August 7, 2014

Invariance of Galileo's Law of Fall under a Change of the Unit of Time

John D. Norton

Department of History and Philosophy of Science

Center for Philosophy of Science

University of Pittsburgh

Pittsburgh PA USA 15260

www.pitt.edu/~jdnorton

Supplement to "A Material Defense of Inductive Inference"

The inductive problem of extending the sequence 1, 3, 5, 7, is solved when these numbers are the ratios of the incremental distances fallen in successive unit times. The controlling fact is Galileo's assumption that these ratios are invariant under a change of the unit of time. It admits few laws and only one is compatible with the two-numbered initial sequence 1, 3.

1. Introduction

Here is a simple problem in inductive inference. You have measured the distances fallen by a body in free fall after times 1, 2, 3 and 4 and find that the total distances fallen are in the ratios 1 to 4 to 9 to 16; and that the incremental distances fallen in each unit of time are in the ratios 1 to 3 to 5 to 7. What is the general rule? Famously, the answer Galileo gave is that that total distances fallen increase with the squares of the time, while the incremental distances increase as the odd numbers.

This formulation of the problem greatly oversimplifies the inductive problem Galileo faced in his discovery of his law of fall. However the formulation is part of the problem. In his *Two New Sciences* (1638, pp. 178-79), Galileo describes an experiment that would measure the distances fallen and the times taken. Instead of a body in free fall, the experiment uses a surrogate system of a ball rolling down a grooved inclined place, so that the times taken are sufficiently slowed to enable precise measurement. Time is measured by weighing water collected from a thin jet during the motion. We cannot know if Galileo himself really performed the experiment or whether it was a fictional artifice of the dialog. Stillman Drake (1978, p. 89), however, has identified an earlier Galileo manuscript in which, Drake believes, Galileo recorded the experimentally measured distances traversed by a body rolling down a grooved inclined plane. Drake conjectures that Galileo tied gut frets across the groove. He then carefully spaced them so that the rolling ball beat out a uniform rhythm. The gut frets are then spaced so that they mark the distances fallen after 1, 2, 3, … units of time.

Let us then posit a Galile0-like inductive problem. Given the experimentally obtained numbers above, how can we infer inductively to the full law, using the resources available to Galileo? In Norton (2014), I have described how this induction can be warranted by two material facts. First is a Platonic assumption: that fall conforms to a rule that may be written simply using the mathematical vocabulary available to Galileo. The second is an invariance assumption: that the law of fall is invariant under a change of the unit of time. I asserted in Norton (2014) that this second assumption places a powerful restriction on the admissible laws of fall. My purpose in this note is to restate the invariance and then to show, using techniques not available to Galileo, just how profoundly this invariance restricts the admissible laws of fall.

2. Invariance under the Change of the Unit of Time

Galileo was clearly aware of the invariance of his law of fall under the selection of a different unit of time. It is explicit in his statement of the incremental form of the law. He wrote in *Two New Sciences* (1638, Third Day, Naturally Accelerated Motion, Thm. II, Prop. II, Cor. I; my emphasis)

> Hence it is clear that if we take *any equal intervals of time whatever*, counting from the beginning of the motion, such as AD, DE, EF, FG, in which the spaces HL, LM, MN, NI are traversed, these spaces will bear to one another the same ratio as the series of odd numbers, 1, 3, 5, 7;…

This is an important property of his law of fall. For Galileo's experiments, in so far as we can reconstruct them, did not employ a single, accurately reproducible unit of time. Rather, he could divide the time of the experiment into equal parts that became a temporary unit. He would then assume that his results were insensitive to the choice of unit. The alternative was the unlikely possibility that the results held only for certain choices of the unit of time and that Galileo had by good fortune implemented them in his experiment.

We can see the invariance of the law Galileo discovered with some arithmetic. In times

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

the total distances fallen will be in the ratios

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100$$

and the incremental distances

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19$$

Now choose a different unit of time that is twice as long as the original. Measured with this new unit, the even numbered subset 2, 4, 6, 8, 10 of the original times are now labeled

$$1, 2, 3, 4, 5$$

and the total distances fallen are

$$\begin{gathered} 4, 16, 36, 64, 100 \\ = 4\text{x}1, 4\text{x}4, 4\text{x}9, 4\text{x}16, 4\text{x}25 \end{gathered}$$

The factor of 4 does not appear in the ratios of the distances fallen, which now once again are in the ratios of 1 to 4 to 9 to 16 to 25. The original square numbers are restored. Correspondingly, the incremental distances are

$$\begin{gathered} 1+3, 5+7, 9+11, 13+15, 17+19 \\ = 4, 12, 20, 28, 36 \\ = 4\text{x}1, 4\text{x}3, 4\text{x}5, 4\text{x}7, 4\text{x}9 \end{gathered}$$

Dropping the factor of 4, the original odd number ratios of 1 to 3 to 5 to 7 to 9 are restored.

