The emergence of resonance dual models: a first look at some early historical prolegomena of a related formal technique

Giuseppe Iurato

University of Palermo, IT

Abstract

We wish to lay out, within its historical context, one of the chief works which has led to the formulation of the early resonance dual models prior to string theory. Indeed, we shall focus on the fundamental Tullio Regge work of 1959, to be precise, on certain mathematical methods handled by him to pursue his original intentions mainly motivated to prove the validity of the so-called Mandelstam representation for the potential scattering of two spinless particles for a given class of generalized Yukawa potentials.

It is well-known that the celebrated Gabriele Veneziano work of 1968 is almost unanimously considered as the first step toward resonance dual models, from which then string theory soon arose. Nevertheless, we are of the historiographical opinion that the prolegomena to resonance dual model is the famous 1959 seminal paper of Tullio Regge, in which a notable formal technique has been introduced, the so-called *Watson-Sommerfeld transform.* Thenceforth, such a formal tool even more played a fundamental role in theoretical particle physics, so that we would like to highlight the main historical moments which led to such a technique, trying to identify the truly early sources of it. The historical course treated by us to pursue this end, comprises two main steps, a first one having an introductory character which is devoted to a very brief outline of the main formal elements of scattering theory, and a second one just centered on such Tullio Regge paper. In pursuing this, a special attention has been paid to those many unavoidable moments concerning history of mathematics and mathematical-physics whose preeminent presence, along this historical route, cannot be evaded. But, let us say it immediately: we will not deal with the wide history of scattering theory, because our main aim is just directly arguing on the rising of 1959 Regge work about the introduction of complex angular momenta and related mathematical methods, pointing out just some historical aspects concerning these latter and that have nevertheless been quite neglected by the usual historical treatments of the subject.

1. Scattering theory: a brief general overview

Most of information concerning quantum systems comes from collision theory. Following[1] (Rossetti 1985, Capitolo I), the early crucial moments of the history of quantum theory has seen basically involved celebrated collision experiments: to mention the main ones only, we recall the 1911 E. Rutherford experiments on elastic collision of α particles on atoms, the 1913-14 J. Franck and G. Hertz inelastic collisions of electrons on gas atoms and molecules, the photoelectric effect, the 1923 A.H. Compton scattering of photons on atomic electrons, the 1927 C. Davisson and L. Germer diffraction experiments and the 1934 E. Fermi experiments on neutron collisions; on the other hand, the modern elementary particle physics is also based on collision phenomena: for instance, the celebrated 1983 C. Rubbia experimental observation of intermediate vector bosons essentially was a proton-antiproton annihilation process, which constituted one of the first experimental confirmations of the Standard Model. In this paper, we are interested only in those historical aspects of collision theory which have led to the dawning of resonance dual models of string theory, with a particular attention to the many intersection points with history of mathematics.

One of the chief notions of collision theory for putting together experimental data and theoretical issues, is that of *scattering cross-section.* The *differential scattering cross-section* $d\sigma = \sigma(\Omega)d\Omega$ is defined to be the number of scattered particles within the infinitesimal solid angle $d\Omega$ (with Ω computed in polar coordinates $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$) per unit of time and per unit of incident beam of target particle. The *total scattering cross-section* is defined to be *Scattering cross-sections*

$$\sigma_t(E) \doteq \int_\Omega \sigma(\Omega)d\Omega \tag{1}$$

which depends only upon the energy E of the incident beam. Following (Landau & Lifšits 1982, Chapter XVII) and (Rossetti 1985, Chapter II), in the center of mass reference frame, the motion of two particles[2] with well-

[1] This textbook is one the main references that we have taken into account in drawing up this paper. Surely, (Rossetti, 1985) is the central reference considered in drawing up the Section 1.

[2] With negligible spin-orbit interaction.

defined energy E and interacting by means of a central potential field $V(\vec{r})$, may be reduced to the motion of a single particle with reduced mass m and relative energy E, which is ruled by the following well-known stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r}) \tag{2}$$

which is the equation of motion of a particle with mass m moving in a central potential field $V(\vec{r})$. If we set $k^2 = (2m/\hbar^2)E$ and $U(\vec{r}) = (2m/\hbar^2)V(\vec{r})$, the equation (2) will assume the form

$$\Delta\psi(\vec{r}) + [k^2 - U(\vec{r})]\psi(\vec{r}) = 0. \tag{3}$$

We suppose $U(\vec{r}) \to 0$ as $\vec{r} \to \infty$ in such a rapid manner that we might speak of so afar regions where the interaction is negligible and where the above parameter k will be equal to the modulus of the wave vector $\vec{k}$ of the relative motion.

We consider an incident beam directed along the positive z axis, as polar axis, oriented according to the related wave vector $\vec{k}$. At infinite distances, the wave function $\psi_{as}(\vec{r})$ will satisfy the following asymptotic wave equation

$$\Delta\psi_{as}(\vec{r}) + k^2\psi_{as}(\vec{r}) = 0 \tag{4}$$

whose solutions are either (progressive and regressive) plane waves of the form $\psi_{as}(\vec{r}) = Ae^{\pm i\vec{k}\cdot\vec{r}}$ and (divergent and convergent distorted) spherical waves of the form $\psi_{as}(\vec{r}) = f(\theta,\varphi)e^{\pm i\vec{k}\cdot\vec{r}}/r$, so that the wave function $\psi(\vec{r})$ describing a scattering process, at infinite distance, is the superposition of an incident plane wave (associated to the incident beam) and of a wave emerging from the scattering center. Therefore, we have the following asymptotic behavior

$$\psi(\vec{r}) \underset{r\to\infty}{\sim} \psi_{as}(\vec{r}) = A\Big\{e^{i\vec{k}\cdot\vec{r}} + f(\theta,\varphi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\Big\} \tag{5}$$

where A is a normalizing constant, $f(\theta,\varphi)$ is a function, called *scattering amplitude* and having a length dimension, which estimates the amplitude of the scattered distorted spherical wave with respect to the amplitude of the incident beam, θ is the planar scattering angle of the emerging beam with respect to the z axis and k is the modulus of the wave vector $\vec{k}$. The geometry of the system implies that $f(\theta,\varphi)$ depends on the scattering angle θ (colatitude) but not on the anomaly φ. In the scattering amplitude are included the most important information on the scattering process, whose knowledge involves exact solutions to the original Schrödinger equation, hence the form of $U(\vec{r})$. Furthermore, as regard the differential scattering cross-section, it is possible to prove that

$$\sigma(\Omega) = \sigma(\theta, \varphi) = |f(\theta, \varphi)|^2 \tag{6}$$

while, as regard the total scattering cross-section, we have

$$\sigma_t(k) = \int_\Omega |f(\theta, \varphi)|^2 d\Omega, \tag{7}$$

in the geometry of central potentials having these formulas themselves but with $f(\theta)$ for $f(\theta, \varphi)$. Therefore, the scattering cross-section may be estimated by means of the scattering amplitude, and vice versa.

In non-relativistic potential scattering theory, we have to consider a central short-range potential field $U(r)$. Usually, the z axis will be the axis for the scattering center, coincident with the origin of the given polar coordinate system, parallel to the wave vector $\vec{k} = \vec{p}/\hbar$ of the incident beam. Therefore, we have a dynamical problem having cylindric symmetry with respect to the z axis, so that almost all the most important physical quantities will be independent from the anomaly φ (like $f(\theta, \varphi) \to f(\theta)$). In this case, under certain amplitude normalization conditions, equation (5) reduces to the following

$$\psi(\vec{r}) \underset{r\to\infty}{\sim} e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{i\vec{k}\cdot\vec{r}}}{r} \tag{8}$$

which is a solution to the following Schrödinger equation

$$[\Delta + (k^2 - U(r))]\psi(\vec{r}) = 0, \tag{9}$$

and is the sum of two terms, an incident plane wave and an emerging spherical wave (scattered wave) that contains the scattering amplitude from which it is possible to deduce all the physical characteristics of the collision process. The equation (9), for central potentials, has a set of elementary solutions given by

$$\psi_{lm}(\vec{r}) = \varphi_l(r) Y_l^m(\theta, \varphi) \tag{10}$$

which forms a complete set of solutions, so that every other solution to (9) will be a linear combination of them. In the geometry of the dynamical problem that we have chosen above, only those spherical functions not depending on the anomaly φ may be considered, that is to say $Y_l^0(\theta, \varphi) =$

$\sqrt{(2l+1)/4\pi}P_l(\cos\theta)$, so that our wave function has a Legendre polynomial series expansion of the type

$$\psi(\vec{r}) = \psi(r,\theta) = \sum_{l=0}^{\infty} C_l \varphi_l(r) P_l(\cos\theta) \tag{11}$$

with $\varphi_l(r)$ radial wave functions which have well-defined behaviors as r varies in $[0,\infty[$. In particular, we are interested in the asymptotic behavior for sufficiently large values of r in which potential and centrifugal terms (due to finite potential barriers) are either negligible, given by

$$\varphi_l(r) \underset{r\to\infty}{\sim} \frac{A_l}{kr}\sin\left(kr - l\frac{\pi}{2} + \delta_l\right) \tag{12}$$

where A_l is a suitable coefficient which goes to 1 as $U(r) \to 0$ and δ_l is an asymptotic phase displacement or shift of the given solution to the radial Schrödinger equation, related to the physical interaction and computed with respect to the free interaction solution. On the other hand, as $U \to 0$, the wave function defined by the asymptotic formula (9) should reduce to the free solution given by the incident plane wave, so that, taking into account (11), we will have

$$\psi(\vec{r}) \underset{U\to 0}{\longrightarrow} \psi_{free}(\vec{r}) \equiv e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \tag{13}$$

where $j_l(kr)$ are the Bessel spherical functions in the argument kr. In turn, (11) might be written as an expansion in partial waves each having a definite angular momentum l, of the type[3]

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} i^l(2l+1)\varphi_l(r)P_l(\cos\theta) \tag{14}$$

where $\varphi_l(r)$ are solutions to the radial Schrödinger equation, regular at the origin and normalized in order to reduce them to Bessel functions $j_l(kr)$ as $U \to 0$.

Inserting (12) into (14), we have

Phase shifts, partial amplitudes

$$\psi(\vec{r}) \underset{r\to 0}{\sim} \sum_{l=0}^{\infty} i^l(2l+1)\frac{A_l}{kr}\sin\left(kr - l\frac{\pi}{2} + \delta_l\right)P_l(\cos\theta) \tag{15}$$

so obtaining an asymptotic partial wave development of the wave function which basically dependent on the phase displacements, or *scattering phases*,

[3] First expansions of this type were provided by W. Gordon in (Gordon 1928).

or *phase shifts,* δ_l, since it is possible to prove that also the constants A_l are function of the latter. In turn, δ_l are determined by the potential $U(r)$ and vice versa: indeed, the knowledge of δ_l will allow the reconstruction of the interaction potential (scattering phase-function method). Therefore, these latter play a very fundamental role in potential scattering theory[4]. They have real values for real values of the potential, whilst, for complex values of the potential, the phases have in general complex values as well. On the other hand, as pointed out above, the scattering cross-section is closely related with the scattering amplitude which, as now we shall see, is related with the asymptotic form of the wave function, hence, ultimately, with the phase displacements. Thus, measuring scattering cross-sections, we shall be able to get information about phase displacements, hence about the interaction potential. Indeed, following (Rossetti 1985, Capitolo II), it turns out to be

$$\begin{aligned} f(\theta) =& \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta) = \\ =& \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}\sin\delta_l\cdot P_l(\cos\theta) = \\ =& \sum_{l=0}^{\infty}(2l+1)a_lP_l(\cos\theta) \end{aligned} \tag{16}$$

having put, as a *partial amplitude* corresponding to the partial wave of angular momentum l

$$a_l \equiv a_l(k) = \frac{1}{2ik}(e^{2i\delta_l}-1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l, \tag{17}$$

so that, being $\sigma(\theta) = |f(\theta)|^2$, we have the following expression for the differ-

[4]Following (Landau & Lifšits 1982, Chapter XVII), the problem how to build up the form of scattering potential from the scattering phases supposed to be known, has a particular relevance and has been solved by I.M. Gelfand, B.M. Levitan and V.A. Marčenko. In this regard, it turns out to be that, for the determination of $U(r)$, it is enough to know $\delta_0(k)$ as function of the wave vector in the region comprised between $k = 0$ and ∞ as well as the coefficients a_n in the following asymptotic expressions

$$\varphi_l \approx a_l e^{(\sqrt{2m|E_l|}/\hbar)r}/r$$

of the radial wave functions φ_l (of (11)) relative to those states corresponding to negative discrete energy levels, if these exist. The determination of $U(r)$ from these latter data involves the resolution of a certain linear integral equation (see (De Alfaro & Regge 1965)). Furthermore, another very interesting method to determine the scattering phase is the so-called *variable phase method* (see (Calogero 1967)).

ential scattering cross-section

$$\sigma(\theta) = \frac{1}{k^2}\Big|\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}\sin\delta_l \cdot P_l(\cos\theta)\Big|^2 \tag{18}$$

whilst for the (elastic) total scattering cross-section we have

$$\sigma_{tot}(k) = 2\pi\int_{-1}^{1}\sigma(\theta)d\cos\theta = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 \tag{19}$$

which, for $\theta = 0$, reduces to the *optical theorem*[5] $\sigma_{tot}(k) = (4\pi/k)\Im f(0)$. From (17), it follows that, if the phase $\delta_l = \delta_l(k)$ assumes, for a certain energy value[6] of k, the value of π, then $a_l(k) = 0$ and therefore the l-th partial wave does not concur to the scattering process (*Ramsauer-Townsend effect*), whilst if the phase δ_l assumes, for a certain energy value of k, the value of $\pi/2$, then the contribution of the l-th partial wave will be maximally exalted in the scattering process (*resonance scattering*). In the latter case, the scattering contribution by the l-th partial wave has the highest influence only when the phase δ_l changes in a neighborhood of the energy E_0 corresponding to that value of k for which δ_l assumes the value of $\pi/2$, in this case being possible to say that there exists an l-th partial wave resonance for $E = E_0$. In such a neighborhood, say $\mathcal{I}(E_0)$, an approximate expression for the l-th amplitude a_l was provided by G. Breit and E.P. Wigner in 1946 (*Breit-Wigner formula*), and given by

Breit-Wigner formula, Born approximation

$$a_l \approx -\frac{1}{k}\frac{\Gamma/2}{E - E_0 + i(\Gamma/2)} \tag{20}$$

where $\Gamma = 2/(d\delta_l(E)/dE)_{E=E_0}$. It provides a contribution to the total scattering cross-section given by

$$\sigma_l(E) \equiv 4\pi(2l+1)|a_l|^2 \approx \frac{2\pi\hbar^2}{m}(2l+1)\frac{1}{E}\frac{\Gamma^2/4}{(E-E_0)^2 + \Gamma^2/4} \tag{21}$$

whose graph in function of E shows a maximum value in the neighborhood $\mathcal{I}(E_0)$ having a width approximately equal to Γ. The appearance of an l-th wave function resonance peak physically corresponds to the production of a *metastable* (or *resonant*) *state*, localized in the interaction domain and with an equal angular momentum l, in which a particle may stay for a sufficiently long time estimated by the mean lifetime given by $\tau = \hbar/\Gamma$, in such a manner

[5]Introduced by Melvin Lax in (Lax 1950a,b).

[6]Indeed $E = \hbar^2k^2/2m$.

to exalt the corresponding scattering cross-section, this being the key essence of a scattering resonance phenomenon. The *Born approximation* for phases and scattering amplitudes respectively are

$$\delta_l = -k\int_0^\infty U(r)[rj_l(kr)]^2 dr, \quad f(\theta) = -\int_0^\infty U(r)\frac{\sin qr}{qr}r^2 dr. \tag{22}$$

We are particularly interested in the complex case. When the potential has real values, then the scattering phases are also real, but when the potential has complex values, then, in general, the scattering phases too are complex. In the case of central potentials, as regard the scattering amplitude $f(\theta)$ given by (16), we consider the following formula

$$\begin{aligned} f(\theta) =& \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta) = \\ =& \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(\cos\theta), \end{aligned} \tag{23}$$

having put $S_l \equiv e^{2i\delta_l}$. As regard the elastic differential cross-section, we have $\sigma_{el}(\theta) = |f(\theta)|^2$, while, taking into account (19), for the elastic total cross-section, we have

$$\begin{aligned} \sigma_{el}^{tot}(k) =& \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_l-1|^2 = \\ =& \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-2e^{-2\Im\delta_l}\cos(e\Re\delta_l)+e^{-4\Im\delta_l}]. \end{aligned} \tag{24}$$

Instead, the inelastic differential scattering cross-section is given by

$$\sigma_{in}(k) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-|S_l|^2] = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-e^{-4\Im\delta_l}]. \tag{25}$$

Therefore, in those scattering processes in which are present many alternative phenomena, like elastic and inelastic collisions, it is needed to consider the following expression for the total scattering cross-section

$$\begin{aligned} \sigma_{tot}(k) =& \sigma_{el}^{tot}(k)+\sigma_{in}(k) = \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-\Re S_l] = \\ =& \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-2e^{-2\Im\delta_l}\cos(2\Re\delta_l)]. \end{aligned} \tag{26}$$

S matrix, optical theorem

The temporal evolution $t_0 \to t$ of a quantum system is defined through a unitary operator $S(t, t_0)$ which acts on an initial state $|s(t_0)\rangle$ producing the final state $|s(t)\rangle = S(t, t_0)|s(t_0)\rangle$. In a scattering process, it is not necessary to know $S(t_0, t)$ for arbitrary values of the time, but it is often enough to consider time intervals for $t_0 \ll 0$ and $t \gg 0$ when one supposes that the interaction is efficient only into a definite time interval centered around $t = 0$; in such a case, we may define a unitary operator S, called *S matrix* (or *scattering matrix*) and introduced by J.A. Wheeler[7] in 1937, as a limit of unitary operators in the following fashion

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} S(t_0, t) \tag{27}$$

where the asymptotic states $t_0 = -\infty$ and $t = \infty$ refer to ideal moments in which the dynamical system is supposed to be into a non-interacting state. The S matrix (together the so-called *T matrix,* defined by $S = \mathbb{I} + iT$) is a formal tool which was mainly set up to study relativistic interactions of elementary subnuclear particles whose related forces were so little known in the form and nature to entail a quantum description through the Schrödinger and Dirac equations almost unusable. Instead, by means of such a tool, the scattering amplitudes could be deduced from general physical and mathematical principles no matter by the knowledge of the involved forces. Furthermore, the proliferation of the impetuous phenomenology of 1950s high-energy physics led to a so great amount of new and unexpected phenomena that the old quantum field theory turned out to be unable to provide a right theoretical basis to them, to make necessary appealing to general formal principles, above all unitarity and analyticity, to theoretically explain these new results. Every element of the S matrix, that is to say $S_{fi} = \langle f|S|i\rangle$, is the probability amplitude to observe a particular final state $|f\rangle$ starting from a given initial state $|i\rangle$, that is to say, the probability amplitude associated to the dynamical process $|i\rangle \to |f\rangle$. Other basic formal properties of the S matrix may be deduced from general physical principles, like relativistic invariants. For instance, as regard collision processes involving four scalar particles of the type $1 + 2 \to 3 + 4$, the possible relativistic invariants are $s = (p_1 + p_2)^2$,

[7]Following (Mackey 1978, Section 21) and (Chew 1962, 1966, Chapter 1), the idea of using the S matrix (more properly, the S operator) to describe the "scattering" of particles by one another in quantum mechanics, was introduced by J.A. Wheeler in 1937 and again independently by W. Heisenberg in 1943 who however lost interest in such a tool because of formal problems related just to its analytic continuation that was required to give to the S matrix dynamical content. Later, C. Møller published two seminal papers, in 1945 and 1946, in which the mathematics underlying Heisenberg ideas were developed, so that sometimes S matrix is also called *Møller wave matrix.* For a deeper historical analysis, see for instance (Chew 1962, 1966) and (Cushing 1986).

