

# Physical properties as modal operators in the topos approach to quantum mechanics

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## Abstract

In the framework of the topos approach to quantum mechanics we give a representation of physical properties in terms of modal operators on Heyting algebras. It allows us to introduce a classical type study of the mentioned properties.

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## Introduction

Quantum mechanics (QM) is unanimously recognized to be one of the most successful physical theories ever, but parallel to this it is also widely acknowledged that many aspects of what quantum theory means remain unexplained and obscure. And, although quite some aspects that originally were considered problems —e.g., the Schrödinger cat situation related to the linearity of the Schrödinger equation which gives rise to the superposition principle—

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are nowadays resources of technical applications, nevertheless there still is a lack of a semantics and a conceptual language for QM that would explain what the theory is talking about.

In the last years several approaches using category theory have been used to search for an adequate and rigorous language for quantum systems. First, both from a neo-realist point of view [18, 24, 27] or not [34], there are attempts that relate algebraic QM to topos theory, either recognizing the associated intrinsic intuitionistic logic or equipping the structure with an external intuitionistic logic. In these approaches, the quantum analogue of classical phase space is captured by the notion of frame; i.e., a complete Heyting algebra. There are also other schemes related to category theory which attempt to describe several aspects of QM. For example, contextuality and non-locality may be modeled using the framework of sheaf theory [6] while monoidal categories can be used for representing processes [5]. This approach also enables a consistent description of compound systems [10], a deep difficulty for standard quantum logic (QL).

In this work we expose some logical characteristics related to the intuitionistic approach to quantum phase spaces when the topos approach to QM is considered. Moreover, we provide a representation of physical properties as modal operators in a Heyting structure. This representation allows us to analyze the classical and quantum aspects of properties in terms of logical consequences. The paper is organized as follows. In Section 1, we recall some notions about Heyting algebras and frames. In Section 2, we review some basic facts about algebraic approaches to QM. In Section 3, we introduce the Heyting algebra associated to the phase space when the topos approach to quantum systems is considered. Section 4 is dedicated to the study of a representation of properties as modal operators in a Heyting algebra. In this framework, we define a type of classical interpretation for these properties. This interpretation will describe semantic aspects related to the intuitionistic logic associated to the phase space of the system. Finally, in Section 5 we provide the conclusions.

## 1 Heyting algebras and frames

We recall from [8, 9] some notions of universal algebra and Heyting algebras that will play an important role in what follows. Let  $A$  be a non-empty set and  $n$  be a non-negative integer. An  $n$ -ary operation on  $A$  is a function of the form  $f : A^n \rightarrow A$ . In this way,  $n$  is said to be the *arity* of  $f$ . A *type* of

algebras is a (possible infinite) sequence of natural numbers  $\tau = \{n_1, n_2, \dots\}$ . An *algebra of type  $\tau$*  is a pair  $\langle A, F \rangle$  where,  $A$  is a non-empty set and  $F = \{f_{n_1}, f_{n_2}, \dots\}$  is a set of operation on  $A$  such that *arity*( $f_i$ ) =  $n_i$ . The operations in  $F$  are called  $\tau$ -*operations*. Let  $A$  and  $B$  be two algebras of the same type  $\tau$ . A function  $\alpha : A \rightarrow B$  is  $\tau$ -*homomorphism* iff it preserves the  $\tau$ -operations. Let  $A$  be an algebra and  $X \subseteq A$ . We denote by  $G_A(X)$  the subalgebra of  $A$  generated by the set  $X$ , i.e., the smallest subalgebra of  $A$  containing  $X$ .

**Proposition 1.1** *Let  $A, B$  algebras of type  $\tau$ . Let  $X$  be a subset of  $A$  and  $f : A \rightarrow B$  be a  $\tau$ -homomorphism. Then,*

1.  $f(G_A(X)) = G_B(f(X))$  [9, Theorem: 6.6 ].
2. *If  $A = G_A(X)$  and  $g : A \rightarrow B$  is a  $\tau$ -homomorphism such that the restrictions  $g|_X, f|_X$  coincides. Then,  $f = g$  [9, Theorem: 6.2 ].*

□

Let  $\mathcal{A}$  be a category of algebras of type  $\tau$  whose arrows are the  $\tau$ -homomorphisms between algebras of  $\mathcal{A}$ .  $\mathcal{A}$  is a *variety* iff its objects form a class defined by equations. It is well known that, if  $\mathcal{A}$  is a variety, then monomorphisms in  $\mathcal{A}$  are exactly injective  $\tau$ -homomorphisms.