This arithmetic argument shows only one case of the invariance in which the old and new time units are in the simple arithmetic ratio of 1 to 2. Will the law of fall remain invariant under arbitrary rescaling of the unit? A simple geometric argument establishes that it will. Extending Galileo's geometric methods slightly, we can represent the law of fall by a triangle, shown in Figure 1.

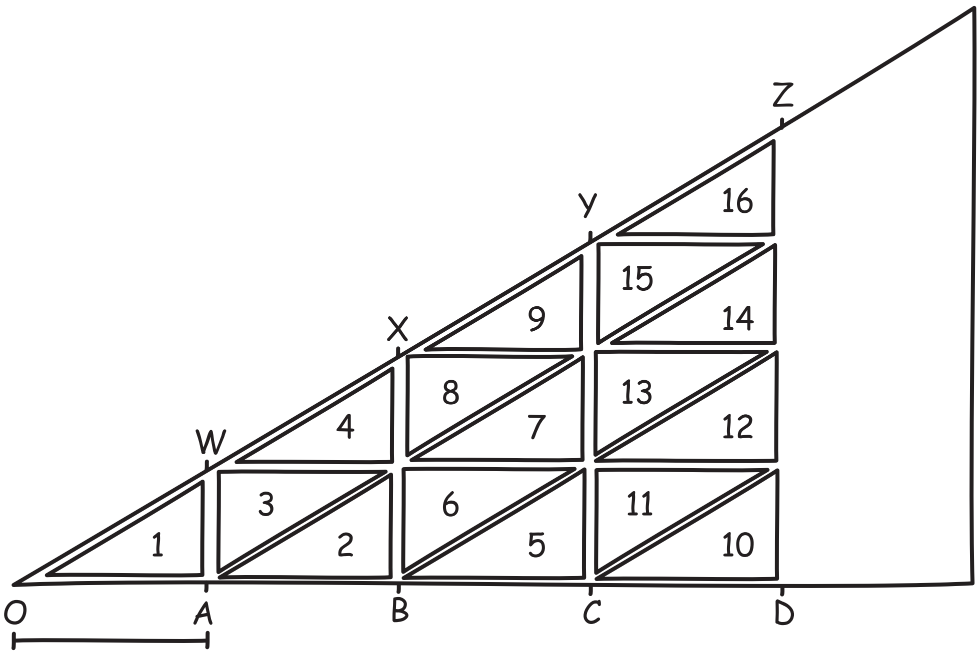

Figure 1. Galileo's Law of Fall Showing Total Distances Fallen

The lengths OA, OB, OC and OD along the base represent times and the areas of the triangles OAW, OBX, OCY and ODZ above the base segments correspond to distances fallen. These triangles increase in area in proportion to the square of the base segments, thereby expressing the law of fall. We can also see both forms of the law of fall in the figure by a counting procedure. We take the triangle OAW as a unit distance and use it to fill the triangle, as shown. We can count the total distances fallen in times OA, OB, OC and OD by the number of unit triangles in the triangles, OAW, OBX, OCY and ODZ. As shown, there are 1, 4, 9 and 16 of these unit triangles, respectively.

