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A Categorial Semantic Representation of Quantum Event Structures

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Abstract The overwhelming majority of the attempts in exploring the problems related to quantum logical structures and their interpretation have been based on an underlying set-theoretic syntactic language. We propose a transition in the involved syntactic language to tackle these problems from the set-theoretic to the category-theoretic mode, together with a study of the consequent semantic transition in the logical interpretation of quantum event structures. In the present work, this is realized by representing categorically the global structure of a quantum algebra of events (or propositions) in terms of sheaves of local Boolean frames forming Boolean localization functors. The category of sheaves is a topos providing the possibility of applying the powerful logical classification methodology of topos theory with reference to the quantum world. In particular, we show that the topos-theoretic representation scheme of quantum event algebras by means of Boolean localization functors incorporates an object of truth values, which constitutes the appropriate tool for the definition of quantum truth-value assignments to propositions describing the behavior of quantum systems. Effectively, this scheme induces a revised realist account of truth in the quantum domain of discourse. We also include an appendix, where we compare our topos-theoretic representation scheme of quantum event algebras with other categorial and topos-theoretic approaches.

Keywords Quantum Event Structures · Boolean Algebras · Topos Subobject Classifier · Kochen-Specker Theorem · Quantum Truth Values · Adjoint Functors · Sheaves · Grothendieck Topos · Realist Account

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1 Introduction

The logic of a physical theory reflects the structure of the propositions describing the behavior of a physical system in the domain of the corresponding theory. The original quantum logical formulation of quantum theory depends in an essential way on the identification of propositions with projection operators on a complex Hilbert space H. In this framework, due to the one-to-one correspondence between the set of all projection operators and the set of all closed subspaces of H, the equivalence of propositions and events is made literal (e.g., Dalla Chiara et al. 2004). In this sense, the Hilbert-space formalism of quantum theory associates events with closed subspaces of a suitable Hilbert space corresponding to a quantum system. Then, the quantum event structure is identified with the lattice of closed linear subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complement of the closed subspaces, thus forming a complete, atomic, orthomodular lattice. A non-Boolean logical structure is effectively induced which has its origin in quantum theory.

On the contrary, the logic underlying the propositional or event structure of classical physics is Boolean, in the sense that the algebra of propositions of a classical system is isomorphic to the lattice of subsets of phase space, a Boolean lattice that can be interpreted semantically by a two-valued truth-function. This means that to every classical mechanical proposition one of the two possible truth values 1 (true) and 0 (false) can be assigned. Thus, the propositions of a classical system are semantically decidable. From a physical point of view, this is immediately linked to the fact that classical physics views objects-systems as bearers of determinate properties. That is, properties of classical systems are considered as being intrinsic to the system and independent of whether or not any measurement is performed on them.

Unlike the case in classical mechanics, however, for a given quantum system, the propositions represented by projection operators or Hilbert space subspaces are not partitioned into two mutually exclusive and collectively exhaustive sets representing either true or false propositions (e.g., von Neumann 1955). This kind of semantic ambiguity constitutes an inevitable consequence of the Hilbert-space structure of conventional quantum mechanics demonstrated rigorously, for the first time, by Kochen-Specker's (1967) theorem. According to this, for any quantum system associated to a Hilbert space of dimension higher than two, there does not exist a two-valued, truth-functional assignment $h : L_H \to \{0, 1\}$ on the set of closed linear subspaces, L_H, interpretable as events or elementary quantum mechanical propositions, preserving the lattice operations and the orthocomplement. In other words, the gist of the theorem, when interpreted semantically, asserts the impossibility of assigning definite truth values to all propositions pertaining to a physical system at any one time, for any of its quantum states, without generating a contradiction.

It should be noted, however, that although the preceding Kochen-Specker result forbids a global, absolute assignment of truth values to quantum mechanical propositions, it does not exclude ones that are contextual (e.g., Karakostas 2007). Here, "contextual" means that the truth value given to a proposition *depends* on which subset of mutually commuting projection operators (meaning "simultaneously measurable") one may consider it to be a member, i.e., it *depends* on which other compatible propositions are given truth values at the same time. Of course, the formalism of quantum theory does not imply how such a contextual valuation might be obtained, or what properties it should possess.

To this end, we resort to the powerful methods of categorical topos theory, which directly captures the idea of structures varying over contexts, thus providing a natural setting for studying contextuality phenomena. Specifically, the research path we propose implements the intuitively clear idea of probing the global structure of a quantum algebra of events in terms of structured multitudes of interlocking local Boolean logical frames. It is probably one of the deepest insights of modern quantum theory that whereas the totality of all experimental/empirical facts can only be represented in a globally non-Boolean structure, the acquisition of every single fact depends on a locally Boolean context. Indeed, we view each preparatory Boolean environment of measurement as a context that offers a "classical perspective" on a quantum system. A classical perspective or context is nothing but a set of commuting physical quantities, or, more precisely, a complete Boolean algebra of commuting projection operators generated by such a set. Physical quantities in any such algebra can be given consistent values, as in classical physics. Thus, each context functions as a "Boolean frame" providing a "local classical viewpoint on reality". No single context or perspective can deliver a complete picture of the quantum system, but, by applying category-theoretic reasoning, it is possible to use the collection of all of them in an overall structure that will capture the entire system. It is also of great importance how the various contexts relate to each other. Categorically speaking, this consideration is naturally incorporated into our scheme, since the category-theoretic representation of quantum event structures in terms of Boolean localization contexts can be described by means of a topos, which stands for a category of sheaves of variable local Boolean frames encoding the global logical information of these localization contexts.

