the most elementary conceptual foundations which lie at the bases of the term mathematical entity, starting from the very remarkable work made by Hermann Weyl in philosophy of mathematics and natural sciences. In what follows, not without a certain obvious fear, we briefly sketch some main philosophical definitions with which we are concerned herein: in doing so, we have considered (Abbagnano 1998), (Brezzi 1994) and (CSFG 1977) as main references thanks to which to have a general and schematic but enough and rigorous sight of what we are discussing.

The term *being*, as known, has two main distinct uses, a *predicative* one and an *existential* one. The former, in turn, distinguishes three uses which entail three distinct doctrines of the predicative being: the *doctrine of the inherence*, the *doctrine of the identity* and the *doctrine of the relation* of the predicative being. In turn, the latter distinguishes between a *subjective* and an *objective* standpoint of the predicative relation identified in the copula. The subjective standpoint starts from a subjective view of the copula which assigns a central role to the thinking subject, hence giving (Cartesian) pre-eminence to the idea as an immediate object of human knowledge. In this context falls the Kantian a priori synthetic judgement seen as a relational synthesis between elements of states of the *same* subject. Hence, also the subjective view highlights the relational nature of the structure of the judgements involving a predicative being, but that remains always centred around the subject, as well as its components relate those objects to which them refer. For instance, in the sentence «Socrates is a man», we do not mean Socrates as the representation of a human being but refers to the real individual to whom the name refers. Instead, the objective standpoint distinguishes a relation between objects from a relation between facts or propositions, so giving rise to the mathematical logic. Furthermore, many types of copula are also classified, reaching the highest level of generality and identifying a functional structure between the elements of a proposition which will become variables. The attention will be turned towards the possible relations between representations, propositions, ideas or concepts, disregarding the role of subject. As regards that historical period herein considered, ranging from mid-1800s to early 1900s, on the one hand we are concerned with the predicative use of the term being when we refer to the proper historical evolution of the logic, while, on the other hand, we are also concerned with the existentialist use of the term being (that, roughly, is centred around the *modes* or *modalities of concernancy* of the being[1]) when we will deal with moments of the history of philosophy involving phenomenological-existentialistic trends. As we shall see later, certain aspects of the appreciated Weyl's work will be the opportunity to fruitfully meet both these two historical moments. *Existence* is, very roughly, that for which a thing there exists, or better a delimitation of the being, a definite *mode* of being. After Descartes, it is important to consider the possible relationships between existence and essence, the latter expressing that intelligible *determination* by which a thing there exists as such. In philosophy, the term existence is usually referred to modalities of being real, whereas, instead, the first definition – as a delimitation of the being, as a definite mode of being – is usually adopted by sciences when one talks about the existence of mathematical entities, the existence of physical entities, and so forth. We refer to the existential use of the term Being, not to the predicative use of it. The last phenomenological-existentialistic trends, above all from Husserl[2] and Heidegger onward, are turned to consider the proper human modalities of being, so that we may say that, according to these recent trends, when one deals with existence issues, then a pre-eminence is assigned to the modalities or manners of being. For this reason, we think that Husserl's phenomenological thought may have interesting implications for the philosophy of mathematics, in particular as regards its ontological questions.

A *thing* refers, in its wider, generic and undetermined sense, to an arbitrary entity or term, a real or unreal, mental or physical, or merely possible *quid*, with which, in any fashion, one has to do. Frequently, it is referred to the external or natural world, so that the thing is an *object*, or else every possible reality, which may exists, in a fictitious or real manner, outside our thought. On the other hand, an *object* refers to what a faculty, a tendency, a practice or an (intentional) act is turned. We may have a *material* object[3], which is the thing considered in itself, and a *formal* object, which is the aspect under which the thing is considered. The material object may be identical, whereas the formal objects may be variable. Though thing is almost always considered as a synonymous or object, the former dates back to ancient epoch (πράγμα), the latter was introduced by Scholastic around 13-th century and is the philosophical term which has the widest and undetermined meaning. Because of that, in the sequel, we will refer to the notion of object in the sense of Husserl's thought, relating it to the notion of intentionality, so reaching, at the end of this paper, to a possible characterization of a mathematic object as entity. Indeed, the term *entity*, introduced by Duns Scoto, refers to an object existing in the predicative sense of the term existence, as provided by a specifically defined modality of being. In the contemporary logic, the term is used to indicate every object of which it is possible to define the related *existential status*, or, as the saying goes, every object with respect to which the linguistic use entails a certain ontological commitment. Rudolf Carnap has insisted on the importance of this term, stressing the fact that the logical entities are not reducible to sensible data, that is to say, they are not real entities, in this being therefore in agreement with Husserl. On the other hand, the term *relation* (τὸ πρὸς) refers to the modality of being or to the modality of behaviour of the objects between them; this is basically what Aristotle wanted to express «as that thing whose being consists in behaving according to a certain modality with respect to something other», analogous to how Charles Sanders Peirce defined the relation «as a fact concerning a certain number of things». Therefore, we return to the modality of behaving with respect to the otherness. The philosophical *vexata quæstio* involving the notion of relation is centred around its reality and objectivity characters, giving rise to three main fundamental doctrines in dependence on the supposition that the relations are objective and real, the relations are neither objective nor real and the relations are objective but not real. The latter is the most modern trend, and is closely related with the meaning of predicative being taken by contemporary logic, so to the notion of entity just above sketched. This last point of view is moreover just what Husserl states about ontology of mathematical entities as, for instance, very well exposed in (Caniglia 2006).

[1] Starting from the well-known Aristotle's statement according to which «the being is said in many manners».
[2] See (Spiegelberg 1960).
[3] For a deeper epistemological analysis of the notion of *thing*, see (Toraldo di Francia 1986).

In conclusion, albeit this framework is very roughed out, we may however already understand that the philosophical notions of object, entity, being, existence and relation have strictly correlated amongst them, the one referring to the other, revolving around the notion of *modalities of being*, that is to say, from Aristotle onwards, the various differences of predication, that is to say, the various fashions with which a predicate is referred to a subject, till to the modalities of concernancy of the being. Thereafter, the notion of modality too undertaken an historical evolution of its own until up reach a wider realm of modalities (epistemic, deontic, etc.) and the modal logic which are nowadays fundamental chapters of the philosophy of science. As concerns, then, a general modern overview of the philosophy of mathematics, we refer to (Lolli 2002) to which we remand for every deepening of what herein is said. In its first part, the author deals with the quite prickly problem of what is philosophy of mathematics, trying to answering, not univocally and not in a definite fashion, in the second part of the text with a general survey of the possible answers, that is to say, outlining the various philosophies of mathematics as historically they have appeared. The variety of answers is mainly due to the rich *problematicity* which characterizes, due to its intrinsic nature, almost every philosophical issue and, in general, every knowledge question. A central problem of the general philosophical reflection is the historical dichotomy between *realism* and *nominalism* about the acceptance or not of the existence of abstract concepts, general terms and universals[4], which, in the case of the philosophy of mathematics, essentially reduces to the dichotomy between the existence or not of mathematical entities. In these last terms, mathematics and metaphysics cannot be disentangled because they are closely interwoven.

1. On the advent of non-Euclidean geometries

Following (Nacinovich 1996, Introduzione), the geometry has been maybe the first knowledge's field which organized as a science in the modern sense of this latter term. Since ancient epoch, geometry has been an inspirational model of great scientific and philosophical syntheses aimed to try to rationally explain the Universe and to understand our own position in it. Due to this, the geometry has taken part to the evolution of human thought, so its history may also be read as in basic dialectic relationship with the culture of western world. Up to the coming of the Kantian criticism, the geometry was considered as a chapter of physics, mainly meant as the formal description of the space in which the phenomena take place as well as of the interactions between material objects. In the *Critique of Pure Reason*, the Euclidean space is the necessary premise to every physical knowledge; space and time are innate pure forms of perception or intuition. Nevertheless, the rising of non-Euclidean geometries around 19-th century raised the crucial question concerning the exact description of the physical world by mathematics, with the well-known measurements made by Gauss to establish what geometry were the true one. But Felix Klein, in his celebrated 1872 prolusion, known as *Erlangen Program*, made explicit a new radical conception of geometry which will deeply influence, in a decisive manner, the next development and content of all the mathematics, starting from the previous 1868 work made by Hermann von Helmholtz who was the first to attempt to found geometry on the properties of the group of motions (see (Weyl 1918) and (Schmidt 1979)); moreover, A.F. Möbius too had already put into evidence the importance of group theory point of view as regards certain geometries. In Klein, the already known real preoccupations to fix an environment in which to give a certain geometry are coexistent with a new intuition that shifts the investigation from the objects in themselves to the relationships between them. This change of perspective will invest all the mathematical edifice, comprises logic[5],

[4] As regards the well-known dispute on universals, we remember the distinction, made by Scholastics, amongst the *ante rem*, *in re* and *post rem* existentialistic precedence modalities of the universal with respect to the object.

[5] It was mainly Charles Sanders Pierce (1839-1914), the founder of pragmatism, and Ernst Schröder (1841-1902), to have opened, on the wake of what previously made both by August De Morgan (1806-1871) and by George Boole (1815-1864), the way to the symbolic logic of relations that will find its rigorous and systematic setting in the later work Bertrand Russell. Almost in the same period, we attend to the crucial passage from the pre-Fregean classical logic (mainly analytics of the proposition, little clearness on the different functions played by the verb 'to be' and

casting an important and innovative bridge between geometric entities and algebraic tools. Therefore, the fundamental subject-matter of geometry becomes the projective space, within which the choice of a transformation group that defines the congruence between the given geometrical entities, allows us to find the various geometries due to Euclid, Lobačewskij, Riemann, and so forth. This formulation explicitly puts at the centre of attention the concept of *transformation group* whose main structure, that of group, had already been worked out by J.L. Lagrange, H.N. Abel, E. Galois, S. Lie, C. Jordan and others. Afterwards, the revolution made by the work accomplished by G.F. Frobenius, W. Burnside, H. Weyl, E. Nöther, I. Schur, F. Peter, É. Cartan, E.P. Wigner and others, since the beginnings of 20-th century, will bring these intuitions to their extreme consequences, choosing as starting point the notion of *group* and building up the space by means of its *realizations* (or its *representations* in the linear case). According to (Weil 1928, Introduction; 1962), the earliest works of art show that the symmetry groups of plane figures were known since antiquity, although the theory of these was only explicitly given in definitive form in the latter part of the 18-th and in the 19-th centuries.

Following (Radicati 1982, 1983), Sophus Lie too, in his celebrated 1893-94 work on continuous transformation groups, had already considered the basic structure of the physical laws as described by the symmetry of the underlying space-time continuum. This Lie's conjecture has continued to be true until up recent time: indeed, before the pioneering work (Shechtman et al. 1984) (see also (Emmer 1991, Chapter II)), it was believed that long-range orientational order crystals cannot and do not exhibit the icosahedral point group symmetry mainly because of the lack of lattice invariance translation imposition, a fundamental law of crystallography since then. But, thanks to the researches made by Shechtman and co-workers, a new type of crystals was found, that of *quasi-crystals*, whose mathematical structure has already been described by the previous work made by Roger Penrose on tessellations. Nevertheless, Penrose himself was sceptic in believing that nature could have such type of symmetries like the pentagonal and icosahedral one. Klein himself had already done important theoretical studies on this last type of symmetry in relation to the resolution of algebraic equations via Galois theory, exposed in the famous 1888 treatise *Lectures on the Ikosahedron and the Solution of Equations of the Fifth Degree*. The icosahedral phase of the metallic compound analyzed in (Shechtman et al. 1984) shows a symmetry intermediates between those of a crystal and a liquid. It differs from other intermediate phases in that it is both solid, like a metallic glass, and that it has long-range orientational order. Many intermediate phases do have orientational order, but usually it is only local, and the transition to such phases is continuous. The possibility of an icosahedral phase with long-range order was inferred from computer simulations, and a first-order liquid-to-icosahedral phase transition was predicted from a mean-field theory. Therefore, also in this case, the theory of groups and their representations provide new sights of the physical world, that is to say, allowing the discovery of new spatial structures having physical relevance, the *quasicrystals*, which break the ordinary translational symmetry. The seminal paper (Shechtman et al. 1984) just begins with the following words

> *«We report herein the existence of a metallic solid which diffracts electrons like a single crystal but has point group symmetry m35 (icosahedral) which is inconsistent with lattice translations. We have observed a metallic phase solid with long-range orientational order, but with icosahedral point group symmetry, which is inconsistent with lattice translations. Its diffraction spots are as sharp as those of crystals but cannot be indexed to any Bravais lattice. The solid is metastable and forms from the melt by a first-order transition».*

Thus, this existence has been possible only by comparison of two different representations of the involved symmetry groups with related symmetry breaking of the translational invariance. This

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understanding of the statement in terms of subject-predicate) to the Fregean one (mainly disambiguation of the verb 'to be' and understanding of the statement in terms of argument-function), with major load given to the relational viewpoint.

main fact has therefore pointed out the existence of another physical entity before undetected by sprung out from symmetry arguments (see also Penrose's lecture in (Chandrasekharan 1986)). Roughly speaking, the theory of the representations of a group G consists in realizing, in many different fashions called *representations*, this structure as a group of linear transformations defined upon different linear spaces V, so that it basically try to homomorphically transpose an abstract group structure into one having a preeminent relational nature, that is to say, a structure of linear transformations. Therefore, a *representation* of G into V is simply a group homomorphism of the type $\rho_V^G: G \rightarrow Aut(V)$. The theory of group representations has been the logical accrual of the 18-th century theory of invariants of binary algebraic forms as due to A. Cayley, J.J. Sylvester, R.F. Clebsch, P. Gordan and others. This revolution has an important conceptual analogy in philosophy with the Edmund Husserl phenomenology (which had influences and insights from the geometric context of the time) and with the work of Bertrand Russell on the logic of relations and in physics with the coming of relativity theory and quantum mechanics[6], basically pointing out that what our physical experience may catch is not the *being in itself* but rather the *symmetries* of nature which manifest themselves only through the *interactions* between the different objects variously involved. At the end of this process, the crisis of the certainty as well as any possible issue on realism are overcome by the capacity of geometry of comprehending that the different objects of physics, with their specific features and properties, have in themselves a geometry of their own which reveals their intrinsic phenomenology.

The theory of group representations may be roughly considered as a generalization of harmonic analysis: indeed, we start with a given Hilbert space of functions upon which acts a certain symmetry group, hence we develop each function into an expansion, formed by component functions which are invariant respect to this symmetry group, in such a way that this decomposition isn't further simplified, that is to say it is irreducible. Therefore, in a certain sense, the set of group representations of a given group characterizes this last, and, in this regards, a celebrated theorem due to A. Cayley simply states that every group is isomorphic to a well-determined transformation group. On the other hand, historical roots of group theory are closely interwoven with the history of transformation groups (see (Caldirola & Loinger 1979, Chapter III)), while, following (Radicati 1982, 1983), in the late 1800s, we also attend to the advent of group theory methods in physics. Indeed, Sophus Lie, a pupil of Felix Klein, around 1890s published his celebrated *Theorie der Transformationsgruppen* for which he wrote a long but meaningful preface where he discussed the origin and the developments of his ideas on group theory. Though Lie never explicitly applied his magnificent theory to physics, he was definitively aware that it would provide the key to a geometrical interpretation of the physical principles on the wake of the influential Klein's *Erlangen Program* (of 1872): indeed, just shortly later, G. Hamel (around 1904) and F. Engel[7] (around 1916) provided important and unexpected group theoretical bases to some main principles of mechanics, which will be later extended and generalized by Emmy Nöther (around 1918) whose general work was later systematized by her pupil B.L. Van der Waerden; finally, the philosophy of Klein's program will be successfully applied to the early developments of 20-th century physics with the dawning of relativity theory. At almost the same time, Pierre Curie published a short but seminal paper where he too forcefully asserted the usefulness of group theory for physics. Klein rightly considered the group concept as the most characteristic one of 19-th century mathematics (see (Weyl 1928, Introduction)). Albeit contemporary, Lie's and Curie's points of view were however quite different: while Lie was interested in the mathematical structure of the fundamental equations and saw in their invariance the origin of the conservation principles (in a certain sense, with a research program analogous to the Galois one, but applied to differential equations rather than to algebraic ones), Curie's more limited aim was to provide a truly simple comprehensive rational explanation of the numerous electromagnetomechanical effects that occupied at that time the

[6] See, above all, the magnificent work (Weyl 1928). See also the *operational* viewpoint of P.W. Bridgman in physics which arose almost contemporaneously to the dawning of the Jan Brouwer *intuitionism*.