$t = (p_1 - p_2)^2$ and $u = (p_1 - p_2)^2$, which satisfy the following functional dependence relation $s + t + u = \sum_{i=1}^{4} m_i^2$ where $m_i^2 = p_i^2 = p_{0i}^2 - \vec{p}_i^2$. Moreover, another fundamental property of S matrix is the so-called *crossing* which relates the scattering amplitude of the initial process $1 + 2 \rightarrow 3 + 4$ with those processes obtained by this replacing one incoming particle with momentum p_i of the former process with its antiparticle having four-momentum $-p_i$, and vice versa. For example, in terms of T matrix, if $T(s, t, u)$ is the scattering amplitude of the process $1 + 2 \rightarrow 3 + 4$, then it is also the scattering amplitude of the process $1 + \bar{3} \rightarrow \bar{2} + 4$ or of the process $1 + \bar{4} \rightarrow \bar{2} + 3$. Therefore, information about different collision processes may be suitably put into analytical relationship between them: for instance, one of these is given by that providing the presence of poles in correspondence to possible bound states or resonances. The scattering amplitude cannot have only poles because them generate only a real amplitude whilst the unitarity condition implies the existence of an imaginary part, say $\Im T(s, u, t)$, proportional to the total scattering cross-section, and that, in turn, contains the various cuts of the corresponding Riemann surface. Therefore, crossing and unitarity link together analytical properties of different collision processes, that is to say, different elements of the S (or T) matrix, a notable fact, this, which was overstated in the 1960s till to think that such a formal scheme[8] could determine all the scattering amplitudes through well-determined spectral representations, amongst which certain dispersion relations[9], involving the various singularities of the analytic S (or T) matrix.

Setting

$$|s_i\rangle = \lim_{t \to -\infty} |s(t)\rangle, \quad |s_f\rangle = \lim_{t \to \infty} |s(t)\rangle, \tag{28}$$

we have

$$|s_f\rangle = S|s_i\rangle, \tag{29}$$

so that into the matrix S are included all the information on the scattering process by which the initial state $|s_i\rangle$ temporally evolves to the final state

[8]Upon which relies the so-called ***bootstrap hypothesis*** according to which, roughly speaking, from a given set of bound states - named "particles" - describing physical states and forces through a well-defined dynamics formulated in such a manner that other states may be self-consistently generated without make reference to initial fundamental states.

[9]Following (Cini 1977), the dispersion relations had a preeminent role in high-energy physics, becoming the prevalent paradigm for strong interactions, within which framework a particular analytical property of the scattering amplitude related to non-relativistic Schrödinger equation was extended, by analogy, to the relativistic context just by Regge's work. See (Cini 1977) for a deep historical account and basic philosophy of science considerations related to dispersion relations.

$|s_f\rangle$ due to the existence of an interaction between the particles constituting the given dynamical system involved into the scattering process. Now, if $|s_i\rangle = |a\rangle$, then the final state has the following expansion into a set of complete orthonormal states $|b\rangle$

$$|s_f\rangle = \int C_{ab}|b\rangle = \int \langle b|s_f\rangle|b\rangle = \int \langle b|S|a\rangle|b\rangle = \int S_{ab}|b\rangle, \tag{30}$$

where the index a refers to the state evolution $|a\rangle \to |b\rangle$, while $S_{ab} = \langle b|S|a\rangle$, thanks to the unitarity of S, provides the probability amplitude, given by $W_{ab} = |S_{ab}|^2$, with which the dynamical system may run along scattering process from $|a\rangle$ to $|b\rangle$. In general, the final state $|b\rangle$ is, roughly speaking, characterized by a continuous variability in such a manner that it is more correct to formally consider a differential probability defined as $dW_{ab} = |\langle b|S|a\rangle|^2 db$ which, therefore, is the transition probability from the initial state $|a\rangle$ into one of the final states $|b\rangle$ having quantum numbers comprised between b and $b + db$. In the time-dependent interaction representation, the S matrix has the following representation

$$S = Te^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} H'_I(t)dt} \tag{31}$$

said to be the *Dyson's formula*, T being the time decreasing operator called *chronological product* and H' the perturbative part of the hamiltonian operator. At the first order perturbative effects, this formula reduces to the following *Born approximation* for S

$$S \approx S_B = \mathbb{I} - \frac{i}{\hbar}\int_{-\infty}^{\infty} H'_I(t)dt. \tag{32}$$

In a scattering process, over each energy surface E, we have the following conservation energy relation

$$\langle b|S|a\rangle = \langle b|S^E|a\rangle\delta(E_b - E_a) \equiv S^E_{ba}\delta(E_b - E_a) \tag{33}$$

where $S^E_{ba} \equiv \langle b|S^E|a\rangle$ is the so-called S matrix element over the energy surface E. Because of the continuous variability of the final states b, we have

$$dW_{ba} = \frac{1}{2\pi\hbar}|\langle b|S^E|a\rangle|^2\frac{db}{dE_b} \overset{\substack{Born\\approx.}}{\longrightarrow} \frac{2\pi}{\hbar}|\langle b|H'_I|a\rangle|^2\rho_f(b) \tag{34}$$

which is said to be the *first Fermi golden rule*, where

$$\rho_f(b) = db/dE_b \tag{35}$$

is the density of the final states. With greater precision, the two above mentioned quantum states, involved in the scattering process and considered satisfying the right normalization conditions, should be denoted as follows $|a, E_a, \vec{n}_a\rangle$ and $|b, E_b, \vec{n}_b\rangle$, where a, b refer to the nature of the system and to the related characterizing quantum numbers, while $\vec{n}_a = \vec{p}_a/p_a = \vec{k}_a/k_a$ and $\vec{n}_b = \vec{p}_b/p_b = \vec{k}_b/k_b$ identify the related momentums whose directions are respectively identifiable by means of the related solid angles $\Omega_a = (\theta_a, \varphi_a)$ and $\Omega_b = (\theta_b, \varphi_b)$. In particular, we have $db = dE_b d\Omega_b$, so that, in the hypothesis $E = E_b = E_a$, the first right hand side of (34) becomes

$$dW_{ba} = \frac{1}{2\pi\hbar}|\langle b, E, \vec{n}_b|S^E|a, E, \vec{n}_a\rangle|^2 d\Omega_b \tag{36}$$

which provides the transition's probability per unit of time, from the initial state $|a\rangle$ to the final state $|b\rangle$ with wave vector having direction $\vec{n}_b$ comprised into the elementary solid angle $d\Omega_b$. Making use of the probability current density associated to a quantum particle, it is possible to prove that the differential scattering cross-section for the elastic and inelastic scattering processes $a \rightarrow b$ is given by

$$\sigma_{ba}(\Omega_b)d\Omega_b = \frac{dW_{ba}}{|\vec{j}_a|} = \frac{4\pi^2}{k_a^2}|\langle b, E, \vec{n}_b|S^E - \mathbb{I}|a, E, \vec{n}_a\rangle|^2 d\Omega_b, \tag{37}$$

where $\vec{j}_a$ is the probability current density associated with the particles of the incident beam. For elastic scattering processes, since $a = b$, we may write

$$\sigma(\Omega) \equiv \sigma(\theta, \varphi) = \frac{4\pi^2}{k^2}|\langle \vec{n}'|S^E - \mathbb{I}|\vec{n}\rangle|^2 \tag{38}$$

where $\Omega \equiv (\theta, \varphi)$ identifies the direction $\vec{n}'$ of the scattered wave vector with respect to the direction of the incident wave vector $\vec{n}$. From (6) and (38), it follows that

$$f(\theta, \vartheta) = \frac{2\pi}{k}e^{i\alpha}\langle \vec{n}'|S^E - \mathbb{I}|\vec{n}\rangle \tag{39}$$

where $e^{i\alpha}$ is a suitable phase factor.

In a central potential problem, the angular momentum $\vec{L}$ is a constant of motion so that, said $|lm\rangle$ an orthonormal basis of eigenstates of the angular momentum operators L^2 and L_z, the S matrix elements are diagonal and we may set

$$\langle b, l'm'|S^E|a, lm\rangle = \delta_{ll'}\delta_{mm'}S^l_{ba} \tag{40}$$

where the reduced S matrix elements $S^l_{ba}(K)$ does not depend on m, seen the arbitrariness of the direction z with respect to which compute L_z, but only on the two states a, b and on the energy E via the modulus k of the relative wave vector. The elements of the operator $S^E - \mathbb{I}$ upon the energy surface E, given by

$$\langle b, \vec{n}_b | S^E - \mathbb{I} | a, \vec{n}_a \rangle, \tag{41}$$

can be estimated as follows

$$\langle b, \vec{n}_b | S^E - \mathbb{I} | a, \vec{n}_a \rangle = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)(S_{ba} - \delta_{ba}) P_l(\cos\theta) \tag{42}$$

where θ is the scattering angle between $\vec{n}_a$ and $\vec{n}_b$, such a formula providing the Legendre polynomial expansion of the matric elements (39), with coefficients which depend on the reduced S matrix elements S^l_{ba}. For elastic scattering processes, $a = b$ so that, setting $S^l_{aa} \equiv S_l$, we have

$$\langle \vec{n}' | S^E - \mathbb{I} | \vec{n} \rangle = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta). \tag{43}$$

But, from (39) for central symmetry problems, we have

$$f(\theta) = \frac{2\pi}{k} e^{i\alpha} \langle \vec{n}' | S^E - \mathbb{I} | \vec{n} \rangle \tag{44}$$

with $\alpha \in \mathbb{R}$ an undefined parameter, so that

$$f(\theta) = \frac{e^{i\alpha}}{2k} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta) \tag{45}$$

which, compared with the following Legendre polynomial expansion

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta), \tag{46}$$

implies $e^{i\alpha} = -i$ and $S_l = e^{2i\delta_l}$, that is to say, the reduced S matrix elements are closely related to the corresponding phase displacements δ_l. Therefore, (39) reduces to

$$f(\theta, \varphi) = -\frac{2\pi i}{k} \langle \vec{n}' | S^E - \mathbb{I} | \vec{n} \rangle \tag{47}$$

which holds too for all the elastic scattering processes, comprised the non-central ones, being θ and φ the polar angles of $\vec{n}'$ with respect to the incident direction $\vec{n}$. In the case of central potential scattering, using (42) in (37), we obtain the following expression for the differential scattering cross-section

$$\sigma_{ba}(\Omega_b) = \frac{1}{4k^2}|\sum_{l=0}^{\infty}(2l+1)(S_{ba}^l - \delta_{ba})P_l(\cos\theta)|^2, \tag{48}$$

whilst for the total scattering cross-section, we have

$$\sigma_{ba}^{tot}(k) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_{ba}^l - \delta_{ba}|^2, \tag{49}$$

both these latter formulas being valid for every scattering process $a \to b$. For elastic scattering processes ($a = b$), they respectively reduce to the following

$$\sigma_{ba}^{el}(\theta) = \frac{1}{4k^2}|\sum_{l=0}^{\infty}(2l+1)(S_l - \mathbb{I})P_l(\cos\theta)|^2 \tag{50}$$

and

$$\sigma_{el}^{tot}(k; a \to b) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_l - \mathbb{I}|^2, \tag{51}$$

whereas, for inelastic scattering processes ($a \neq b$), we respectively have

$$\sigma_{ba}^{inel}(\theta) = \frac{1}{4k^2}|\sum_{l=0}^{\infty}(2l+1)S_{ba}^l P_l(\cos\theta)|^2 \tag{52}$$

and

$$\sigma_{inel}^{tot}(k; a \to b) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_{ba}^l|^2. \tag{53}$$

Often, a sum over all the possible scattering channels $a \to b$ is considered as follows

$$\begin{aligned}\sigma_{inel}^{tot}(k) =& \sum_{a\neq b}\sigma_{ba}^{tot}(k) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sum_{a\neq b}|S_{ba}^l|^2 = \\ =& \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1 - |S_l|^2).\end{aligned} \tag{54}$$

Furthermore, in both elastic and inelastic scattering process case, we have $S_l = e^{2i\delta_l}$ with $|S_l| \leq 1$, so that $|S_l| = |e^{2i\delta_l}| = e^{-2\Im\delta_l}$ and $\Im\delta_l \geq 0$. As regard the total scattering cross-section, we have

$$\sigma_{tot}(k) = \sigma_{el}^{tot}(k) + \sigma_{inel}^{tot}(k) = \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-\Re S_l). \tag{55}$$

On the other hand, taking into account (46), we have

$$\Im f(0) = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)(1-\Re S_l), \tag{56}$$

that, compared with (55), gives rise to the *optical theorem* in its most general form

$$\Im f(0) = \frac{k}{4\pi}\sigma_{tot}(k). \tag{57}$$

In analogy with (47), which links together the elastic scattering amplitude with the elements of the S matrix, we may define a *generalized scattering amplitude* as follows

$$f_{ba}(\vec{n}_b, \vec{n}_a) \equiv f_{ba}(\Omega_b) = -\frac{2\pi i}{k}\langle b\vec{n}_b|S^E - \mathbb{I}|a\vec{n}_a\rangle \tag{58}$$

where $\Omega_b = \Omega \equiv (\theta, \varphi)$ is the solid angle which identifies the direction of $\vec{n}_b$ with respect to $\vec{n}_a$, this last chose parallel to the direction of the polar axis in such a manner that the differential scattering cross-section for the channel $a \to b$, is given by $\sigma_{ba}(\Omega_b) = |f_{ba}(\Omega_b)|^2$. In the case of multichannel processes, for simplicity's sake we consider a three channel process whose directions are $\vec{n}, \vec{n}'$ and $\vec{n}''$, by which we have

$$f(\vec{n}', \vec{n}) - f^*(\vec{n}, \vec{n}') = \frac{ik}{2\pi}\int f^*(\vec{n}'', \vec{n}')f(\vec{n}'', \vec{n})d\Omega'' \tag{59}$$

that is the general condition to which undergoes the scattering amplitude for purely elastic processes, due to unitarity of S, and that, for central collisions, reduces to the optical theorem. Indeed, in such a case, $f(\vec{n}', \vec{n})$ depends on $\cos\theta = \vec{n}\cdot\vec{n}'$ only, and since we have too $f(\vec{n}, \vec{n}') = f(\vec{n}', \vec{n})$, it follows that (58) reduces to

$$\Im f(\vec{n}, \vec{n}') = \frac{k}{4\pi}\int f^*(\vec{n}'', \vec{n}')f(\vec{n}'', \vec{n})d\Omega'' \tag{60}$$

that is an extension of the optical theorem because, for $\vec{n} = \vec{n}'$, we have $\theta = 0$, so that (60) reduces further to

$$\Im f(\vec{n}, \vec{n}) = \frac{k}{4\pi} \int |f(\vec{n}'', \vec{n})|^2 d\Omega'' = \frac{k}{4\pi} \sigma_{tot} \tag{61}$$

that is to say, the optical theorem (57). It is also interesting to notice that (60) allows, at least in principle, to rebuild up the scattering amplitude when it is known its modulus, that is to say, it is known the differential cross-section, excepts the indeterminacy arising from the invariance of (60) with respect to the change $f(\theta) \to -f^*(\theta)$.