An algebra  $A \in \mathcal{A}$  is *injective in  $\mathcal{A}$*  iff, for every monomorphism  $f : B \rightarrow C$  and every homomorphism  $g : B \rightarrow A$ , there exists a homomorphism  $h : C \rightarrow A$  such that  $g = hf$ .

We shall focus our interest in two varieties, the variety of bounded distributive lattices and the variety of Heyting algebras. The following result will be used in Section 5.

**Theorem 1.2** [8, V.9.-Theorem: 3] *An algebra is injective in the variety of bounded distributive lattices iff it is a complete Boolean algebra.* □

Heyting algebras provide an algebraic semantics for the intuitionistic propositional calculus presented by Heyting in his 1930's papers [25, 26]. A *Heyting algebra* [8] is an algebra  $\langle A, \vee, \wedge, \rightarrow, 0 \rangle$  of type  $\langle 2, 2, 2, 0 \rangle$  satisfying the following equations:

H1  $\langle A, \vee, \wedge, 0 \rangle$  is a lattice with universal lower bound 0,

H2  $x \wedge y = x \wedge (x \rightarrow y)$ ,

$$\text{H3 } x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z)),$$

$$\text{H4 } z \wedge ((x \wedge y) \rightarrow x) = z.$$

We denote by  $\mathbb{H}$  the variety of Heyting algebras. In agreement with the usual Heyting algebraic operations, we define the negation  $\neg x = x \rightarrow 0$  and  $1 = \neg 0$ .

In each Heyting algebra  $A$ , the reduct  $\langle A, \vee, \wedge, 0, 1 \rangle$  is a bounded distributive lattice. The lattice order, expressed in terms of the operation  $\rightarrow$ , is equivalent to  $a \leq b$  iff  $1 = a \rightarrow b$ . Moreover, for  $a, b \in A$ ,  $a \rightarrow b = \bigvee \{x \in A : x \wedge a \leq b\}$ . Boolean algebras are Heyting algebras satisfying the equation  $x \vee \neg x = 1$ . In this case, the operation  $\rightarrow$  satisfies that  $x \rightarrow y = \neg x \vee y$ .

Let  $A$  be a Heyting algebra and  $x \in A$ .  $x$  is said to be *regular* iff  $\neg\neg x = x$ . We denote by  $Reg(A)$  the set of regular elements of  $A$ .  $Reg(A)$  is a Boolean algebra under the operations  $x \vee_R y = \neg\neg(x \vee y)$ ,  $x \wedge_R y = x \wedge y$  and  $x \rightarrow_R y = \neg\neg(\neg x \vee y)$  [8, IX.5.-Theorem: 3]. In general  $Reg(A)$  is not a subalgebra of  $A$ .  $x$  is said to be *central* iff  $x \vee \neg x = 1$ . The set  $Z(A)$  of central elements of  $A$  constitutes a Boolean subalgebra  $A$ . Note that  $Z(A) \subseteq Reg(A)$ . In particular  $Z(A) = Reg(A)$  iff the equation  $\neg(x \wedge y) = \neg x \vee \neg y$  is satisfied in  $A$ . The following result is well known:

**Proposition 1.3** *Let  $A$  be a Heyting algebra, then  $A$  is a Boolean algebra iff  $A = Reg(A)$ .  $\square$*

We have a special interest in the class of complete Heyting algebras, i.e., Heyting algebras that are complete when considered as lattices. Complete Heyting algebras are the objects of different categories. They differ in their arrows, and thus get distinct names. One of them is the category *frames*; i.e., complete Heyting algebras whose arrows, called *frame homomorphisms*, are functions preserving all joins, all finite meets, 0 and 1. The Heyting operation  $\rightarrow$  is not generally preserved by frame homomorphisms. We also note that frames are very important structures since they allow to study topological spaces in terms of their open-sets lattices [28].

## 2 Algebraic approaches to quantum mechanics

In classical physics every system can be described by specifying its actual properties. Mathematically, this happens by representing the state of the system by a point  $(p, q)$  in the corresponding phase space  $\Gamma$  and its properties by subsets of  $\Gamma$ , with a structure of operations compatible with the

usual mathematics of set theory. Consequently, the propositional structure associated with the properties of a classical system follows the rules of classical logic. In the orthodox formulation of QM, a pure state of a system is represented by a ray in the Hilbert space  $\mathcal{H}$  and its physical properties by closed subspaces of  $\mathcal{H}$ , which with adequate definitions of meet and join operations give rise to an orthomodular lattice [31]. This lattice, denoted by  $\mathcal{L}(\mathcal{H})$ , is called the Hilbert lattice associated to  $\mathcal{H}$  and motivates the standard QL introduced in the thirties by Birkhoff and von Neumann [7].