[7] Friedrich Engel (1861-1941) was a German mathematician, pupil of Klein and co-worker of Lie.

attention of most physicists. He was therefore primarily concerned not with the symmetry of the equations, but rather with that of their solutions, i.e. with the symmetry of the physical states. This phenomenological approach led him naturally to emphasize the role of asymmetry rather than that of symmetry, so giving rise to the notion of *symmetry breaking* which will play a very fundamental role in the 20-th century physics until up nowadays. The introduction of group theory methods in physics were then employed, in all their full relevance, both by Weyl and by Eugene P. Wigner (see (Wigner 1931)), in the early 1900s (see (Iurato 2013)).

2. On the axiomatic method and other

Following (Dieudonné 1970, Capitolo I), since ancient times (e.g., Aristotle) every science bases itself upon what that it is possible to call as the *principle of the voluntarily incomplete knowledge*: indeed, abstraction or generalization means systematically ignore certain aspects of the objects under consideration, and paying attention to other ones. The axiomatic method is nothing but an application of this principle, enumerating, in a exhaustively manner, those properties that one intentionally or explicitly wants put into evidence, in relation to the objects under examination, through *axioms*, and, as saying goes, forbidding to appealing to anything other that isn't such properties and logic rules. Therefore, axiomatic method provides a necessarily incomplete knowledge framework. Following the Italian introduction to (Hilbert 1970) due to Carlo Felice Manara (1916-2011), the rising of the non-Euclidean geometries stimulated the philosophical reflection on the nature and on the objects of a mathematical theory, as well as on the related and crucial existential questions mainly through logical compatibility and coherence arguments. According to Hilbert, the existence of a mathematical entity relies only on coherence and compatibility conditions, no matter possible intuitive or constructive features of the entity (as required by Brouwer) which must be built up only axiomatically. On the basis of the rigorous setting given by Hilbert to geometry, Manara refers too about a possible origin of a mathematical entity as due to a sort of *explicitation* of what *implicitly* given or inferred by a certain coherent and compatible enunciated set of syntactic rules, amongst which axioms: for instance, the axioms may give an implicit definition of certain mathematical entities similarly to the case in which the rules of a given card game (like Poker) provide an implicit definition of the playing cards themselves used. Therefore, according to this last viewpoint, it seems that a certain pre-eminence should be given to the syntactic moment rather than to the semantic one in questions inherent the existence of mathematical entities: for instance, a quite similar situation might be descried in the dawning of the notion of the structure of differentiable manifold (see (Iurato 2012a,b)). Likewise, an analogous situation might be identified in the rising of the structure of group, which seems searchable out in the attempts, first due to Lagrange and Galois, to find the possible relationships existent between the roots of an algebraic equation. An brief but meaningful historical account of the occurrence of the just above mentioned dialectic relationship between implicit and explicit knowledge regarding group structure is given by (Miller 1931). On the other hand, just this dialectic opposition between implicit and explicit isn't in no way a mere void rhetoric performance but has a deep and valid epistemological status: this stance is, for instance, stressed in (Medvedev 1991) about the historical evolution of the notion of function according to which it had already present, as saying goes *in nuce*, since ancient epoch, having done, as known, its definitive explicit appearance in the 17-th century with the dawning of infinitesimal analysis, even if, as pointed out in (Ponte 1992) and (Eves 1997), already in the Middle Age could be identified prolegomena to the modern notion of function. Finally, also Hermann Weyl and Moritz Schick pointed out the importance of an implicit definition (as just above briefly accounted) not only for mathematics but for the all sciences in general (see (Weyl 1949, Chapter 1, Section 4)): in this regards, for instance, the term *Lie algebra* appears first in a 1934-35 Weyl lectures on the structure and representation of continuous groups delivered by Weyl at Princeton but was explicitly suggested by Nathan Jacobson (who edited such lectures) even if the concept was already implicitly present very early in the theory mainly under the usual terminology of ''Lie infinitesimal group'' (see Borel lecture in (Chandrasekharan 1986) and (Weyl 1990, Preface by A. Loinger, 4.).

Again going on with group theory, after the pioneering work of Ruffini, Abel and Galois[8], the existence of roots (obtainable for resolution by radicals) is just closely related to properties of a well-definite group of transformations (said to be *Galois group*) connected with the coefficients of the equation under examination. To be precise, following (Weyl 1946, Chapter II, Section 1) and (Straumann 1998), decisive for the development of group theory was the use by Galois (around 1832) made of groups of permutations for the investigation of algebraic equations; he genially recognized that the relation of a field algebraic extension K of its ground field k is to a large extent determined by the group of automorphisms. His ingenious theory may be described as the algebraic relativity theory of a finite set of numbers which are given as the roots of an algebraic equation. Nevertheless, Galois' very brief allusions remained for a long time a booklet of seven seals. After the previous work made by Joseph Liouville around the mid-1800s, only Camille Jordan, in his famous *Traité des substitutions et des équations algébriques* (of 1870), was the first mathematician to have explicitly recognized and developed what preciously made by Galois. Also in this last case, however, we reconnect with what said above around the fundamental role played by the theory of groups and their representation, in mathematical ontology issues, in part already foreshadowed in (Iurato 2013) where the wide emergence of group theory in physics, with its deep epistemological implications, is highlighted. Indeed, in almost all context of physics, from the classical one (see, for instance, the celebrated work made by Pierre Curie on symmetry principles mentioned above[9]) to the modern physics (relativity theory, quantum and statistical mechanics, particle physics, supersymmetry theory, and so forth), it is inconceivable does not deal with group and symmetry questions just in connection to existential questions: just to mention very few elementary instances, in statistical physics the existence of phase transitions is ascertained by means of group theory methods (see, for instance, (Iurato 2012c,d) and references therein); in quantum mechanics, there are different formal representations of a physical observable, like the Schrödinger, the Heisenberg and the Dirac representations[10]; in mechanics, there are two main representations of the motion of a continuum medium, the Lagrangian representation and Eulerian one; in statistical mechanics, basically there are three chief representations in dependence on the Gibbs ensemble representation chosen; and, in fundamental physics, an elementary particle is a particular unitary irreducible representation of the Poincaré group, characterized by two parameters, namely mass $m \geq 0$ and spin s or helicity λ. The violation of symmetry principles are at the grounding of the contemporary fundamental physics: in this regards, it is enough to look at precious booklet (Yang 1961).The comparison between all these possible representations, as well as symmetry arguments, are strictly related to crucial existential questions concerning physical reality: for instance, just gauge invariance arguments and themes are called into question about the existence of the so-called *Aharonov-Bohm effect* – see (Bocchieri 1985) and later on. Following (Sellerio 1935), (Michel 1980), (Radicati 1983), (Gieres 1997), (Straumann 1998) and references therein, the formal and conceptual tools of symmetry breaking theory rely on group theory. Symmetry breaking entails asymmetry which, in turn, since Curie's pioneering works, is what creates a phenomenon, that is to say, it lies, as a necessary condition[11], at the basis of the existence of a phenomenon (see also

[8] See (Lombardo-Radice 1965, Capitolo IV) and (Purkert & Wussing 1994, Section 1.1).

[9] See also (Caldirola & Loinger 1979, Appendix) and references therein.

[10] It is noteworthy to recall what Dirac said in the Preface to the 1930 first edition of his celebrated work *The Principles of Quantum Mechanics*, namely that, as regards the mathematical line to be followed to describe the new quantum theory, two main trends were available, the *symbolic method*, which directly treats, in a thoroughly abstract manner, the physical quantities having fundamental importance (the invariants of transformations, and so forth), and the *coordinate* or *representation* method, mainly adopted in *wave mechanics* (if one considers the *states* of a physical system as physical entities having pre-eminence) and in *matrix mechanics* (if one gives pre-eminence to the *dynamical variables* as physical entities). Dirac said too that Weyl, in his famous 1928 work (see (Weyl 1928)), was the first one to have used the former method.

[11] Which, in general, is not sufficient too.

(Radicati 1983), (Nicolle 1950, Chapter V), (Whyte 1949, (D)) and references therein): as examples, from a purely speculation stance, the P.M.S. Blackett's equations connecting the magnetic field and the angular momentum of massive rotating bodies satisfy one Curie's necessary condition for the *presence* (or else, following Heidegger, the mode of being of the things, distinguished by the mode of being of the human being) of a magnetic field, i.e. the *absence* (as not presence) of the axial symmetry or the presence of an axial vector; the *Magnus effect* of fluid-dynamics and the *Wiedemann effect* of electromechanics, are further very simple instances of the existence of a physical phenomenon due to the occurrence of an asymmetric cause; likewise, the molecule of ozone O_3 there exists because of the action of *Jahn-Teller effect* which implies a distortion of the initially alleged equilateral geometrical disposition of the three oxygen atoms of this molecule into an isosceles triangular arrangement (see (Bersuker 2007)). Furthermore, from what has been said above, one might argue that, on the one hand, the breakdown of a physical and/or of a geometrical symmetry, which often underlies the occurrence of a physical phenomenon, might be drawn near the manifestation or the revelation of the continuous *Eraclitean becoming* or *dynamicity* of the being, whereas, on the other hand, the physical invariance, as revealing objective features of the being, might be drawn near the *Parmenidean fixity* or *staticity* of the being; and this latter will be just what, in part, Weyl himself stands up.

3. On Husserl's phenomenology: very brief outlines of some its main points

Edmund Husserl (1859-1938) studied mathematics with Karl Weierstrass, then he passed to study philosophy and psychology with Franz Brentano. Following (Hartimo 2010, Chapter VI), Husserl was fascinated by the Weierstrass' manner of doing and teaching mathematics, basically consisting in a mixture of rational thought and irrational instinct and tact into a purely rational thought. The Weierstrass' *ethos* in mathematics was characterized by an aspiration to a descriptive completeness, to capture everything there is to say about the domain of a theory. This ethos greatly influenced Husserl and his view of *Definithein*, as well as other mathematicians like Hilbert himself. Herein, we shall give a very brief account of the basic points of Husserlian thought needed for what follows, considering (Abbagnano 1995) as a main reference.

To renew philosophy either from a worldly attitude (like that of traditional empiricism and of natural sciences[12]) and from an extremely abstract and inconclusive idealism, Husserl introduced his new trend centred around a new conception of reality meant as the outcome of that unavoidable and indistinguishable relation between subjectivity or consciousness (which gives sense) and external objectivity (which gives content): indeed, the former wouldn't have any meaning without some objective content (because the consciousness is always awareness of …), while the latter would have neither sense nor value if it received not, from consciousness, either a well-determined aspect and form. Therefore, by means of such a concept, said to be *intentionality*[13], we reject, on the one hand, every objectivistic reality conception (materialism) and, on the other hand, each subjectivistic reality conception (idealism), so reaching to a *relationistic* position for which has no importance the reality's being in itself (which, in any case, is incognizable for the human being) but rather the importance of the this being for each individual, that is to say, her or his lived experience. The meaning is an intentional correlate of consciousness. The nature of a knowledge fact consists in a mere relation between a subjective and real pole (noesis) and an objective and ideal pole (noema), so trying to overcome the alternative realism-idealism into the reciprocal dynamicity of a relation

[12] Whose knowledge aim was mainly turned towards the practical uses and purposes of things.

[13] Following (Brezzi 1994), the general term *intentionality* refers to the *otherness*, to relationship oneself with other from self, as well as to the characteristic of every act of the human being to be inclined towards something other beyond itself, either having practical value (e.g., a loving act towards the lower) and theoretical nature (e.g., the idea towards the thought thing). It is a notion which dates back to Scholastic (not by chance, in the same period in which the concept of function began to gradually become even more explicit), standing for the conceptual or mental reference to the thing. Thereafter, such a term was quite neglected, to being retaken later by Franz Brentano to characterize psychic phenomena, in contraposition to the physical ones.

which guarantees both the independent giveness of the datum and its indeclinable connection with the intuiting consciousness. A central point of the Husserlian thought is the consideration of the essences of the modalities of knowledge. Phenomenology is not a science of facts but a science of essences (an *eidetic science*) which are immediately picked up through a suspension judgement, called *phenomenological epoché* or *phenomenological reduction*, which is turned towards the essential content of a phenomenon, suspending any reality acquisition or acknowledgement, with all its practical interests, biases and superstructures, and implicitly contained into every natural knowledge inclination of human being (comprised every scientific attitude), to become an observer who is interested only to try grasping or catching the *essence* of those acts by means of consciousness is in relationship with the reality and gives meaning to this[14]. Following what says Sofia Vanni Rovighi in (CSFG 1977), the intuition of essences is the root of the a priori; logic and mathematics are made by a priori propositions which express relations between essences. Besides an empirical ego, ingenuously interested in the world and its superstructures, a phenomenological ego takes place, as a disinterested spectator through which an immediate apprehension of the essences is possible, putting aside biases and previous knowledge and inter-subjective contents, which constitute those preconceived and often fideistic rigid *mental grids* with which we judge the world, putting aside as well the thesis of the existence of the world. Only in this manner, we may immediately to be aware of the data in themselves, in their pure quality, elevated to the level of pre-giveness.

Another central point of the Husserlian framework is the notion of intentionality. Since consciousness is always awareness *of* something (every *cogito* has a *cogitatum* of its own), an arbitrary analysis of consciousness is the analysis of the acts with which the consciousness puts itself into relationship with its objects, or else, it is the analysis of the modalities with which such objects are offering to the consciousness. These latter consciousness acts or modalities of the giveness with which the objects give themselves to consciousness, form the so-called *intentionality* of consciousness. According to Husserl, consciousness is (but not only) a flow (*Erlebnisstrom*) of lived experiences (*Erlebnisse*) each of which has an essence of its own, to which the intentioned object is offered through a precise modality. Nevertheless, this object does not belong to the lived experiences but it becomes an intentioned thing of consciousness through an opposition of two main aspects, namely a subjective aspect formed by those real acts which are aimed to catch the object, said to be *noesis*, like the perceiving, the remembering, the imaging, and so forth, and an objective aspect, like the perceived, the remembered, the imagined, and so forth, said to be *noema*, this latter not being the object itself, but rather it is what is present into consciousness, that towards which the consciousness is opening: for instance, in perceiving a tree, the object if the tree but the noema of this perception is the set of all the predicates or of all the being's modalities given by experience, that is to say its giveness: the green tree, the illuminated tree, the leafless tree, the perceived tree, the remembered tree, and so on. The real object is a pole of identity always having a preconceived sense which must be realized; in each moment of consciousness, it is the index of a noetic (or noema making) intentionality which is the meaning of this object, and that may be problematized and becoming explicit. Therefore, the object constitutes the pole around which revolve and clustering the various noemata of the lived experiences. We perceive something because it presents to us under different and various aspects, whereas a consciousness fact does not occur to us by aspects. According to Husserl, there is coincidence amongst intuition, evidence and verity with which either an abstract (or ideal) object and a concrete one are given, although with different modalities, to consciousness. The basic phenomenological relationship between subject and object in terms of intentionality opens the way to the realism: indeed, so as the subject who intentions the object does not become part of the latter, so too the intentioned object does not become part of the subject. For instance, feelings and sensations belong to consciousness, are elements of the Erlebnisse, whereas sounds and colours are, instead, intentional objects.