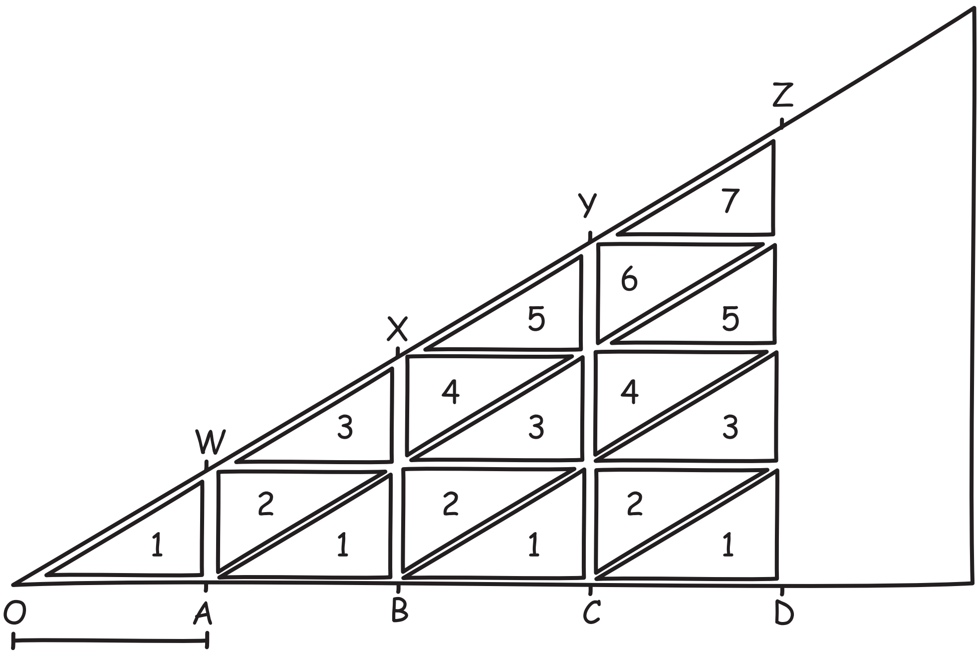

Figure 2. Galileo's Law of Fall Showing Incremental Distances Fallen

The incremental distances fallen in the time intervals OA, AB, BC and CD are represented in Figure 2 by the areas of the columns above the unit times, OA, AB, BC and CD. There are 1, 3, 5 and 7 of the unit triangles in the columns.

The selection of a new unit merely involves replacing the old selection of OA by a new one OA', otherwise keeping the original triangle the same, that is, keeping the original motion unchanged. It is immediately clear that any new selection—such as shown in Figure 3—will be geometrically similar to the original construction and will return the same ratios for the total and incremental distances fallen.

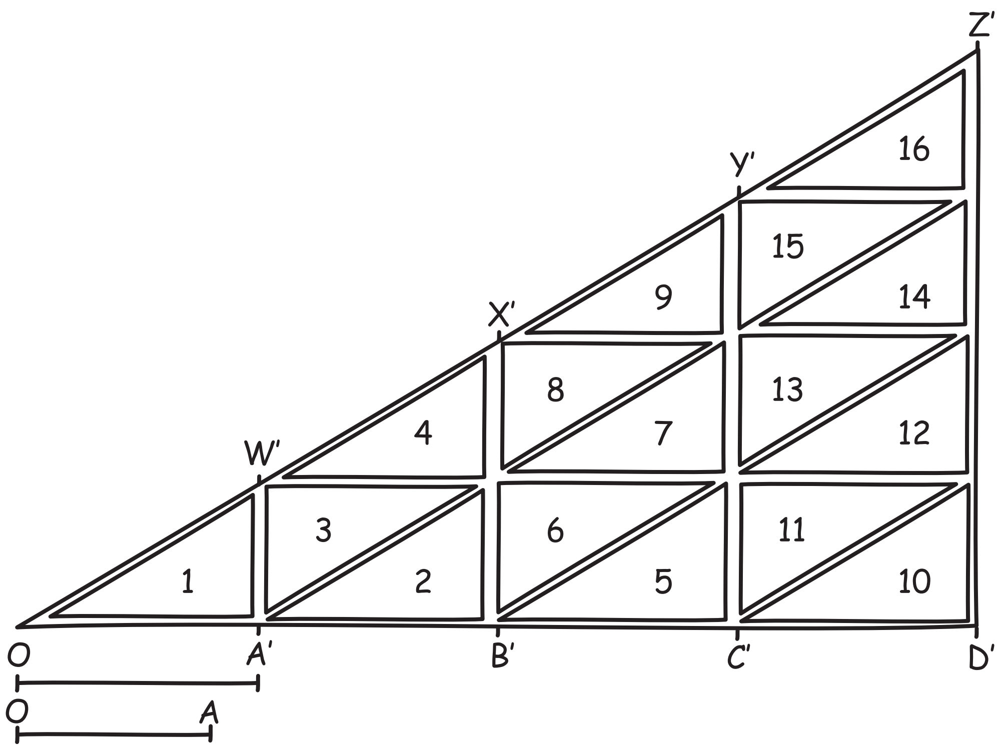

Figure 3. Galileo's Law of Fall with a New Unit of Time

The simplest argument for the generality of the invariance is algebraic. If s(t) is the total distance fallen by t and d(t)=s(t)-s(t-1) the incremental distance fallen between t-1 and t, then Galileo's law of fall asserts (up to multiplicative constant) that

$$s(t) = t^2 \qquad d(t) = t^2 - (t-1)^2 = 2t-1 \tag{1}$$

Adopting any new unit of time corresponds to rescaling t to t'=rt, for any positive r. The corresponding rescaled laws are

$$s'(t') = (t'/r)^2 = (1/r^2)\, t'^2 = \text{constant } t'^2$$

$$d'(t') = (t'/r)^2 - ((t'-1)/r)^2 = (1/r^2)\, (t'^2 - (t'-1)^2)$$

$$= (1/r^2)\, (2t'-1) = \text{constant } (2t'-1)$$

The constants do not affect the ratios, so the original law is preserved. With the new time unit, the ratios of total distance grows with the square of time and the incremental distances as the odd numbers, as before.