The traditional version of QL needs careful consideration for several reasons. From an algebraic point of view, QL is founded on the orthomodular lattice structure. But it is well known that the variety of orthomodular lattices is strictly larger than the variety generated by the Hilbert lattices. Thus, standard QL does not fully capture the concept of the Hilbert lattice. From a physical point of view there are several features which must be carefully considered: if  $P$  represents a proposition about the system, in general there are superposition states in which it is wrong to say that either  $P$  or its negation  $\neg P$  hold in accordance with the association of the join operation with the smallest closed subspace including the projection represented by  $P$  and its orthocomplement instead of with their set theoretical union. However, the orthomodular structure satisfies the equation  $P \vee \neg P = 1$  which is a kind of law of the excluded middle. Thus, as discussed in [12], it seems necessary to distinguish the logical law of excluded middle from the semantic principle in which the truth of the disjunction implies the truth of at least one of the members.<sup>1</sup> Moreover, in spite of the fact that the meet of its elements is well defined in the lattice, there are conjunctions of (actual) properties that make no sense because the corresponding operators do not commute. Thus, the orthomodular structure shows a kind of conflict with the underlying physical content of the theory. There is also a well known difficulty with traditional forms of QL in relation to composite systems, namely the lack of a canonical formalism for dealing with the properties of the whole system when given the description of its components. In fact, if  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are the representatives of two systems, the postulates of QM say that the tensor product  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  stands for the representative of the composite. But the naive construction of the lattice of propositions of the whole as the tensor product of the lattices of the individuals fails due to the lack of a product of lattices [3, 4, 33], or even posets [19]. Mathematically, this is the expression of the fact that the category of Hilbert lattices

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<sup>1</sup>For a discussion about contradiction and superposition states see [11].

as objects and lattice morphisms as arrows does not possess a categorical product due to the failure of orthocomplementation [1, 2, 22]. Attempts to vary the conditions that define the product of lattices have been made [32], but in all cases it results that the Hilbert lattice factorizes only in the case in which one of the factors is a Boolean lattice or when systems have never interacted, rendering the construction either trivial or physically useless. For a complete review, see [19].<sup>2</sup> In view of the mentioned characteristics of orthomodular systems of propositions, there have been attempts to obtain “more tractable” structures (see for example [5, 10, 18, 23, 24]).<sup>3</sup>

The algebraic formulation of QM usually starts with the  $C^*$ -algebra of observables. This is a complex algebra  $A$  that is complete in a norm  $\|\cdot\|$  satisfying  $\|xy\| \leq \|x\|\|y\|$  and has an unary involutive operation  $*$  such that  $\|x^*x\| = \|x\|^2$ . In this way, a quantum system is mathematically modeled by a  $C^*$ -algebra. If  $\mathcal{H}$  is a Hilbert space, the algebra  $B(\mathcal{H})$  of all bounded operators of  $\mathcal{H}$ , equipped with the usual norm and adjoint is an example of  $C^*$ -algebra. By the Gelfand-Naimark theorem [21], any  $C^*$ -algebra is isomorphic to a norm-closed self-adjoint subalgebra of  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .

A von Neumann algebra  $N$  is a special case of  $C^*$ -algebra  $N \subseteq B(\mathcal{H})$  equal to its own bicommutant. More precisely, if  $N'$  is the set of all bounded operators on  $\mathcal{H}$  that commute with every element of  $N$  then  $N'' = N$ . Whereas  $C^*$ -algebra are usually considered in their norm-topology, a von Neumann algebra carries in addition a second interesting topology, called the weak-topology, in which it is complete as well. In this topology, one has convergence  $x_n \rightarrow x$  iff, for each density operator  $\rho$ ,  $tr\rho(x_n - x) \rightarrow 0$  in  $\mathcal{H}$  where  $tr$  is the trace. A general  $C^*$ -algebra may not have any nontrivial projections while a von Neumann algebra is generated by its projections, i.e., elements satisfying  $p^2 = p^* = p$ . In a von Neumann algebra, the projections are in natural correspondence with the closed subspaces of a Hilbert space. In this way, projections of a von Neumann algebra form a complete orthomodular lattice. A state in a von Neumann algebra  $N$  is a

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<sup>2</sup>For the construction of a lattice using convex sets instead of rays as states, see [16].