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[14] This idea will be also retaken later by mathematical *intuitionism*.

But consciousness field is larger than intentionality that characterizes the Erlebnisse which, therefore, turn out to be not intentioned objects, because they are given as experiences immediately, originally and indubitable lived or experienced by the human being which give rise to the *absolute being* sphere, and of which consciousness has an *immanent* perception, the perception that consciousness has of its own experiences, put against to the *transcendental* perception that consciousness has of those things belonging to that *presumptive* reality which may be put into Cartesian doubt differently from the *being-there* (or *being-now*) of the *Erlebnis* which is indubitable. In turn, lived experiences require a necessary subject, the *Ego*, to which Husserl gives a preeminent ontological role, on the wake of the Cartesianism. Following (Lolli 2002, Part II, Chapter 4), the human knowledge, included the mathematical one, is characterized by intentionality, that is to say knowledge is always turned towards something, is aimed by the interest for something, the intentional object. Then, and this is a central point, the cognitive acts have a perspective nature and never can catch all together the various perspectives or views of the intentional object, that is to say, we never have experience of something through a unique act. Furthermore, every cognitive act is related to the essences of the intentional object through categories. These ideal essences are gathered or caught, by (eidetic) intuition, in a well-determined knowledge pre-categorical stage, which may become, through successive refinements, even more precise and clear as time goes on, until up reach the categorical stage. Nevertheless, the clarification work is always current and necessary, the process of categorical knowledge being always a work in progress because we never have a complete, perfect and definitive consciousness of the essences of the things, since the more the time goes on, the more we have an improved knowledge of them. We use the expression «eidetic intuition of the essences» because, when we reflect on our own experience (*Erlebnis*), an essence is given to us in an immediately manner which precedes our analysis of it as well as the comparison of it with the others. The phenomenology is a science of essences (an eidetic science) or ideas, not of facts. Mathematics may also fall into the realm of phenomenology if one considers the mathematical entities basically as ideal essences. Husserl criticized both mathematical Platonism and realism because they do not put attention to the mathematical essences because believed out of any possible experience, considered as objects in themselves. Instead, we can approach them thanks to intention, that is to say, we are intentionally oriented toward essences.

Very little attention has been paid by philosophy of science (and of mathematics, in particular) to the Husserlian phenomenology[15]. Nevertheless, Gödel and G.C. Rota have made some interesting reflections on it. We are mainly interested, for our purposes, to what Gödel briefly states to this end. In particular, Gödel points out the fundamental role played by laws and eidetic rules in building up an ideal object, included the mathematical entity. The *eidetic rules* basically are the *perspective visions*: only through them, that is to say only putting an object under a series of *eidetic variations*, it will be possible to identify the essence of such an object. This is a very basic and central philosophical matrix – drawn from (Lolli 2002, Part II, Chapter 4) – of the wider Husserlian system, which will turn out to be of fundamental importance for our scopes. Following (Uehlein 1992) and (Hartimo 2010, Chapter IV) as concerns the role of eidetic variations on mathematics, the *eidetic reduction* consists in applying a sequence of eidetic variations upon the intentioned object until up reach that phenomenological residue, said to be *eidetic essence*, which is that key invariant and universal datum, apodictically obvious, as a result of the application of a series of repeatedly eidetic variations upon the intentioned object, just characterizing this last. Therefore, following what is said in *Erfahrung und Urteil – Untersuchungen zur Genealogie der Logik* (1948), the Husserl's method of eidetic reduction consists in finding at first one model, for instance drawn from an experiential or ideal objectivity, hence puts it at an abstract level in which the Ego may work out new variations upon this initial model which, therefore, will be transformed to obtain new images of it from which sequence it turns out to be manifest their essence, the *eidos*, an invariable quid. Husserl states that such a sequence of variations is however finite, that every element must be put in comparison

[15] See also (Israel 2011).

amongst them to be congruent towards the eidos. Husserl more times tries to pursuing a method (the phenomenological method) which resembles the pureness of mathematics, on the wake of what already made by Kant in the second book *Transcendental Doctrine of Method* of the *Critique of Pure Reason*, considering this last as ‘’an organon of the Being’’ (following an expression due to T.W. Adorno in his famous 1960s *Philosophische Terminologie* – see (Adorno 1975) – who was too an eminent scholar of the Husserlian thought). However, differently from Kant, Husserl tried to apply mathematical method to philosophy. In this regards, we are interested towards the possible influences exerted by 19-th century mathematics on this ambitious Husserlian program which will lead to the phenomenological method; in particular, on the basis of what has been said above, we focus on the possible role played by Klein’s Erlangen Program about the dawning of the Husserl’s phenomenological reduction method as above briefly described, and on the outcomes of its action. In any case, already there exists literature which confirms what notable role played the theory of representation of space in the early developments of Husserlian phenomenological thought: see, above all, (Hartimo 2008) and references therein, where many points here discussed, first of all the influence of group-theoretic methods, find a confirmation.

To be precise, if one carries on with this phenomenological reduction, until up its extreme consequences, we reach the consciousness in itself, with the intentionality’s character of its own. This latter is the real nature of consciousness. Hence, the phenomenological residue of the epoché resists to every attempt of methodical doubt, is apodictically evident. Furthermore, following (Abbagnano 1998), within the wider and general notion of *representation*, we pick up the use made by Husserl of this notion on the legacy left by Brentano. Namely, Husserl retaken the basic distinction between *Vorstellung* and *Repräsentation* (or *Darstellung*[16]) inherent an *intentional Erlebnis*: the former is the full act of the mere representation in the sense that every thought is offered to consciousness in a purely intuitive and immediate fashion free of every form of previous knowledge, while the latter is that additional component which makes determinate the reference to

[16] It is well-known the distinction between *Vorstellung* and *Darstellung* first identified by A. Schopenhauer, who didn’t consider synonymous such two terms, in turn pointing out too on the semantic distinction between each member of the two pairs *vorstellung/Vorstellung* and *darstellung/Darstellung*. It seems just that the prefixes *vor-* and *dar-* provide such a different semantic function (see (Foschi Albert 2011)): the former falls into the conceptual and mental domain and is used to express a meaning of variability and versatility inherent spatial and temporal multiform relationships, while the latter refers to the comprehending modalities with which a concrete or abstract thing may be exposed or realized, as well as its image (Schopenhauer takes, as example, an artistic representation). An interesting historical fact is that some contemporary poets, like Ludwig Tieck (1773-1853), make use just of these terms also in reference to mathematical context. On the other hand, Tieck and Schopenhauer were friends and often talked about many philosophical issues (see (Safranski 1989, Book Two)), as well as Schopenhauer’s thought on mathematics and sciences in general, exerted a great influence on the early developments of Husserl’s thought (see (Vandenabeele 2012, Part I, Chapter 3)), above all his celebrated two-part 1818 treatise *The World as Will and Representation* (hereafter, WWR I-II) in which the just above mentioned terms are discussed by Schopenhauer and in which intrinsically interesting philosophy of science considerations are exposed. Husserl, at his time, was fully immersed into the general cultural context of the epoch, hence feeling the Romanticism influences of then, which, among other, also exerted their not indifferent role in formulating that strict and inseparable relation between *pre-categorical* and *categorical* moments of thought, explaining as well that pre-eminence given by Husserl to the first respect to the second moment (see (Geymonat 1981, Volumes VI, VII)). Schopenhauer’s thought on mathematics, logic and natural sciences is quite original, unique and very interesting (in this regards, see the very valuable work of Dale Jacquette entitled *Schopenhauer’s Philosophy of Logic and Mathematics* and comprised in (Vandenabeele 2012, Part I, Chapter 3)), deserving much more attention than what, at a first sight, might not seem. In particular, reading that part of Schopenhauer’s thought concerning the passage from intuitive to abstract knowledge in mathematics, we notice strong conceptual analogies with what Husserl will do later, above all as regards the notion of apprehension of the object, the pre-eminence given to the intuitive perception of the object respect to the abstract knowledge of this, the former providing insight into the ground of being upon which the latter depend on. He furthermore classifies all knowledge as intuitive (or perceptual) and abstract representations which must be put into reciprocal comparison. In WWR I, Schopenhauer stresses the fact that it is the immediate intuitive grasp of mathematical relations (to which he gives epistemological pre-eminence) rather than abstract rules of an axiomatic mathematics, on which every mathematical judgment is properly grounded. This basically two-level mathematical knowledge conception of Schopenhauer, surely will influence next Husserl’s thought. On the other hand, it is also noteworthy, from a historical standpoint, to notice that either Hermann von Helmholtz and Bernhard Riemann work too influenced Husserl’s thought (see (Hartimo 2008) and references therein).

the object (that is to say, gives content, ''matter''), through a certain concernancy's modality. This last characterization (also taking into account what said in the footnote [16]) has an analogical parallelism with the conceptual foundation of group representation theory in which, roughly, an abstract group, immediately and intuitively conceived as characterizing the symmetry properties of a given entity, should correspond to the Husserl's Vorstellung, whereas the next representation of it, through a certain linear space of transformations, should correspond to the Repräsentation (or Darstellung).

4. Some historical-epistemological considerations: firstly

4.1. We consider two very simple cases of existentialistic issues in elementary mathematics which, however, both refer to a basic underlying relational character.

A first merely existentialistic result is, for instance, provided by the proof of the following very elementary result.

Theorem. *The set of transcendental real numbers $\mathbb{R}_t$ is not empty and has the continuum power*.

Proof. We have $\mathbb{R} = \mathbb{R}_a \cup \mathbb{R}_t$ with $\mathbb{R}_a \cap \mathbb{R}_t = \emptyset$ and $\mathbb{R}_a$ set of algebraic real numbers, so that $\mathbb{R}_t = \mathbb{R} \setminus \mathbb{R}_a$. But $\mathbb{R}$ has the continuum power greater the denumerable one, whereas $\mathbb{R}_a$ has the denumerable power, so that $\mathbb{R}_t$ must have too the continuum power, hence it is not empty and equipotent with $\mathbb{R}$. QED

This is a purely existential proof of the non-emptiness of $\mathbb{R}_t$, which has consisted in proving that the only set $\mathbb{R}_a$ couldn't complete $\mathbb{R}$, even without explicitly know any transcendental number, but, however, referring to other two already known results concerning cardinality of sets, namely putting into reciprocal comparison already known facts concerning the cardinalities of $\mathbb{R}$ and $\mathbb{R}_a$, so appealing to a basically relational argument. Instead, a constructive proof (also in the intuitionistic sense) would consist in explicitly finding a real number which is proved to be a transcendental one: this effectively was what historically happened when F. Lindemann proved, after two millennium of failed attempts, the transcendentality of π in 1882.

As said above, another remarkable existentialistic case regards group theory, where a special attention should be paid to the Galois theory according to which an algebraic equation is solvable by radicals if and only if the related Galois group is solvable: this is another very interesting existential question closely related to group theory methods. As briefly said above, the advent of group theory methods, with 1872 Felix Klein work, has revolutionized either mathematics and physics. Now, following (Van Dalen 1984), (Lolli 2002), (Bernet et al. 2005), (Tieszen 2005), (Mancosu 2010, Part 3) and (Embree & Nenon 2013, Part I, Chapter 7), Husserl studied at Halle and Göttingen universities where respectively Cantor and Hilbert taught just in the period in which Husserl attended his studies. Klein was also active in Göttingen just in the same period, becoming head of that Faculty of Philosophy that regrouped philosophers and mathematicians together; Hermann Minkowski joined this Faculty in this period, where he formulated those bases of space-time theory which will be the dawning of the Einstein relativity theory, as well as Ernest Zermelo, so that the mathematical interests weren't only relegated to the mere formalistic work but also embracing a large land of philosophical questions: in particular, Hilbert was working on his celebrated *Grundlagen der Geometrie*, while Zermelo was carrying on with his notable researches on the foundations of set theory. Starting from the previous work on algebraic structures made by Klein, Dedekind and Kronecker in the late 1800s, Hilbert will throw the bases for an algebraic theory of the formal structures which emphasize the relational nature between objects no matter their nature; this work will be carried on by Emmy Nöther and Emil Artin as well as will form the central core of the next Boubakist program. In this fervid and active atmosphere of Göttingen, Husserl, as extraordinary professor, held lessons on various philosophical teachings regarding foundations of mathematics, followed by many students some of whom will become leading scientists: amongst them, Max Born, Ernst Hellinger and Dénes König. Husserl, therefore, was into touch with many leading scientists and mathematicians of the time, amongst whom Hilbert, Klein, Paul Bernays, Ernst Zermelo, Frege, so that mathematics has surely played a very deep role in Husserl's thought training. But, a particular student of Husserl was Helene Joseph who will marry Hermann Weyl, maybe the most famous pupil of Hilbert, who will deeply mark the course of 20-th century mathematics and physics. By means of Joseph, Weyl became aware of Husserl work, becoming so interested by Husserl work on phenomenology that it will play a very fundamental role in formulating and developing his notable work on mathematics and physics. The acquaintance amongst Weyl and Husserl began since the latter was chair of the valuation committee of Weyl's dissertation, which carried on later also in an epistolary manner: in particular, in a 1918 letter, Husserl expresses his appreciation that a mathematician, like Weyl, recognized the importance of a phenomenological treatment of grounding scientific concepts. Following Husserl, Weyl thinks that a priori concept of space in physics (with its notions of congruence, etc.) needs to be aligned with the phenomenological conception of lived space. Indeed, already in his *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie* (of 1913), Husserl talks about the profound phenomenological problem of the idea of the origins of the idea of space in relation to the experience of things as near or far, oriented in a certain way with regards to us. He then returns on this problem in *Die Ur-Arche Erde bewegt sich nicht* (of 1934) and in *Ursprung der Geometrie* (of 1936), where Husserl argues on the manner in which, for example, surfaces experienced in everyday life become selected for various practical purposes (like smoothness, and so forth) and then become idealized into the concept of a two-dimensional surface without a third dimension of depth; then, this concept of a two-dimensional surface is, through an idealizing abstraction process, constituted as an object in itself with essential properties to be determined by its own science, the one of the essences (that is to say, the intrinsic properties of the given mathematical object). On Weyl, we shall return later. *En passant*, we notice that, around the same period of the Hilbert formalization, besides Husserlian phenomenology, also gestaltist analyses sprung up from Brentano's philosophy (see (Albertazzi 2006, Chapter 7)), where a crucial rule is played by gestaltist switches or slippings between figure and background, hence still having a basically relational nature of comparison between two distinct objects. On the other hand, *Gestalt* theory has received great influences and notable insights just from phenomenology.