In a well defined sense, topos theory provides us with the first natural examples of global multi-valued functional truth structures. By definition, a topos, conceived as a category of sheaves for a categorical topology, is equipped with an internal object of truth values, called a subobject classifier, which generalizes the classical binary object of truth values used for valuations of propositions. As explained below, this generalized object of truth values in a topos is not ad hoc, but reflects genuine constraints of the surrounding universe of discourse. We will show, in particular, that the topos-theoretic representation scheme of quantum event algebras by means of variable local Boolean frames induces an object of truth values, or classifying object, which constitutes the appropriate tool for the definition of quantum truth-value assignments, corresponding to valuations of propositions describing the behavior of quantum systems. This, in effect, characterizes the novelty of our approach and its fruitfulness for a revised realist account of truth in the quantum domain in comparison to a multiplicity of various other approaches on the foundations of quantum physics.

2 Category-Theoretic Scheme of Truth Value Assignment in Quantum Mechanics

As indicated in the introduction, the global semantic ambiguity of the non-Boolean logical structure of quantum mechanics, expressed formally by the Kochen-Specker theorem, does not exclude local two-valued truth-functional assignments with respect to complete Boolean algebras of projection operators on the Hilbert space of a quantum system. More precisely, each self-adjoint operator representing an observable has associated with it a Boolean subalgebra which is identified with the Boolean algebra of projection operators belonging to its spectral decomposition. Hence, given a set of observables of a quantum system, there always exists a complete Boolean algebra of projection operators, viz. a local Boolean subalgebra of the global non-Boolean event algebra of a quantum system with respect to which a local two-valued truth-functional assignment is meaningful, if and only if the given observables can be simultaneously measurable. Consequently, the possibility of local two-valued truth-functional assignments of the global non-Boolean event algebra of a quantum system points to the assumption that complete Boolean algebras play the role of local Boolean logical frames for contextual true/false value assignments. The modeling scheme we propose in order to implement this idea in a universal way, so that the global structure of a quantum system to be modeled categorically in terms of a topos of sheaves of local Boolean frames, uses the technical apparatus of categorical sheaf theory (Mac Lane and Moerdijk 1992, Awodey 2010).

It is not possible to provide here a concise account of category theory. For a general introduction to this well-developed mathematical framework, topos theory and categorial logic, the reader may consult Lawvere and Schanuel (2009), Bell (1988/2008) and Goldblatt (1984/2006).

2.1 Conceptual Framework

The basic ideas pertaining to the proposed semantic interpretation of quantum event structures along category-theoretic lines may be summarized as follows: Firstly, we introduce the notion of a topological covering scheme of a quantum event algebra (Zafiris 2006a) consisting of epimorphic families of local Boolean logical frames. These frames provide local covers of a quantum event algebra in terms of complete Boolean algebras. The local Boolean covers capture individually complementary features of a quantum algebra of events and provide collectively its categorical local decomposition in the descriptive terms of Boolean logical frames. Technically, this is described by an action of a category of local Boolean frames on a global quantum event algebra, forming a presheaf. Secondly, we define appropriate compatibility conditions between overlapping local Boolean covers. This is necessary since it enforces an efficient, uniquely defined pasting code between different local covers of a quantum algebra of events. Technically, this is described by the notion of a Boolean localization functor, or equivalently, by a structure sheaf of Boolean coefficients of a quantum event algebra. Thirdly, we establish the necessary and sufficient conditions for the isomorphic representation of quantum event algebras in terms of Boolean localization functors.

The major technical and semantical method used in order to establish these conditions is based on the existence of a pair of adjoint functors between presheaves of Boolean logical frames and quantum event algebras. This pair of adjoint functors formalizes categorically the process of encoding and decoding information between Boolean frames and quantum event algebras respecting their distinctive structural form. In general, the existence of an adjunction between two categories gives rise to a family of universal morphisms (called unit and counit of the adjunction), one for each object in the first category and one for each object in the second. In this way, each object in the first category induces a certain property in the second category and the universal morphism carries the object to the universal for that property. Most significantly, every adjunction extends to an adjoint equivalence of certain subcategories of the initially correlated categories. It is precisely this category-theoretic fact which determines the necessary and sufficient conditions for the isomorphic representation of quantum event algebras in terms of sheaves of Boolean coefficients.

The notion of a sheaf incorporates the requirements of consistency under extension from the local Boolean to the global quantum level, and inversely, under reduction of the global quantum to the local Boolean level. The functional dependence implicated by a categorical sheaf relativizes the presupposed rigid relations between quantum events with respect to variable local Boolean frames conditioning the actualization of events. The category of sheaves of variable local Boolean frames encoding the global logical information of Boolean localization functors constitutes a topos providing the possibility of applying the powerful logical classification methodology of topos theory with reference to the quantum universe of discourse.