Regge poles, Jost solutions and functions

Many fundamental results of scattering theory may be deduced without make reference to a specific form of the related interaction, only through the basic properties of the S matrix like, for instance, its unitarity from which we have information about the scattering amplitude. Further properties may be deduced from other analyticity properties of the elements of the S matrix, which are able to describe a collision process. In what follows, we shall refer to elastic collisions in a central field, where the S matrix elements are given by the functions $S_l \equiv S_l(k)$ - that is to say, S matrix elements computed over a surface of energy[10] k with values related to states of definite angular momentum l - depending on the scattering phases through

$$S_l(k) = \exp(2i\delta_l(k)). \tag{62}$$

For instance, for an elastic scattering process in a central field, if one considers (16) wrote in the form

$$f(k, \theta) = \sum_{l=0}^{\infty} (2l+1) a_l(k) P_l(\cos\theta), \tag{63}$$

where the partial amplitudes $a_l(k)$ are given by (17) and written as follows

$$a_l(k) = \frac{1}{2ik}(S_l(k) - 1), \tag{64}$$

then it follows that properties of $f(k, \theta)$ are consequence of the properties either of $S_l(k)$ and of the Legendre polynomials $P(\cos\theta)$. The latter formulas (63) and (64) resolve, d'après H. Faxen and J. Holtsmark works of 1927, the problem how to express the scattering amplitude by means of the scattering phases. Moreover, from the orthogonality properties of these latter, an inversion of (63) gives rise to

$$a_l(k) = \frac{1}{2} \int_{-1}^{1} f(k, \theta) P_l(\cos\theta) d\cos\theta \tag{65}$$

[10] Recall that $E = \hbar^2 k^2 / 2\mu$.

hence, from (64), it also follows

$$S_l(k) = 1 + ik\int_{-1}^{1} f(k,\theta)P_l(\cos\theta)d\cos\theta, \tag{66}$$

so that there exists a certain equivalence between the informations given by the scattering amplitude $f(k,\theta)$ and those given by the set of functions $S_l(k)$. Therefore, it is of fundamental importance to study the analytical properties of the functions $S_l(k)$ considered as function either of l and k, though only for $k \in \mathbb{R}^+$ and $l \in \mathbb{N}_0$ they have a direct physical meaning for scattering processes. Nevertheless, as we will see later, the consideration of these functions also for complex values of both these two variables, gives rise to interesting and unexpected results and interpretations. For instance, the knowledge of the properties of analyticity in the variable k of the functions $S_l(k)$, allows, inter alia, to show that the scattering amplitude verifies certain dispersion relations in the variable k, whereas the knowledge of the properties of analyticity in the variable l of the functions $S_l(k)$, provides a representation for the scattering amplitude which is useful to study the asymptotic properties for great transferred impulses, so giving rise to the so-called *Regge poles.* Therefore, for pursuing this, it is preliminarily need to define $S_l(k)$ for arbitrary values of l and k, and this may be done, in potential theory[11], through the following reduced radial Schrödinger equation

$$u_l'' + [k^2 - \frac{l(l+1)}{r^2} - U(r)]u_l = 0 \tag{67}$$

which, mathematically, has always a meaning for every complex value of k and l. The relation (64) has been obtained by taking into consideration a wave function with an asymptotic behavior given by (5) and developed in partial waves as follows

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} i^l(2l+1)\frac{\chi_l(k,r)}{r}P_l(\cos\theta) \tag{68}$$

where $\chi_l(k,r)$ are the physical solutions to (67) which are zero in the origin of the given reference frame, and having the following asymptotic behavior

$$\chi_l(k,r) \underset{r\to\infty}{\sim} \frac{e^{i\delta_l(k)}}{k}\sin[kr - l\frac{\pi}{2} + \delta_l(k)] \tag{69}$$

[11] Hence, within the non-relativistic quantum mechanics framework where an interaction's potential is definable, so that the dynamical equation governing the collision process is the Schrödinger equation.

which, in absence of any interaction, reduces the following free solutions

$$\chi_l^0(k,r) = r j_l(k,r) \tag{70}$$

where $j_l(k,r)$ are the Bessel spherical functions. The formula (69) implicitly defines the scattering phase $\delta_l(k)$, hence $S_l(k)$ via (62), through $\chi_l(k,r)$ for each l,k. Thus, the analyticity properties of the functions $S_l(k)$ will be known once we know those of the functions $\chi_l(k,r)$ as solutions to (67). For simplicity's sake, we shall consider simple solutions to (67) as, for instance, those obtained under certain boundary (in the origin or at infinity) conditions which are very simple and which are dependent only on one of the two variable k and l, from which, therefore, to accordingly derive fundamental analyticity properties of $S_l(k)$ in the variable k (with l fixed at a given physical value) or in the variable l (with k fixed and belonging to $\mathbb{R}^+$). If we set $\lambda = l + 1/2$, then (67) reduce to the following more symmetric form

$$u'' + [k^2 - \frac{\lambda^2 - 1/4}{r^2} - U(r)]u = 0 \tag{71}$$

whose solutions are of the form $u = u(\lambda, k, r)$, or else $u = u_l(k,r)$ when we wish to refer to physical values of l. We search for solutions to (71) through suitable hypotheses on the interaction potential $U(r)$ which, nevertheless, allow us to have an enough degree of generality to leave aside from the particular type of potential so chosen. Such hypotheses regard the behavior of the potential either in the origin of the reference frame and at infinity. To be precise, we consider short-range potentials $U(r)$ such that

$$\begin{aligned} \int_0^a r|U(r)|dr &= M(a) < \infty \quad \forall a \in \mathbb{R}^+, \\ \int_b^\infty |U(r)|dr &= N(b) < \infty \quad \forall b \in \mathbb{R}^+. \end{aligned} \tag{72}$$

The hypothesis $(72)_1$ implies that the point $r = 0$ is a singular point of Fuchsian type for the equation (71), with characteristic exponents $\lambda + 1/2 \equiv l + 1$ and $-\lambda + 1/2 \equiv -l$, so that surely there exist two solutions to (71) which behave like to $r^{\pm\lambda+1/2}$ as $r \to 0$. Instead, the hypothesis $(72)_2$ implies that (71) has solutions which have an asymptotic behavior similar to the one of the solutions to the equation $u'' + k^2 u = 0$, so that surely there exist solutions to (71) which behave like to $\exp(\pm i\vec{k} \cdot \vec{r})$ as $r \to \infty$. Therefore, through such asymptotic behaviors, namely $r^{\pm\lambda+1/2}$ as $r \to 0$ and $e^{\pm i\vec{k}\cdot\vec{r}}$ as $r \to \infty$, we are able to find solutions to (71).

The solution $u = \phi(\lambda, k, r) \underset{r\to 0}{\sim} r^{\lambda+1/2}$, with the boundary condition

$$\lim_{r\to 0} r^{-\lambda-1/2}\phi(\lambda, k, r) = 1, \tag{73}$$

is said to be the *regular solution* to (71), and, as such, it turns out to be given by $\phi(\lambda, k, r) \equiv \phi_l(k, r) = C_l(k)\chi_l(k, r)$. Likewise, the function $\phi(\lambda, k, r)$, in absence of interaction, must reduce to the free solution $\phi^0(k, r)$, that is to say, to the solution to the free equation

$$u'' + \Big(k^2 - \frac{\lambda^2 - 1/4}{r^2}\Big)u = 0 \tag{74}$$

which has the same asymptotic behavior of $\phi(\lambda, k, r)$ in $r = 0$. For $l \in \mathbb{N}_0$, we have

$$\phi^0(k, r) = \frac{(2l + 1)!!}{k^l} r j_l(k, r) \tag{75}$$

where $j_l(k, r)$ are the spherical Bessel functions. Therefore, to may carry out an extension to every value of l, we should consider the functional generalization of the semi-factorial, hence the Euler gamma function, so that we have

$$\phi^0(\lambda, k, r) = 2^\lambda k^{-\lambda}\Gamma(1 + \lambda)r^{1/2} J_\lambda(kr) \tag{76}$$

where $J_\lambda(kr) = J_{l+1/2}(kr)$ are the spherical Bessel functions of the first kind, defined for arbitrary values of l. To find analytical properties of $\phi(\lambda, k, r)$ as a solution to (71), we may construct a suitable integral equation of the Volterra type to which such a ϕ must satisfy under the boundary conditions (72), and this, in turn, may be accomplished by means of an auxiliary equation that takes into account the same boundary conditions (72), equivalent to

$$\phi \underset{r\to 0}{\sim} r^{\lambda+1/2}, \qquad \phi' \underset{r\to 0}{\sim} (\lambda + 1/2)r^{\lambda-1/2}, \tag{77}$$

and that has known solutions having the same behavior, in the origin of the reference frame, of the searched solutions to (71), that is to say, having a behavior as $r^{\pm\lambda+1/2}$. As a possible auxiliary equation to which ϕ ought to be satisfied as $r \to 0$, we choose the following one

$$u'' - \frac{\lambda^2 - 1/4}{r^2}u = 0, \tag{78}$$

which has the following pair of fundamental solutions

$$u_+ = r^{\lambda+1/2}, \qquad u_- = r^{-\lambda+1/2}. \tag{79}$$

It is possible to prove that such an integral equation is as follows

$$\begin{aligned}\phi(\lambda,k,r) &= u_+(r)+ \\ &+\frac{1}{2\lambda}\int_0^r [k^2-U(x)][u_+(x)u_-(r)-u_-(x)u_+(r)]\phi(\lambda,k,x)dx,\end{aligned} \tag{80}$$

so that, the integral equation of the Volterra type to which the regular solution $\phi(\lambda,k,r)$ must satisfy is

$$\begin{aligned}\phi(\lambda,k,r) &= r^{\lambda+1/2}+ \\ &+\frac{1}{2\lambda}\int_0^r [k^2-U(x)]\sqrt{rx}\Big\{\Big(\frac{x}{r}\Big)^\lambda-\Big(\frac{r}{x}\Big)^\lambda\Big\}\phi(\lambda,k,x)dx\end{aligned} \tag{81}$$

whose formal solution is

$$\phi(\lambda,k,r)=\sum_{n=0}^{\infty}\phi_n(r) \tag{82}$$

with $\phi_0(r)=r^{\lambda+1/2}$ and

$$\phi_n(r)=\frac{1}{2\lambda}\int_0^r [k^2-U(x)]\sqrt{rx}\Big\{\Big(\frac{x}{r}\Big)^\lambda-\Big(\frac{r}{x}\Big)^\lambda\Big\}\phi_{n-1}(x)dx, \quad \forall n\in\mathbb{N}, \tag{83}$$

which provides too an effective solution to (80) when it is uniformly convergent. To study the related convergence, we distinguish between the two cases $\Re\lambda \geq 0$ and $\Re\lambda < 0$. Putting $\lambda=\mu+i\sigma$, in the case $\Re\lambda=\mu\geq 0$, which comprises the physical case of angular momentum values (which are semi-integers), it is possible to prove that

$$|\phi_n(r)| \leq \frac{r^{\mu+1/2}}{|\lambda|^n}\frac{[P(r)]^n}{n!}, \quad \forall n\in\mathbb{N}, \tag{84}$$

with $P(r) \doteq \int_0^r x|k^2-U(x)|dx \leq (1/2)|k^2|r^2+M(r)$, so that we have

$$|\phi(\lambda,k,r)| \leq \sum_{n=0}^{\infty}|\phi_n(r)| \leq r^{\mu+1/2}\exp(P(r)/|\lambda|), \tag{85}$$

a condition which guarantees convergence. Now, a celebrated theorem, due to H.J. Poincaré, which roughly states that a solution to a differential equation, like (71), whose coefficients are entire functions of a certain parameter, is also an entire function of the same parameter when defined through boundary conditions which are independent from this parameter, may be applied to

(81). Indeed, (81) is, in every finite domain, a uniformly convergent series of polynomial in the variable k^2, so that it is an entire function of k^2 as well, in agreement with the just above mentioned Poincaré theorem. Moreover, it is possible to prove that, at least in the region $\Re\lambda > 0$, each term of (81) is an analytic function in the variable λ, so that also $\phi(\lambda, k, r)$ is analytic there. The analytic continuation of $\phi(\lambda, k, r)$ for negative values of $\Re\lambda$ entails more detailed information about the potential $U(r)$; in particular, if one supposes that

$$U(r) \underset{r\to 0}{\sim} cr^{-2+\epsilon}, \qquad \epsilon > 0, \tag{86}$$

then it is possible to prove that $\phi(\lambda, k, r)$ is analytic in λ in the region $\Re\lambda > -\epsilon/2$, so that there exists, at least, a region of the λ plane, i.e. the strip $|\Re\lambda| < \epsilon/2$, in which the functions $\phi(\lambda, k, r)$ and $\phi(-\lambda, k, r)$ are contemporaneously defined and constitute a fundamental pair of solutions to (71) for $\lambda \neq 0$. Therefore, for each λ arbitrarily fixed, $\phi(\lambda, k, r)$ is an entire function of k^2, at least for $\Re\lambda > 0$, or else for $\Re\lambda > 0$ if (86) holds, as well as it is an even function of k, i.e. $\phi(\lambda, -k, r) = \phi(\lambda, k, r)$; furthermore, for each k, $\phi(\lambda, k, r)$ is an analytic function of λ at least for $\Re\lambda > 0$ or else for $\Re\lambda > -\epsilon/2$ if (86) holds, for each finite value of k arbitrarily fixed. Instead, as concerns the behavior of $\phi(\lambda, k, r)$ as $k \to \infty$, it is possible to prove that, for any direction by which $k \to \infty$, $\phi(\lambda, k, r)$ reduces to the following free solution

$$\phi_0(\lambda, k, r) = \Big(c_1 \frac{e^{i\vec{k}\cdot\vec{r}}}{k^{\lambda+1/2}} + c_2 \frac{e^{-i\vec{k}\cdot\vec{r}}}{k^{\lambda+1/2}}\Big)(1 + O(1/k)) \tag{87}$$

where c_1, c_2 are independent from k. Finally, we notice that $\phi(\lambda, k, r)$ is real when k and r are real (under the obvious hypothesis that $U(r)$ is also real) because of the boundary conditions (73), so that, due to the H.A. Schwarz reflection principle, the following Hermiticity condition holds $\phi^*(\lambda, k, r) = \phi(\lambda^*, k^*, r)$ which, for physical values of $l = \lambda - 1/2$, implies $\phi_l^*(k, r) = \phi_l(k^*, r)$.

We have seen that the S matrix elements, as well as the scattering phases, are determined by the behavior of the wave function as $r \to \infty$, so that a primary role is played by boundary conditions at infinity, like (72); in particular, the condition $(72)_2$ says that $U(r)$ goes to zero faster than $1/r$ as $r \to \infty$, and guarantees the existence of two independent solutions to (71), which, as $r \to \infty$, behave as $\exp(\pm i\vec{k} \cdot \vec{r})$. Let $f(\lambda, k, r)$ be the exact solution to (71) with the following boundary condition

$$\lim_{r\to\infty} e^{i\vec{k}\cdot\vec{r}} f(\lambda, k, r) = 1, \tag{88}$$

so that

$$f(\lambda, k, r) \underset{r\to\infty}{\sim} e^{-i\vec{k}\cdot\vec{r}}, \tag{89}$$

which is said to be the *Jost solution.* Moreover, due to the inversion symmetry invariance of (71) with respect to the variable k, it follows that $f(\lambda, -k, r)$ too is a solution to (71), with

$$f(\lambda, -k, r) \underset{r\to\infty}{\sim} e^{i\vec{k}\cdot\vec{r}}. \tag{90}$$

Therefore, for every $k \neq 0$, we shall refer to the pair of Jost solutions $f(\lambda, \pm k, r)$ as two independent fundamental solutions to (71). The free Jost solution is given by

$$f_0(\lambda, k, r) = \sqrt{\frac{\pi k r}{2}} e^{-i(\lambda+1/2)\pi/2} H_\lambda^{(2)}(kr) \tag{91}$$

where $H_\lambda^{(2)}$ is the second-kind Hankel function. Since (88) does not depend on λ, the above mentioned Poincaré theorem states that, for each k fixed, $f(\lambda, k, r)$, where is defined, is an entire function of λ^2, hence everywhere analytic over the finite plane of λ. As an entire function of λ^2, $f(\lambda, k, r)$ is even in the variable λ, that is to say, $f(-\lambda, k, r) = f(\lambda, k, r)$. Indeed, if one proceeds as made above, the analytical properties of $f(\lambda, k, r)$ may be deduced transforming the differential equation to which it satisfies, i.e.

$$f'' + \Big[k^2 - \frac{\lambda^2 - 1/4}{r^2}\Big] f = 0 \tag{92}$$

with the boundary condition (88), into an appropriate integral equation of the Volterra type, through the following auxiliary equation

$$u'' + k^2 u = 0 \tag{93}$$

and its solutions

$$u_+ = e^{i\vec{k}\cdot\vec{r}}, \qquad u_- = e^{-i\vec{k}\cdot\vec{r}}. \tag{94}$$

Such an integral equation is

$$\begin{aligned} f(\lambda, k, r) =& e^{-i\vec{k}\cdot\vec{r}} + \\ &+ \frac{1}{k}\int_r^\infty \sin[k(x-r)]\Big[U(x) + \frac{\lambda^2 - 1/4}{x^2}\Big] f(\lambda, k, x)dx \end{aligned} \tag{95}$$

from which it follows that $f(\lambda, k, r)$, for each $k \neq 0$, is an entire function of λ^2 reasoning as made above by means of a series expansion whose general n-th term is a polynomial of degree n in the variable λ^2, at least in the region $\Im k < 0$. Likewise, it is possible to prove that, for any direction by which $|k| \to \infty$, we have

$$f(\lambda, k, r) \underset{|k|\to\infty}{\sim} f_0(\lambda, k, r). \tag{96}$$

The analyticity region in k may be extended if, for instance, one considers potentials $U(r)$ such that

$$\int_0^\infty rU(r)e^{\nu r}dr < \infty \qquad \text{for } \nu < m, \tag{97}$$

where m is an arbitrary positive number, then it is possible to prove that $f(\lambda, k, r)$ is analytic in k in the region $\Im k < m/2$, except $k = 0$, which is a polydromy point having kinematical nature because it is also shared by $f_0(\lambda, k, r)$, besides to become a pole of order l for physical angular momenta ($l \in \mathbb{N}_0$). Taking into account this, it is usual to cut the complex plane[12] of the variable k along the imaginary axis between the origin and the point $im/2$, so giving rise to the so-called *kinematical cut* (because independent from the potential). Therefore, under condition (97), the function $f(\lambda, k, r)$ is analytically defined in the region $\Im k < m/2$, hence $f(\lambda, -k, r)$ in the region $\Im k > -m/2$. Furthermore, for Yukawian potentials of the type

$$U(r) = \int_m^\infty e^{-\mu r}c(\mu)d\mu, \tag{98}$$

it is possible to prove that Jost solution $f(\lambda, k, r)$ is, for every λ arbitrarily fixed, everywhere analytic in the variable k, except in the branch point in $k = 0$ and in a cut in the positive imaginary semi-axis made between $im/2$ and $i\infty$, and said to be the *dynamical cut*[13]. Therefore, to sum up, the

[12] Following (Eden et al., 1966, Preface), one of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane. From the early days of dispersion relations, the analytic approach to the subject has proved to be one of the most useful tools.