4.2. Werner Heisenberg has always been interested to philosophy of science problems, contributing with remarkable epistemological reflections. In the final discussion on the conference held by Heisenberg in (Heisenberg et al. 1980), Giacomo Devoto (1897-1974), one of the most influential Italian linguist of 20-th century, puts forward some interesting hints and remarks about the crucial relationships between science and philosophy. First, Devoto says that philosophy is something that always precede science; for centuries, the former oscillates between a realistic vision of the world and an idealistic one, the discoveries of science having could influence philosophy course in a direction or in another one, but never in a decisive manner. Thereafter, Devoto makes a close parallelism between the structure of the science's development through 19-th century to 20-th century and the evolution of a language; to be precise, if the main epistemological change of science in this period has mainly had a linguistic change character, then the chief paradigmatic features of the latter should be found too in the former. This linguistic character of science is inherent the crucial relationship between the physical *givenness* and the related mathematical interpretation whose terms should be respectively put in comparison with the analogous relation between the diachronic observation of a language (i.e., its history) and its actual pedagogical application by the grammarian (like a mathematician) who tries synchronically to state and to describe the everyday conventions and rules used into a given linguistic environment at a given historical moment. Therefore, Devoto considers the historical retrospective view of a language as providing a kind of phenomenological reality from which pick up data upon which to work. Thus, the mathematics is a manner to formally describe the physical phenomena as well as the grammar is a manner to formally describe a linguistic phenomena; nevertheless, the grammatical rules aren't the language simply because are neglected the other two linguistic dimensions, semantics and pragmatics.

This discussion may be laid out within the wider discussion regarding the relationships between ontology and language. Following (Lolli 2002, Part I, Chapter 2), likewise to what has been said above, the grammar's rules govern the usage and meaning of words, that is to say, it seems that syntax precedes semantic and pragmatic which, nevertheless, should be meant all intertwined amongst them at least at a linguistic level, whereas, at a pure mathematical level, one can only state that semantics and syntax are closely linked between them (see (Lolli 2000)): there not exist any semantics which does not refer to a certain syntax whereas, in turn, every syntax is always in searching for a semantics. It turns out that abstract words with adjectival function (like beautiful) or without denoting function, seem unavoidable and that it is not possible to speak without them; such abstract terms would correspond to the universals and seems too that the adjectival function has a precedence in the appearance of linguistic functions. Nevertheless, it is not completely correct to consider numbers as universals understood in the latter sense, because they have not denoting character as the common words: for instance, a child, in front of four dolls, may spell out in series 'bear cub', 'bear cub', 'bear cub', 'bear cub', and once attached a label to an object, the child is unable to change it with another one or to assign two different labels to the same object. Instead, numeric labels may be associated to different objects, that is to say, they have a basic abstract character. Therefore, the rules for numbers are basically different from those of common names, so that a certain oppositional language discrepancy between mathematics and linguistic analysis holds, notwithstanding their strict link: for instance, it is no possible to disregard the linguistic dimension when one gives a mathematical definition, so that mathematical existential questions regard multilevel linguistic analysis.

4.3. Hermann Weyl (1855-1955) was one of the most outstanding and influential mathematicians and mathematical-physicists of 20-th century, a pupil of Hilbert in Göttingen. We are interested to certain moments of the Weyl's thought devoted to philosophical interests, in this regards referring to (Mancosu 2010, Part 3, Chapters 9 and 10) for a most complete treatment. The relations between mathematics and phenomenology were a central and constant concern for Weyl since his doctoral days, differently from Hilbert and Bernays who found their philosophical background in Kantian and neo-Kantian tradition. Weyl deliberately said that the encounter with phenomenology, around 1912, was a liberation from his previous positivistic allegiances, thanks to which he started to deal with the relationships between intuition and formalization both in mathematics and physics. Already in *Das Kontinuum* (1918), Weyl acknowledges his debit towards Husserl's philosophy, quoting the above mentioned 1913 Husserlian lessons *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie* as well as the 1901 *Logische Untersuchunge*. In such famous work, Weyl expressly says does not want to avoid the unavoidable philosophical questions as instead usually made by most of mathematicians. Weyl had, as interlocutors, Husserl and one of his pupils, Oskar Becker (1889-1964). Following M. Franchella[17], (Gethmann 2003), (Scholz 2004) and (Mancosu 2010, Part 3, Chapter 10, Section 10.5), Becker, a mathematician who subsequently switched to study philosophy with Husserl, given notable contributions to the philosophy of mathematics and mathematical physics; amongst others, Jürgen Habermas studied too with him. In 1927, Becker published his opus magnum, namely the magnificent work *Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene* contemporaneously to the publication of *Existenz und Zeit* of Heidegger with whom he was into touch and attended many of his seminaries and lessons of that and past period. Heideggerian thought influenced philosophy of mathematics ideas of Becker which, nevertheless, was quite neglected because of his Nazism

[17] See http://www.filosofia.unimi.it/infofranchella/ecds/a/printer/fid/285.html

likings; in particular, in spite of everything[18], Heideggerian influences may be clearly identified in *Mathematische Existenz*, where, on the basis of Weyl, Husserl and Brouwer contributions, Becker works out some interesting considerations on the existence of mathematical entities above all based on the Heideggerian reflections on time as well as on the facticity of *Dasein*, referring thus the existence of a mathematical entity to *ontic* levels, to concrete aspects of human live, so giving rise, for first, to an anthropological grounding of mathematics. Therefore, the first Heideggerian hermeneutical phenomenology has also exerted a non-indifferent influence in mathematical ontology, via the notable but little known Becker's work; Weyl too put attention to the Heidegger reflections of 1927 *Sein und Zeit* (see (Weyl 2009, Introduction)). As known, temporal dimension has constituted a crucial point of the whole Heideggerian existentialistic ontology[19]. Becker, as a philosopher and historian of science, had too correspondences with notable scientists, amongst whom, through Husserl, Weyl himself since the 1923 PhD thesis of Becker entitled *Investigations of the Phenomenological Foundations of Geometry and their Physical Applications*. With Becker, also Felix Kaufmann (1895-1949) worked on the phenomenological treatment of formal sciences. Becker assiduously attended to Fribourgian Heidegger lessons whose content influenced so much his philosophy of mathematics, roughly conceiving a mathematical entity as a concrete outcome of the being-there (Dasein), a kind of 'precipitate' of the Dasein, and having an independent existence from the actualizing description or modality (of as something is got) of the concrete Dasein[20]. It is clear that these last important considerations deserve a much more detailed analysis and a deepening that in this place cannot be pursued simply because we are mainly interested to search for the Husserlian phenomenological bases of part of the Weyl's work. The work of Becker was then reconsidered by A. Heyting, who developed the intuitionism from a phenomenological standpoint, as well as by Gödel who, in the second part of his life, turned his attention towards phenomenology. Weyl, in a few words, was a realist in mathematics, giving – *d'après* Husserl – priority to the (unique) intuition of a mathematical situation respect to the multiplicity of its next representations (see (Weyl 1977, translator's note No. 8)).

[18] Here, we refer to the well-know criticisms moved by Heidegger to scientific disciplines. Nevertheless, as pointed out in (Volpi 1997, Chapter VII, Section 11), (Giannetto 2010) and (Von Weizsäcker 1967), Heidegger himself indirectly contributed to better understanding the epistemological status of science: for instance, Carl Friedrich Von Weizsäcker (1912-2007), an eminent theoretical physicist and philosopher, pupil of Heisenberg and a friend of Heidegger, claimed that the natural science should re-appropriate of the Heideggerian reflection on being and time, re-finding those necessary ontological presuppositions upon which relies science. See also the interesting dialogue between Heisenberg and Heidegger, who were friends, reported in (Chiodi 1956), as well as see (Giannetto 2010). *En passant*, we recall that Von Weizsäcker was also a profound philosopher of science whose wide work has concerned many aspects of knowledge besides the strictly scientific context: here, we remember only one point of his many interests, that regarding the so-called *theory of ur-objects* (see (Von Weizsäcker 1971) and (Lyre 1995)) that is to say archetypical objects, based on a chief dialectical opposition between empirically, binary alternatives, theory which resembles, in its conceptual nature, parts of Jungian theory (see (Card 1996)); see also (Von Weizsäcker 1994).

[19] On the other hand, Heidegger wasn't fully unaware of natural and exact sciences, he having been one of the main protagonists of that rich and variegated German cultural environment of the early 20-th century, world-wide leader in physics and mathematics. For a new, revised and in a certain sense unorthodox sight of the relationships between Heidegger and science, see (Giannetto, 2010). Moreover, as regards further interesting links between Weyl's work and other philosophical contexts (like Fichte's idealism and his general philosophical ideas), see (Weyl 2009, Chapter 9), (Scholz 2004) and references therein. On the other hand, Weyl himself, in his celebrated *Das Kontinuum* (see (Weyl 1918)), extensively refers to the Husserl work; this Weyl's monograph has a deep philosophical styling that, besides the well-known intuitionist predicative logic aspects, is also rich of interesting viewpoints and foreshadowed sights which will be developed later by other scholars. For instance, already Weyl himself, in (Weyl 1918, Chapter I, Section 1), would have wished to treat some aspects of metaphysics, even reaching to consider Fichte's thought, but giving up in this difficult enterprise to avoid compassion's laughters of the mathematicians! In particular, on the basis of Husserl's work, Weyl stresses the well-known logical passage from the *predicative* to the *relational* judgements, considering the former as particular cases of the latter, hence reaching to the consideration of the so-called *existential* judgements.

[20] In a certain sense, what has just been said might find a conceptual analogy with the consideration of the *numeral* (meant as the highest abstraction of a mere quantity – see (Devoto 1935)) as a participated universal, i.e. it is distributed or present in each single *number*, as an outcome of the concrete participation of the Dasein. Nevertheless, some numerals are primitive, while others are derived from these latter in dependence on the numeration system considered.

4.4. To further highlight the significance of the *relational viewpoint* from a proper ontological stance, we report some considerations drawn from a theological-metaphysical philosophical sight relying on the two distinct but inseparable levels of the theological and metaphysical stances, staring from the *principle of creatio ex-nihilo*. To be precise, following (Sciacca 1972, Chapter II, Sections 1 and 2), there not exist two concepts of existence that, therefore, as a mere existence considered (for abstraction) independently from all its predicates, it is uniquely predicated of the created things and of the creating Being, predicate which is common both to the Creation and to the Creator. But this common predication confirms that none of the Creatures is the Being which is present with a different modality in everyone of the former: the essence of God is infinite, whereas that of every Creature is finite. Due to the ontological constrain (between being and Being – see below), the being of every entity participates of the Being not according to a universal mode but rather according to a well-determined modality that takes place along that *creative act* which provides the determination of the pre-existent essences which assume an ontological status of existence and subsistence just thanks to this Creating principle, to this determinant act. This is the *principle of participation* of the finite crated being with the infinite creating Being, and in order to there be a some analogical relation between them, with the finite being having a being of its own, this participation principle cannot be disjoint from the one of creative determination act. There is no uniqueness but only analogy, such an analogical relation being possible because of the uniqueness of the predication of the pure existence. But, since the finite modality of the being would have could not be, it follows its contingency, which entails that all the entities are created from the Nothing, that is to say, receive the being from the Being through a creative act which is neither any of these entities nor their totality: in this sense, the contingent being *is* not the being but rather *has* the being. That is to say, the being predicates the infinite Being with the copulative *is*, whereas predicates the finite being with the copulative *has*. The finite entity would not exist without the Being, from which it has the being. But, once that an entity *has* all the being which competes it inasmuch finite, it *is* such a being in the given real finite form, so acquiring an ontological status of its own. Therefore, the finite entity would not be without the Being from which it *has* the being, and to which it remains ontologically linked (*ontological constrain*); but, once had its being, it is all its being. The finite modality of being is related to the absolute Being and entails the multiplicity of the entities, this latter implying, in turn, the various determinations: therefore, the modality of every created entity is the set of all the determinations that confines or restricts it. Nevertheless, the created finite entity does not exist because it *has* that essence (or being) provided to it by the Being, but rather the various determinations of its finite modality to relate itself with its being, provide that being by which it *is*. The finite entity is unthinkable and inexistent without the Being that it is for itself, so that it is relative. Between the infinite Being and the finite being there holds an essential relation which passes for the latter and without which it would have not been and would cease to be. Therefore, the system of relations is a characteristic of the created world, it is the order of the finite, the one that the creating Being has given to it. Having a multifarious and composite nature, the world is a system of relations between entities, all in a dialectic position or in relation amongst them, as well as each and in their totality, in relation with the Being. It is not possible to conceive entities without the essential (also triadic and Trinitarian) relations between them. The world is rationale because *relationem habet*, so that it is multifarious but not simple, relative but not absolute, contingent but not necessary. The Being is absolute, where the existence is identical to the essence; it is one, simple and necessary, without limitations and conditions, without relations, free from the system of relations that governs the world created by it, that is to say, the rational real.

These are very few considerations made in (Sciacca 1972), where more and more times the author[21] refers to the Aquinas' and Rosmini's thought; to be precise, the pointing out of the

[21] We remember that Michele Federico Sciacca (1908-1975), founder, in 1946, of the renowned *Giornale di Metafisica*, was one of the major thinkers of the Christian spiritualism who, starting from the Plato-Augustine tradition with Rosminian influences, reaches a neo-Christian conception of human being.

relational foundation of the metaphysical-theological ontology is just taken from the Aquinas' thought, coherently with what has been said above about the early origins, more or less tacitly, of the notion of function in mathematics that certain historiography would want dating back to Middle Age (see, for instance, the well-known names of Oresme, St. Augustine, Aquinas, and so forth). Now, all these considerations might seem have an extreme metaphysical and idealistic nature, relegated only to the mere theological context. Nevertheless, from history of mathematics and logic, we must not forget that the theological disciplines have, in a very capillary fashion, permeated the whole cultural world for many centuries, included mathematics. For instance, theology has given notable contributions to the general ontological framework which, from the metaphysical side, surely has meet points with ontology of mathematics. Therefore, it is not meaningless to try to give a look at on the possible historical influences that surely theological disciplines have exerted too on sciences, at least as concerns their metaphysical counterparts. Instead, also in the light of the recent studies of the philosophy of science (see, for instance, (Scholz 2004) and (Giannetto 2010), a revaluation of this possible line of research is reasonable, if nothing else for those idealistic trends of philosophy of mathematics which are very close to this program that has, however, a methodological justification of its own simply because it is not historiographically correct to consider a certain knowledge discipline avulsed from the general and wider cultural context in which it grew up. In any case, what has been said above is another confirmation, coming from a discipline (the metaphysical-theological ontology) apparently very far from the scientific context, of the main fact that the various *relational* ideas underlying both the basic Triadic-Trinitarian structure conception of the Creating Being and the corresponding as many basic triadic-trinitarian structure conception of the created being analogically moulded upon the former, and of which it participates thanks to the Creation principle, were already present, in a manifestly fashion, in this metaphysics context (see also (Sciacca 1972, Chapter IV, Sections 1, 2 and 3)) since Middle Age, mainly with St. Augustine and Aquinas. Again following (Sciacca 1972), few languages, amongst which the Italian and the French one, distinguish between *entity* and *being*: in this sense, the entity refers to anything which has the act to be for which it exists, so distinguishing between the *act* for which a thing there exists, and the intelligible *determination* for which it is such a thing. In doing so, we reach the traditional Rosminian distinction between the *being* and the *determination of the being*, so that the entity will be formed by two basic constitutive elements: the *act* of being and the *determination* which warrants that it is just this entity. All that which exists must, if it is something, participate of the being, otherwise it would be the nothing and not only it might not exist but would be even unthinkable. The idea of entity so contains in itself all the differences not only specific but also individualizing. On the other hand, it is not possible does not see conceptual analogies or links between such notions of act and, above all, of determination, as belonging to the notable and wide Rosminian ontological framework, with the general gnoseological notion of *representation*. The act is the realization of the essence of the being, the being is act for essence, while the essence is that of which the being is act: therefore, a strict connection does hold between act and essence.