2.2 Basic Structures in the Functorial Approach to Quantum Mechanics

A *Boolean categorical event structure* is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. The objects of $\mathcal{B}$ are σ-Boolean algebras of events and the arrows are the corresponding Boolean algebraic homomorphisms.

A *quantum categorical event structure* is a locally small co-complete category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras. The objects of $\mathcal{L}$ are quantum event algebras and the arrows are quantum algebraic homomorphisms. A quantum event algebra L in $\mathcal{L}$ is defined as an *orthomodular σ-orthoposet* (Zafiris 2006a), that is, as a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \to L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

We recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice. The σ-completeness condition, meaning that the join of countable families of pairwise orthogonal events exists, is required in order to have a well defined theory of quantum observables over L (Zafiris 2004). In the sequel, the measure-theoretic σ-completeness condition is not going to play any particular role in the exposition of the arguments, so the interested reader may drop it and consider complete Boolean algebras and complete orthomodular lattices instead.

The *functor category of presheaves on Boolean event algebras*, denoted by $\mathbf{Sets}^{\mathcal{B}^{op}}$, has objects all functors $\mathbf{P} : \mathcal{B}^{op} \to \mathbf{Sets}$, and morphisms all natural transformations between such functors, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$. Each object $\mathbf{P}$ in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a contravariant set-valued functor on $\mathcal{B}$, called a *presheaf* on $\mathcal{B}$, defined as follows: For each Boolean algebra B of $\mathcal{B}$, $\mathbf{P}$(B) is a set, and for each Boolean homomorphism $f : C \to B$, $\mathbf{P}(f) : \mathbf{P}(B) \to \mathbf{P}$(C) is a set-theoretic function such that if $p \in \mathbf{P}$(B), the value $\mathbf{P}(f)(p)$ for an arrow $f : C \to B$ in $\mathcal{B}$ is called the restriction of p along f and is denoted by $\mathbf{P}(f)(p) = p \cdot f$. We notice that each Boolean algebra B of $\mathcal{B}$ gives rise to a contravariant Hom-functor $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$. This functor defines a presheaf on $\mathcal{B}$ for each B in $\mathcal{B}$. Concomitantly, the functor $\mathbf{y}$ is a full and faithful functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$, viz. $\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$, defining an embedding $\mathcal{B} \hookrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$, which is called the Yoneda embedding (Mac Lane and Moerdijk 1992, Awodey 2010).

The category of elements of a presheaf $\mathbf{P}$, denoted by $\int(\mathbf{P}, \mathcal{B})$, has objects all pairs (B, p), and morphisms $(\acute{B}, \acute{p}) \to (B, p)$ are those morphisms $u : \acute{B} \to B$ of $\mathcal{B}$ for which $p \cdot u = \acute{p}$, that is the restriction or pullback of p along u is $\acute{p}$. Projection on the second coordinate of $\int(\mathbf{P}, \mathcal{B})$ defines a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \to \mathcal{B}$, called the *split discrete fibration* induced by $\mathbf{P}$, where $\mathcal{B}$ is the base category of the fibration as in the diagram below. We note that the fibers are categories in which the only arrows are identity arrows. If B is an object of $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}$(B), although its elements are written as pairs so as to form a disjoint union.

$$
\begin{tikzcd}
\int(\mathbf{P}, \mathcal{B}) \arrow[d, "\int_{\mathbf{P}}"'] & \\
\mathcal{B} \arrow[r, "\mathbf{P}"] & \mathbf{Sets}
\end{tikzcd}
$$

The *Boolean realization functor of a quantum categorical event structure* $\mathcal{L}$ is defined by:

$$\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}},$$

where the action on a Boolean algebra B in $\mathcal{B}$ is given by:

$$\mathbf{R}(L)(B) := \mathbf{R}_L(B) = Hom_{\mathcal{L}}(\mathbf{M}(B), L).$$

The functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ is called the *functor of Boolean frames* of L, where $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ is a *Boolean modeling functor* of $\mathcal{L}$. The action on a Boolean homomorphism $D \xrightarrow{x} B$ in $\mathcal{B}$, for $v : \mathbf{M}(B) \longrightarrow L$ is given by:

$$\mathbf{R}(L)(x) : Hom_{\mathcal{L}}(\mathbf{M}(B), L) \longrightarrow Hom_{\mathcal{B}}(\mathbf{M}(D), L)$$

$$\mathbf{R}(L)(x)(v) = v \circ x.$$

The crucial conceptual and technical distinguishing feature of the proposed categorical modeling scheme of quantum event structures and their truth-objects in comparison to other categorical approaches (see Appendix A.1) is that it is based on the existence of a *categorical adjunction* between the categories $\mathbf{Sets}^{\mathcal{B}^{op}}$ and $\mathcal{L}$. More precisely, there exists *a pair of adjoint functors* $\mathbf{L} \dashv \mathbf{R}$ as follows (Zafiris 2004):

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{L} : \mathbf{R}.$$

The *Boolean frames-quantum adjunction* consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints, as well as the natural bijection:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L).$$