[13] For certain types of potential, like the Yukawian one (see later), such a dynamical cut may degenerate into a series of poles which are usually said to be *false poles* of the S matrix, and that, in general, are poles of $f_l(k)$ in the region $\Im k > 0$ where, usually, $f_l(k)$ is not analytic. Nevertheless, we are interested in the analytical properties of the scattering amplitude $f(k, \theta)$ that, in potential theory, does not have such a dynamical cut in the region $\Im k > 0$, so that every pole of $f(k, \theta)$ is there always associated with a bound state having a definite value of l (see later).

Jost solution is defined by the boundary condition (88), it is a holomorphic function in the finite plane of λ (at least, in the strip $|\Re\lambda| < \epsilon/2$), is an even function in λ and an entire function in λ^2. As regard its dependence on k, such a function is analytic in the region $\Im k < 0$ and continuous in the real axis except a polydromy point in the origin which becomes a pole of order l for physical values of $\lambda = l+1/2$. It is analytic in the region $\Im k < m/2$ if the potential verifies (98); in particular, if $m = 0$, it turns out to be continuous in the real axis (except $k = 0$), while if $m = \infty$, the unique finite singularity is the point $k = 0$. For Yukawian potentials, it is analytic in the whole of the plane k cut between the origin and $i\infty$; for physical angular momentum values, such a cut reduces to the dynamical one between $im/2$ and $i\infty$, plus a pole in $k = 0$. Moreover, thanks to (89), $f(\lambda, k, r)$ is, for real λ and pure imaginary $k = i\tau$, to a real function of λ and τ, so that, due to the Schwarz reflection principle, it has the following Hermiticity property

$$f^*(\lambda, k, r) = f(\lambda^*, -k^*, r) \tag{99}$$

that, for physical values of the angular momentum, reads as

$$f_l^*(k, r) = f_l(-k^*, r). \tag{100}$$

Therefore, in the above mentioned regions, the four fundamental solutions to (71), that is to say, the two regular solutions $\phi(\pm\lambda, k, r)$ (which are also functionally independent for $\lambda \neq 0$) and the two Jost solutions $f(\lambda, \pm k, r)$ (which are also functionally independent for $k \neq 0$), are all well-defined, so that the ones are expressible as linear combination of the remaining others. We shall consider the regular $\phi(\lambda, k, r)$ as a linear combination of the Jost solutions $f(\lambda, k, r)$ because, through it, we shall be able to have a link with the S matrix elements. To be precise, we consider the following combination

$$\phi(\lambda, k, r) = \frac{1}{2ik}[f_J(\lambda, k)f(\lambda, -k, r) - f_J(\lambda, -k)f(\lambda, k, r)] \tag{101}$$

where

$$\begin{aligned} f_l(k) \equiv f_J(\lambda, k) =& w[f(\lambda, k, r), \phi(\lambda, k, r)] = \\ =& f(\lambda, k, r)\phi'(\lambda, k, r) - f'(\lambda, k, r)\phi(\lambda, k, r) \end{aligned} \tag{102}$$

is the Wronskian between $f(\lambda, k, r)$ and $\phi(\lambda, k, r)$, which define a new function said to be the *Jost function*. From (101), it is possible to prove that the asymptotic behavior of the regular solution is given by

$$\phi(\lambda, k, r) \underset{r\to\infty}{\sim} \frac{1}{2ik}[f_J(\lambda, k)e^{i\vec{k}\cdot\vec{r}} - f_J(\lambda, -k)e^{-i\vec{k}\cdot\vec{r}}]. \tag{103}$$

Moreover, by means of the boundary behaviors of the regular solutions before considered, it will be possible to attain the following expressions for such a Jost function[14], namely

(104)
$$
\begin{aligned}
f(\lambda, k) =& 2\lambda \lim_{r\to 0} r^{\lambda-1/2} f(\lambda, k, r) \qquad (\Re\lambda > 0),\\
f(\lambda, k) =& 2ik \lim_{r\to \infty} e^{-i\vec{k}\cdot\vec{r}} \phi(\lambda, k, r) \qquad (\Im k < 0),
\end{aligned}
$$

thanks to which it will be possible to infer the properties of $f(\lambda, k)$ from the boundary behaviors and the properties of the regular solutions. For instance, the analyticity properties of the Jost function, in the variable λ, are the same of those of the regular solution $\phi(\lambda, k, r)$, whereas the analyticity properties of the Jost function, in the variable k, are the same of those of the Jost solutions $f(\lambda, k, r)$. In particular, we have the following Hermiticity condition $f^*(\lambda, k) = f(\lambda^*, -k^*)$ (which holds for real values of $U(r)$ and r) that, for physical values of the angular momentum, implies $f_l^*(k) = f_l(-k^*)$. In absence of interaction, from the free expressions for regular and Jost solutions, we deduce the following free Jost function

(105)
$$
\begin{aligned}
f_0(\lambda, k) =& \sqrt{2/\pi}\, 2^\lambda \Gamma(1+\lambda) e^{-i(\lambda-1/2)\frac{\pi}{2}} k^{-\lambda+1/2} =\\
=& 2\lambda \lim_{r\to 0} r^{\lambda-1/2} f_0(\lambda, k, r).
\end{aligned}
$$

We have above considered the regular solution $\phi(\lambda, l, k)$ that, for physical values of $\lambda = l + 1/2$, that is to say, for semi-integer values of λ, it vanishes in the origin of the reference frame as r^{l+1}, so that it turns out to be proportional to the physical solution $\chi_l(k, r)$, that is to say $\phi(\lambda, k, r) \equiv \phi_l(k, r) = C_l(k)\chi_l(k, r)$, where χ_l is the so-called physical solution, in the origin, to (67), defined by (69) that here we re-write in the following form

$$
\chi_l(k, r) \underset{r\to\infty}{\sim} \frac{e^{i\delta_l}}{2ik}[e^{i(\delta_l - l\pi/2)} e^{i\vec{k}\cdot\vec{r}} - e^{-i(\delta_l - l\pi/2)} e^{-i\vec{k}\cdot\vec{r}}] \tag{106}
$$

so that we have the following asymptotic behavior

$$
\phi_l(k, r) \underset{r\to\infty}{\sim} \frac{\tau_l(k)}{2ik}[e^{i(\delta_l - l\pi/2)} e^{i\vec{k}\cdot\vec{r}} - e^{-i(\delta_l - l\pi/2)} e^{-i\vec{k}\cdot\vec{r}}], \tag{107}
$$

[14] Often, the expression $f(0, k)$ is also considered as a Jost function (see (Umezawa & Vitiello 1985, Chapter 7, Section 7.8)).

having put $\tau_l(k) \equiv C_l(k)e^{i\delta_l}$. Considering complex values of $l = \lambda - 1/2$ and of k, we re-write (107) as follows

$$\phi_l(k,r) \underset{r\to\infty}{\sim} \frac{\tau(\lambda,k)}{2ik}[e^{i(\delta(\lambda,k)-(\lambda-1/2)\pi/2)}e^{i\vec{k}\cdot\vec{r}} - \\ + e^{-i(\delta(\lambda,k)-(\lambda-1/2)\pi/2)}e^{-i\vec{k}\cdot\vec{r}}], \tag{108}$$

and comparing this asymptotic behavior with the one given by (103), we have the following identities

$$\begin{aligned} f(\lambda,k) &= \tau(\lambda,k)e^{i(\delta(\lambda,k)-(\lambda-1/2)\pi/2))}, \\ f(\lambda,-k) &= \tau(\lambda,k)e^{-i(\delta(\lambda,k)-(\lambda-1/2)\pi/2))}, \end{aligned} \tag{109}$$

which can be considered as a parametrization of the Jost functions $f(\lambda, \pm k)$. Making the ratio between the two formulas of (109), we see that the function $S(\lambda,k) \equiv e^{2i\delta(\lambda,k)}$ is linked to the Jost functions $f(\lambda, \pm k)$ by the simple following relation

$$S(\lambda,k) = e^{i(\lambda-1/2)\pi}\frac{f(\lambda,k)}{f(\lambda,-k)} \tag{110}$$

that, for physical values of l, reduces to

$$S_l(k) = (-1)^l\frac{f_l(k)}{f_l(-k)} \tag{111}$$

which represents the analytical continuation, to complex values of k and l, of the elements $S_l(k)$ of the S matrix. Therefore, the S matrix elements $S(\lambda,k)$, as functions of the complex variables λ and k, are determined by the Jost functions $f(\lambda,\pm k)$. In particular, we are concerned with the properties of $S(\lambda,k) = S(l+1/2,k) \equiv S_l(k)$ for fixed physical values of l (and arbitrary k), and for fixed arbitrary values of k (and arbitrary λ), as well as in their implications for scattering amplitudes. As concern some first analyticity properties of $S(\lambda,k)$, we may immediately say that $S(\lambda,k)$ is, for each fixed λ with $\Re\lambda > 0$, a meromorphic function of k at least in the region $|\Im k| < m/2$ (in the hypothesis that $U(r)$ goes to zero as $r \to \infty$ at least like e^{-mr}) except the point $k = 0$, while $S(\lambda,k)$ is, for each fixed k (in particular, for physical values of k), a meromorphic function of λ at least in the region $\Re\lambda > 0$. Considering $S_l(k)$ as a function of the complex variable k with a fixed l equal to one of its possible physical value, we have that (111) holds, from which it follows that, in the k plane, there may be two types of singularities: the first type corresponding to zeros of $f_l(-k)$, the second type corresponding to the poles of $f_l(k)$ in the region $\Im k > 0$ (or $\Im k > m/2$). But what we want to highlight in this regard is the main fact that the zeros of the Jost function, hence some poles of $S_l(k)$, are determined by the nature of the dynamical system involved in the collision process, hence by its potential. For bound states, having a certain angular momentum l and defined by a radial wave function regular in the origin of the reference frame, say $\phi_l(k, r)$, we have that the latter is a square-integrable function (condition characterizing a bound state that, physically, corresponds to a particle which is relegated near the scattering center) only for k not real but purely imaginary, that is to say, for $k = i\kappa$ with $\kappa \in \mathbb{R}^+$ (take into account the symmetry of $\phi_l(k, r)$ with respect to $k \to -k$). From (103), it follows that

$$\phi_l(i\kappa, r) \underset{r\to\infty}{\sim} -\frac{1}{2k}[f_l(i\kappa)e^{-\kappa r} - f_l(-i\kappa)e^{\kappa r}] \tag{112}$$

so in order to have a bound state for $k = i\kappa$ with energy $E = -\hbar^2\kappa^2/2\mu$, it is necessary that the coefficient of $e^{\kappa r}$ is zero, that is

$$f_l(-i\kappa) = 0, \tag{113}$$

so that

$$\phi_l(i\kappa, r) \underset{r\to\infty}{\sim} -\frac{1}{2k}f_l(i\kappa)e^{-\kappa r} \tag{114}$$

which is normalizable, whence it describes a bound state. Thus, (113) provides a necessary and sufficient condition for the existence of a bound state having angular momentum l and energy $E = -\hbar^2\kappa^2/2\mu$, that is, a bound state corresponds to a zero of the Jost function in the half-plane $\Im k < 0$ on the negative imaginary semi-axis $\Re k = 0$. In such a case, the wave function $\phi_l(i\kappa, r)$ describing such a state is a real function because, taking into account the parity symmetry $k \to -k$, we have

$$\phi_l^*(i\kappa, r) = \phi_l(-i\kappa, r) = \phi_l(i\kappa, r). \tag{115}$$

Therefore, as regard the S matrix elements, the bound states manifest themselves both as poles of $S_l(k)$ in $\Im k > 0$, due to the zeros of $f_l(-k)$, and as zeros (placed symmetrically with respect to the poles) of $f_l(k)$ in $\Im k < 0$ and placed into the semi-axis $\Re k = 0$. It is also possible to prove that, for bound states, the latter are simple, so that the corresponding poles of $S_l(k)$ are simple too. If one denotes with $\dot{f}_l(k)$ the first derivative of $f_l(k)$ with respect to k, then it is possible to prove that, for bound states, the following relation holds

$$\int_0^\infty [\phi_l(k_0, r)]^2 dr = \frac{1}{4ik_0^2}\dot{f}_l(k_0)f_l(-k_0), \tag{116}$$

while, an expression for the residues of $S_l(k)$, with respect to the variable $k = i\kappa$ ($\kappa > 0$), into the poles corresponding to bound states, is as follows

$$(117) \qquad R_l(i\kappa) = -i(-1)^l \frac{N_l^2(\kappa)}{\displaystyle\int_0^\infty \phi_l^2(i\kappa, r)dr}$$

where $N_l(k) \equiv -f_l(i\kappa)/2\kappa$ is the normalization constant of the asymptotic wave function given by (114). It follows that such a residue is purely imaginary, while, if one uses the variable $E = \hbar^2k^2/2\mu$ instead of k, in regard to a bound state corresponding to $k = i\kappa$, the residue itself computed with respect to $E = E_\kappa = -\hbar^2\kappa^2/2\mu$ is given by

$$(118) \qquad r(E_\kappa) = \lim_{E\to E_\kappa}(E - E_\kappa)S_l(k) = \frac{i\kappa\hbar^2}{\mu}R_l(i\kappa)$$

from which it turns out to be real.

Besides the above mentioned poles in the half-plane $\Im k > 0$, with respect to the parity symmetry $k \to -k$, $S_l(k)$ has further poles also in the half-plane $\Im k < 0$, corresponding to the possible zeros of $f_l(-k)$ in such a half-plane, where, in general, it is not analytic. In such poles, the wave functions $\phi_l(k, r)$ have an explosive behavior and, therefore, do not represent bound states. In the region $\Im k < 0$, we may have poles either with $\Re k \neq 0$ and $\Re k = 0$. In the first case, if $k = k_0$ is a pole of $S_l(k)$ in the region $\Im k < 0$, the case $\Re k_0 \neq 0$ (which cannot to take place for bound states due to the related square-summable finiteness condition) implies the existence of pairs of poles symmetrically placed with respect to the axis $\Re k = 0$. The poles of $S_l(k)$ as zeros of $f_l(-k) = 0$ in the region $\Im k < 0$ with $\Re k \neq 0$, are usually called *resonance,* while the poles of $S_l(k)$ as zeros of $f_l(-k) = 0$ in the same region but with $\Re k = 0$, are said to be *virtual states* (or *antibound states* or *redundant zeros*[15]). For a resonance, that is to say, for a pole of $S_l(k)$ of the type $k = h + ib$ with $b \leq 0$, we never have $b = \Im k = 0$ since, if it were $f_l(-k) = 0$ for $k \in \mathbb{R}$, due to the Hermiticity condition $f_l^*(k) = f_l(-k^*)$, we would have too $f_l^*(-k) = f_l(k) = 0$, so that the wave function $\phi_l(k, r)$ would turn out to be zero as well. We give the name resonance to such a type of pole because, when b is small, its presence is manifested in the scattering amplitude through the appearance, in the l-th wave function component, of a resonance, centered around an energy value $E = E_0 = \hbar^2h^2/2\mu$ and having a width Γ proportional to $|b|$. This last reasoning holds too for all the poles of $S_l(k)$ in $\Im k < 0$ with $\Re k \neq 0$, for every physical value of l. Instead, an

Bound states, resonances, virtual states

[15] The redundant zeros were first discussed by S.T. Ma in (Ma 1946; 1947a,b).

antibound state meant as a zero of $f_l(-k)$ in the region $\Im k < 0$ with $\Re k = 0$, say $k = ib$ with $b < 0$, may manifest with an exaltation of the scattering amplitude at the threshold, if $|b|$ is very small, even if this last phenomenon is also due to the presence of a bound state. For what follows, in studying analytical properties of $S_l(k)$ for physical values of l, it is better to choose $E \equiv \hbar^2 k^2/2\mu$ as a fundamental variable rather than

$$k = \sqrt{E}\sqrt{2\mu/\hbar^2}. \tag{119}$$

Therefore, to get full variability of k, E must variate over all the two sheeted Riemann surface of $\sqrt{E}$, made by two sheets with branch point at the origin except for a cut on both sheets extending along $\Re k \leq 0$. In such a case, it is a common practice to cut the plane of the complex variable E along the positive real semi-axis, calling *physical sheet* (or *first sheet*) of such a Riemann surface the one corresponding to the positive determination of $\sqrt{E}$ (hence that corresponding to $\Im k > 0$), and *non-physical sheet* (or *second sheet*) the one corresponding to the negative determination of $\sqrt{E}$ (hence corresponding to $\Im k < 0$). Thus, for $S_l = S_l(E)$, we have that bound states are poles of $S_l(E)$ placed on the negative real semi-axis of the physical sheet (being, for bound states, $\Im k > 0$ and $\Re k \leq 0$); the resonances are poles of $S_l(E)$ which lie on the non-physical sheet (being $\Im k < 0$) and the nearer they are to the cut, the stronger their effects will be on the scattering cross-section; and the antibound states are poles of $S_l(E)$ which lie on the negative real semi-axis of the non-physical sheet (being $\Im k < 0$ and $\Re k \leq 0$) and the nearer they are to the origin of the reference frame, the stronger their effect will be on the scattering cross-section in threshold; in this latter case, moreover, we have trajectories which behave for a while as bound states but finally break-up into two particles moving apart (see (Mackey 1978, Section 21)). Therefore, it is meaningful to speak of either *close* polar singularities with respect to the physical region $E > 0$, represented by the cut, and *far* polar singularities, the first ones being those that mainly contribute to determine the form of the scattering cross-section. Furthermore, if the pole of $S_l(k)$ is at the point $k = a+ib$, then it represents our unstable bound state of energy a and angular momentum (or spin) l; the larger b is, the shorter the particle lifetime will be, and when the lifetime is so short that the bound state decomposes before it can travel a measurable distance, then we have a resonance. Of course, there is no sharp distinction between resonances and unstable particles. It is interesting that resonances and unstable particles not only have energies and angular momenta but behave like stable particles in other respects. One can even discuss their interactions with one another (see (Mackey 1978, Section 21)).