Above, we have briefly sketched, as pertaining instances, only few aspects – those concerning the ontotheology – of the rich and long historical course of the notion of relation which embeds its deep roots in ancient times, since we think that, above all in that historical period of inextricable connections amongst cultural contexts of the middle age in which predominated just the theological questions, they have more or less indirectly contributed (at least, as regards that *vexata quæstio* concerning the reality and objectivity characters of the relations) to the accrual of the logical counterpart of the historical-philosophical evolution of term relation. Furthermore, we think that all these metaphysical-theological conceptualizations ought be retaken with the right historiographical consideration in order to be put into comparison them with the history of mathematical ideas as well as of logic. Indeed, it is unconceivable to look at history of scientific thought without an albeit minimal reference to the philosophical trends of a certain period under historical-critical examination. In another place, therefore, we shall try to pursue this in a deeper manner.

5. Some historical-epistemological considerations: secondly

5.1. The chief aim of the present paper is try to identify some main Husserl's phenomenology ideas underlying some aspects of the Weyl's work. In pursuing this, an unavoidable reference is (Weyl 1949), in particular its Chapter 4. This begins with a rapid survey of the various main notions of space and time from antiquity until up 20-th century developments, above all pointing out on those achieved by relativity theory. After mid-1800s, both the philosophical and scientific reflection was turned towards a re-evaluation of the notions of space and time, reaching unification attempts of them: in this regards, see, for instance (Knopff 1923) and (Schmidt 1922), where are included very interesting philosophical discussions on the new relativity's discoveries. But the point that Weyl mostly stresses, is the four-dimensionality of the continuum space-time, hence the philosophical implications of this new and unprecedented conceptual unification. Surely, the previous advent of group theory methods with the related invariant theory have also promoted certain philosophical conclusions, above all from the epistemological side. This was also favoured by the close relationships that there existed between philosophers and scientists if nothing else for the fact that they lived and worked in the same Faculty: it is enough to mention only the Riemann figure to bear an emblematic example of this useful cultural exchange which was, however, a widespread social-cultural custom. Then, Weyl's reflection focuses on the fundamental relationships between *subject* and *object*, where the presence of Husserl's thought is very imposing. Weyl, starting from a re-examination of the sensible qualities, reaches to consider the mathematical symbolism construction from the Husserlian phenomenology viewpoint. Indeed, the mathematical framework is seen, by Weyl, as the result of a kind of 'distillation' of the objective world which is symbolically representable from what is immediately given to our intuition. Such a process takes place only gradually through a sequence of different knowledge's levels from whose reciprocal comparison springs out a knowledge content. For example, a thing may appear having a solid form just because this last is the common origin of all its own perspective visions, which is possible to identify because of the multiplicity of its infinite representations from whose reciprocal comparison such a common (or fixed, or invariant) form (i.e., the solid one) arises.

Then, Weil considers the case of a given electric field $\vec{F}(P)$, probed by a test particle with charge q and placed in a point P, to determine the intensity of the field, say $\vec{E}(P)$, in such a way that $\vec{F}(P) = q\vec{E}(P)$, where the field $\vec{E}(P)$, as a point function, *there exists* independently by the state of the test particle, while the charge q of this last is a scalar factor which depends only by the internal structure of the particle itself. Weyl stresses on the *existence* of $\vec{E}(P)$ no matter if one measures or not the force that such a field exerts upon an arbitrary test particle. The above equation $\vec{F}(P) = q\vec{E}(P)$ does not define the intensity of the electric field, but it is simple a law of nature which determines the ponderomotive force $\vec{F}$, and that is undergone to circumstantial changes. The test particle exclusively allows the access to the measuring and to the observation of the given electric field. In this last case, the analogy with the previous example given by a solid object, is easily made: indeed, the test particle with its charge respectively correspond to the observer and its position; then, the force exerted by the field upon the test particle, which varies either with the position and the charge of the latter, corresponds to the solid form of the observed object. The light, for example, is roughly given by rapid oscillations of electric and magnetic fields, that our eyes perceive not by means of their ponderomotive effects: the electric field intensity $\vec{E}$ is introduced in a merely symbolic form, without explanations, working out only a posteriori the laws ruling such a field. Therefore, there must be another way to rationally define such an entity, and the group theory turns out to be useful just to this end. Indeed, the first exact formalizations of electric and magnetic fields arose in the late of 1800s, just thanks to the new group theory methods, above all those introduced by Lie, with the well-known 1890s Pierre Curie works on crystallographic symmetries, later retaken and deepened in the early 1900s (see, for instance, (Wigner 1931) as well as (Weyl 1928)). At almost the same time, the tensor analysis made its appearance with the early developments of absolute differential calculus of invariant forms, providing further useful formal tools for symmetry arguments. This new theory pointed out the main role played by the reference frame, that is to say, a coordinate system. In particular, if M is an abstract space with dimension n, an arbitrary point $P \in M$ will be identified, respect to a general curvilinear coordinate system Ξ, by an n-tuple of the type $(x_1, \dots, x_n)$; if Σ is another curvilinear coordinate system, the same point P will have curvilinear coordinates of the form $(\bar{x}_1, \dots, \bar{x}_n)$ such that $\bar{x}_k = \bar{x}_k(x_1, \dots, x_n)$ $k = 1, \dots, n$ where the functional form of these last formal expressions will depend on the coordinate system change $\Xi \to \Sigma$, said to be a *coordinate transformation* according to the *passive* viewpoint. Alternatively, in the *active* viewpoint, the functional system $\bar{x}_k = \bar{x}_k(x_1, \dots, x_n)$ $k = 1, \dots, n$ may be considered as defining a transformation of points respect to a unique and fixed coordinate system. Accordingly, all the various other quantities which can be defined respect to a coordinate system (like vector fields, and so forth) undergo such rules: in our case, we are interested to the electric field, to be precise, we shall see that just thanks to what has been said above, it will be possible to define the existence of the electric field intensity according to Husserl's phenomenology.

On the basis of the previous Weyl and Wigner work, following (Picasso 1999, Chapter 1, Section 6), let $\rho(\vec{x})$ be a charge distribution generating an electric field intensity $\vec{E}(\vec{x})$, and let $\psi(\vec{x}) = R\vec{x} + \vec{a}$ an isometry, with $R^t R = I$ and $det\ R = \pm 1$. Therefore, from the active viewpoint, we have a new charge distribution that, due either to charge conservation law and to the hypothesis of homogeneity and isotropy of space, is given by $\rho(\vec{x}') = \rho\big(\psi(\vec{x})\big) = \rho'(\vec{x})$, that is to say, the new distribution $\rho' = \rho \circ \psi$ is the same of the previous one ρ which rigidly undergoes the same transformation ψ acting upon the points of the environment space, i.e. $\vec{x} \to \psi(\vec{x})$. Accordingly, also the electric field intensity changes with ψ, that is to say, it is given by the electric field intensity generated by ρ under the action of ψ, to be precise $E_i'\big(\psi(\vec{x})\big) = \sum_j R_{ij} E(\vec{x})$ for each i. Under the more restricted hypothesis $\rho'(\vec{x}) = \rho\big(\psi(\vec{x})\big) = \rho(\vec{x})$, we have $E_i\big(\psi(\vec{x})\big) = \sum_j R_{ij} E(\vec{x})$, that is to say, if the field sources are invariant respect to a given transformation, then the electric field intensity is also invariant under the same transformation. The symmetry arguments related to the various possible transformations ψ, allow us to build up the electric field intensity as follows. Indeed, the set of transformations ψ such that $\rho\big(\psi(\vec{x})\big) = \rho(\vec{x})$, forms a group, subgroup of the isometry group, said to be the *invariance group* of the sources $\mathcal{G}$. The more the group $\mathcal{G}$ is structurally rich, the more are the information which we can have on the related electric field generated by such sources. These last information mainly consist in formal relationships between the electric field determined in the various points belonging to a given orbit computed respect to the action of the group $\mathcal{G}$ upon the points of the space: an orbit $O_{\mathcal{G}}(P)$ on a point P is given by $\{g(P); g \in \mathcal{G}\} \subseteq \mathcal{S}$ if the action of $\mathcal{G}$ on the given environment space $\mathcal{S}$ (usually, a numerical real space, like $\mathbb{R}^3$ or $\mathbb{R}^4$, with a certain metric structure) is a map of the type $(g, P) \to g(P)$ for each $g \in \mathcal{G}$ and $P \in \mathcal{S}$. Therefore, the relation $E_i\big(\psi(\vec{x})\big) = \sum_j R_{ij} E_j(\vec{x})$ simply says us that the knowledge of the electric field intensity in a given point P_0 determines it in every other point $P = \psi(P_0)$ of the orbit $O_{\mathcal{G}}(P_0)$ of P_0; in particular, the intensity of the electric field is the same in every point of a given orbit. Now, for each point P, let $\mathcal{G}_P$ be the set of transformations ψ which leaves P invariant (fixed point), that is to say $\mathcal{G}_P = \{\psi \in \mathcal{G}; \psi(P) = P\}$; it is a subgroup of $\mathcal{G}$, said to be the *isotropy group* (or *stabilizer*, or *little group*) of P. If $\mathcal{G}_P$ is not trivial, then much information can be drawn about the direction of the electric field intensity because $\vec{E}(P)$, but, in general, not about orientation and modulus of it because of the linearity and homogeneity of the following invariant relation $E_i(P) = \sum_j R_{ij} E_j(P)$ for each $R \in \mathcal{G}_P$, which expresses the invariance of $\vec{E}(P)$ respect to all the transformations of $\mathcal{G}_P$, that is to say, it is an eigenvector of each of these latter. $\mathcal{G}_P$ may contain only rotations or reflections because the translations do not leave fixed any point; it is often completely determined by well-precise its subgroups. $\mathcal{G}_P$ is, in general, enough to determine the direction of $\vec{E}$ in the point P, hence in all the points of $O_{\mathcal{G}}(P)$. For instance, in the case of a homogeneous charge distribution having spherical symmetry, we have $\rho = \rho(\vec{r})$ and the invariance group of the sources is the group O_3 of the rotations and reflections. It is enough to use its subgroup of rotations. Let P be an arbitrary point. If P is the centre O of the charge distribution, then $\mathcal{G}_P = O_3$, so that, if $\vec{E}(O)$ is defined, then it must be $E_i(O) = \sum_j R_{ij}E_j(O)$ that, being a linear and homogeneous equation, implies that $\vec{E}(O) = 0$; for instance, if a point charge is placed in O, then it follows that $\vec{E}(O)$ cannot be defined, *does not exists*, because of the constrain imposed by the above linear and homogeneous relation between the values of this entity seen from those various, different perspectives (provided by the action of group transformations) along an orbit. Instead, if $P \neq O$, then $\mathcal{G}_P$ is the group of rotations around the line OP, so that $\vec{E}(P)$ is invariant only if it is parallel to such a line, that is to say, if it is radial; $O_{\mathcal{G}}(P)$ is the spherical surface with centre O passing through P, over which $\vec{E}$ has a constant modulus and is internally or externally directed. In conclusion, we must have $\vec{E}(\vec{r}) = E_r(\vec{r})\vec{r}/r$. In conclusion, we have seen, through some very elementary examples drawn from basic physics, that the notable relational viewpoint offered by group theory methods introduces fundamental *existential* issues, according to Husserl's phenomenology main principles. For more information about role and use of symmetry principles in electromagnetic theory from a more technical standpoint, see also (Ghigi 1999, Chapter 3, Section 3).

Again following (Weyl 1949, Chapter 4, Section 17), we discuss the case of the measurement of the distance between two stars from a point-eye O, that we suppose to be our percipient consciousness, whose universe line is B. Let Σ be the universe lines of the two stars. The posterior light cone K emerging from O in a given instant of the life of the point-eye O in the Minkowski four-dimensional world, meets the universe lines Σ in two well-determined and unique points of the posterior light cone centred in the universe instant O, which perceives them through those two universe lines Λ identified by the light rays which go from the two stars towards O. From these data, it will be possible to compute, in a merely formal manner, the angle θ under which these two stars appear to the observer O. This construction, with the same positions, is invariant under certain geometrical transformations applied to this formal setting, again obtaining the same result θ that, therefore, gives the form with which the constellation appears to our eyes in dependence on the metric field generated by the two stars, on the spatial position of the observer O in the Minkowski universe as well as by its dynamical state given by the direction of the universe line B through O (said to be the *stellar aberration*). Therefore, θ provides the form with which the given stellar constellation appears to our sight. It has no objective correspondent in a description of the related objective data but may be only experienced, in a directly manner, if one accepts the presupposition, likewise without any objective correspondent, that, in turn, I am the point-eye placed in O. If another stellar constellation The objective world simply *is*, it does not happen: only to the sight of my consciousness, which is in movement along the universe line that represents the life of my body, a spatial-temporal *section* of this universe may *offer* to me as an image which fluctuates in the space and that continuously change in the time. In the construction of the angle θ, an important role is played by the *separation* of space and time in the Minkowski four-dimensional world, made by my consciousness: to be precise, if e_0, e_1, e_2, e_3 are the components of a four-vector $\vec{e}$ which refers to the direction of B in O, then my close spatial neighbourhood is characterized by the set of the linear elements (dx_0, dx_1, dx_2, dx_3) emerging from O and orthogonal to the four-vector $\vec{e}$, hence verifying the equation

$$\sum_{i,k=0}^{3} g_{ik} dx_i\, e_k = 0, \quad g_{ik} = g_{ik}(O).$$

It follows that, on the one hand, the immediate perceived experience is subjective and absolute, and given in such a manner and not otherwise insofar as nebulous and it can be, while, on the other hand, the objective situation is necessarily *relative* because the objective world can be represented with well-defined objects (numbers, symbols, and so forth) only through a preliminary but arbitrary definition of reference frame in such a manner that to every variation into our subjective experience there corresponds a variation in the underlying objective situation; furthermore, such a variation is invariant respect to an arbitrary change of the reference frame. The body of our Ego, is a physical object. From Max Born epistemological work, Weyl states that this pair of opposites, namely, *subjective-absolute* and *objective-relative*, has a profound epistemological meaning: everyone wishes the absolute must place oneself on the subjectivity and egocentrism plane; everyone wishes the objectivity must deal with relativity.