Hence, the Boolean realization functor of $\mathcal{L}$, realized for each L in $\mathcal{L}$ by its functor of Boolean frames, viz. by

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{M}(B), L),$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit (inductive limit)

$$\mathbf{L}(\mathbf{P}) = Colim\{\int(\mathbf{P}, \mathcal{B}) \xrightarrow{\int_{\mathbf{P}}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\}.$$

Consequently, we obtain immediately that the modeling functor $\mathbf{M}$ evaluated at a Boolean algebra B, viz. $\mathbf{M}(B)$, is characterized as the colimit of the representable presheaf $\mathbf{y}[B]$ on the category of Boolean event algebras $\mathcal{B}$, as follows:

$$\mathbf{Ly}[B](B) \cong \mathbf{M}(B).$$

In order to obtain a clear intuitive idea of the function of the left-adjoint functor by the colimit construction it is instructive to compute it explicitly for the case of interest, where the functor on which it acts is the functor of Boolean frames of L. For simplicity, we carry out the construction using set-theoretic arguments. The general case is presented in detail in Appendix A.2.

For this purpose we define the set of *pointed Boolean frames* of a quantum event algebra L as follows:

$$\mathbf{Y}(\mathbf{R}_L) = \{(\psi_{\mathbf{M}(B)}, q)/(\psi_{\mathbf{M}(B)} : \mathbf{M}(B) \to L), q \in B\}.$$

Note that the morphisms $\psi_{\mathbf{M}(B)} : \mathbf{M}(B) \to L$ denote Boolean frames of L, encoded as elements in the category of Boolean frames of L, viz. $\int(\mathbf{R}_L, \mathcal{B})$.

We notice that if there exists a Boolean morphism $u : \acute{B} \to B$ such that: $u(\acute{q}) = q$ and $\psi_{\mathbf{M}(\acute{B})} = \psi_{\mathbf{M}(B)} \cdot u$, then we may define a transitive and reflexive relation $\Re$ on the set $\mathbf{Y}(\mathbf{R}_L)$. Of course the inverse also holds true. Thus, we have:

$$(\psi_{\mathbf{M}(B)} \circ u, q)\Re(\psi_{\mathbf{M}(B)}, u(\acute{q}))$$

for any Boolean morphism $u : \acute{B} \to B$ in the base category $\mathcal{B}$. The next step is to make this relation also symmetric by postulating that for pointed Boolean frames ζ, η in $\mathbf{Y}(\mathbf{R}_L)$, where ζ, η denote pairs in the above set, we have

$$\zeta \sim \eta,$$

if and only if $\zeta\Re\eta$ or $\eta\Re\zeta$. Finally, by considering a sequence $\xi_1, \xi_2, \ldots, \xi_k$ of elements of the set $\mathbf{Y}(\mathbf{R}_L)$ and also ζ, η such that:

$$\zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta,$$

we may define an equivalence relation on the set $\mathbf{Y}(\mathbf{R}_L)$ if there exists a path of Boolean transition morphisms as follows:

$$\zeta \bowtie \eta := \zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta.$$

Then, for each pair $\zeta = (\psi_{\mathbf{M}(B)}, q) \in \mathbf{Y}(\mathbf{R}_L)$, we define the equivalence class at pointed Boolean frame ζ as follows:

$$Q_\zeta = \{\iota \in \mathbf{Y}(\mathbf{R}_L) : \zeta \bowtie \iota\}.$$

We finally define the *quotient set*:

$$\mathbf{Y}(\mathbf{R}_L)/\bowtie := \{Q_\zeta : \zeta = (\psi_{\mathbf{M}(B)}, q) \in \mathbf{Y}(\mathbf{R}_L)\},$$

and use the notation $Q_\zeta = \|(\psi_{\mathbf{M}(B)}, q)\|$, where $\|(\psi_{\mathbf{M}(B)}, q)\|$ denotes the equivalence class at pointed Boolean frame $\zeta = (\psi_{\mathbf{M}(B)}, q)$. The quotient set $\mathbf{Y}(\mathbf{R}_L)/\bowtie$ defines the colimit (inductive limit) in the category of Boolean frames of the functor $\mathbf{R}_L$, that is

$$\mathbf{Y}(\mathbf{R}_L)/\bowtie = \mathit{Colim}\{\int(\mathbf{R}_L, \mathcal{B}) \to \mathcal{B} \to \mathcal{L}\},$$

by noticing that it is naturally endowed with a quantum event algebra structure as follows:

[1]. The orthocomplementation is defined by $Q_\zeta^* = \|(\psi_{\mathbf{M}(B)}, q)\|^* = \|(\psi_{\mathbf{M}(B)}, q^*)\|$.

[2]. The unit element is defined by $\mathbf{1} = \|(\psi_{\mathbf{M}(B)}, 1)\|$.