Now, we consider the scattering amplitude $f(k,\theta)$ (as, for instance, given by (63)) as a function of the energy $E=\hbar^2k^2/2\mu$ and of the scattering angle θ, hence denoted by $f(E,\theta)$. We may consider E as a complex variable. For simplicity's sake, we consider natural units, so that $\hbar=2\mu=1$, whence $k=\sqrt{E}$. Therefore, considering $f(E,\theta)$ as a function of E at each fixed value of θ, from (63) we have the following partial wave development *Dispersion relations*

$$f(E,\theta)=\sum_{0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta) \tag{120}$$

where

$$a_l(k)=\frac{1}{2ik}(S_l(k)-1) \tag{121}$$

and $k=\sqrt{E}$. Therefore, the singularities of the partial amplitudes a_l are the same of $S_l(k)$, and since it is possible to prove that the dynamical cut of $S_l(k)$ does not appear in the scattering amplitude, we have that the unique singularities of $f(E,\theta)$, in the variable E, are poles corresponding to the possible bound states, resonances and virtual (or antibound) states, besides the possible polydromy points of $\sqrt{E}$, i.e., origin and point at infinity. We restrict to consider $f(E,\theta)$ over the physical sheet of $\sqrt{E}$, where the unique possible singularities, for what has been said above, are the points of the cut along the positive real semi-axis as well as possible bound states having a definite angular momentum l and placed into the non-positive real semi-axis. Moreover, the singularities of the scattering amplitude $f(E,\theta)$ on the physical sheet coincide with the spectrum of the Hamiltonian of the dynamical system constituted by the interacting particles, including the allowed bound states as poles of $f(E,\theta)$ for $\Re E<0$, which are in a finite number for every potential decreasing faster than $1/r^2$ (hence belonging to the discrete spectrum), and all the states of the cut of $f(E,\theta)$ which correspond to all values of $\Re E>0$ (hence belonging to the continuous spectrum). Then, as $E\to\infty$ by means of any direction along the physical sheet, it is possible to prove that $f(E,\theta)\underset{E\to\infty}{\to}f_B(\theta)$ in a uniform manner, being $f_B(\theta)$ the Born scattering amplitude obtained considering the Born approximation $(22)_2$ which holds when high energy values are called into question. To this point, it is important to show that the scattering amplitude $f(E,\theta)$ undergoes a certain energy dispersion relation computed with respect to a fixed value of the angle θ. To get it, we consider the Cauchy integral representation of $f(E,\theta)-f_B(\theta)$ extended along a closed path of the E' plane this last being cut along the $\Re E'>0$, having poles along $\Re E'\leq 0$, surrounding the point E but no one of the singularities of the integrating function. Thanks to Cauchy theorem, this integration path, say Υ, may be homotopically deformed into a bigger circle, with radius $R \to \infty$, closely embracing the cut $\Re E' > 0$ plus small neighborhoods each of which centered around every pole of $\Re E' \leq 0$. As we have said before, $f(E,\theta) - f_B(\theta)$ uniformly tends to zero, so that the contribution of the integral

$$f(E,\theta) - f_B(\theta) = \frac{1}{2\pi i} \oint_\Upsilon \frac{f(E',\theta) - f_B(\theta)}{E' - E} dE', \tag{122}$$

along this circle, goes to zero as well, so that we may write

$$\begin{aligned} f(E,\theta) =& f_B(\theta) + \\ &+ \frac{1}{2\pi i} \int_0^\infty \frac{f(E' + i\epsilon, \theta) - f(E' - i\epsilon, \theta)}{E' - E} dE' + \sum_n \frac{y_n(\theta)}{E - E_n}, \end{aligned} \tag{123}$$

the first contribution being extended to that part of the path Υ embracing $\Re E' > 0$ along a narrow strip of width $\epsilon \to 0$, whilst the second one comprehends each contribution given by the neighborhood of every pole of $\Re E' \leq 0$, where $y_n(\theta) = \mathrm{Res}\{f(E,\theta)\}_{E=E_n}$, for each n, is the residue of the scattering amplitude in the pole $E = E_n$ (which represents a bound state having a definite angular momentum l) and the summation is over all the allowed bound states. For $\theta \in \mathbb{R}$, we have $f^*(E,\theta) = f(E^*,\theta)$, whence $f(E - i\epsilon, \theta) = f^*(E + i\epsilon, \theta)$, so that (123) reduces to

$$f(E,\theta) = f_B(\theta) + \frac{1}{\pi} \int_0^\infty \frac{\Im f(E',\theta)}{E' - E} dE' + \sum_n \frac{y_n(\theta)}{E - E_n} \tag{124}$$

where $\Im f(E',\theta)$ is evaluated upon the upper boundary of the cut. The dispersion relation (124) is due the classical form of the *Khuri's dispersion relation* (see (Khuri 1957)). This last expression is nothing but a dispersion relation for the scattering amplitude which provides all the possible singularities and characteristics: namely, the cut along the positive real semi-axis and the discontinuities of $f(E,\theta)$ in it (which are $2i\Im f(E,\theta)$), as well as the the poles corresponding to bound states for $E = E_n$ ($\Re E_n < 0$) and the related residues $y_n(\theta)$; furthermore, it provides the behavior of $f(E,\theta)$ as $E \to \infty$, being $f(E,\theta) \to f_B(\theta)$. As regard, then, the residues $y_n(\theta)$, these can be evaluated through the corresponding functions $S_l(k)$ as follows

$$y_n(\theta) = (2l+1) \frac{r_l(E_n)}{2\sqrt{-E_n}} P_l(\cos\theta) \in \mathbb{R} \tag{125}$$

where $r_l(E_n)$ is the residue of $S_l(k)$ in $E = E_n$, basically given by (117) and (118). Nevertheless, the (124) not always turns out to be useful because it might be quite difficult to have information on $\Im f(E,\theta)$, so that, through the optical theorem (57) in the energy E with respect to $\theta = 0$, another formula is often used instead of (124), namely the following one

$$\Re f(E,0) = f_B(0) + \frac{P}{4\pi^2}\int_0^\infty \frac{\sqrt{E'}\sigma_t(E')}{E'-E}dE' + \sum_n \frac{y_n(0)}{E-E_n} \tag{126}$$

which is said to be the *forward dispersion relation* of the scattering amplitude[16], connecting the real part of $f(E,0)$ with an integral over its imaginary part as well as with the position and residues of the poles. It was provided by D. Wong and N.N. Khuri in 1957.

Regge poles and trajectories

We reconsider $S(\lambda,k)$, as given by (110), as a function of the complex variable λ, for any $k \in \mathbb{R}^+$ arbitrarily fixed. For $k \in \mathbb{R}^+$ (positive energy), we have the following Hermiticity condition $S^*(\lambda,k) = [S(\lambda^*,k)]^{-1}$ which implies, as regard the scattering phases δ of $S(\lambda,k) = e^{2i\delta(\lambda,k)}$, the relation $\delta^*(\lambda,k) = \delta(\lambda^*,k)$, so that they are real when λ is also real. From what has been said above, $S(\lambda,k)$ is a meromorphic function in the variable λ at least in the region $\Re\lambda > 0$, where $f(\lambda,\pm k)$ are both analytic when k is real, so that, in the region $\Re\lambda > 0$, due to zeros of $f(\lambda,-k)$, poles of $S(\lambda,k)$ may exist, which are universally known as *Regge poles*; they are confined in the region $\Im\lambda > 0$ and there exist independently of the potential $U(r)$, that is, their existence has a pure kinematical nature. If one considers further restriction on the potential $U(r)$, then the localization of the Regge poles may be more accurately identified. For instance, for purely imaginary values of r, if $|U(i\rho)| < M/\rho^2$ with $M \in \mathbb{R}^+$, then the Regge poles are confined, in the half-plane $\Re\lambda > 0$ and $\Im\lambda > 0$, within the region delimited by the hyperbole branch $\Re\lambda\cdot\Im\lambda < M/2$ and by their asymptotes $\Re\lambda = 0$ and $\Im\lambda = 0$. Instead, if $|U(i\rho)| < N/\rho$ with $N \in \mathbb{R}^+$, then the Regge poles are confined into the region $\Re\lambda < N/k$, whereas, for fixed values of $k \in \mathbb{R}^+$, the half-plane on the right hand side of the line $\Re\lambda = N/k$, is a region in which $S(\lambda,k)$ is holomorphic. Moreover, for Yukawian potentials verifying $|U(i\rho)| < M/\rho^2$, it is possible to prove that the real part of the Regge poles is upperly limited by $\Re\lambda = M'$ so that the half-plane $\Re\lambda > M'$ is a holomorphic zone for $S(\lambda,k)$. We have also need to know the behavior of $S(\lambda,k)$ as $|\lambda| \to \infty$ in the region $\Re\lambda > 0$. To this end, it is possible to prove that the following relation holds

$$\lim_{|\lambda|\to\infty} S(\lambda,k) = 1 \tag{127}$$

at least into certain regions of the λ plane. Such a relation may be also

[16] We have $1/(E'-E-i\epsilon) = P/(E'-E) + i\pi\delta(E'-E)$.

deduced by some estimates like the following ones. If $S(\lambda, k) = e^{2i\delta(\lambda,k)}$, then

$$|S(\lambda, k) - 1| = O(\lambda^{-1/2} e^{-\alpha\lambda}) \tag{128}$$

for potentials $|U(r)| < ce^{-mr}/r$ such that $\cosh\alpha = 1 + m^2/2k^2$. For complex values of λ, at least when the real part of the poles of $S(\lambda, k)$ is upperly bounded, for $|\lambda|$ great enough and such that $|\arg\lambda| < \pi/2$, we have

$$|S(\lambda, k) - 1| < ce^{-\alpha\Re\lambda}, \tag{129}$$

so that $S(\lambda, k) - 1$ exponentially tends to zero as $\lambda \to \infty$ in the region $\Re\lambda > 0$. It is desirable that many results of potential theory, when it is possible, may be extended to the relativistic case. In doing so, with respect to natural units, it is better to replace the fundamental kinematical variables, the relative wave vector k and the scattering angle θ, with the following two other variables, namely $s = E = k^2$ and the new parameter

$$t \equiv q^2 = 2k^2(1 - \cos\theta), \tag{130}$$

which respectively represent the relative energy of the incident particle beam and the square of the transfer momentum during the scattering process. So, the scattering amplitude $f(s, t)$, expressed in these two new variables, has a physical meaning only when $s \in \mathbb{R}^+$ and $t \in \mathbb{R}^- \cap\ ]-4s, 0[$. But, from these values, it is possible to analytically prolong such a function, as well as its asymptotic behavior, to all complex values of s and t, whose analytical properties are often obtainable by means of certain dispersion relations involving such a function, like (124), which is a dispersion relation at a fixed θ (hence at a fixed t), or the so-called *double representation* (in s and t) of S. Mandelstam (see (Chew 1962)), which allows to determine the scattering amplitude without make use of any wave equation. The asymptotic behavior of $f(s, t)$ as $t \to \infty$ has no physical meaning in the non-relativistic context but only in the relativistic one where the particle/antiparticle symmetry, which entails invariance with respect to the exchange between the two variable s and t, provides physical meaning to such a behavior. Indeed, at relativistic level, the so-called *substitution law* holds, according to which, together a diffusion process of the type

$$a + b \to a' + b', \tag{131}$$

it is needed to consider the following *cross-diffusion* process

$$a + \bar{a}' \to \bar{b} + b' \tag{132}$$

where $\bar{a}$ denotes the antiparticle of a, and so forth. In a few words, the substitution law states that, in a diffusion process, an incoming particle (like $\bar{a}'$) is equivalent to its outgoing antiparticle (like a). The quantity $t \equiv q^2$, is the transfer momentum into the direct channel (131) as well as the energy of the cross channel (132). t is a limited quantity for the former, being $0 \leq t \leq 4k^2$, whilst isn't for the cross channel for which t may take any value in $\mathbb{R}^+$, so that it is possible to consider the asymptotic behavior of the scattering amplitude

$$f(k,\theta) = f\Big[\sqrt{E}, \arccos\big(1 - \frac{t}{2k^2}\big)\Big] \equiv F(E,t) \tag{133}$$

as $t \to \infty$. If we work at a fixed value of energy, that is to say $k \in \mathbb{R}^+$ is arbitrarily fixed, then, by (130), it follows that $t \to \infty$ is equivalent to $\cos\theta \to -\infty$, so that it is needed to search for a representation of the scattering amplitude $f(k,\theta)$, which is valid for physical values of θ and that may be analytically prolonged in a open region of the plane of $\cos\theta$ in which it is possible bringing $\cos\theta$ to $-\infty$.

In pursuing this, we start from the following partial wave expansion of the scattering amplitude

$$f(k,\theta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta) \tag{134}$$

which defines $f(k,\theta)$ as an analytic function of $\cos\theta$ in that region of the plane of the complex variable $z \equiv \cos\theta$ in which the series at the right hand side of (134), converges. Such a convergence region depends on the l-th term of the series as $l \to \infty$, so that we should study the behavior of $a_l(k)$ and $P_l(\cos\theta)$ as $l \to \infty$. To this end, we have the following asymptotic relations

$$P_l(\cos\theta) = O\Big(\frac{e^{l|\Im\theta|}}{\sqrt{l}}\Big), \qquad |a_l(k)| = O\Big(\frac{e^{-\alpha l}}{\sqrt{l}}\Big) \tag{135}$$

where α is a positive real constant as above defined by $\cosh\alpha = 1 + m^2/2k^2$, so that the l-th term of the series (134) exponentially goes to zero as $l \to \infty$ in the region $|\Im\theta| < \alpha$ which, therefore, identifies the convergence region of the series (134) in the θ plane. To determine the strip $|\Im\theta| < \alpha$ in the plane of $z \equiv \cos\theta$, let us set $x = \Re z$, $y = \Im z$, $\theta_1 = \Re\theta$ and $\theta_2 = \Im\theta$, whence

$$z \equiv x + iy = \cos\theta \equiv \cos(\theta_1 + i\theta_2) = \cos\theta_1\cosh\theta_2 - i\sin\theta_1\sinh\theta_2 \tag{136}$$

from which it follows that

$$x = \cos\theta_1\cosh\theta_2, \qquad y = -\sin\theta_1\sinh\theta_2, \tag{137}$$

so the boundary of the convergence region of (134) is the boundary of the strip $|\Im\theta| < \alpha$, that is to say $\theta_2 \equiv \Im\theta = \pm\alpha$, and inserting these values of θ_2 into (137), hence eliminating θ_1, we get the equation of the boundary of the convergence region of (134) in the plane of $z \equiv \cos\theta$, that is the ellipse

$$\frac{x^2}{\cosh^2\alpha} + \frac{y^2}{\sinh^2\alpha} = 1, \tag{138}$$

said to be the *Lehmann ellipse,* (see (Lehmann 1958)) having major radius $\cosh\alpha = 1 + m^2/2k^2$, minor radius $\sinh\alpha = \sqrt{\cosh^2\alpha - 1}$ and foci in the points $\cos\theta = \pm 1$, the extremes of the physical region related to (134), which identifies the largest ellipse in the $\cos\theta$ plane where the scattering amplitude is analytic. It is possible to prove that, through (137), the strip $|\Im\theta| < \alpha$ is the interior of the region identified by (138), so that the convergence region of (134), in the plane of $\cos\theta$, is the interior of the Lehmann ellipse. In the plane $t = 2k^2(1 - \cos\theta)$, the Lehmann ellipse remains an ellipse, with foci in $t = 0$ and $t = 4k^2$, the point $t = -m^2$ is the extreme of the major radius corresponding to $\cos\theta = 1 + m^2/2k^2$, its two semi-axes depends on the energy through the above expressions of $\cosh\alpha$ and $\sinh\alpha$, and the more the energy grows up, the smaller and the more restricted such an ellipse becomes, at last reducing to the physical region given by the segment $[-1, 1] \subset \Re z$ of the plane of $z = \cos\theta$ as $k \to \infty$. Therefore, (134) is a representation of the scattering amplitude which converges within the Lehmann ellipse where it turns out to be an analytic function of $\cos\theta$. Nevertheless, this representation is unable to study the behavior of the scattering amplitude as $t \to \infty$ or equivalently as $\cos\theta \to -\infty$ that, however, may be suitably re-written to accomplish this purpose transforming the series into an integral over the complex plane of the above considered variable $\lambda = l + 1/2$, in such a manner to have an analytic continuation of $f(k, \theta)$ over an open domain of the plane of $\cos\theta$; such a particular integral transformation is usually called *Watson-Sommerfeld transformation.* To introduce this, we need to consider the Legendre function $P_l(z) \equiv P_{\lambda-1/2}(z)$ for arbitrary complex values of λ, which is an even entire function of λ^2. For $z = \cos\theta$, we have the following estimate

$$|P_{\lambda-1/2}(\cos\theta)| \leq c|\sin\theta|^{-1/2}|\lambda|^{-1/2}e^{|\Re\theta||\Im\theta| - \Im\theta\cdot\Re\lambda}. \tag{139}$$