5.2. Weyl then argues on possible comparisons between *realism* and *idealism*. The general principle according to which a difference which lies upon the perceptions offered to us, always stays upon a difference in the real conditions (H. von Helmholtz), is the basis of the realism trend. The physicist J.H. Lambert reformulated this principle stating that ''an appearance is the same when the same eye is stroke in the same manner''. Until up I do not go other the immediate datum, an objective world structure is not need. Even if I included my personal memory as, in principle, a valid evidence plus, possibly, the contents of other witnesses provided to me by interpersonal relationships, then I couldn't still proceed towards a scientific construction as usually made because, as indispensable *methodological element*, I have needed for those transformations which mediate amongst them all the images of these different (present and past) consciousnesses. This is what, on the other hand, had already made Leibnitz himself with his monadology: instead to construct the perspective sight of a given thing from the different perspective images of it, we might do not consider this thing in itself e go on as follows. Let $A, B, C, \ldots$ be different consciousnesses placed in different body-points, and let K be a solid thing within their visual field. Then, the above mentioned methodological element consists in describing the laws of the geometrical relationships between the images that each person $A, B, C, \ldots$ has of K, together the position or placement of the remaining two persons respect to the third one who has each image. Apart the effective operative difficulties of this procedure, this is, however, the principle of the method of the realistic view of science according to which objective reality is immediately given. Instead, according to the idealistic trend, the objective reality is not given, but it is built up and that, from the scientific viewpoint, it can be construct only in dependence on a reference frame choices within a given symbolic domain. The chief thesis of idealism is summarized by the following principle, in a certain sense inverse to the one above mentioned about realism: the objective image of the world cannot accept any difference which cannot manifest itself by means of a perceptive difference, that is to say, it is not possible to admit an existence principle that is fully inaccessible to perception. For instance, Leibnitz points out that it is impossible to identify any kind of motion without ascertain some observable change. In short, an arbitrary difference which cannot be caught from our perception, is not existent.

Between the real universe and the one given to own perception at a certain moment and in a given manner, there is a correspondence, a relation given by mathematical representations through which, according to Husserl, it will be possible to work out interpretations. Again following Husserl, what can be truly ascertained comes from taking into account *all* the possible sensible images that, put in reciprocal comparison amongst them: only doing so, we can identify possible abnormal conditions. This series (often having a temporal dimension) of variable and multiform representations, provided by intuition, notwithstanding the necessary corrections which are required to overcome the unavoidable occurrence of errors, should contain a sufficient number of concordant (or congruent) elements towards a certain, unique conceptual invariant (or universal) which shall characterize as real that given thing to which such a series refers. In this regards, Weyl reports a simple but meaningful example drew from linear algebra. Indeed, a n-dimensional vector $\vec{v}$ may be represented only through a decomposition of it respect to a given vector basis $\{\vec{e}_i; i = 1, \ldots, n\}$ of a certain linear space $V_{\mathbb{K}}$ over a given scalar field $\mathbb{K}$, of the type $\vec{v} = \sum_{i=1}^{n} \lambda_i \vec{e}_i$. Such a vector becomes a *geometrical entity* only when its component set undergoes certain rules provided by basis changes which, in turn, correspond to reference frame changes formally given by linear transformations. Every set of scalar components respect to a prefixed vector basis corresponds to the appearances of our intuition (giveness) while the reference frame corresponds to the observer: for instance, if one considers the canonical basis, then we might speak of a privileged or absolute observer. Only when, along a series of different representations, such basis change rules are satisfied, then we can identify a mathematical entity classified as a vector. The decomposition $\vec{v} = \sum_{i=1}^{n} \lambda_i \vec{e}_i$ in itself does not define any entity called vector if it does not verify the above mentioned basis change rules, that is to say, if it is not an invariant respect to the corresponding linear transformations. Therefore, the scalars λ_i are mere appearances of a possible entity respect to the privileged observer $\{\vec{e}_i\}$ which, nevertheless, may lead to an entity only when all the possible relations (i.e., reference changes) provide an invariant object respect to which we have an objective equivalence of the various observers. In this case, we are more close to Leibnitz than Descartes with whom, by means of his *cogito ergo sum*, gives to the reality of the Ego a privileged position respect to the external world, while, instead, it provides only a unique point of view with respect to the multiple plethora of possible observers. Then, from the further analysis of all the possible changes of reference frame, it will be possible to identify certain constant properties which will characterize that entity which is invariant under such a series of reference frame changes, that is to say, it will be possible to identify those *axioms* or *postulates* which implicitly will define such an entity. This is also the case of the structure of that entity called differentiable manifold as historically considered in (Iurato 2012a,b). In this sense, also the existentialism, notwithstanding the aversion by positivism (for which the being is a non-sense) contributes to the foundations of science because any *axiomatic*, as an unavoidable scheme of every rigorous systematization of the logic of any rational thought, cannot be theoretically built up without make reference to a some theory of being (see also (CSFG 1977, see items Being, Entity, Ontological Difference)).

As said above, Kant considered space as an a priori intuition, a mere form not having empirical character, so considering Euclidean geometry as an a priori synthetic judgement science, as many a priori valid until up the coming of non-Euclidean geometries. But already Gauss guessed that whereas number is a mere product of our mind, instead space has a reality that transcends our mind and of which we cannot completely predetermine its rules.

6. Some historical-epistemological considerations: thirdly

6.1. Since the preface to the first edition of the celebrated work (Weyl 1918), the author points out his wish to present the new Einstein's theory of relativity *«as an illustration of the intermingling of philosophical, mathematical and physical thought, a study which is dear to my heart. This could be done only by building up the theory systematically from the foundations, and by restricting attention throughout to the principles»*. Therefore, Weyl has always paid, like no other, very much attention to the philosophical issues underlying the scientific aspects and mathematical foundations of relativity theory, and the Introduction to (Weyl 1918) is just devoted to explain such philosophical bases as regards space, time and matter categories. This rightly famous Weyl's treatise marked a turning point in both scientific and epistemological literature of the time. In any case, throughout the whole Weyl's treatise, the philosophical issues and efforts are always present where they need for. In doing so, the influences of Husserl's thought are very tangible and explicitly acknowledged by the author. Weyl starts from space and time categories

> *«as commonly regarded as the forms of the existence of the real world, matter as its substance. A definite portion of matter occupies a definite part of space at a definite moment of time. It is the composite idea of motion that these three fundamental conceptions enter into intimate relationship. Since the human mind first wakened from slumber and was allowed to give itself free rein, it was never ceased to feel the profoundly mysterious nature of time consciousness, of the progression of the world in time, of becoming».*

Again Weyl says that

«philosophy, mathematics and physics have each a share in the problem to treat these conceptions. The task of shedding philosophical light on to these questions is none the less an important one, because it is radically different from that which falls to the lot of individual sciences. This is the point at which the philosopher must exercise his discretion. If he keep in view the boundary lines determined by the difficulties inherent in these problems, he may direct, but must not impede, the advance of sciences whose field of inquiry is confined to the domain of concrete objects».

Therefore, Weyl was fully aware of the fundamental importance to philosophically handle technical questions inherent exact and natural sciences, in this case, relativity theory. Hence, he carries outlining a general philosophical framework within which to build up his personal sight of the Einstein's theory of space-time-matter. To sum up, immediately Weyl begins his philosophical reflections recalling the main lines of Husserl's thought. Weyl states that

«as human beings engaged in the ordinary activities of everyday life, we find ourselves confronted in our acts of perception by material things. These material things are immersed in, and transfused by, a manifold, indefinite in outline, of analogous realities which unite to form a single ever-present world of space to which I, with my own body, belong. Let us here consider only these bodily objects, and not all the other things of a different category, with which we as ordinary beings are confronted: living creatures, persons, objects of daily use, values, such entities as state, right, language, etc. Philosophical reflection probably begins in every one of us who is endowed with an abstract turn of mind when he first becomes sceptical about the world-view of naïve realism».

Then, Weyl passes to examine the role of the recognition of the subjectivity of the qualities of sense as given by the immediate perception and that constitutes the basis of the world-view naïve realism of above. With Galilei, on the wake of Descartes and Hobbes, starts to predominate that principle underlying the constructive mathematical method of modern physics which repudiates qualities. Hence, Weyl goes on saying that

«In the field of philosophy, Kant was the first to take the next decisive step towards the point of view that not only the qualities revealed by the senses, but also space and spatial characteristics have no objective significance in the absolute sense; in other words, that space too is only a form of our perception. In the realm of physics, it has been perhaps only the theory of relativity to have made quite clear that the two essences, space and time, entering into our intuition have no place in the world constructed by mathematical physics. For instance, colours are thus really not even æther-vibrations but merely a series of values of mathematical functions in which occur four independent parameters corresponding to the three dimensions of space, and the one of time. Expressed as a general principle, this means that the real world, and everyone of its constituents with their accompanying characteristics, are, and can only be given as, intentional objects of acts of consciousness. The immediate data which I receive are the experiences of consciousness in just the form in which I receive them. They are not composed of the mere stuff of perception, as many positivists assert, but we may say that in a sensation, an object, for example, is actually physically present for me – to whom that sensation relates – in a manner known to everyone, yet, since it is characteristics, it cannot be described more fully. Following Brentano, I shall call it the ''intentional object''.

<u>In experiencing perceptions, I see this chair, for example. My attention is fully directed towards it. I ''have'' the perception, but it is only when I make of this perception in turn the</u>

intentional object of a new inner perception (a free act of reflection enables me to do this) that I ''know'' something regarding it (and not the chair alone), and ascertain precisely what I remarked just above. In this second act, the intentional object is immanent, i.e. like the act itself, it is a real component of my stream of experiences (Erlebnisse)*, whereas in the primary act of perception the object is transcendental, i.e. it is given in an experience of consciousness, but it is not a real component of it. What is immanent is absolute, i.e. it is exactly what is in the form in which I have it, and I can reduce this, its essence, to the axiomatic by acts of reflection. On the other hand, transcendental objects have only a phenomenal existence; they are appearances presenting themselves in manifold ways and in manifold ''gradations''. One and the same leaf seems to have such and such size, or to be coloured in such and such a way, according to my position and the conditions of illumination. Neither of these modes of appearance can claim to present the leaf just as it is ''in itself''* […].

''Pure consciousness'' is the seat of that which is philosophically a priori. On the other hand, a philosophic examination of the thesis of truth must and will lead to the conclusion that none of these acts of perception, memory, etc., which present experiences from which I seize reality, gives us a conclusive right to ascribe to the perceived object an existence and a constitution as perceived. This right can always in its turn be over-ridden by rights founded on the other perceptions, etc. It is the nature of a real thing to be inexhaustible in content; we can get an ever deeper insight into this content by the continual addition of new experiences, partly in apparent contradiction, by bringing them into harmony with one another. In this interpretation, things of the real world are approximate ideas. From this arises the empirical character of all our knowledge of reality. Time is the primitive form of the stream of consciousness. It is a fact, however obscure and perplexing to our minds, that the contents of consciousness do not present themselves simply as being (such as conceptions, numbers, etc.), but as being now *filling the form of the enduring present with a varying content. So that one does not say this* is *but this is* now*, yet now no more. If we project ourselves outside the stream of consciousness and represent its content as an object, it becomes an event happening in time, the separate stages of which stand to one another in the relations of* earlier *and* later*. Just as time is the form of the stream of consciousness, so one may justifiably assert that space is the form of external material reality. All characteristics of material things as they are presented to us in the acts of external perception (e.g. colour) are endowed with the separateness of spatial extension, but it is only when we build up a single connected real world out of all our experiences that the spatial extension, which is a constituent of every perception, becomes a part of one and the same all-inclusive space. Thus space is the* form *of the external world. That is to say, every material thing can, without changing content, equally well occupy a position in space different from its present one. This immediately gives us the property of the homogeneity of space which is the root of the conception of* congruence. […] *All beginnings are obscure. Inasmuch as the mathematician operates with his conceptions along strict and formal lines, he, above all, must be reminded from time to time that the origins of things lie in greater depths than those to which his methods enable him to descend. Beyond the knowledge gained from the individual sciences, there remains the task of* comprehending*. In spite of the fact that the views of philosophy sway from one system to another, we cannot dispense with it unless we are to convert knowledge into a meaningless chaos».*

Weyl, therefore, stresses the fundamental relationships established by relativity theory about space and time intervals, which are closely linked among them, also in causal connection, within the general Minkowski's four-dimensional world of space-time (*chronòtope*) upon which a metric may be suitably defined. The conception of *being-now* also stressed by Weyl, might be also put in a certain conceptual analogy with the various well-known phenomenological and existentialistic notions of *being-there* (in this regards, see also (Giannetto 2010) and (Von Weiszsäcker 1994)). In this regards, see also what will be said later at the end of next section 7.

6.2. Following (Weyl 1949, Chapters 1 and 3, Sections 4 and 13), after having taken into account what history of culture says us, the *problem of relativity* arose when it was further examined the role played by objectivity in our knowledge, which can be attained only when, once explicitated all its characterizing elements, factors or variables, the truth stated by a certain proposition is not influenced by the free variations of them. Neglecting this, it is possible to have a lack of objectivity. Highlighting, on the one hand, the difficult task of philosophy in defining objectivity, on the other hand Weyl says that this problem has been plainly laid out through a well-precise mathematical framework which explicitly stresses the *relational* viewpoint of mathematics and logic that already Hilbert pointed out in his famous rigorous formulation of the geometrical setting, putting attention to a restrict number of fundamental geometrical relations (like congruencies) between points, hence abstracting from their nature, and remanding from one system or model to another one. This inevitably leads us to the notion of *isomorphism*, which has a fundamental importance in epistemology. Very roughly, when we have two systems, say Σ, Σ', respectively formed by points P, P' and relations R, R', such that to corresponding points $P \stackrel{\psi}{\to} P'$ undergoing certain relations, we have too $R \stackrel{\psi}{\to} R'$, then we say that Σ and Σ' are *isomorphic* through ψ, setting $\Sigma \cong_{\psi} \Sigma'$, so that they identify the same structure with the same properties. The axiomatic method starts from these latter which will be erected to axioms of an abstract entity or theory and considering an axiomatic system as a kind of 'logic mould' (syntactic structure or *Leerform*) of possible sciences, which may have different interpretations (*semantics*) when one proceeds to the *designation* of the names of the fundamental concepts in such a manner that the axioms become truth statements (*model*). Every science remains indifferent with respect to the essence of own objects: for instance, we can know (*kennen*) the difference between the real points of space from their scalar (numerical) coordinates, as well as from other geometrical representations, only through the direct intuitive perception that, nevertheless, is always in continuous dialectic movement towards knowledge (*erkennen*). It is not possible that knowledge reveals something of the secret essence of the things, hidden behind what is manifestly given by intuition. The notion of isomorphism marks the insurmountable frontier of knowledge. The correspondence between the absolute world of things and the phenomenological world is, at least, univoque in the direction *thing in itself* ⇝ *phenomenon*, that is to say, everything intentionally reveals to us only in a phenomenological manner, simply because when different perception are offering to us, then we may infer that as well different are the underlying real conditions (H. von Helmholtz). Therefore, though we do not recognize the things in themselves, we nevertheless know of them exactly what we know of the related phenomena. In this last sense, with an extraordinary foresight, Weyl had already descried what fundamental role would have played symmetry and related breakdown phenomena in epistemology.