[3]. The partial order structure on the quotient $\mathbf{Y}(\mathbf{R}_L)/\bowtie$ is defined by $\|(\psi_{\mathbf{M}(B)}, q)\| \preceq \|(\psi_{\mathbf{M}(C)}, r)\|$ if and only if $d_1 \preceq d_2$ where we have made the following identifications: $\|(\psi_{\mathbf{M}(B)}, q)\| = \|(\psi_{\mathbf{M}(D)}, d_1)\|$ and $\|(\psi_{\mathbf{M}(C)}, r)\| = \|(\psi_{\mathbf{M}(D)}, d_2)\|$, with d_1, $d_2 \in \mathbf{M}(D)$, such that $\beta(d_1) = q$, $\gamma(d_2) = r$, where $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$, and $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$ is the pullback of $\alpha : \mathbf{M}(B) \to L$ along $\lambda : \mathbf{M}(C) \to L$ in the category of quantum event algebras.

$$\begin{tikzcd} \mathbf{M}(D) \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathbf{M}(B) \arrow[d, "\alpha"] \\ \mathbf{M}(C) \arrow[r, "\lambda"] & L \end{tikzcd}$$

The physical meaning of the adjunction between presheaves of Boolean logical frames and quantum event algebras is made transparent if we consider that the pair of adjoint functors formalizes the process of encoding and decoding information relevant to the structural form of their domain and codomain categories. If we think of $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the categorical universe of variable local Boolean frames modeled in **Sets**, and of $\mathcal{L}$ as the categorical universe of quantum event structures, then the functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ signifies a translational code from the level of local Boolean algebras to the level of global quantum event algebras, whereas the Boolean realization functor $\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$ a translational code in the inverse direction. In general, the structural content of the information is not possible to remain completely invariant with respect to translating from one categorical universe to another and conversely. However, there remain two alternatives for a variable set over local Boolean frames $\mathbf{P}$ to exchange information with a quantum algebra L. Either the content of information is transferred in quantum terms with the inductive limit in the category of elements of $\mathbf{P}$ translating, represented as the quantum morphism $\mathbf{LP} \to L$, or the content of information is transferred in Boolean terms with the functor of Boolean frames of L translating, represented correspondingly as the natural transformation $\mathbf{P} \to \mathbf{R}(L)$. Then, the natural bijection corresponds to the assertion that these two distinct ways of information transfer are equivalent. Most significantly, the totality of the structural information included in quantum event algebras remains invariant under Boolean encodings, corresponding to local Boolean logical frames, if and only if, the adjunctive correspondence can be appropriately restricted to an equivalence of the functorially correlated categories. For this purpose, we need to localize the category of presheaves of Boolean logical frames and concomitantly define a functorial covering scheme of quantum event algebras induced by these local Boolean frames.

A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a subfunctor $\mathbf{S}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L,

$$\mathbf{S} \hookrightarrow \mathbf{R}(L).$$

A functor of Boolean coverings for an L in $\mathcal{L}$ is equivalent to an algebraic ideal or sieve of quantum homomorphisms $\mathbf{S} \triangleright \mathbf{R}(L)$, defined by the requirement: For each B in $\mathcal{B}$, $\mathbf{S}(B) \subseteq [\mathbf{R}(L)](B)$ is a set of quantum homomorphisms of the form $\psi_B : \mathbf{M}(B) \to L$, called *Boolean covers of* L, satisfying the following property:

$\langle$ If $[\psi_B : \mathbf{M}(B) \to L] \in \mathbf{S}(B)$, and $\mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathbf{M}(B)$ in $\mathcal{L}$, for $v : \acute{B} \to B$ in $\mathcal{B}$, then $[\psi_B \circ \mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathcal{L}] \in \mathbf{S}(B)$ $\rangle$.

A family of Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, is the *generator of an ideal of Boolean coverings* $\mathbf{S}$, if and only if, this ideal is the smallest among all that contains that family. The ideals of Boolean coverings for an L in $\mathcal{L}$ constitute a partially ordered set under inclusion. The minimal ideal is the empty one, namely $\mathbf{S}(B) = \emptyset$ for all B in $\mathcal{B}$, whereas the maximal ideal is the functor of Boolean frames $\mathbf{R}(L)$ of L itself.

The *pasting or gluing isomorphism* of the Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) \longrightarrow L$, $\acute{B}$ in $\mathcal{B}$, is defined as follows:

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})) \longrightarrow \psi_{B\acute{B}}(\mathbf{M}(B) \times_L \mathbf{M}(\acute{B}))$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ \psi_{\acute{B}B}{}^{-1}$$

where $\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})$, together with the two projections $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, is the *pullback or categorical overlap of the Boolean covers* $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) \longrightarrow L$, $\acute{B}$ in $\mathcal{B}$, with common codomain a quantum event algebra L, as shown in the following diagram:

$$\begin{tikzcd}
\mathbf{M}(B) \times_L \mathbf{M}(\acute{B}) \arrow[r, "\psi_{B,\acute{B}}"] \arrow[d, "\psi_{\acute{B},B}"] & \mathbf{M}(B) \arrow[d, "\psi_B"] \\
\mathbf{M}(\acute{B}) \arrow[r, "\psi_{\acute{B}}"] & L
\end{tikzcd}$$

An immediate consequence of the previous definition is the satisfaction of the following *Boolean coordinate cocycle conditions* for injective Boolean covers: $\Omega_{B,B} = 1_B$, $\Omega_{B,\acute{B}} \circ \Omega_{\acute{B},\acute{\acute{B}}} = \Omega_{B,\acute{\acute{B}}}$ and $\Omega_{B,\acute{B}} = \Omega^{-1}{}_{\acute{B},B}$ whenever they are defined. Thus, the pasting isomorphism assures that the Boolean covers $\psi_{\acute{B}B} : (\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})) \to L$ and $\psi_{B\acute{B}} : (\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})) \to L$ cover the same part of L compatibly.