<sup>3</sup>In this line, we have built in a previous paper a QL that arises from considering a sheaf over a topological space associated to the Boolean sublattices of the ortholattice of closed subspaces of  $\mathcal{H}$  [14]. To do so, we defined a valuation that respects contextuality (first translating the Kochen-Specker (KS) theorem to topological terms [14, Theorem 4.3]) and a frame for the Kripke model of the language. As frames are complete Heyting algebras, the resulting logic is an intuitionistic one—with restrictions on the allowed valuations arising from the KS theorem—, thus it has “good” properties as the distributive lattice structure and a nice definition of the implication as a residue of the conjunction.

linear functional  $s : N \rightarrow \mathbb{C}$  that is continuous in the weak topology and such that  $s(x^*x) \geq 0$  and  $s(1) = 1$ .

### 3 Intuitionistic approach to phase spaces

In the topos approach to QM [18, 27, 24] it is possible to encode physical properties in a Heyting algebra. This provides an intuitionistic description for the phase space of the system. More precisely, in a quantum system represented by a von Neumann algebra  $N$ , the abelian subalgebras of  $N$  represent *contexts* in which, restricted to the context, the rules of classical logic hold (see for discussion [13]). Let  $N$  be a von Neumann algebra and  $\mathcal{V}(N)$  be a family of commutative subalgebras of  $N$  which share the unit element with  $N$ . Consider the partial ordered set  $\langle \mathcal{V}(N), \subseteq \rangle$  viewed as the small category whose arrows are defined by the partial order  $\subseteq$ . In the topos approach the system is modeled in the category of presheaves

$$\widehat{\mathcal{V}(N)} = \mathbf{Set}^{\mathcal{V}(N)^{op}}$$

Thus, the category  $\widehat{\mathcal{V}(N)}$  can be seen as a category of sets fibred over the contexts. Let  $N$  be an abelian von Neumann algebra. A *multiplicative state* is a state  $s$  such that  $s(xy) = s(x)s(y)$ . We denote by  $\Sigma(N)$  the set of multiplicative states in  $N$  and the weak\* topology is considered in  $\Sigma(N)$ . We recall that if a classical system is modeled as an abelian von Neumann algebra  $N$ ,  $\Sigma(N)$  represents the phase space of the system.

To model a quantum system, the *spectral presheaf* defined as the functor:

$$\Sigma : \mathcal{V}(N)^{op} \rightarrow \mathbf{Set}$$

such that,  $\mathcal{V}(N) \ni A \mapsto \Sigma(A)$  and, for each arrow  $f : A \rightarrow B$ , (i.e.,  $f$  is the inclusion  $A \subseteq B$ ),  $\Sigma(f)$  is the function  $\Sigma(f) : \Sigma(B) \rightarrow \Sigma(A)$  such that  $(\Sigma(f))(s) = s|_A$  is naturally chosen as the state space.

Let  $N$  be an abelian von Neumann algebra and  $\mathcal{P}(N)$  be the set of projections. Let  $P \in \mathcal{P}(N)$ . It is well known that the set  $C_P = \{s \in \Sigma(N) : s(P) = 1\}$  is clopen when the weak\* topology is considered in  $\Sigma(N)$ . Moreover, if we consider the set  $Clp(\Sigma(N))$  of all clopen sets in  $\Sigma(N)$ , the function  $C : \mathcal{P}(N) \rightarrow Clp(\Sigma(N))$  such that  $C(P) = C_P$  is a bijection. A *clopen subobject* of the spectral presheaf  $\Sigma$  is a subfunctor  $T$  of  $\Sigma$  such that for each  $A \in \mathcal{V}(N)$ ,  $T(A) \in Clp(\Sigma(N))$ .

When considering  $Sub_{cl}(\Sigma)$ , the set of clopen subobjects of  $\Sigma$ , we can see that,  $Sub_{cl}(\Sigma)$  is a bounded distributive lattice where the operations  $\vee, \wedge$

over clopen subobjects are defined pointwise in each subalgebra of  $\mathcal{V}(N)$ ,  $0$  is the empty subobject and  $1 = \Sigma$ . In [17, §2.3] and [18, Theorem 2.5] the following result is proved:

**Theorem 3.1**  *$Sub_{cl}(\Sigma)$  is a complete Heyting algebra.* □

In a classical system, represented by a commutative von Neumann algebra, the subsets of the phase space with usual set operations define the logical (Boolean) structure of the system. For a quantum system, represented by a von Neumann algebra whose phase space is modeled by the spectral presheaf  $\Sigma$ ,  $Sub_{cl}(\Sigma)$  represents the logical structure of the system which is intuitionistic. We will refer to  $Sub_{cl}(\Sigma)$  as the *the algebra of propositions associated to the spectral presheaf  $\Sigma$* .