The pure mathematics, from a modern standpoint, is mainly a hypothetical-deductive theory of relations, developing a theory of logical moulds without to be constrained to one of these or to one of its interpretations. This last Weylian epistemological consideration is taken from the Husserlian *Erfahrung und Urteil – Untersuchungen zur Genealogie der Logik,* where such an extreme formalization, which leaves aside from any semantic model, together the coherence requirement, is the only requisite to which must undergo mathematics. Weyl again highlights the fundamental importance of group theory as a ''purely intellectual mathematics'' which is the most suitable one to reach the Leibnizian *mathesis universalis*. Like Klein, also Weyl confirms that the notion of group is the most important one of modern mathematics. Coming back to what has been just said above, we are interested to the notion of *automorphism*, which is simply an isomorphism of the type $\Sigma \cong_{\psi} \Sigma'$ with $\Sigma = \Sigma'$, hence simply a bijective correspondence on a given set Σ which preserves the point relations, that is to say, if a point relation of the type $R(P, P', \ldots)$ does hold, then $R(\psi(P), \psi(P'), \ldots)$ does hold too, and vice versa. In terms of automorphisms, the Leibnizian figure's similarity must be rigorously meant. A relation between points is said to be *objective* if it is invariant respect to a given group of automorphisms G. When one proceeds to the study of the real space, neither axioms nor fundamental primitive relations are given. On the contrary, inverting the natural course of our ideas, in any attempt to axiomatize the geometry we choice certain relations between points which we feel to be having a fundamental importance or nature, or else an objective meaning. We therefore start from a certain transformation group G, which estimates the degree of objectivity of the set of all possible relations there definable: a relation is said to be *objective* when it is invariant with respect to any element of G, in such a case speaking of G*-invariance*. This is the main idea underlying the philosophy of the Erlangen program, so relegating to the margins every axiomatization attempt, because the primary interest is to find those objective relations which turn out to be G-invariant. Weyl points out that the early historical origin of the relativity problem and group theory is findable in the work of Galois on algebraic equations whose objective relations are just these last, that is to say algebraic relations with rational coefficients, from which it follows existential questions about their possible roots (see also (Straumann 1998)). In any case, it is not possible to leave aside from a reference system which has the main role to assign names to objects. The inevitability of a reference frame corresponds to the impossibility to fully eliminate the Ego and its life of direct intuitions, so it is linked with phenomenological reduction process. Therefore, given a certain group of transformations G, we consider a geometry whose only objective entities to be studied are all and only all those relations which are G-invariant.

Riemann was the first mathematician to formally analyze the notion of a real n-dimensional manifold V^n whose points are locally represented by n-uple of real numbers $(x_1, \dots, x_n)$ subjects to well-precise differentiable coordinate transformations. Let $P = (x_1, \dots, x_n)$ be an arbitrary point of V^n and $P^* = (x_1 + dx_1, \dots, x_n + dx_n)$ an arbitrary point infinitely near to P. On the basis of the previous work made by Gauss, Riemann stated that, in an arbitrary but infinitesimal neighbourhood of P, there holds Euclidean geometry, so that

$$ds^2 = \sum_{ik} g_{ik}(x_1, \dots, x_n) dx_i dx_k$$

is an invariant quadratic linear differential form which defines the local metric of the manifold through a covariant *metric field*, say Ξ_P, given by the functions $g_{ik}(x_1, \dots, x_n)$ which continuously depend only on P. In physics, it is often need to consider differentiable manifolds simply because of the locality of the various physical interactions. Riemann, aware of this primary fact, had an original and innovative idea of space and of its metric properties. He supposes that Ξ_P has causal relations with matter, so that such a geometrical field changes too in dependence on the changes of the latter. Therefore, he assigns, for first, a dynamical nature to Ξ_P mainly due to the changes of the matter field, at least in the continuum case. He therefore brings back the space structure to the action of physical forces which determine it. In turn, the metric field Ξ_P manifests itself through the physical effects that it exerts on the physical environment in which it is embedded. Only and only through these last physical interactions, it will be possible to ascertain the existence as well as the quantitative state of the field Ξ_P. Therefore, the latter cannot be placed at an ontologically higher independent and rigid geometrical level overlying the physical dynamicity of matter and avulsed from this last. Einstein developed this ingenious idea considering a four-dimensional manifold including time, so reaching to a magnificent theory of gravitation, working out a detailed formal equations expressing all these precious ideas. In few words, geometry and physics have a very close interdependent relations having a two-way nature. In (Weyl 1949, Chapter 3, Section 14) and (Weyl 1918, Fourth Edition of 1922, Chapter II, Section 18), the author discusses the comparison between the group of physical automorphisms of the ordinary three-dimensional space, that is to say Δ, the group of orthogonal transformations, and the group of the geometrical automorphisms, say Γ, which is the normalizator of Δ. Since $\Delta \subset \Gamma$, it follows that physics never can be reduced to geometry, contrarily to what wished Descartes and Einstein. Instead, as regards four-dimensional Riemannian manifold of space-time $\mathcal{M}_4$, let $\Omega_{\mathcal{M}_4}$ be the group of all differential continuous coordinate transformations (*differential group* of $\mathcal{M}_4$) of the type

$$y_i = \psi_i(x_1, \dots, x_n), \qquad i = 1, \dots, n.$$

which we suppose to form a functional system having a non-null Jacobian determinant in every point $P \in \mathcal{M}_4$. In this case, differently from that regarding three-dimensional space, according to Riemann and Einstein the group of physical automorphisms and the group of geometrical automorphisms are both equal to $\Omega_{\mathcal{M}_4}$; furthermore, now the metric field is considered as a physical entity which acts on the matter, and vice versa. Nevertheless, the Euclidean geometry is considered to be valid in every neighbourhood of each $P \in \mathcal{M}_4$, so that, for a suitable choice of the coordinate reference frame, we have the following Pythagorean-Euclidean expression for the local metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 \qquad (\text{Pythagorean} - \text{Euclidean metric}),$$

so that we might state that the *nature* of the metric, but not its *orientation*, is the same in every point $P \in \mathcal{M}_4$, even if the local coordinate system in which it is considered in general changes with P. When we speak of a metric having same nature but different orientation in each point, we might think, as an analogical situation, the one given by a series of cubes having the same nature but differing only for their orientation due to the matter action. So, the nature of the metric is always the same and is absolute but it is also undergone to a continuous variation depending on the matter distribution, hence by the gravitational forces involved and ruled by Einstein's equations, which determines the quantitative variation of the metric field (again ruled by the Einstein's equations themselves) and that may be estimated only making immediate intuitive reference to reality. Hence, we pass on from an absolute metrical structure at each single point to a (relative) metrical relationship between a point and its immediate neighbourhood by a congruent transference which is not determined by the nature of the space, nor by the mutual orientation of the groups of rotation at the various points of the manifold; such a metrical relationship is dependent rather on the disposition of the material content that, in turn, is capable of any virtual change. Weyl states that the Ego centred in P *perceives* the local intuitive Euclidean space in a neighbourhood of P (like a tangent plane over a sphere), arguing that the related metric structure must necessarily and locally have a Pythagorean-Euclidean metric. And the physics, *d'après* Einstein, confirms this, so locally perceiving a physical space which coincides with the (Euclidean) geometrical one; the more one departs from the centre-Ego P, the higher is the difference between the physical space and the intuitively perceived space. Finally, *en passant*, we also point out that all what has been said above is closely related to the dimension.

From all what has been said so far, it is clear that Weyl always bore in the right consideration the main points of Husserl's phenomenology in pursuing his mathematical working. As a further prove of this, we sketchily report herein the main idea which was at the basis of the early origins of the fundamental notion of a *gauge theory*, that is to say, that principle which constituted the dawning of all the unified field theories (see Yang lecture in (Chandrasekharan 1986) and (Cheng & Li 1988) for a resource paper). Weyl, in formally characterizing the physics of a rotating electron, considers the n-dimensional tangent space in P, say T_P, as its configuration space in which relies the velocity field. Let $\vec{e}_1, \dots, \vec{e}_n$ be an arbitrarily chosen orthonormal basis of T_P, assigning coordinates x_i to every point of a neighbourhood of P, with respect to which acts the group Δ_P of metric-preserving Euclidean rotations, and let $\mathfrak{f}_P$ be a reference frame centred in P with respect to which we consider the local structure, provided by T_P, of the given n-dimensional manifold within a neighbourhood of P. Therefore, let e_{ij} be the scalar components of $\vec{e}_i$ with respect to $\mathfrak{f}_P$. The $n \times n$ matrix $\|e_{ij}\|$ provides the orientation of the basis $\{\vec{e}_i\}$ with respect to $\mathfrak{f}_P$. We have general functional laws of the type $e_{ij} = e_{ij}(P, \mathfrak{f}_P)$ which characterize the metrical field in P. The laws of nature in a neighbourhood of P must be invariant with respect both to a change of coordinates x_i (that is to say, a change of the basis $\{\vec{e}_i\}$) and to a rotation of $\mathfrak{f}_P$: therefore, we have a double invariance, to be precise, with respect to transformations of the differential group Ω (change of x_i), and with respect to rotations of the group Δ_P (change of $\{\vec{e}_i\}$) that may change with the point P. That that is changed with the passage from special to general relativity concerns group theory: in the former, we have invariance with respect to the group of physical automorphisms, that is to say the Poincaré group $O(3,1) \otimes \mathcal{T}$ given by the semidirect product of the Lorentz group and of the translation group, whereas in the latter we have replaced the translation group with the differential group Ω of all the differentiable continuous coordinate transformations while the rotations are remained Euclidean rotations but freely changeable with respect to P. The space, that is to say the mean in which material world extends, is the place of the group Ω, while the group Δ seems to be linked with fundamental particles of matter. Therefore, the scalar quantities e_{ij} play the role of intermediary between space and matter. But what is natural, from an epistemological side, to ask is why does nature have just choose the Euclidean rotation group as a homogenous linear transformation group? As suggested by O. Becker, a partial answer seems to come from the Helmholtz work on the problem of the nature of space, according to which such a group Δ is characterized by the axiom of free mobility, but this does not provide an enough answer. In any case, Weyl states that some linear representations of this abstract group Δ, provide useful physical properties: for instance, in the case of a rotating electron, a vector orthogonal representation of Δ regards the general dynamics of the electron, a tensorial representation of Δ concerns the electromagnetic field of the electron, while a spinorial representation of Δ concerns the wave field of the electron. From this Weyl started to work out the bases of his celebrated *gauge principle* (*Eichininvarianz*), mainly based on the invariance under a change of the scale, in the unfortunately failed attempt to unify electromagnetic and gravitational fields. As known, there holds a general principle of gauge invariance in nature, but it does not connect the electromagnetic potential φ_i with Einstein's gravitational potentials g_{ik}, but rather it ties them to the four components of the electron wave field ψ, just through the above mentioned various possible representations of Δ. Only the (linear) representations of an abstract group (like the Euclidean rotation group) may allow quantitative determinations (which are the operational basis of any physical theory) – see (Weyl 1990, 5th lesson). Nevertheless, when Weyl proposed, for the first time, his gauge principle in 1919, we had a very poor formal knowledge of the wave field of a quantum particle which reached its first rigorous formal description around 1920s, above all with the pioneering work by Dirac. Then, another related, fundamental problem was the so-called *problem of space* (or *Raumproblem* – see (Loinger 1988)) concerned the search for the motivations to the Pythagorean-Euclidean nature of local metric (also in the Riemann-Einstein infinitesimal geometry), which finds its more meaningful expression just in the Euclidean rotation group. Starting from the previous works made by Helmholtz, Klein, Riemann and Hilbert, this problem was successfully accomplished by Weyl himself with those advanced group theory tools settled by Lie, starting from general relativity (as made in his celebrated work (WEyl 1918)) and group theory. Following the Preface to the Italian translation of (Weyl 1990), already Klein, in his Erlangen Program, recalled that «it is always need take into account the criterion according to which a mathematical object should be considered completely legitimized only when it is *conceptually evident*», thus in agreement with Husserl as well. Just to make conceptually evident the privilege possessed by the Pythagorean-Euclidean metric, Weyl solved this issue in the 1920s: for a complete treatment of it, see (Weyl 1990).

7. Some historical-epistemological considerations: fourthly

Following (Weyl 1946, Chapters I and II) (see also (Scholz 2011)) and (Weyl 2009, Chapter 9), 1872 Klein's *Erlangen Program* opened the *problem of relativity* as one of the main pillar of the whole knowledge edifice. Just the as many celebrated work (Weyl 1946) was conceived closely following the philosophical bases underlying such a *Program*. Indeed, following (Weyl 1946, Chapter I, Section 4), with the 1872 pioneering work due to Klein, the notion of group was at the grounding of the idea of *relativity*, in particular in geometry. To explain the underlying idea, Weyl considers the Euclidean point space as an example. With respect to a Cartesian reference frame, say $\mathfrak{f}$, each point P is represented by its coordinates (x_1, x_2, x_3), a column of three real numbers. The coordinates are objectively individualized reproducible symbols, while the points are all alike. There is no distinguishing objective property by which one could tell apart one point from all the other ones; fixation of a point is possible only by a demonstrative act as indicated by terms like ''this'', ''here'', etc. All Cartesian reference frames are equally admissible; any *objective* geometric property possessed by one of them is shared by all others, and so it must be if such a property has an objectivity character. Then Weyl recalls the above mentioned distinction between the *passive* and *active* viewpoint in changing frames, also pointing out as, in general, different measurement scales there exist in different reference frames. Roughly, the passive viewpoint considers the same point from different perspectives (at least, two different frames, that is to say, two different observers or two distinct Ego-centres), whereas the active viewpoint consider a unique frame from which to observe two different points which will be put into reciprocal comparison. Now, starting from the previous work made by Leibnitz on the similarity of geometrical figures, Weyl points out that only the active viewpoint is capable to leave invariant possible relations between points, that is to say, it is the only viewpoint which allow to have *consciousness* of the possible relations between points, hence to identify its objective character which must remain unchanged under the action of a certain automorphism group. Therefore, only through the invariants of certain symmetry groups it is possible to identify objective properties which, in turn, may be quantified only through suitable representations of such groups. Following what textually reported in (Weyl 2009, Chapter 9) where, among other, also Fichte's philosophy is involved, we read as follows

> *«Speaking of the I, Fichte says: ''The I demands, that it comprise all reality and fill up infinity. This demand is based, as a matter of necessity, on the idea of the infinite I, simply posited by itself; this is the absolute I (which is not the I given in real awareness). The I has to reflect on itself; that likewise lies in its meaning''. Referring to the I in this role as the practical I, Fichte now argues that from it as the sole source flows the order of what ought to be, the order of the ideal. Confinement of this unending striving by an opposing principle, the not-I, leads to the order of the real; here the I becomes cognitive intelligence. Yet he says of this opposing force, of the not-I, that finite beings can feel but never know it. ''All possible realizations which can occur of this force of the not-I for all times to come in our consciousness, the Philosophy of Science guarantees to derive from the defining powers of the I''. A geometric analogy will, I think, be helpful in clarifying the problem with which Fichte and Husserl are struggling, namely, to bridge the gap between immanent consciousness which, according to Heidegger's terminology, is evermine, and the concrete man that I am, who was born of a mother and who will die. The objects, the subjects, and the way an object appears to a subject, I model by the points, the coordinate systems, and the coordinates of a point with reference to a coordinate system in geometry. Relative to a system S of coordinates in a plane, consisting of three non-collinear points, there will be defined for each point p a triplet of number x_1, x_2 and x_3, with the sum equal to 1 (gravicentric coordinates). Here objects (points) and subjects (coordinate systems = triplets of points) belong to the same sphere of reality. The appearances of an object, however, lie in another sphere, in the realm of numbers. Naive realism (or dogmatism, as Fichte calls this philosophical viewpoint) accepts the points as something which exists as such. Yet it is possible to build up geometry as an algebraic structure which makes use only of these number appearances (modelling the experiences of pure consciousness). A point, so one defines it forthrightly, is simply a triplet of numbers x which add to 1; a coordinate system consists of three such triplets; algebraically, one explains how such a point p and such a system of coordinates S, determine*

three numbers ξ as the coordinates of p with reference to S. This triple ξ coincides with the triplet X which defines point p, if the system of coordinates S is the absolute one, which consists of the three triplets $(1,0,0,),(0,1,0,)$ and $(0,0,1,)$. This coordinate system, therefore, corresponds to the absolute I, for which object and appearance coincide. In this argument, we never leave the sphere of numbers—or in the analogy, the immanent consciousness. After the fact, we can also do justice to the equivalence of all I's which must be required in the name of objectivity, by declaring that only such numerical relations are of interest as remain unchanged under passage from the absolute to an arbitrary coordinate system or, what amounts to the same, which remain invariant under an arbitrary linear transformation of the three coordinates. This analogy makes it understandable why the unique sense-giving I, when viewed objectively, i.e., from the standpoint of invariance, can appear as just one such subject among many of its kind. (Incidentally, a number of Husserl's theses become demonstrably false when translated into the context of this analogy – something which, it appears to me, gives serious cause for suspecting them). Beyond this, it is expected of me that I recognize the other I – the you – not only by observing in my thought the abstract norm of invariance or objectivity, but absolutely: you are for you, once again, what I am for myself: not just an existing but a conscious carrier of the world of appearances. This is a step which we can take in our geometric analogy only if we pass from the numerical model of point geometry to an axiomatic description. Now, we are not treating the points either as actual realities or as facts, nor have we established from the start an absolute coordinate system by identifying them with number triplets. Instead, the concept of point and the basic geometric relation, according to which a point p and a coordinate system S (i.e., triplet of points) determine a number triplet ξ, are introduced as undefined terms for which certain axioms are valid. This reveals that, above the viewpoints of naive realism and of idealism, it is possible to define, as a third one, a standpoint of transcendentalism which postulates a transcendental reality but which is satisfied with modelling it in symbols. This is the viewpoint to which the axiomatic construction of geometry corresponds in our analogy».