Now, given an ideal of Boolean coverings for an $L \in \mathcal{L}$, we call it *a functor of Boolean localizations of* L, or a *structure sheaf of Boolean coefficients of* L, if and only if the Boolean coordinate cocycle conditions are satisfied.

For any presheaf functor $\mathbf{P}$ in the topos $\mathbf{Sets}^{B^{op}}$, the unit of the Boolean frames-quantum adjunction is defined as follows:

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}.$$

On the other side, for each quantum event algebra L in $\mathcal{L}$ the counit is defined as follows:

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L.$$

The representation of a quantum event algebra L in $\mathcal{L}$, in terms of the functor of Boolean frames $\mathbf{R}(L)$ of L, is full and faithful, if and only if the counit of the Boolean frames-quantum adjunction is a quantum algebraic isomorphism, that is structure-preserving, injective and surjective. In turn, the counit of the

Boolean frames-quantum adjunction is a quantum algebraic isomorphism, if and only if the right adjoint functor is full and faithful. In the latter case we characterize the Boolean modeling functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ as a proper or dense modeling functor. We can show that the Boolean realization functor is full and faithful if it corresponds to a functor of Boolean localizations of L (Zafiris 2004). Thus, the counit of the Boolean frames-quantum adjunction is an isomorphism if it is restricted to an ideal of Boolean localizations of L. Using the more precise terminology of Grothendieck sites (Zafiris 2006a), we may consider the category of Boolean event algebras as a generating subcategory of the category of quantum event algebras. Then, we may endow the base category of Boolean event algebras with a Grothendieck topology (called the topology of epimorphic families), by asserting that a sieve S on a Boolean algebra B in $\mathcal{B}$ is to be a covering sieve of B, when the arrows $s : C \to B$ belonging to the sieve S (Boolean coverings) together form an epimorphic family in $\mathcal{L}$. This requirement may be equivalently expressed in terms of a map

$$G_S : \coprod_{(s:C \to B) \in S} \mathbf{M}(C) \to \mathbf{M}(B)$$

being an epimorphism in $\mathcal{L}$. We note that this is a subcanonical Grothendieck topology, and thus all representable presheaves on $\mathcal{B}$ are sheaves. Then, the presheaf functor of Boolean frames becomes a sheaf with respect to every covering sieve in this Grothendieck topology. As a corollary the counit of the Boolean frames-quantum adjunction is an isomorphism restricted to every covering sieve of L.

From the above, we deduce that the representation of an L in $\mathcal{L}$, in terms of $\mathbf{R}(L)$ of L, is full and faithful, if the Boolean frames-quantum adjunction is restricted to a functor of Boolean localizations (covering sieve) of L. As a corollary, we obtain that $\mathcal{L}$ is a reflection of the topos of presheaves $\mathbf{Sets}^{B^{op}}$ on the base category of Boolean frames, and the total information content of a quantum event algebra L in $\mathcal{L}$ is preserved by some ideal of Boolean covers, if and only if this ideal forms a Boolean localization functor of L.

For reasons of completeness, we note that together with a logical event structure, there always exists a corresponding probabilistic structure, defined by means of convex sets of measures on that logic. In this sense, the probabilistic structure of a classical system is described by convex sets of probability measures on the Boolean algebra of events of this system, whereas the probabilistic structure of a quantum system is described by convex sets of probability measures on the quantum logical event structure of that system. More accurately, in the case of quantum systems, each quantum probability measure, called quantum probabilistic state, is defined by a measurable mapping:

$$p : L \to [0, 1],$$

such that the following conditions are satisfied: $p(1) = 1$ and $p(x \vee y) = p(x) + p(y)$, if $x \perp y$, where, x, $y \in L$. Then, we may define the categories of quantum probabilistic states and Boolean probabilistic states by passing from the logical categories to the corresponding probabilistic categories by slicing over $[0, 1]$ and respecting the measure-theoretic requirements. Finally, using analogous arguments we can show that quantum probabilistic states are represented as equivalence classes of local Boolean probabilistic states with respect to epimorphic families of covering systems induced by Boolean probabilistic frames (Zafiris 2006b).

2.3 The Quantum Truth-Object

Since $\mathcal{L}$ is a reflection of $\mathbf{Sets}^{\mathcal{B}^{op}}$, it is a complete category and monic arrows are preserved by the right adjoint Boolean realization functor $\mathbf{R}$. In particular, there exist a terminal object and pullbacks of monic arrows (Mac Lane and Moerdijk 1992). Thus, there exists a subobject functor for a quantum categorical event structure $\mathcal{L}$ equipped with Boolean localization functors.