## 4 Physical properties as modal operators

In this section we study a class of classical interpretations for quantum properties when the topos approach to quantum systems is considered. For this purpose, we use the theory of modal operators on Heyting algebras. In the orthodox approach, a classical proposition is usually represented by a Boolean (also called central) element of an orthomodular lattice [14, 15]. In particular, propositions about classical systems are represented by a Boolean algebra. Suppose that  $\mathcal{L}$  is a lattice representing the propositional structure associated to a quantum system. A classical interpretation of  $\mathcal{L}$  implies assuming that each  $x \in \mathcal{L}$  has a classical complement  $\neg_c x$  and satisfies distributivity conditions in this interpretation. Then, if  $x$  is not a classical proposition in  $\mathcal{L}$ , a classical interpretation of  $\mathcal{L}$  must, at least, endow a complement for  $x$ . Thus, a natural way to algebraically represent classical interpretations are embeddings of  $\mathcal{L}$  into Boolean algebras, preserving lattice order structure.

When the properties of a quantum system are encoded in  $Sub_{cl}(\Sigma)$ , we propose the following general formalization of the concept of classical interpretation for quantum properties:

**Definition 4.1** Let  $Sub_{cl}(\Sigma)$  be the algebra of propositions associated to the spectral presheaf  $\Sigma$ . A *classical interpretation* of the properties about the system is a lattice embedding  $\mathcal{C} : Sub_{cl}(\Sigma) \hookrightarrow B$  where  $B$  is a Boolean algebra.

Thus  $\mathcal{C} : Sub_{cl}(\Sigma) \hookrightarrow B$  preserves  $\vee, \wedge, 0, 1$ . To study this type of classical interpretation, we introduce the notion of logical consequence and the notion of modal operator on a Heyting algebra. We note that the theory of modal operators on Heyting algebras has its main application in the theory of topoi and sheafification [20, 29].

Let  $A$  be a Heyting algebra and  $a, b \in A$ . We say that  $b$  is a *logical consequence* of  $a$  iff  $a \leq b$  or equivalently  $1 = a \rightarrow b$ . We denote by  $[a]$  the set of logical consequences of  $a$ . We remark that  $[a]$  is the principal filter associated to  $a$  in  $A$ .

**Definition 4.2** Let  $A$  be a Heyting algebra. A modal operator on  $A$  [30] is a unary operation  $j$  such that

$$x \leq j(x), \quad jj(x) = j(x), \quad j(x \wedge y) = j(x) \wedge j(y).$$

Let  $A$  be a Heyting algebra and  $a \in A$ . The operation  $\diamond_a(x) = a \vee x$  defines a modal operator. Modal operators of this form are known as *closed*. The operation  $\diamond_{a \rightarrow}(x) = a \rightarrow x$  defines another modal operator on  $A$  and modal operators of this second form are known as *open*.

**Proposition 4.3** Let  $A$  be a Heyting algebra and  $a \in A$ . Then:

1.  $Imag(\diamond_a) = [a]$
2.  $\diamond_{\neg a}(x) \leq \diamond_{a \rightarrow}(x)$
3.  $Imag(\diamond_{a \rightarrow}) \subseteq Imag(\diamond_{\neg a}) = [\neg a]$
4.  $a$  is a Boolean element in  $A$  iff  $\diamond_{\neg a} = \diamond_{a \rightarrow}$

*Proof:* Let  $x \in A$ .