Now, we consider the function

$$G(\lambda) \doteq \lambda\frac{S(\lambda, k) - 1}{\cos\pi\lambda}P_{\lambda-1/2}(-\cos\theta) \tag{140}$$

which is a meromorphic function of λ in $\Re\lambda > 0$, with poles in the zeros of $\cos\pi\lambda$ (in correspondence to the physical values of $\lambda = l + 1/2$) plus further possible poles in $\Im\lambda > 0$ given by $S(\lambda, k)$. The residue of $G(\lambda)$ in a generic pole in the variable $\lambda = l + 1/2$, with l having physical values, is

$$\text{Res}\{G(\lambda)\}_{\lambda=l+1/2} = -\frac{1}{2\pi}(2l+1)[S_l(k) - 1]P_l(\cos\theta). \tag{141}$$

If one defines the integral

$$\int_{\mathcal{C}} G(\lambda)d\lambda \tag{142}$$

where $\mathcal{C}$ is the path surrounding all and only all the poles of the real semi-axis $\Re\lambda > 0$, then, from the residue theorem, we have

$$\int_{\mathcal{C}} G(\lambda)d\lambda = -2\pi i \sum_{l=0}^{\infty} \text{Res}\{G(\lambda)\}_{\lambda=l+1/2} = -2kf(k,\theta) \tag{143}$$

whence we have the following integral representation

$$f(k,\theta) = -\frac{1}{2k}\int_{\mathcal{C}} \lambda \frac{S(\lambda,k) - 1}{\cos\pi\lambda} P_{\lambda-1/2}(-\cos\theta)d\lambda. \tag{144}$$

We would like to homotopically deform the path $\mathcal{C}$ in such a manner it coincides with the imaginary axis. First, we suppose do not exist poles of $S(\lambda, k)$ in $\Re\lambda > 0$. We notice that, as $|\lambda| \to \infty$ along any direction of the half-plane $\Re\lambda > 0$, from (139) with $\cos\theta \to -\cos\theta = \cos(\pi - \theta)$, it follows

$$\Big|\frac{\lambda P_{\lambda-1/2}(-\cos\theta)}{\cos\pi\lambda}\Big| \leq c(\sin\theta)^{-1/2}|\lambda|^{1/2}e^{-|\Re\theta\cdot\Im\lambda| + \Im\theta\cdot\Re\lambda}, \tag{145}$$

and, if $|S(\lambda, k) - 1| < ce^{-\alpha\Re\lambda}$, then, from (144), it follows that

$$|G(\lambda)| < c|\lambda|^{1/2}e^{-|\Re\theta\cdot\Im\lambda| + (\Im\theta - \alpha)\Re\lambda} \tag{146}$$

whence, in turn, it follows that $G(\lambda)$ goes to zero uniformly as $|\lambda| \to \infty$ along any direction in $\Re\lambda > 0$, provided that $|\Im\theta| < \alpha$, which is a condition surely satisfied within Lehmann ellipse. Therefore, in these conditions (above all, in absence of poles of $S(\lambda, k)$), the deformation of $\mathcal{C}$ to the imaginary axis $\Im\lambda$ is allowed, so obtaining the following integral representation of the scattering amplitude

$$\begin{aligned} f(k,\theta) &= -\frac{1}{2k}\int_{-i\infty}^{i\infty} G(\lambda)d\lambda = \\ &= -\frac{1}{2k}\int_{-i\infty}^{i\infty} \lambda\frac{S(\lambda,k) - 1}{\cos\pi\lambda}P_{\lambda-1/2}(-\cos\theta)d\lambda \end{aligned} \tag{147}$$

which is said to be the *Watson-Sommerfeld transform* of $S(\lambda, k)$; it expresses the scattering amplitude $f(k,\theta)$ as a continuous superposition of *conical functions,* that is to say, Legendre functions of the type $P_{it-1/2}(x)$ with $t \in \mathbb{R}$. Following (De Alfaro & Regge 1965, Chapter 13, Section 13.2), the formula (147) looks like an infinite level Breit-Wigner formula. In (147), the variables λ and $\cos\theta$ are conjugate of each other by means of the so-called *Melher-Fock inversion formulas*[17] through which it is possible to consider $S(\lambda, k)$ in terms of $f(k,\theta)$; therefore, the properties of $f(k,\theta)$ with respect to the variable $\cos\theta$, may be deduced by the properties of $S(\lambda, k)$ with respect to the variable λ, and vice versa. Of course, further questions of convergence and related areas of convergence of the integral of (147) occur: in any case, what is important is to notice that there exists a common convergence region in which both representations (134) and (147) are the analytic continuation of each other; in particular, the representation (147) comprises the asymptotic region $\cos\theta \to -\infty$, so that the Watson-Sommerfeld transform is suitable for studying the behavior of the scattering amplitude as $\cos\theta \to -\infty$, that is, as $t \to \infty$, whose properties in function of $t = 2k^2(1-\cos\theta)$ are obtainable from those in $\cos\theta$.

It is interesting to consider the Watson-Sommerfeld transform of $S(\lambda, K)$ in presence of Regge poles, that is to say, in the case in which $S(\lambda, k)$ has a certain number of poles in the quadrant $\Re\lambda > 0, \Im\lambda > 0$. If, for each fixed value of k, $S(\lambda, k)$ has a poles of the type $\lambda = \alpha_n(k) + 1/2$ with residue $\sigma_n = \sigma_n(k)$, then the residue of the function $G(\lambda)$, given by (140), computed in this pole, will be

$$\{\text{Res } G(\lambda)\}_{\lambda=\alpha_n+1/2} = -(\alpha_n + 1/2)\frac{\sigma_n(k)}{\sin \pi\alpha_n} P_{\alpha_n}(-\cos\theta). \tag{148}$$

Therefore, in applying the Watson-Sommerfeld transform along the deformation of the original path $\mathcal{C}$ to the imaginary axis, we must further consider a closed path which surrounds every single pole of such a type when we apply (143) in pursing this, each of which provides the following contribution

$$\frac{\beta_n(k)}{\sin \pi\alpha_n(k)} P_{\alpha_n(k)}(-\cos\theta) \tag{149}$$

where

$$\beta_n(k) = \frac{i\pi}{k}[\alpha_n(k) + \frac{1}{2}]\sigma_n(k), \tag{150}$$

[17]See (Mehler 1881) and (Fock 1943), as regard the original papers.

so that, in presence of Regge poles, the Watson-Sommerfeld transform assumes the following expression

$$f(k,\theta) = -\frac{1}{2k}\int_{-i\infty}^{i\infty} \lambda\frac{S(\lambda,k)-1}{\cos\pi\lambda}P_{\lambda-1/2}(-\cos\theta)d\lambda + \\ + \Sigma_n \frac{\beta_n(k)}{\sin\pi\alpha_n(k)}P_{\alpha_n(k)}(-\cos\theta) \tag{151}$$

the summation being over all the existent Regge poles. This is the fundamental formula to study the asymptotic behavior of the scattering amplitude $f(k,\theta) = F(E,t)$ as $\cos\theta \to -\infty$, that is, as $t \to \infty$, which we now will carry out only in the case of a unique Regge pole $\lambda = \alpha(k) + 1/2$, in such a manner that (151) reduces to

$$f(k,\theta) = -\frac{1}{2k}\int_{-i\infty}^{i\infty} \lambda\frac{S(\lambda,k)-1}{\cos\pi\lambda}P_{\lambda-1/2}(-\cos\theta)d\lambda + \\ + \frac{\beta(k)}{\sin\pi\alpha(k)}P_{\alpha(k)}(-\cos\theta) \tag{152}$$

with $k = \sqrt{E}$ and $-\cos\theta = (t/2k^2) - 1 = (t/2E) - 1$. Now, from the asymptotic behavior of Legendre functions, it is possible to deduce that the integral at the right hand side of (152) goes to zero as $t \to \infty$, which is the main reason for which it is often called the *background integral*; furthermore, it is also possible to prove that Regge pole behaves like $t^{\alpha(k)}$, so that, in the case of a unique Regge pole with $\Re\alpha > -1/2$, this dominates the background integral, so obtaining the following asymptotic expansion of the scattering amplitude given by (133)

$$F(E,t) \underset{t\to\infty}{\sim} c(E, \text{Res } G(\alpha(k)))t^{\alpha(k)} \qquad (\textit{Regge's theorem}) \tag{153}$$

where $c(E, \text{Res } G(\alpha(k)))$ is a constant which depends on the energy and on the residue of the Regge pole. The asymptotic formula (153) still holds for a finite number of Regge poles in which α will be replaced by that pole α_n with highest real part, whereas, in general, it does not hold for an infinite number of poles. Attempts to extend toward relativistic regimes - or, however, to high energy contexts - the relation (153) have been made: in this latter case, the (153), thanks as well to the substitution law, shows that the high-energy behavior of the scattering amplitude is mainly ruled by the related Regge poles involved in the given collision process, besides certain *Regge cuts* of the quadrant $\Re\lambda > 0, \Im\lambda > 0$ whose existence seems to be suggested by various physical models and that contribute as well to the asymptotic behavior of the scattering amplitude. We finally will try to shed light on the possible physical interpretation of the Regge poles.

A single Regge pole provides the following contribution to the scattering amplitude,

$$\frac{\beta}{\sin \pi\alpha} P_{\alpha}(-\cos\theta) \equiv f_R(k,\theta) \tag{154}$$

where the position of the pole $\alpha(k) + 1/2$ depends on the energy $E = k^2$, while β is a residue which, in turn, depends on the energy as well. From (65) and (154), as well as taking into account certain properties of the Legendre functions, it follows that the contribution of this pole to the l-th partial amplitude is

$$a_l^R(k) = \frac{1}{2}\int_{-1}^{1} f_R(k,\theta) P_l(\cos\theta) d\cos\theta = \frac{1}{\pi}\frac{\sin\pi\alpha}{(l-\alpha)(l+\alpha+1)} \tag{155}$$

with $\alpha = \alpha(E), \beta = \beta(E)$, from which it turns out that this contribution of the pole to the l-th partial amplitude is very relevant when $\alpha \approx l$: for instance, if, whilst E is varying, $\alpha(E)$ takes, for $E = E_0$, a value, say α_0, much near to an integer one, say l, then the l-th partial wave shows a resonance in correspondence at $E = E_0$. This last argument may be also supported by the fact that (155) is closely related to the Breit-Wigner formula (20) because it is possible to prove that it reduces to a formula of Breit-Wigner type. In fact, we suppose that $S(\lambda, k)$ has a Regge pole for $\lambda = \alpha(E) + 1/2$ and that, for $E = E_0$, let $\alpha(E_0) = \alpha_0$ be much near to an integer value l, that is to say $\alpha(E_0) = \alpha_0 = l + \eta_0 + i\sigma_0$ where[18] $|\eta_0| \ll 1, \sigma_0 \ll 1$. But, if $S(\lambda, k)$, for $k = k_0$ (that is to say, for $E = E_0$), has a pole for $\lambda = \alpha(E_0) + 1/2$, then the Jost function $f(\lambda, -k)$ is zero in these values, that is $f(\alpha(E_0) + 1/2, -k_0) = 0$. In a neighborhood of $E = E_0$, the position of the pole in the plane λ will be given by $\lambda = \lambda(E) \equiv \alpha(E) + 1/2$, where the function $\alpha(E)$ is defined as a solution to the equation $f(\alpha(E) + 1/2, -k) = 0$. But, since $f(\lambda, -k)$ is locally analytic in the variables λ and k, with $\partial f/\partial\lambda = \partial f/\partial\alpha \neq 0$, it follows that $\alpha(E)$ will be, in turn, locally analytic in the given neighborhood of $E = E_0$, so that we may write $\alpha(E) \approx l + \eta_0 + i\sigma_0 + \alpha_0'(E - E_0)$ where $\alpha_0' = (d\alpha/dE)_{E=E_0} \equiv \eta_0' + i\sigma_0'$ and σ_0', η_0' real numbers. Therefore, in the neighborhood of $E = E_0$, due to $|\eta_0'| \ll 1, \sigma_0 \ll 1$, we have $(l+\alpha+1) \approx 2l+1$ and $l - \alpha \approx -\alpha_0'[E - E_0 + (\eta_0 + i\sigma_0)/\alpha_0'] = -\alpha_0'(E - E_0 + \Delta E + i\Gamma/2)$ where $\Delta E = (\eta_0\eta_0' + \sigma_0\sigma_0')/|\alpha_0'|^2$ and $\Gamma = 2(\sigma_0\eta_0' - \eta_0\sigma_0')/|\alpha_0'|^2$, so that (155) assumes

[18] The Regge pole $\alpha(E_0) = \alpha_0 = (l + \eta_0) + i\sigma_0$, has positive imaginary part σ_0 because every Regge pole lies in the region $\Im\lambda > 0$.

the following expression

$$a_l^R(k) \approx -\frac{1}{\pi(2l+1)} \frac{\beta/\alpha_0'}{E - E_0 + \Delta E + i\Gamma/2}. \tag{156}$$

Under the hypotheses $|\eta_0| \ll 1$ and $\sigma_0 \ll 1$, we also have $\Delta E \ll E_0$ and $\Gamma \ll E_0$, whence, in these conditions, the l-th partial wave basically will coincide with a_l^R, so that the contribution of the l-th wave, given by

$$\sigma_l(k) = 4\pi(2l+1)|a_l(k)|^2, \tag{157}$$

to the total scattering cross section, will be

$$\sigma_l(k) \approx \frac{4|\beta/\alpha_0'|^2/[(2l+1)\pi]}{(E - E_0 + \Delta E)^2 + \Gamma^2/4} \tag{158}$$

which is a formula of Breit-Wigner type, showing that, under the above hypotheses, we have a resonance, in the wave l, for an energy $E = E_0 - \Delta E$, with width Γ. The number ΔE represents the small variation between the real energy E_0, with which the Regge pole goes through nearest possible to the physical value $l + 1/2$ in the plane of λ, and the real part $E_0 - \Delta E$ of the energy with which the resonance appears in the plane of E, for l having a physical value.

Now, so as, in the E plane, the imaginary part - proportional to the width Γ - of the pole associated with a resonance, has a well-determined physical meaning related to the mean lifetime $\tau = 1/\Gamma$, likewise it is possible to give, in the λ plane or plane of the angular momentum, a direct physical meaning also to σ_0, i.e. the imaginary part of the pole associated with the same resonance, as follows. If one imagines, from a classical standpoint, a resonance as a (semi-bound) metastable system in which the related constituents rotate around each other for a time τ, then σ_0 is related with the angular lifetime of such a resonance. Indeed, if $\lambda = \lambda(E)$ is a value of λ for which a Regge pole appears, then we have

$$f(\lambda(E), -k) = 0 \tag{159}$$

and the regular solution $\phi(\lambda(E), k(E), r)$ undergoes the asymptotic behavior given by (103), that, from (159), reduces to

$$\phi \underset{r\to\infty}{\sim} \frac{1}{2ik} f(\lambda(E), k) e^{i\vec{k}\cdot\vec{r}}. \tag{160}$$

If $\Im k > 0$, we consider the following equations

$$\begin{aligned} &\phi'' + \Big(E - \frac{\lambda^2(E) - 1/4}{r^2} - U(r)\Big)\phi = 0 \\ &\dot{\phi}'' + \Big(E - \frac{\lambda^2(E) - 1/4}{r^2} - U(r)\Big)\dot{\phi} = -\phi + \frac{d\lambda^2(E)}{dE}\frac{\phi}{dE} \end{aligned} \tag{161}$$

where $\dot{\phi} = \partial\phi/\partial E$. From them, it follows that

$$\frac{d\lambda^2}{dE} = \frac{\displaystyle\int_0^\infty \phi^2 dr}{\displaystyle\int_0^\infty \frac{\phi^2}{r^2} dr} \tag{162}$$

so, when E and λ are quasi-real numbers[19], then also ϕ^2 will be a quasi-real and non-negative function, hence also $d\lambda^2/dE$ will be a quasi-real and non-negative function as well, to be precise equal to R^2 if R is the classical radius of the orbit of the system, so that, if $\alpha' = d\alpha/dE = d\lambda/dE = (1/2\lambda)(d\lambda^2/dE)$, then we have $\alpha' \approx R^2/2\lambda$. Furthermore, σ_0 is linked to the width Γ of the resonance by the approximate relation $\Gamma/2 \approx \sigma_0/\alpha'_0$, so that we have $\sigma_0 \approx R^2\Gamma/4\lambda$ that, in natural units, reduces to $\sigma_0 \approx R\Gamma/2v$ because the angular momentum approximately is $\lambda \approx (1/2)vR$. Therefore, if Δt is the revolution period, then we have $\Delta t = 2\pi R/v$, and being $\tau = 1/\Gamma$, we have too $\sigma_0 \approx (1/4\pi)(\Delta t/\tau)$, so that $\sigma_0/2$ represents the angular mean lifetime $\Delta\theta$ of the system, that is $1/2\sigma_0 \approx 2\pi(\tau/\Delta t) \equiv \Delta\theta$, by which it follows that the lower the imaginary part of the Regge pole is, the greater $\Delta\theta$ will be. Accordingly, we may conclude that a Regge pole, for $\lambda = \alpha(E)+1/2 = \alpha_l(E)+1/2$, describes, at varying the energy E, a trajectory in the plane λ; each time that a *Regge trajectory* $\alpha_l = \alpha_l(E)$ goes through very near to an integer number l, then a resonance will appear. Thus, the function $\alpha_l(E)$ itself may give rise to many resonances, which may also be assembled into families, each of which associated with a single representing trajectory $\alpha = \alpha_n(E)$, this last idea having given rise to to many other applications and phenomenological speculations in the field of high and low energy physics. Therefore, to sum up following (Eden et al. 1966, Chapter 3) and (Mackey 1978, Section 21), bound states of Schrödinger equation for a spherically symmetric potential fall into families characterized by increasing angular momentum and decreasing binding energy, such a family appearing as a sequence of poles occurring in the successive partial wave amplitudes $a_l(s)$ $l \in \mathbb{N}_0$, at increasing values of s. As already mentioned above, following a previous analytic continuation technique employed by S. Mandelstam in 1958 in the case of $f(k,\theta)$, the theory of complex angular momentum was introduced by Tullio Regge, in 1959, thanks to which it was possible to show that one could extend $S_l(k)$ in a natural way so as to be defined for all positive real l and that then $l \to S_l(k)$ could be regarded as a boundary value of an analytic function of two complex variables, the famous Regge poles being the poles of $l \to S_l(k)$, with respect to l and for fixed real values of k, located in the half-plane $\Re l > -1/2$ where

[19]See, for instance (Kontolatou 1993) and (Goodman & Hawkins 2014, Definition 2.2).