Nevertheless, the two above mentioned viewpoints, i.e. the passive and active one, are equivalent between them in the following sense. With respect to a reference frame $\mathfrak{f}$, the point space is mapped upon a field of reproducible symbols or coordinates $p \to x$ or $x = x(p)$. We suppose that the *coordinatization* sets up a one-to-one correspondence between the points p and the coordinates x. There is no objection to regarding as the frame of reference this coordinatization itself. By means of an automorphism $\sigma: p \to p' = \sigma(p)$, we can define a new coordinatization given by $x'(p) = x(p') = x(\sigma(p))$ which is equivalent to the first one in no way objectively distinguishable from it. Both are linked by the transformation S given by $x' = S(x)$ with $x = x(p)$ and $x' = x(p')$, which describes the automorphism σ in term of the frame $\mathfrak{f}$ or coordinate system x. The various transformations S expressing the several automorphisms σ in terms of the given frame $\mathfrak{f}$, form a certain group isomorphic to the initial (abstract) group of automorphisms, that is to say, the former is a *realization* of the latter, through which it is possible to made *quantification* (or quantitative) processes. At the same time, the group of the S describes the transitions between the various equivalent frames. The utmost we can hope for, is to define objectively a class of equally admissible frames such that any two frames within that class will be equivalent. This is the gist of the (*mathematical*) *relativity problem*: to fix objectively a class of equivalent coordinatizations, hence to ascertain the group of transformations S mediating between them. In philosophical terms, the equivalence between the active and passive viewpoint should respectively correspond to the multifaceted perspectives experienced by a same observer and to the collective comparison among what experienced by multiple observers, so that a *reality* statement is reachable only when such last equivalence condition between point of views there holds, that is, a certain agreement has to be between what singly felt (active stance) and what collectively believed (passive stance[22]).

However, Weyl highlights that not only points are required to be represented by reproducible symbols, like S, but also every other kind of geometric entity, and when passing to physics all sorts of physical quantities like velocities, forces, field strengths, wave functions, and what not, expect a similar symbolic treatment. One often acts as though once the points have been submitted to it by fixing a frame of reference for them, all these other things will follow suit without necessitating further provisions. This is certainly not quite true; at least, further units of measurements have to be fixed at random so as to make the scheme of reference complete. Without prejudicing the situation beforehand, we may then talk of a transformation for the symbols describing a given sort of *entity* (points or electromagnetic field strengths) relatively to the frames depends on the particular entity under consideration. The automorphism group will then be an abstract rather than a transformation group, this seeming to be a natural step beyond Klein's own formulation of his program. The abstract group characterizes the *geometry* in Klein's sense, while the *type* of a variable quantity in that geometry is characterized by its transformation laws. Each element s of the abstract group describes the transition from one frame to another. The transformation law states how the symbol or coordinate representing any arbitrary value of the quantity under consideration with respect to a frame $\mathfrak{f}$ changes under transition to another frame $\mathfrak{f}'$ by means of s; it is therefore a realization $\mathcal{H}$ of the abstract group through transformations $s \to S = A(s)$ in the field of coordinates. Then Weyl gives a systematization putting in comparison a *symbolic* part with a *geometric* part, and vice versa. To be precise, the symbolic part deals with *group elements* $s, t, \dots$ of an *abstract group* γ and

[22] We report textual words of (Heelan 1987), according to whom *«the mutual involvement of subject (you, the observer) and object (it, the observed) in the process of perception can be understood in the following reconstruction that brings out incidentally the influence of Klein's conception of geometry on Husserl's conception of lived space. Imagine two scenarios: in the first of passive scenario, the object plays out its dramatic role before your eyes without your intervention. You are the audience, the passive spectator of this show in which the object exhibits a continuous sequence of transitions among its profiles, each transition generated by a sample of its transformation group (the transformations that act on a profile to produce another profile constitutive of a group); under these transformations the object remains the same for the observer throughout the changes of appearance. In the alternative or active scenario, you play an active role. For every sequence of profile changes associated with a certain transformation of the object, there are actions you could perform that would have the equivalent effect. If the object is turned around in a clockwise direction, this brings into your view the same sequence of profiles as you would see if you chose to move around the object in the anticlockwise direction. In natural perception, no instruments are used, but there is nothing precluding the use of a common repertory of standard instruments, such as clocks, rulers, and even more complicated instruments to change the profile of the object. Such an analysis is familiar to theoretical physicists, particularly in elementary particle physics. They would call the transformation group of the object the active transformation group, and the transformation group of the observer (here, the viewing subject) the passive transformation group. They are identically the same group looked at from the point of view of the observer and of the object. Each will have a set of invariants that define on the one hand the horizon (technically, the ''representation'') of the object or its essential kind and on the other hand the horizon (technically, the ''representation'') of its counterpart in the subject. In this way subjectivity and objectivity,* noesis *and* noema*, mutually ''mirror'' one another. How is the spatial-temporality of a body ''mirrored'' in the* Leib *of the perceiving subject? Husserl takes it to be through the program of the passive scenario; this program is Husserl's meaning (*Sinn*). This is a program for practical action, like a musical score, capable of directing the living body (*Leib*) in how to use its bodily kinaesthesia and the resources of the environment (including, let me add, technologies) in order to bring successively into perceptual view the themes or melodies of profiles that define the spatiotemporal invariance of the perceptual object. The spatial-temporal representation of the object then within the subject – that is, how it is ''mirrored'' – is not like a typical picture of the object, rather it is the competence to enact or to receive in a particular case the active and passive scenarios through which a body exhibits itself to a mobile observer in characteristic sequences of its spatial-temporal shapes and figures.* […] *The Erlangen Program sees geometry as essentially the study of forms that are invariant under group transformations of the mathematical space. Husserl, though not a geometer, would certainly have assimilated at Göttingen the central notion of the Erlangen Program. His account of a perceptual* eidos *as an invariant under a group of transformations within a space of pragmatic-perceptual manipulations clearly reflects the influence of the Erlangen Program. Moreover, Husserl shared with the members of the Göttingen group the view that mathematics and natural science were intimately related»*. Furthermore, according to (Heelan 1987), besides Weyl, there were too mutual influences between Hilbert and Husserl.

coordinates x, hence let there be given a set (or field) of elements, called scalar *coordinates* x and a *realization* $\mathcal{H}: s \to S = A(s)$ of the group γ, by means of one-to-one correspondences $x \to x' = S(x) = A(s)(x)$ within that field; when $A(s)$ is an arbitrary linear operator with respect to a scalar field $\mathbb{K}$, then we speak of a *representation* with respect to a linear space $V_{\mathbb{K}}$ with scalar field $\mathbb{K}$. The geometric part, instead, deals with *frames* and *quantities*, namely, any two frames $\mathfrak{f}, \mathfrak{f}'$ determines a group element s, called the *transition* from $\mathfrak{f}$ to $\mathfrak{f}'$, and vice versa, that is to say, each group element s carries a frame $\mathfrak{f}$ in a uniquely determined frame $\mathfrak{f}' = s\,\mathfrak{f}$ such that the related transition is $(\mathfrak{f} \to \mathfrak{f}') = s$, while a *quantity* q of the type $\mathcal{H}$ is capable of different values. Relatively to an arbitrary fixed frame $\mathfrak{f}$, each value of q determines a coordinate x such that $q \to x$ is a one-to-one mapping of the possible values of q on the field of coordinates. The coordinates x' corresponding to the same arbitrary value q in any other frame $\mathfrak{f}'$, is linked to x by the transformation $x' = Sx$ associated with the transition $(\mathfrak{f} \to \mathfrak{f}') = s$ by the given realization $\mathcal{H}$. The mathematician does not hesitate to identify the values of the quantity q with their respective coordinates x, and the requirement that only such relations matter or have objective significance as stay unaltered when x is replaced by $x' = Sx$ (with $s \to S$ in $\mathcal{H}$) for every s, will mean to her or him a mere convention by which he or she proclaims that he or she will study no other relations. The word quantity shall be reserved to the case in which the realization is a representation in a scalar field $\mathbb{K}$. To be precise, a *quantity* q of *type* $\mathcal{H}$ is characterized by a *representation* $\mathcal{H}$ of γ with respect to an n-dimensional linear space $V_{\mathbb{K}}$ (hence, having degree n), and given by $s \to A(s)$. Each value of q, relatively to a frame $\mathfrak{f}$, determines an n-vector $(x_1, \dots, x_n) \in \mathbb{K}^n$ such that the scalar components x_i of q transform under the transition s to another frame $\mathfrak{f}'$ according to $A(s)$. Finally, following (Caldirola & Loinger 1979, Chapters II, III and Appendix), the structure of an abstract group γ is basically determined by its internal composition law which acts upon its *group manifold* (roughly, the set of all the elements of γ). As said above, the structure of γ is isomorphic to the structure of the corresponding automorphism group, so that, generalizing this last standpoint, given an arbitrary *mathematical entity*, say $\mathfrak{E}$, then its structure will be revealed by the corresponding group of symmetry, say $\mathcal{G}_{\mathfrak{E}}$, that is to say, its group of automorphisms. Therefore, for what has been said above, we might say that, in its highest generality, the (*epistemological*) *relativity problem* mainly consists in considering and comparing all the possible realizations of that given entity under (gnoseological) examination. At the symbolic level, all that has been said so far has started from the initial Klein's *Program* related to geometry, by means of a conceptual development characterized by an ever more high generalization degree. From then onwards, both mathematics and physics have been deeply revolutionized by group theory methods: just to mention another example besides the previous ones, the automorphic functions, as roughly analytic functions having the same value on congruent points with respect to a certain discontinuous group[23], sprung out of the application of the principle of the method of Erlangen Program to study algebraic properties of elliptic and modular functions (see (Weyl 1946, Chapter III, Section 1)), by Poincaré and Klein himself.

The philosophical implications of the *theory of relativity*, as here meant (that is to say, from the birth of group theory onwards), do not have to do with the various philosophical trends whose names, in a certain manner, might refer to the noun *relativity* used in the former theory. Instead, as rightly pointed out in (Abbagnano 1998), the profound innovative epistemological implications of this theory ought be rather understood as belonging to the wider realm of the methodological questions inherent knowledge, in this sense relativity theory having provided precious and new sights about possible gnoseological modalities. We have simply outlined the close and unavoidable historical-foundational links between the development of the theory of relativity and the presence of the imposing edifice of Husserlian phenomenology, by means of the re-evocation of some main moments of the history of physics and mathematics through the remembrance, together his appreciated work, of the as many imposing figure of Hermann Weyl, one of the most outstanding contemporary mathematician, physicist and philosopher of science, whose notable work just in this

[23] See (Ford 2004, Chapter IV, Section 39).

last field has been quite neglected (for a recent re-evaluation, see (Weyl 2009)). In particular, we have wanted highlight what primary role played the Husserlian ideas in the rich, innovative and stimulating research program carried forward by Weyl in mathematical physics, in which group theory has exerted a central leading role which has revolutionized twentieth century mathematics and physics. In this sense, having for first taking into account phenomenological ideas and usefully applied them to the general philosophy of science (see (Heelan 1987)), Weyl has inverted what made by Husserl himself in building up his magnificent philosophical framework, in which a stimulating role was played by ninetieth century scientific knowledge as a epistemological model, attaining to a new seeing of physics and mathematics with their crucial and prickly relationships and epistemological implications. In this sense, therefore, we should meant the remarkable contribution given by group theory to exact and natural sciences as well as to philosophy, once again confirming what precious role has the latter also in the development of the former (see (Weyl 2009, Chapter 9)): in particular, he stresses the primary role and effort played by philosophical reflection during some crucial moments of his work on unified field theory, until up to touch deep metaphysical-religious issues as well as general theological questions, which surely exerted (also unconsciously) a certain influence in his training: as Weyl himself said, in approaching any scientific question or problem, before to choice and to use technical or formal tools and methods, he always performed a preliminary and deep philosophical analysis and reflection upon the basic concepts and notions involved in the given problem under examination. All that is refindable in what Peter Pesic states in (Weyl 2009, Introduction), in which the extraordinary and polyhedral geniality and ability of Weyl is more times highlighted, which was always characterized by taking into the right account the suitable philosophical view along his everyday scientific working, so always reaching profound and new sights and results. Indeed, Pesic says that

> *«Both here and throughout his life, Weyl used philosophical reflection to guide his theoretical work, preferring ''to approach a question through a deep analysis of the concepts it involves rather than by blind computations'', as Jean Dieudonné put it. Though others of his friends, such as Einstein and Schrödinger, shared his broad humanistic education and philosophical bent, Weyl tended to go even further in this direction. As a young student in Göttingen, Weyl had studied with Edmund Husserl (who had been a mathematician before turning to philosophy), with whom Helene Weyl had also studied. Weyl's continuing interest in phenomenological philosophy marks many of his works, such as his 1927 essay on ''Time Relations in the Cosmos, Proper Time, Lived Time, and Metaphysical Time''* […]. *The essay's title indicates its scope, beginning with his interpretation of the four-dimensional space-time Hermann Minkowski introduced in 1908, which Weyl then connects with human time consciousness (also a deep interest of Husserl's). Weyl treats a world point not merely as a mathematical abstraction but as situating a ''point-eye'', a living symbol of consciousness peering along its world line. Counterintuitively, that point-eye associates the objective with the relative, the subjective with the absolute».*

In any case, a deepening of the influences of Husserl phenomenological thought on the Weyl work would deserve to be made elsewhere. For instance, going on along this line concerning relativity theory, it would be of a certain epistemological interest to analyze as well, besides Weyl's work, the phenomenology of the well-known *FitzGerald-Lorenz contraction* of special relativity from the Husserl's phenomenological viewpoint, within which, in particular, to account for that emblematic and apparently paradoxical result obtained either by Roger Penrose and James Terrell in 1959 about the apparent shape of a relativistically moving sphere which instead keeps always its spherical outline (see (Penrose 1959), (Teller 1959); see also (Synge 1965, Chapter V) as well as (Frappier et al. 2013)) and, in general, about the observability or not of this relativistic effect, which led, amongst other things, to the so-called *Penrose-Terrell effect*.

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