1)  $x \in Imag(\diamond_a)$  iff  $x = a \vee t$  for some  $t \in A$  iff  $x = a \vee x$  iff  $x \in [a]$ .  
2) Note that  $a \wedge (\neg a \vee x) = (a \wedge \neg a) \vee (a \wedge x) = a \wedge x \leq x$ . Then  $\diamond_{\neg a}(x) = \neg a \vee x \leq a \rightarrow x = \diamond_{a \rightarrow}(x)$ .  
3) Since  $a \wedge \neg a \leq x$ ,  $\neg a \leq a \rightarrow x$ . Thus  $a \rightarrow x \in [\neg a]$  and  $Imag(\diamond_{a \rightarrow}) \subseteq Imag(\diamond_{\neg a}) = [\neg a]$ .  
4) Suppose that  $a$  is a Boolean element, i.e.,  $\neg a \vee a = 1$ . On the one hand,  $x \leq a \rightarrow x$  and  $\neg a \leq a \rightarrow x$ . Then  $\neg a \vee x \leq a \rightarrow x$ . On the other hand, suppose that  $t \wedge a \leq x$ . Then  $\neg a \vee x \geq \neg a \vee (t \wedge a) = (\neg a \vee t) \wedge (\neg a \vee a) = (\neg a \vee t) \wedge 1 = \neg a \vee t$ . In particular  $\neg a \vee (a \rightarrow x) = \neg a \vee \bigvee_{t \wedge a \leq x} t \leq \neg a \vee x$ . Since  $\neg a \leq a \rightarrow x$ , we have that  $a \rightarrow x \leq \neg a \vee x$ . Hence  $\diamond_{a \rightarrow}(x) = a \rightarrow x = \neg a \vee x = \diamond_{\neg a}(x)$  and  $\diamond_{\neg a} = \diamond_{a \rightarrow}$ . Now we suppose that  $\diamond_{\neg a} = \diamond_{a \rightarrow}$ . Then  $\neg a \vee a = \diamond_{\neg a}(a) =$

$\diamond_{a \rightarrow}(a) = a \rightarrow a = 1$ . Hence  $a$  is a Boolean element in  $A$ . □

The set  $M(A)$  of all modal operators on  $A$  is partially ordered by the relation  $j_1 \leq j_2$  iff  $j_1(x) \leq j_2(x)$  for all  $x \in A$ . If  $A$  is a complete Heyting algebra, this partial order defines a complete Heyting algebra structure on  $M(A)$  [30, Theorem 2.3] where  $\bigwedge_i j_i$  is given by the operation  $(\bigwedge_i j_i)(x) = \bigwedge_i j_i(x)$ . The implication  $j_1 \rightarrow j_2$  is given by the operation  $(j_1 \rightarrow j_2)(x) = \bigwedge \{j_1(y) \rightarrow j_2(y) : y \geq x\}$ . Joins in  $M(A)$  are defined as  $j_1 \vee j_2 = \bigwedge \{j \in M(A) : j_1, j_2 \leq j\}$ .

**Theorem 4.4** [28, § 2.6, § 2.7] *Let  $A$  be a complete Heyting algebra and  $a \in A$  then:*

1.  $\diamond_a$  is a Boolean element in  $M(A)$  and  $\diamond_{a \rightarrow}$  is its complement in  $M(A)$ .
2. The map  $a \mapsto \diamond_a$  defines an injective frame homomorphism  $A \rightarrow \text{Reg}(M(A))$ .
3.  $a \mapsto \diamond_a$  is an isomorphism iff  $A$  is a Boolean algebra.

□

In general,  $a \mapsto \diamond_a$  does not preserve the operation  $\rightarrow$  except in the case in which  $A$  is a Boolean algebra.

**Definition 4.5** Let  $A$  be a complete Heyting algebra. We define the algebra  $A^\diamond$  as the Boolean subalgebra of  $\text{Reg}(M(A))$  generated by  $\{\diamond_a, \diamond_{a \rightarrow} : a \in A\}$ .

When considering the properties of the system encoded in  $\text{Sub}_{cl}(\Sigma)$ , the lattice embedding  $\mathcal{C}_0 : \text{Sub}_{cl}(\Sigma) \rightarrow \text{Sub}_{cl}(\Sigma)^\diamond$  such that  $\mathcal{C}_0(a) = \diamond_a$  can be seen as a classical interpretation of the quantum properties. We are interested in giving a meaning to this classical interpretation. To do so, we use the concept of logical consequence presented before Definition 4.2.