$S_l(k)$ is meromorphic with respect to l. To be precise, the partial scattering amplitude $a_l(k)$ undergoes an *analytic interpolation* from integer values[20] to complex values that, for a certain class of Yukawian potentials, is uniquely determined by the asymptotic behavior of $\Im l \to \infty$, thanks to Carlson's theorem. At the same time, with this analytic continuation of $S_l(k)$, considering $s = k^2$ with its negative and positive determinations, we may also analytically extend $f(s,t)$ by means of a sum of partial scattering amplitudes $a_l(k)$ via the Watson-Sommerfeld transformation as given by (151) and the relation (64) which links together scattering phases (i.e., $S_l(k)$) with partial scattering amplitudes (i.e., $a_l(k)$). Varying $k \in \mathbb{R}$, we obtain a family of curves in the l plane, say $S_l(k_n)$, hence a family of poles, say $\alpha_l(k_n)$, which therefore change with k, so obtaining a discrete family of functions (Regge trajectories) $k \to \alpha_l(k)$ at varying of k, where $\alpha_l(k)$ is a pole of $l \to S_l(k)$ for each k fixed, which identifies its relative position on the given Regge trajectory. For $s \in \mathbb{R}^-$, $\alpha_l(k)$ is a real increasing function with respect to k; then, whenever $\alpha_l(k)$ passes through a positive integral value l_0, then the corresponding value of k is a pole of $S_{l_0}(k)$ corresponding to a bound state with negative energy; for $k \in \mathbb{R}^+$, $\alpha_l(s)$ is instead a complex function, with positive imaginary part, so that when k passes from negative values to positive ones, the Regge trajectory leaves the real axis and enters into the upper half-plane $\Im l > 0$ of the complex l plane. Therefore, the bound states of the dynamical system lie on Regge pole trajectories, several times on each trajectory, so that the bound states may be grouped into families, all members of the same family lying on a single Regge trajectory. Again, following (Eden et al. 1966, Chapter 3), bound states of Schrödinger equation for a spherically symmetric potential fall into families characterized by increasing angular momentum and decreasing binding energy, such a family appearing as a sequence of poles occurring in the successive partial wave amplitudes $a_l(s)$ $l \in \mathbb{N}_0$, at increasing values of s. The Regge theory pictures this sequence as due to the presence of a single pole whose position varies continuously with l and which is relevant to the physics of bound states only when l takes non-negative integral values. This idea has given meaning by the construction of an interpolating amplitude $a_l(s)$, defined for non-integral, and indeed complex, values of l, which coincides with the physical amplitudes $a_l(k)$ when $l \in \mathbb{N}_0$. This function $a_l(s)$ is an analytic function of its arguments except for certain singularises. Among these singularities will be a pole (Regge pole) corresponding to each of the bound state families, the location, or trajectory, of such a pole being given

[20] Because such a function is initially defined, in the variable l, only for an infinite discrete set of values, that is $\mathbb{N}_0$, so that it is more correct to speak of an analytic interpolation rather than an analytic continuation.

by the equation $l = \alpha(s)$, the bound state energies corresponding to values of s which make l take the values $0, 1, 2, \ldots$. One can deal similarly with unstable bound states and resonances: indeed, given any pole k_0 of $\alpha_{l_0}(k)$ in the complex k plane, we may obtain a nearby pole in the complex k plane for each value of l near l_0. Thus, we obtain a two-dimensional submanifold of the direct product of the complex l plane with the complex k plane, having the property that each pair (l, k) in this submanifold is a pair such that k is a pole of $\alpha_l(k)$. These submanifolds will be finite or countable in number, and we may group together resonances when they lie on the same one, speaking of *Regge recurrences* of a given bound state or resonance. To be precise, if there exist positive values of k in which $\alpha_l(k)$ has an integer real part and a very small imaginary part, say $\alpha(k) = L + i\alpha'(k)$ with $0 < \alpha'(k) \ll 1$, then a resonance with angular momentum L and energy k, does appear. The greater the value of k, the more the Regge trajectories depart out from the real axis, towards the plane $\Re l > 0$, which, for attractive potentials, form clusters or families of bound states and resonances.

Duality

Therefore, to summarize, the connection between singularities of the scattering amplitudes, considered as a complex function analytically prolonged with respect to the energy or the angular momentum, and the presence of stable (bound states) or unstable (resonances) particles, played a fundamental role in those theoretical approaches of elementary particle physics in which a central role is given by the scattering amplitudes and their analytical properties; moreover, the phenomenology of the 1950s high-energy physics plainly recalled the attention on the importance of singularities of S matrix theory with their physical meaning, pointing out on the notable contribution of bound states and resonances till to suppose that non-resonant contributions could be described by means of suitable averages of resonances. This last idea is the conceptual key of the so-called *duality,* which, in turn, was at the basis of the *string theory,* whose early origins may be retraced in the late 1960s works of Gabriele Veneziano, and whose main study's object is the so-called *relativistic string,* a concept generalizing that of particle associated to a given quantum field, understanding this as represented by the vibrational modes of certain geometrical structures, having a well-defined dimension, so providing a new theoretical framework for the strong interactions whose phenomenology of the time shown a prevalence of either diffractive phenomena and resonances which, therefore, entailed the consideration of a mathematical structure generating infinite states, like a vibrating string. Following (Marchesini et al. 1976), at the beginnings, attempts to apply quantum field theory for treating strong interactions were tempted, going beyond the initial and limited Yukawian work on the mesonic field. The EM analysis of the structure of hadrons has moreover identified a complex structure formed by more elementary components, called *partons,* to which it will be allowed to apply possibly the ordinary quantum theory of fields. Therefore, from this standpoint, hadrons should be seen as bound states of partons, while the collisions between hadrons should be considered as collisions between bound states. Nevertheless, the resulting theoretical framework presented difficult formal problems like the one regarding the confinement of partons within an hadron, so that a necessary way to follow for overcoming this type of problems consisted in finding non-conventional field theories or even completely new formal sights, like the one which revolves just around S matrix theory, of which one of the most promising formal model was that based on the above notion of duality as meant by Veneziano. If we consider a general exchange process[21] of the type $a + b \rightarrow a' + b'$, and assume to be valid the hypothesis of duality, then the related scattering amplitude may be given by the sum of two main contributions: a low-energy contribution given by the Breit-Wigner formula (20), say a_l^{BW}, related to a decay of resonances of the type $a + b \rightarrow (R) \rightarrow a' + b'$, and an high-energy contribution given by the Regge behavior according to (153), say $a_l^R \approx (k^2)^\alpha$, so that we should have $a_l \approx a_l^{BW} + a_l^R$. The phenomenological analysis of such a question has entailed that, at least in the average approximation, the related imaginary parts should be equal amongst them, that is $\Im a_l \cong \Im a_l^R \cong \Im a_l^{BW}$, so that, roughly, the Regge behavior and the formation of resonances basically are dual descriptions of the same phenomenon. Following (Martin & Collins 1984, Chapter 7), one of the main characteristic properties of hadron interactions is that the scattering amplitude may be given as a sum of resonance contributions of the type $a_l \approx \sum_{res} a_l^{BW}$ with a_l^{BW} given by the Breit-Wigner formula. In the limit of small resonance width Γ, we may also write this last sum as a sum of resonance contributions of mass m_i along the direct channel as follows

$$a_l(s,t) = \sum_i \frac{1}{s - m_i^2} c_i(t) \tag{163}$$

where $s = E^2 = k$ and t is defined as above and proportional to the square of the four-momentum. But, in the limit $\Gamma \rightarrow \infty$, from what has been said above, we have that the scattering amplitude may be also written as a sum of Regge contributions as follows $a_l \approx \sum_{res} a_l^R$, so that, along the crossed

[21] It is a process in which, for instance, a condition of the following type holds, i.e., the quantum numbers of a and a' are different between them; this is the case which happens, for example, when the exchange of quarks and gluons between the hadrons is involved (see (Collins & Martin 1984, Chapter 7)).

channel, we have too

$$a_l(s,t) = \sum_i \frac{1}{t - m_i^2} c_i(s), \tag{164}$$

so that duality implies that the scattering amplitude may be equivalently written both as a sum of resonances related to the direct channel (equation (163)) and as a sum of resonances related to the crossed channel (equation (164)), from which it follows another of the most characteristic properties of hadron interactions, that is to say the crossing symmetry in the variables s and t (*duality*), so that each of these two formulas expresses a single equivalent contribution to the scattering amplitude, this last property having been elegantly expressed by the mathematical properties of following formula for the scattering amplitude

$$a_l(s,t) = \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}, \tag{165}$$

proposed by G. Veneziano in 1968, where $\alpha(x) = \alpha_0 + \alpha' x$ is a linear Regge's trajectory, Γ is the Euler gamma function with poles in $\mathbb{N}_0$ such that $\Gamma(z) \approx z^z$. The symmetries of (165) take into account the duality principle of above; furthermore, it has poles of resonance type for $s = m_i^2(i - \alpha_0)/\alpha'$, with highest spin $\alpha_i(m_i^2)$, that, for great values of s, gives rise to the Regge asymptotic behavior $a_l(s,t) \approx s^{\alpha(t)}$. The interest of (163) and (164) is that, the duality concept underlying them, may be extended to the case of scattering amplitudes related to an arbitrary number of particles when one is able to identify the right variables generalizing s and t, so that such a duality hypothesis (with $\Gamma \to 0$) determines the form of every hadronic scattering amplitude, so casting the bases of a theoretical framework of hadronic interactions. Nevertheless, not always the basilar condition $\Gamma \to 0$ has a reality meaning because, in general, Γ grows together with the resonance mass.

2. On some historical aspects of 1959 Tullio Regge paper

In the monograph (De Alfaro & Regge 1965, Chapter 1, Section 1.2), the authors give a compendious historical outline of potential theory. In particular, they point out what usefulness has their approach in reconstructing the scattering amplitude via Born approximation method, which uses techniques based on the introduction of a unique analytic interpolation of partial scattering amplitudes for complex values of the angular momentum, following 1959 Regge work. Then, in the next Chapter 2, the authors summarize the main mathematical tools used throughout the text, amongst which a Poincaré theorem already quoted in the previous section 1, and that they say play a key role in potential scattering. This theorem concerns general linear second order homogeneous differential equations of the form

$$\frac{d^2 f(x)}{dx^2} + p(x)\frac{df(x)}{dx} + q(x)f(x) = 0 \tag{166}$$

which, following (Whittaker & Watson 1927, Chapter X, Section 10.1), can be written in the form

$$\frac{d^2\varphi(x)}{dx^2} + J(x)\varphi(x) = 0 \tag{167}$$

where

$$f(x) = \varphi(x)\exp\Big(-\frac{1}{2}\int_b^x p(y)dy\Big) \tag{168}$$

and

$$J(x) = q(x) - \frac{1}{2}\frac{dp(x)}{dx} - \frac{1}{4}p^2(x). \tag{169}$$

For example, Schrödinger equation for partial waves may be written in the form (167). The above Poincaré theorem (see (Poincaré 1884a,b)) has to do with equations of the form (167) depending on a parameter η through a function J of the type $J(x,\eta)$, supposed to be an analytic entire function of η. Take now a solution $\psi(x)$ defined by a boundary condition independent on η in an ordinary point $P(x = c)$. The theorem states that $\psi(x,\eta)$ for fixed x, as a function of η, is also an entire function. The condition that P should be an ordinary point, can be relaxed provided the boundary conditions are still η-independent. Hence, De Alfaro and Regge say that many of the theorems which will be established later in the text, are really generalizations of Poincaré's theorem, this fact witnessing the importance of this result in the formal framework of potential scattering. For instance, we have used

Poincaré theorem to prove analyticity properties of a regular solution to (71). In (Regge 1959), it was considered, for the first time, the possibility of introducing the angular momentum as a complex variable, showing the convenience of this procedure in proving the Mandelstam representation for potential scattering. Regge starts with the consideration of the Schrödinger equation written in the form

$$\psi'' - \Big(\frac{\lambda^2 - (1/4)}{x^2} - 1 + U(x)\Big)\psi = 0 \tag{170}$$

having chosen dimensionless variables, by putting $x = kr$, where r is the distance from the origin, and k the wave number (fixed). Here λ is a generalized complex orbital momentum, and when it assumes positive half-integer values (hereafter referred to as the physical values), then we shall write $\lambda = j + 1/2$. The equation (170) is even in λ, and further restrictions are assumed for $U(x)$ (see (72)). Then Regge considers a solution to (170), say $F(\lambda, \eta, x)$, whose parameter η may variate in such a manner to comprehend other solutions to (170) (see (Regge 1959, (1.3) and (1.4)); moreover, a slight generalized form of the above mentioned theorem of Poincaré, states that $F(\lambda, \eta, x)$, for fixed η and x, is an entire function of λ. At the point 6. of his paper, Regge establishes some results in the field of dispersion relations. As well-known these relations are statements of analyticity of the scattering amplitude as a function of the energy and of the transmitted momentum. Although the energy is kept fixed, it is still possible to derive, for special classes of potentials, enough properties as to guarantee for the existence of such relations. As we have seen in the previous section, analyticity in $\cos\theta$ is known to subsist within the Lehmann ellipse (138), whose corresponding representation is unable for studying the behavior of the scattering amplitude letting $t \to \infty$ or equivalently as $\cos\theta \to \infty$. But Regge cleverly and ably overcomes this formal problem transforming the series of the partial wave expansion of the scattering amplitude given by (134) (i.e., the expansion (6.3) of (Regge 1959)), into an integral over the complex plane of the variable $\lambda = l + 1/2$ in such a manner to have an analytic interpolation of the scattering amplitude over an open domain of the $\cos\theta$-plane, so obtaining the integral (147) (i.e., the integral (6.4) of (Regge 1959) or the integral (9.19) of (De Alfaro & Regge 1965, Chapter 9, Section 9.3)), with an artifice that Regge says as due to G.N. Watson and used by A. Sommerfeld in some wave propagation problems, but without obviously giving other historical informations in this regard. In (De Alfaro & Regge 1965, Chapter 13, Section 13.2), the authors say that, strangely enough, formulas of the type (147) for general singular potential were first historical treated in connection with the propagation of waves around the Earth where the potential $U(r)$ was a truly hard-core potential, i.e., $U(r) = \infty$ for $r < R$, where R stood for the radius of the Earth supposed to have a spherical form and perfectly conducting. The resulting pole expansion was seen to converge very rapidly for large k but there is no indication that this result may be extended to cases of significance in nuclear physics, said then De Alfaro and Regge in 1960s.