2. Laws of Fall Invariant under a Change of Unit of Time

Galileo's law of fall is invariant under a change of unit of time. Which others laws also respect this invariance? The result to be proven shortly is that there are very few. If s(t) is the total distance fallen by t and d(t)=s(t)-s(t-1) the incremental distance fallen in time t-1 to t, then the *only* laws of fall with this invariance are

$$s(t) = Kt^p \qquad d(t) = K(t^p - (t-1)^p) \tag{2}$$

for any real p >0 and an arbitrary constant K>0.[1] This is highly restrictive, especially if we make the natural choice for Galileo and restrict p to natural numbers. Then (setting K=1) the admissible laws reduce to:

$$\begin{array}{ll} s(t) = t^2 & d(t) = 2t-1 \\ s(t) = t^3 & d(t) = 3t^2-3t+1 \\ s(t) = t^4 & d(t) = 4t^3-6t^2+4t-1 \\ s(t) = t^5 & d(t) = 5t^4-10t^3+10t^2-5t+1 \\ \ldots & \ldots \end{array}$$

Restricting the law to whole number values of p is unnecessary, however. For the law has just one free parameter, p. (The constant of proportionality K does not affect the ratios of s(t) and the ratios of d(t), so its value need not be determined or can be set arbitrarily.) It follows immediately from (2) that, once Galileo has collected very little data, it is possible to show that his law of fall is the only law of fall with this invariance. For example, all that is needed are the first two values, s(1) and s(2). For we have from (2) that

$$s(2)/s(1) = 2^p/1^p = 2^p$$

But since the measured s(2)/s(1) = 4/1, it follows immediately that p=2.

---

[1] The only additional antecedent condition is that there has to be at least value of t>0 at which the function s(t) is differentiable. In physical terms, that means that, in the course of the fall, there has to be at least one moment at which the instantaneous velocity is defined. The proof below cannot preclude the possibility of additional laws of fall that respect the invariance but are differentiable at no moment of time at all. That is, for them, at no moment is an instantaneous velocity defined.

We have a most striking conclusion. Given the above background and, in particular, invariance under the unit of time, the only admissible continuation of the sequence of total distances s(t) with just two members 1, 4, is the sequence of squares. The only admissible continuation of the sequence of incremental distances d(t) with just two members, 1, 3, is the odd number sequence.

3. Proof

To prove that (2) follows from the invariance, we need to write the condition that a general law of fall s(t) is invariant under a change of the unit of time. Changing the unit of time corresponds to replacing the time variable t by a new time variable

$$t' = r\,t$$

where r is any positive, real rescaling factor. When we rescale the time variable, the distance function s will transform to a new function s'(t'). Since we are only relabeling times, the distance fallen by t as recorded by s(t) must be the same as the distance fallen by t'=rt as recorded by s'(t'). That is,

$$s'(t') = s(t) \tag{3}$$

There is a further condition. The ratios s(1) to s(2) to … to s(T) must be the same as the ratios s'(1) to s'(2) to … to s'(T), for any T>0. This expresses the rule's invariance under a change of the unit of time. It follows that

$$\frac{s'(T)}{s'(1)} = \frac{s(T)}{s(1)} \quad \text{and} \quad s'(T) = \frac{s'(1)}{s(1)} s(T) = f(r) s(T) \tag{4}$$

where the ratio s'(1)/s(1) can only be a function of r and is written as f(r). Combining (3) and (4) and setting T to be t'=rt we have

$$s'(t') = s(t) = f(r)\ s(rt) \tag{5}$$

Setting t=1 in (5) entails that

$$f(r) = s(1)/s(r)$$

Hence we can rewrite (5) as

$$s(1)\ s(rt) = s(r)\ s(t) \tag{6}$$

We proceed with the simplifying assumption that s(t) is differentiable. It turns out that we need assume much less. In the appendix it is shown that, if s(t) is differentiable at just one value of t>0, then it is differentiable at all t>0.