Suppose that  $a$  is a quantum property encoded in  $\text{Sub}_{cl}(\Sigma)$ . Then, by Proposition 4.3-1, the classical interpretation of  $a$ , given by the modal operator  $\diamond_a$ , makes reference to the logical consequences of  $a$  in  $\text{Sub}_{cl}(\Sigma)$ . The Boolean complement of  $a$  in  $\text{Sub}_{cl}(\Sigma)^\diamond$  given by  $\diamond_{a \rightarrow}$ , by Theorem 4.4-1 and Proposition 4.3-3, makes reference only to the consequences of  $\neg a$  in  $\text{Sub}_{cl}(\Sigma)$  that have the form  $a \rightarrow x$ . Note that, had  $a$  been a property that

commuted with all other properties,  $a$  would have been a Boolean element in  $Sub_{cl}(\Sigma)$  and, by Proposition 4.3-4, the logical consequences of  $\neg a$  would have been of the form  $a \rightarrow x$ ; i.e., the following identification would have hold:  $\neg a \approx \diamond_{\neg a} = \diamond_{a \rightarrow}$ . This means that thinking of  $a$  as a classical property forces us to only consider as the consequences of  $\neg a$  those of the form  $a \rightarrow x$ .

A first conclusion is that in the encoding of physical properties in  $Sub_{cl}(\Sigma)$ , by Proposition 4.3-4, a classical property is distinguished from a non classical one via the form of the logical consequences of its negation in  $Sub_{cl}(\Sigma)$ . The following example may help to make our assertion more clear:

**Example 4.6** Suppose that  $a, b \in Sub_{cl}(\Sigma)$  and  $\diamond_b \geq \diamond_{a \rightarrow}$ . This means that the property  $b$  is a logical consequence of the complement of  $a$  in the classical interpretation  $Sub_{cl}(\Sigma)^\diamond$ . Taking into account the definition of  $\diamond_b$  and  $\diamond_{a \rightarrow}$ , the classical meaning of  $\diamond_b \geq \diamond_{a \rightarrow}$  is that the logical consequences of  $\neg a$  of the form  $a \rightarrow x$  have as logical consequence, the logical consequences of  $b$  of the form  $b \vee x$ . We remark the difference from the case in which  $a$  would have been a classical property. In this case —in view of Proposition 4.3-4,  $\diamond_{a \rightarrow} = \diamond_{\neg a}$  and the meaning of  $\diamond_b \geq \diamond_{a \rightarrow}$ —, if  $p$  is a logical consequence of  $\neg a$  (i.e.,  $p$  is necessarily of the form  $p = a \rightarrow x$  for some  $x$ ) then  $b \vee x$  is a logical consequence of  $p$ . Clearly there exists a subtle difference between both interpretations that could lead to contradictions when interpreting  $a$  classically without taking into account the distinction  $\diamond_{\neg a}$  and  $\diamond_{a \rightarrow}$ .

Until now we have studied the natural meaning of the classical interpretation  $\mathcal{C}_0 : Sub_{cl}(\Sigma) \rightarrow Sub_{cl}(\Sigma)^\diamond$ . But in fact  $\mathcal{C}_0$  plays an important role since it is present in each possible classical interpretation  $\mathcal{C} : Sub_{cl}(\Sigma) \rightarrow B$  in the sense of Definition 4.1. The following theorem formally describes this fact:

**Theorem 4.7** *Let  $B$  be a Boolean algebra and  $f : Sub_{cl}(\Sigma) \rightarrow B$  be a classical interpretation. Then there exists a unique injective Boolean homomorphism  $\hat{f} : Sub_{cl}(\Sigma)^\diamond \rightarrow B$  such that the following diagram is commutative:*

$$\begin{array}{ccc}
 & & f \\
 & & \nearrow \\
 Sub_{cl}(\Sigma) & \longrightarrow & B \\
 \mathcal{C} \downarrow & \equiv & \nearrow \\
 & & \hat{f} \\
 Sub_{cl}(\Sigma)^\diamond & & 
 \end{array}$$

*Proof:* Let  $f : Sub_{cl}(\Sigma) \rightarrow B$  be an injective lattice homomorphism. Since  $B$  is a Boolean algebra,  $B$  can be embedded into a complete Boolean algebra  $B^*$ . Thus we can see  $B$  as a Boolean subalgebra of  $B^*$ . By Theorem 1.2,  $B^*$  is injective in the variety of bounded distributive lattices. Then there exists a bounded lattice homomorphism  $\hat{f} : Sub_{cl}(\Sigma)^\diamond \rightarrow B^*$  such that the following diagram is commutative:

$$\begin{array}{ccc} Sub_{cl}(\Sigma) & \xrightarrow{f} & B \xrightarrow{1_B} B^* \\ \mathcal{C} \downarrow \equiv & & \nearrow \hat{f} \\ Sub_{cl}(\Sigma)^\diamond & & \end{array}$$