Well, because of the importance of this last crucial formal passage with respect to the whole Regge's work achieved in (Regge 1959), as well as for the notable role played by it in the later developments of high-energy physics, we briefly devote a lot of time to deepen just these latter historical aspects of this 1959 Regge's seminal paper. But, as early as in (De Alfaro & Regge 1965, Chapter 9, Section 9.3), some further historical clarifications have been included. Indeed, the authors recall that the form of the scattering amplitude (147) is essentially derived by the work of H.J. Poincaré in diffraction of electromagnetic waves and exposed in (Poincarè 1910) which, nevertheless, is not quoted in the original Regge paper of 1959. In (Poincaré 1910), the author considers applications of his theorem quoted above to certain differential equations of the type (170) (like, for example, the equation (2) of Section 12 or the equation (1) of Section 11), as well as Legendre polynomial expansions similar to (134) (like, for example, the series (2) of Section 6) and integral expression similar to (147) (like, for example, the expressions (3) or (13) of Section 13 or the expression (3) of Section 15; see also Sections 17 and 19). As his usual style, Poincaré does not quote any reference, except a mention to Max Abraham (1875-1922), at page 170, and to Arnold Sommerfeld (1868-1951), at page 171. Afterwards, De Alfaro and Regge quote too the works of John William Nicholson (1881-1955), namely (Nicholson 1910a,b; 1911), which are centered around mathematical problems inherent diffraction phenomena of electromagnetic waves sent by a Hertzian oscillator round the surface of the Earth considered as a sphere of perfect conductivity, taking into account the previous work made by Poincaré on this subject where the electromagnetic forces involved are expressed as an integral of Fredholm's type which will be, later, expanded into a series of zonal harmonics. In Nicholson's works, the main formal aspect of his treatment of the related physical problem is just centered on the technique of replacing a series with an integral extended to certain infinite regions (see, above all, (Nicholson 1910b) where integral expression of the magnetic forces are deduced from their series expansions) whose values are then determined through to the calculus of residues. Then, De Alfaro and Regge refer that such a mathematical techniques was extensively used by George Neville Watson (1886-1965) in (Watson 1918) (see also (Watson 1944) and references therein), where Poincaré and Nicholson works are quoted, together that of H.M. Macdonald (see (Macdonald 1914)), H.W. March, M. von Rybczynski and A.E.H. Love (see (Love 1915) and references therein). In any case, ever since the time, about 1902, when G. Marconi first succeeded in sending wireless signals across the Atlantic, the question of explaining the mechanism of such transmission has attracted the attention amongst mathematicians and physical-mathematicians, many of whom are quoted in (Love 1915), amongst whom H.M. Macdonald, J. Zenneck, H.W. March, M. von Rybczynski, L.W. Austin, J.L. Hogan and W.H. Eccles, and others (see (Love 1915) and references therein), ever since early 1900s. But, just following what is said in (Love 1915, No. 10), for the infinite series which represents the effect of curvature without resistance, methods of summation have been devised by Macdonald (see (Macdonald 1911)), Poincaré and Nicholson (see their above quoted works in addition to (Poincaré 1903)); and still another method has been devised by Macdonald (see (Macdonald 1911)). All these methods depend upon a transformation of the series into a definite integral, and an approximate evaluation of the integral. Poincaré, in (Poincaré 1910), did not press his method so far as to tabulate numerical results, but concluded that the expression for the electric force normal to the surface, at an angular distance θ from the originating doublet, should contain a factor of the form $e^{-\Lambda\theta}$. Nicholson, in (Nicholson 1910a,b; 1911), went further in the same direction, obtained a formula for the magnetic force containing such an exponential as a factor, and deduced definite numerical results. Macdonald, in (Macdonald 1911), also obtained definite numerical results which cannot be reconciled with those of Nicholson. The discrepancy was discussed by Nicholson himself, who traced it to an alleged flaw in the analysis used by Macdonald in (Macdonald 1911), and it was discussed also by Macdonald in (Macdonald 1914), who pointed out a difficulty in the analysis used by Poincaré in (Poincaré 1910) and Nicholson in (Nicholson 1910a,b; 1911). Fresh numerical results were deduced by Macdonald in (Macdonald 1914) from a new method of summing the series, but they do not agree with those previously found by Nicholson, or with those previously found by Macdonald himself in 1911. Nevertheless, many of the authors and related works quoted in (Love 1915), which have to do with the above crucial passage from a series to an integral, are not instead mentioned in (De Alfaro & Regge 1965).

The 1918 paper of Watson, however, put into reciprocal comparison mainly the works of Nicholson and Poincaré on the one hand, with the works of Macdonald on the other hand, reaching to a major clarification either in physical-mathematics consequences and formal aspects of the original physical problem under examination. Under advice of B. Van der Pol, who asked him to further analyze the problem with the chief purpose to clarify why there subsist the various discrepancies found by different authors in dealing with this physical problem, Watson first highlights, from a further comparison between H.W. March and M. Von Rybczynski works with that of A.E.H. Love and with the one of Poincaré and Nicholson, what follows (see (Watson 1918, No. 2))

«The essential advance in this paper is closely connected with the fundamental error of March and Rybczynski which was pointed out by Love. In dealing with an oscillator on the positive half of the axis of harmonics, those writers express a Hertzian function by an integral of $P_s(\cos\theta)$*, the integration being carried out with regard to the degree* s *of the Legendre function; such an integral has a line of singularities along the line* $\theta = \pi$*, and is regular along the line* $\theta = 0$*. The fact is that, when harmonics of non-integral degree are introduced, the appropriate function to use is not* $P_s(\cos\theta)$ *but* $P_s(-\cos\theta)$*; this fundamental point is somewhat obscured by the equation*

$$P_n(-\cos\theta) = (-1)^n P_n(\cos\theta),$$

which holds between the functions whose degrees are integers. The failure of convergence of an integral involving $P_s(\cos\theta)$ *along the line* $\theta = 0$ *(when an oscillator is placed on the positive half of the axis of harmonics) is strictly analogous to the failure of convergence of the series* $1 + z + z^2 + \ldots$*, all round the circle* $|z| = 1$ *on account of the singularity of the function* $1/(1-z)$ *at the point* $z = 1$*. A simple electrostatic example is afforded by the potential of a unit charge at distance* a *from the origin. The potential near the origin is*

$$(\star_1) \qquad V = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta) =$$

$$= \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{a}\right)^n P_n(\cos\theta) =$$

$$(\star_2) \qquad = \frac{1}{2ia}\int\left(\frac{r}{a}\right)^s P_s(\cos\theta)\frac{ds}{\sin s\pi}$$

where the contour starts from $+\infty$ *and returns to* $+\infty$ *after encircling the points* $s = 0, 1, 2, \ldots$ *which are poles of the integrand. On swinging round the contour so as to surround the other poles of the integrand, and evaluating the residues, we find the series for* V *in descending powers of* r*, valid when* $r > a$*».*

Watson, in discussing the swinging of the contour of integration of $(\star_2)$, makes reference to a paper of Ernest William Barnes (1874-1953) (see (Barnes

1908)) in which the so-called *Mellin-Barnes integral* is defined, as well as he quotes a Laplace's formula for $P_s(\cos\theta)$ to prove the convergence of the integral for $|s|$ large (see (Watson 1944, Chapter 6, Section 6.5)). Therefore, it is more probable that Watson, in achieving the integral ($\star_2$), considered what Barnes made in this regard in defining his integral which is, roughly, a contour integral of products involving gamma function and exponential factors (see also (Barnes 1910)). On the other hand, recent history of mathematics research tells us that it was Salvatore Pincherle (1853-1936), in his works of generalized hypergeometric functions, to have early applied the elements of the technique of contour integration methods to special functions (to be precise, to hypergeometric functions) with the introduction of first forms of integrals of Mellin-Barnes type. Indeed, following (Mainardi & Pagnini 2003, Section 1),

«In Vol. 1, p. 49 of Higher Transcendental Functions *of the Bateman Project, we read "Of all integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle, in 1888* [see (Pincherle 1888; 1965)]*; their theory has been developed in 1910 by H. Mellin* [see (Mellin 1910)] *(where there are references to earlier work) and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes* [see (Barnes 1908)]. *In the classical treatise on Bessel functions by Watson* [see (Watson 1944, Chapter 6, Section 6.5)] *we read: "By using integrals of a type introduced by Pincherle and Mellin, Barnes has obtained representations of Bessel functions which render possible an easy proof of Kummer's formula* [...]. *Here we point out that the 1888 paper (in Italian) of S. Pincherle on the Generalized Hypergeometric Functions led him to introduce the afterwards named Mellin-Barnes integral to represent the solution of a generalized hypergeometric differential equation investigated by Goursat in 1883. Pincherle's priority was explicitly recognized by Mellin and Barnes themselves, as reported below. In 1907 Barnes* [see (Barnes 1907)], *wrote: "The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle, whose suggestive paper was the starting point of the investigations of Mellin (1895) though the type of contour and its use can be traced back to Riemann". In 1910 Mellin* [see (Mellin 1910)], *devoted a section (§ 10: Proof of Theorems of Pincherle) to revisit the original work of Pincherle; in particular, he wrote "Before we are going to prove this theorem, which is a special case of a more general theorem of Mr. Pincherle, we want to describe more closely the lines L over which the integration preferably is to be carried out"».*

Finally, Sommerfeld, in (Sommerfeld 1949, Appendix II to Chapter V, and Appendix to Chapter VI), retakes the above mentioned work of Watson in computing the solution u to the wave equation $\Delta u + k^2 u = 0$, of which a first form is obtained as a series expansion of the type[22]

$$(*) \qquad u = \frac{k}{4\pi i}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\frac{\zeta_n(kr)}{\xi_n(ka)}$$

whose convergence, however, is very poor, and limited to the domain given by $a < r < \infty$ and $0 \leq \theta \leq \pi$. This series expansion has been studied either by P. Debye in 1908-09 as well as by P. Frank and R. von Mises in the second volume of the well-known *Die Differential- und Integralgleichungen der Mechanik und Physik,* edited by P. Frank with the assistance of H. Faxen, R. Fürth, Th. von Kârmân, R. von Mises, Fr. Noether, C. W. Oseen, A. Sommerfeld, and E. Trefftz, Braunschweig, Vieweg, 1927-35, whose second volume is a revised and enlarged edition of the second part of the celebrated 1910 Riemann-Weber treatise *Die Differential- und Integralgleichungen der Mechanik und Physik.* Hence, Sommerfeld transforms $(*)$ into a complex integral, and, to this end and on the basis of the relation $P_n(\cos\theta) = (-l)^n P_n(-\cos\theta)$, which is valid for integral (and only for integral) values of n, he first rewrites the series in $(*)$ in the following form

$$(*_1) \qquad u = \sum_{n=0}^{\infty}(2n+1)(-1)^n P_n(-\cos\theta)\frac{\zeta_n(kr)}{\xi_n(ka)},$$

hence, he replaces n by a complex variable ν and traces a loop A, around the real axis of the ν-plane in such a manner it surrounds all the points $\nu = 0, 1, 2, 3, ..., n, ...$ in a clockwise direction. Over this loop, Sommerfeld takes the integral

$$(*_2) \qquad u = \int \frac{2\nu+1}{2i\sin\nu\pi}P_\nu(-\cos\theta)\frac{\zeta_n(kr)}{\xi_n(ka)}d\nu$$

which is obtained from the general term in $(*_1)$ by interchanging n and ν, suppressing the factor $(-1)^n$, and appending the denominator $\sin\nu n$. As on (Sommerfeld 1949, Chapter V, Appendix II), P_ν does not stand for the Legendre polynomial, but for the hypergeometric function

$$P_\nu(x) = F\Big(-\nu, \nu+1, 1, \frac{1-x}{2}\Big)$$

[22] Here, $\zeta_n(x) \doteq \sqrt{\pi x/2}H^{(2)}_{n+1/2}(x)$ (where $H^{(2)}_\alpha(x)$ is the Hankel function of the second kind), is the so-called *Riccati-Bessel function,* while $\xi_n(x) \doteq \zeta_n(x) + x\zeta'_n(x)$.

which is identical with the Legendre polynomial only for integral ν. In doing so, Sommerfeld resembles 1888 Pincherle's method. Then, Sommerfeld suitably deforms, in many ways, the path A taking into account poles and zeros of the integrand of $(*_2)$ and applying Cauchy's residue theorem, hence discussing the related consequences in relation to $(*_2)$ until up to obtain the following expression

$$(*_3) \qquad u = \pi \sum_{\nu=\nu_0,\nu_1,\dots} \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\theta) \frac{\zeta_n(kr)}{\eta_n(ka)}$$

where, in a neighborhood of the m-th pole of the integrand of $(*_2)$, $\eta_\nu = (\partial\xi_\nu/\partial\nu)_{\nu=\nu_m}$ and $\xi_\nu(ka) = (\nu-\nu_m)\eta_\nu(ka)$ if one considers the zeros of $\xi_\nu(ka) = 0$ for $\nu = \nu_0, \nu_1, \dots$ as the poles of the integrand of $(*_2)$. Now, except for a sign and for a constant factor, the integral $(*_2)$ is identical with the solution $(*_1)$ of the sphere problem. Hence the series $(*_3)$ also represents the solution of the sphere problem, and suppressing the immaterial constant factor, Sommerfeld finally writes

$$(*_4) \qquad u = \sum_{\nu} \frac{2n+1}{\sin\nu\pi} P_n(-\cos\theta) \frac{\zeta_\nu(kr)}{\eta_\nu(ka)},$$

so that the passage from the series $(*_1)$, which is summed over integral n, to the series $(*_4)$, which is summed over the complex ν, is obtained by forming residues in a complex integral twice. This, finally, explains why De Alfaro and Regge, in (De Alfaro & Regge 1965, Chapter 9, Section 9.3) (see also (De Alfaro et al. 1973, Chapter 1, Section 7)) speak of *Watson-Sommerfeld transform* (in short, *WS-transform* or *SW-transform*), even if, in this paper, we have tried to deepen what have been the real historical roots of this transform, identifying them in the previous works of E.W. Barnes and S. Pincherle.

In conclusion, in this section we have only pointed out some little known historical perspectives which underlie the introduction of WS-transform, identifying the early roots upon which it relies. As we have seen in the previous Section 1, after the pioneering use of the WS-transform involving complex angular momenta for studying resonance phenomena of elementary particles mainly motivated to prove the validity of the Mandelstam representation for the potential scattering of two spinless particles for a certain class of generalized Yukawa potentials, it was possible to work out the notions of Regge pole and Regge trajectory, from which Gabriele Veneziano started with his as many pioneering work of 1968 (see (Veneziano 1968)) in which resonance dual models are introduced, for the first time, in fundamental physics, simply beginning with the proposing of a quite simple expression for the relativistic scattering amplitude (today known as *Veneziano amplitude*) that obeys the requirements of Regge asymptotics and crossing symmetry in the case of linearly-rising trajectories, containing automatically Regge poles in families of parallel trajectories with residue in definite ratios, which furthermore satisfies the conditions of superconvergence and exhibits in a nice fashion the duality between Regge poles and resonances in the scattering amplitude, essentially based on symmetry properties of Euler beta function (see (Green et al. 1988, Chapter 1, Sections 1.1 and 1.2)). This is the incipit of a coherent S-matrix theory of strong interactions which then will led to resonance dual models in high-energy particle physics, so, as is well known, marking the rising of string theory. Indeed, following (Di Vecchia 2008), once that Regge discovered that the Schrödinger equation allowed continuation in angular momentum for complex values, and linked resonances with different spin, one of the basic ideas that led to the construction of an S matrix was that it should include resonances at low energy and at the same time give Regge behavior at high energy. But the two contributions of the resonances and of the Regge poles should not be added because this would imply double counting. This was called *Dolen-Horn-Schmidt duality,* discovered around 1967, while another idea (of about 1968) that helped in the construction of an S matrix was the so-called *planar duality* that was visualized by associating to a certain process a duality diagram where each meson was described by two lines representing the quark and the antiquark. Finally, also the requirement of crossing symmetry played a very important role, and starting from these ideas Veneziano was finally able to construct an S matrix for the scattering of four mesons that, at the same time, had an infinite number of zero width resonances lying on linearly-rising Regge trajectories and Regge behaviour at high energy. In any case, after the Regge 1959 seminal paper, the WS-transform was extensively used in field theory as, for instance, done in (Schwarz 1973, Section 1), where, on the basis of an axiomatic scheme (amongst which the assumption that all mesons and baryons, together related poles, lie on a Regge trajectory), it is considered the following dispersion relation

$$A(s,t)=\sum_{n=0}^{\infty}\frac{R_n(t)}{n-\alpha(s)},$$

to which the scattering amplitude must satisfy, together the SW-transform of it, namely

$$A(s,t)=\frac{1}{2\pi i}\int_{\mathcal{C}}\tilde{R}(t)\frac{\Gamma(-n)}{n-\alpha(s)}dn$$

whose SW-contour $\mathcal{C}$ will be suitably deformed to avoid the pole at $n = \alpha(s)$. Through a suitable choice of the function $\tilde{R}(t)$, Schwarz derives Veneziano amplitude just by means of such a method based on SW-transform. Likewise for the paper (Mandelstam 1974), where, although not explicitly mentioned, the SW-transform plays a fundamental role since the beginnings. Furthermore, following (Cushing 1990, Chapter 5, Section 5.4), the asymptotic behavior of $f(k,\theta)$ as $\cos\theta \to \infty$, was settled for potential scattering by Regge in 1959. Although a beautiful piece of classical mathematical analysis in its own right, this paper was to have its greatest impact on the S-matrix program for the conjectures to which it would lead. Only (Collins 1977, Chapter 2, Sections 2.7, 2.9 and 2.10; Chapter 4, Section 4.6; Chapter 7, Section 7.1; Chapter 9, Section 9.3) has devoted the right attention to the SW-transform, there called *Mandelstam-Sommerfeld-Watson transform,* d'après the improvement of original SW-transform by Mandelstam himself in 1962, recalling as well the links between it and the Mellin transform; in this regard, see also (Omnès & Froissart 1963, Chapter 4, Sections 4-2 and 4-3), where interesting historical notes on Sommerfeld original work have also been inserted. In (Frautschi 1963, Chapter X), as regard Sommerfeld-Watson transformation, it is said that this method had a history stretching over several decades and had been used to study rainbows, propagation of radio waves around the Earth, and scattering from various potentials, the author specifying that the related transformation was used by Poincaré and Nicholson in 1910 in connection with the bending of electromagnetic waves by a sphere, the transformation having been introduced in its present form by Watson in 1918 and later resurrected by Sommerfeld. Regge's original contribution lays in understanding the special features of complex angular momenta for scattering from superpositions of Yukawa potentials - the type of potential believed relevant to relativistic scattering - and calling these features to the attention of high-energy physicists. In the (Frautschi 1963, Chapters X and XIII), besides further and interesting its applications in high-energy physics, are also quoted who has later improved and extended WS-transformation.

In conclusion, with this paper, we have pointed out a single historical aspect underling one the most important works of 20th century physics, that is to say, the 1959 seminal paper of Tullio Regge. To be precise, as is seen above, this work has mainly a technically fashion in which new formal methods and techniques have been introduced, besides to have introduced new pioneering and fruitful ideas and notions having physical nature. Among these technique, the so-called Watson-Sommerfeld transform, on which we have focussed our historical sight with the main purpose to identify the early origins and sources upon which relied such a complex transform. In pursuing this aim, we have found early sources of this formal method in some works dating back the early 1900s and mainly due to H.J. Poincaré, J.W. Nicholson, H.M. Macdonald, and E.W. Barnes, until to reach to descry some prolegomena of the WS-method in some previous works of S. Pincherle of the late 1880s. Because of the notable importance played by this WS-transform in the formal framework of fundamental physics, we think that our historical work, albeit centered on a single aspect, is not altogether superfluous if nothing else for the simple fact that such a historical clarifications have not been treated in a whole and deep manner if not fully neglected.

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