Differentiating (5) with respect to r and separately with respect to t and considering parameter values t>0 and r>0, we recover two equations

$$s(1)\frac{ds(rt)}{d(rt)}t = \frac{ds(r)}{dr}s(t) \quad \text{and} \quad s(1)\frac{ds(rt)}{d(rt)}r = s(r)\frac{ds(t)}{dt}$$

Eliminating s(1) ds(rt)/d(rt) from the two equations, we have

$$\frac{ds(r)}{dr}\frac{s(t)}{t} = \frac{s(r)}{r}\frac{ds(t)}{dt}$$

Rearranging we have

$$\frac{r}{s(r)}\frac{ds(r)}{dr} = \frac{t}{s(t)}\frac{ds(t)}{dt} = p \tag{7}$$

The first quantity is a function of the variable r only and the second is a function of the variable t only, yet they are equal. Therefore they must be a constant, independent of both r and t. That constant is labeled p. The second equality is an easily solved differential equation:

$$\frac{1}{s(t)}\frac{ds(t)}{dt} = \frac{d\ln s(t)}{dt} = \frac{p}{t}$$

Integrating we have

$$\ln s(t)\Big|_{t_1}^{t_2} = \ln\left(\frac{s(t_2)}{s(t_1)}\right) = p\int_{t_1}^{t_2}\frac{1}{t}dt = p\ln\left(\frac{t_2}{t_1}\right) = \ln\left(\frac{t_2}{t_1}\right)^p$$

This simplifies to

$$s(t) = Kt^p$$

We have as an initial condition that s(0) = 0, so that p>0.[2] The incremental form of the law,

$$d(t) = s(t) - s(t-1) = K(t^p - (t-1)^p)$$

follows immediately. Equation (2) above has now been recovered. Finally we have

$$f(r) = s(1)/s(r) = 1/t^p$$

Appendix. Relaxing the Requirement of Differentiability

The proof above assumed that s(t) is everywhere differentiable for t>0. That expansive differentiability can be derived from (6) if we assume in addition that there is just one value of t at which s(t) is differentiable. To see this, write (6) with t replaced by t+Δt. We then have

---

[2] We preclude the degenerate case of K=0 since then s(t)=0.

$$s(1)s(r(t+\Delta t)) = s(r)\, s(t+\Delta t)$$

Subtracting (6) and dividing by $r\Delta t$, we have

$$s(1)\frac{s(rt+r\Delta t)-s(rt)}{r\Delta t} = \frac{s(r)}{r}\frac{s(t+\Delta t)-s(t)}{\Delta t}$$

A corresponding manipulation in which r is replaced by $r+\Delta r$ in (6) yields

$$s(1)\frac{s(rt+t\Delta r)-s(rt)}{t\Delta r} = \frac{s(t)}{t}\frac{s(r+\Delta r)-s(r)}{\Delta r}$$

We now require Δt and Δr to satisfy

$$r\Delta t = t\Delta r \tag{8}$$

With this restriction, it follows that

$$\frac{s(r)}{r}\frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{s(t)}{t}\frac{s(r+\Delta r)-s(r)}{\Delta r}$$

Rearranging, we recover

$$\frac{t}{s(t)}\frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{r}{s(r)}\frac{s(r+\Delta r)-s(r)}{\Delta r} \tag{9}$$

Let the particular value of $t>0$ in (9) be the one value of t for which we assume that s(t) is differentiable. Call it t^*. It now follows that the limit of the left hand side of (9) as Δt goes to zero must exist and be equal to a quantity containing the derivative ds(t)/dt:

$$Lim_{\Delta t\to 0}\frac{t^*}{s(t^*)}\frac{s(t^*+\Delta t)-s(t^*)}{\Delta t} = \frac{t^*}{s(t^*)}\left.\frac{ds(t)}{dt}\right|_{t=t^*}$$

Since we have from (8) that $\Delta r = (r/t)\Delta t$, it now follows that the limit on the right hand side of (9) also converges and is equal to

$$Lim_{\Delta r\to 0}\frac{r}{s(r)}\frac{s(r+\Delta r)-s(r)}{\Delta r} = \frac{r}{s(r)}\frac{ds(r)}{dr}$$

But r is any value greater than zero. Hence it follows that s(r) is differentiable for any $r>0$.

References

Drake, Stillman (1978) *Galileo at Work: His Scientific Biography*. Chicago: University of Chicago Press. Repr. Mineola, NY: Dover, 2003.

Galilei, Galileo (1638) *Dialogues Concerning Two New Sciences*. Trans. Henry Crew and Alfonso de Salvo. MacMillan, 1914; repr. New York: Dover, 1954.

Norton, John D. "A Material Defense of Inductive Inference" http://www.pitt.edu/~jdnorton