We first prove that  $\hat{f}$  preserves complements. Let  $x \in Sub_{cl}(\Sigma)^\diamond$ . Since  $1 = \hat{f}(x \vee \neg x) = \hat{f}(x) \vee \hat{f}(\neg x)$  and  $0 = \hat{f}(x \wedge \neg x) = \hat{f}(x) \wedge \hat{f}(\neg x)$  then  $\hat{f}(\neg x)$  is the complement of  $\hat{f}(x)$  in  $B^*$ . Hence  $\hat{f}(\neg x) = \neg \hat{f}(x)$ . Thus  $\hat{f}$  is a Boolean homomorphism. Now we prove that  $Imag(\hat{f}) \subseteq B$ . If  $a \in Sub_{cl}(\Sigma)$  by the commutativity of the diagram  $f(a) = \hat{f}(\diamond_a) \in B$ .  $\hat{f}(\diamond_{a \rightarrow})$  is the complement of  $\hat{f}(\diamond_a)$  in  $B^*$  and  $B$  is a Boolean subalgebra of  $B^*$  containing  $f(\diamond_a)$ . Then the complement of  $f(\diamond_a)$  in  $B$  coincides with  $\hat{f}(\diamond_{a \rightarrow})$  because the complement is unique in a bounded distributive lattice. Thus  $\hat{f}(\diamond_{a \rightarrow}) \in B$ . Note that  $\{\diamond_a, \diamond_{a \rightarrow} : a \in Sub_{cl}(\Sigma)\}$  generates  $Sub_{cl}(\Sigma)^\diamond$ . Then, by Proposition 1.1-1,

$$\begin{aligned} \hat{f}(Sub_{cl}(\Sigma)^\diamond) &= \hat{f}(G_{Sub_{cl}(\Sigma)^\diamond}(\{\diamond_a, \diamond_{a \rightarrow} : a \in Sub_{cl}(\Sigma)\})) \\ &= G_{B^*}(\{\hat{f}(\diamond_a), \hat{f}(\diamond_{a \rightarrow}) : a \in Sub_{cl}(\Sigma)\}) \end{aligned}$$

Since  $\{\hat{f}(\diamond_a), \hat{f}(\diamond_{a \rightarrow}) : a \in Sub_{cl}(\Sigma)\} \subseteq B$  then the subalgebra of  $B^*$  given by  $G_{B^*}(\{\hat{f}(\diamond_a), \hat{f}(\diamond_{a \rightarrow}) : a \in Sub_{cl}(\Sigma)\})$  is a Boolean subalgebra of  $B$  and  $Imag(\hat{f}) \subseteq B$ . It proves that

$$\begin{array}{ccc} Sub_{cl}(\Sigma) & \xrightarrow{f} & B \\ \mathcal{C} \downarrow \equiv & & \nearrow \hat{f} \\ Sub_{cl}(\Sigma)^\diamond & & \end{array}$$

By Proposition 1.1-2,  $\hat{f}$  is the unique Boolean homomorphism that makes commutative the diagram.  $\square$

The classical interpretation  $\mathcal{C}_0 : Sub_{cl}(\Sigma) \rightarrow Sub_{cl}(\Sigma)^\diamond$  may be associated to a piece of the classical language that describes some facts regarding the logical consequences of the propositions about the system. Theorem 4.7 expresses the fact that any other classical interpretation in the sense of Definition 4.1 only represents a classical enrichment of the language associated to  $Sub_{cl}(\Sigma)^\diamond$ . In other words, a classical interpretation would represent an increasing of the expressive power of the piece of the classical language associated to  $Sub_{cl}(\Sigma)^\diamond$  that describes aspects of the logical consequences of the propositions. Thus we may say that the classical interpretations in the sense of Definition 4.1 only describe semantic aspects of the logic of phase spaces.

## 5 Conclusions

The categorical approach to QM in the sense of [18, 27, 24] allows to establish classical interpretations of quantum properties. We have rigorously described these interpretations in terms of modal operators on Heyting algebras. When  $a$  is a quantum property, its classical interpretation given by the modal operator  $\diamond a$ , makes reference to the logical consequences of  $a$  in  $Sub_{cl}(\Sigma)$ . Its complement in  $Sub_{cl}(\Sigma)^\diamond$  is given by  $\diamond_{a \rightarrow}$  and makes reference only to the consequences of  $\neg a$  that have the form  $a \rightarrow x$ . Had  $a$  been a classical property, these would have been all the consequences. But when  $a$  is a genuine quantum property some of its consequences are lacking.

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