Definition : The *subobject functor* of $\mathcal{L}$ is defined as follows:

$$\mathbf{Sub} : \mathcal{L}^{op} \to \mathbf{Sets}.$$

The functor $\mathbf{Sub}$ is a contravariant functor by pulling back. Composition with a proper Boolean modeling functor defines a presheaf in $\mathbf{Sets}^{\mathcal{B}^{op}}$, called the *Boolean frames modeled subobject functor* of $\mathcal{L}$, as follows:

$$\mathbf{Sub} \circ \mathbf{M} : \mathcal{B}^{op} \to \mathcal{L}^{op} \to \mathbf{Sets}.$$

In a compact notation we obtain:

$$\Upsilon_{\mathbf{M}} := \Upsilon(\mathbf{M}(-)) := \mathbf{Sub} \circ \mathbf{M} : \mathcal{B}^{op} \to \mathbf{Sets},$$

such that,

$$\mathcal{B}^{op} \ni B \mapsto \{[Dom(m) \hookrightarrow \mathbf{M}(B)]\} \in \mathbf{Sets}$$

where the range denotes the set of subobjects of $\mathbf{M}(B)$, viz. the set of equivalence class of monic quantum homomorphisms m from $Dom(m)$ to $\mathbf{M}(B)$.

The set $\Upsilon_{\mathbf{M}}(B) = \Upsilon(\mathbf{M}(B))$ is defined as the set of all *subobjects* of $\mathbf{M}(B)$, for every B in $\mathcal{B}$, in the category $\mathcal{L}$. Notice that the set $\Upsilon(\mathbf{M}(B))$, for every B in $\mathcal{B}$, is a partially ordered set under inclusion of subobjects of $\mathbf{M}(B)$.

A natural question arising in this categorical setting is the following: Is the subobject functor representable in $\mathcal{L}$ by means of a concrete quantum event algebra Ω in $\mathcal{L}$ that special object which would play the role of a classifying object in $\mathcal{L}$? The representation of the subobject functor in a quantum categorical event structure $\mathcal{L}$ is significant because it would allow to interpret the concrete classifying object Ω as a truth values object in $\mathcal{L}$, [1] in a sense similar to the role played by the two-valued Boolean object

[1] It is instructive to note that in an arbitrary topos, the existence of a classifying object or a subobject classifier Ω takes the role of the set $\{0,1\} \cong \{false, true\}$ of truth values. If B is an object in the topos, and A denotes a subobject of B, then, there is a monic arrow (monomorphism) $A \to B$, generalizing categorically the inclusion of a subset into a larger set. Like in the familiar topos, $\mathbf{Sets}$, of sets and functions, we can also characterize A as a subobject of B by an arrow from B to the subobject classifier Ω. Intuitively, this "characteristic arrow", $B \to \Omega$, describes how A "lies in" B; in $\mathbf{Sets}$, this arrow is the characteristic function $\chi_S : X \to \{0,1\}$ classifying whether a point $\chi \in X$ lies in S or not. In general, the elements of the subobject classifier, understood as the arrows $1 \to \Omega$, are the truth values, just like "false" and "true", the elements of $\{false, true\}$, are the truth values available in $\mathbf{Sets}$.

$\mathbf{2} := \{0, 1\}$ in characterizing the logic of propositions referring to the behavior of classical systems. In this case, subobjects of a quantum event algebra should be characterized in terms of characteristic functions, which take values not in $\mathbf{2}$, but precisely in the truth values object Ω in $\mathcal{L}$. Most importantly, in that case the category of quantum event algebras $\mathcal{L}$ is endowed with a subobject classifier, defined as follows:

Definition : The *subobject classifier* of the category of quantum event algebras is a universal monic quantum homomorphism,

$$T := \mathit{True} : 1 \hookrightarrow \Omega$$

such that, to every monic arrow, $m : K \hookrightarrow L$ in $\mathcal{L}$, there is a unique characteristic arrow ϕ_m, which, with the given monic arrow m, forms a pullback diagram:

$$\begin{tikzcd} K \arrow[r, "!"] \arrow[d, "m"] & 1 \arrow[d, "T"] \\ L \arrow[r, "\phi_m"] & \Omega \end{tikzcd}$$

This is equivalent to saying that every subobject of L in $\mathcal{L}$ is uniquely a pullback of the universal monic T.

From the general definition of the notion of representability of a **Sets**-valued functor in $\mathcal{L}$, we deduce the following: The functor $\Upsilon_{\mathbf{M}}$ is representable in $\mathcal{L}$, if and only if there exists a classifying or truth values object Ω in $\mathcal{L}$, viz. if and only if there exists an isomorphism for each Boolean frame B in $\mathcal{B}$, that is a natural isomorphism as follows:

$$\Upsilon(\mathbf{M}(-)) \simeq \mathbf{R}(\Omega) := Hom_{\mathcal{L}}(\mathbf{M}(-), \Omega).$$

Proposition : The Boolean frames modeled subobject functor $\Upsilon_{\mathbf{M}}$ of $\mathcal{L}$ is representable in the category of quantum event algebras $\mathcal{L}$, if and only if the evaluation of the unit of the Boolean frames-quantum adjunction at $\Upsilon_{\mathbf{M}}$ restricted to a functor of Boolean localizations of L for every L in $\mathcal{L}$ is an isomorphism.

Proof : The counit of the Boolean frames-quantum adjunction, for each L in $\mathcal{L}$, is

$$\epsilon_L : \mathbf{LR}(L) \to L.$$

The counit evaluated at L, viz. ϵ_L, is an isomorphism if it is restricted to a functor of Boolean localizations of L. For any presheaf $\mathbf{P} \in \mathbf{Sets}^{\mathcal{B}^{op}}$, the unit is defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}.$$

It is easy to see that if we consider as $\mathbf{P} \in \mathbf{Sets}^{\mathcal{B}^{op}}$ the subobject functor $\Upsilon(\mathbf{M}(-))$, we obtain the following natural transformation:

$$\delta_{\Upsilon(\mathbf{M}(-))} : \Upsilon(\mathbf{M}(-)) \to \mathbf{RL}\Upsilon(\mathbf{M}(-)), \quad \text{that is,}$$
$$\delta_{\Upsilon(\mathbf{M}(-))} : \Upsilon(\mathbf{M}(-)) \to Hom_{\mathcal{L}}(\mathbf{M}(-), \mathbf{L}\Upsilon(\mathbf{M}(-)).$$

Hence, by inspecting the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ evaluated at $\Upsilon(\mathbf{M}(-))$, we conclude that the Boolean frames modeled subobject functor becomes representable in $\mathcal{L}$ if and only if, given that the counit ϵ_L for every L in $\mathcal{L}$ is an isomorphism, the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ is also an isomorphism. Thus, $\Upsilon_{\mathbf{M}}$ becomes representable in $\mathcal{L}$ if and only if the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ restricted to a functor of Boolean localizations of L, called the *localized unit* at $\Upsilon_{\mathbf{M}}$, is an isomorphism.

Proposition : If the evaluation of the localized unit at $\Upsilon_{\mathbf{M}}$ is an isomorphism, then the quantum truth values algebra Ω is given by the colimit (inductive limit) taken in the category of elements of the Boolean frames modeled subobject functor $\Upsilon_{\mathbf{M}}$, according to:

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)) = Colim\{\int (\mathbf{\Upsilon}(\mathbf{M}(-)), \mathcal{B}) \longrightarrow \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\}$$

Proof : We may prove this proposition immediately by noticing that if the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ is an isomorphism restricted to a functor of Boolean localizations of L for every L in $\mathcal{L}$, then the quantum truth values algebra Ω is constructed by application of the left adjoint functor of the Boolean frames-quantum adjunction on the localized unit $\delta_{\Upsilon(\mathbf{M}(-))}$, viz.:

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)).$$

This is actually the case because

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)) \simeq \mathbf{L}[\mathbf{RL}\Upsilon(\mathbf{M}(-))] \simeq \mathbf{LR}\Omega$$

is precisely an expression of the counit isomorphism for the quantum event algebra Ω.

As a corollary, we obtain that if the evaluation of the localized unit at $\Upsilon_{\mathbf{M}}$ restricted to a functor of Boolean localizations of L for every L in $\mathcal{L}$ is an isomorphism, then the following diagram is a classifying pullback square in $\mathcal{L}$ for each quantum algebraic homomorphism

$$[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda} : \mathbf{M}(B) \to \mathbf{L}\Upsilon(\mathbf{M}(-)) := \Omega$$

from a Boolean domain modeled object $\mathbf{M}(B)$, such that λ is a subobject of $\mathbf{M}(B)$:

$$\begin{tikzcd}
Dom(\lambda) \arrow[r, "!"] \arrow[d, "\lambda"] & 1 \arrow[d, "T"] \\
\mathbf{M}(B) \arrow[r, "{[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda}}"] & \mathbf{L}\Upsilon(\mathbf{M}(-)) := \Omega
\end{tikzcd}$$

2.4 Tensor Product Representation of Quantum Truth Values

From the preceding we have concluded that if the evaluation of the localized unit at $\Upsilon_{\mathbf{M}}$ is an isomorphism, then the subobject functor $\mathbf{Sub} : \mathcal{L}^{op} \to \mathbf{Sets}$ of $\mathcal{L}$ is representable in $\mathcal{L}$ by $\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-))$ and $\mathcal{L}$ is endowed with a subobject classifier defined by a universal monic quantum homomorphism, $T := True : 1 \hookrightarrow \Omega$. It is important now to provide an explicit representation of the quantum truth values.

Proposition : The elements of the quantum truth values algebra $\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-))$ are equivalence classes represented in tensor product form as follows:

$$[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda}(b) := \Delta^{\lambda}(b) = \lambda \otimes b, \quad \text{where,}$$

$$[\lambda * v] \otimes \acute{b} = \lambda \otimes v(\acute{b}), \quad \lambda \in \Upsilon(\mathbf{M}(B)), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b,$$

and $[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda} := \Delta^{\lambda}$ denotes a local Boolean cover of Ω in the Boolean localization functor $[\delta_{\Upsilon(\mathbf{M}(-))}]^{(-)}$ of Ω using the unit isomorphism.

Proof : The quantum truth values object Ω is given by the colimit in the category of elements of the Boolean frames modeled subobject functor, viz.:

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)) = Colim\{\int(\boldsymbol{\Upsilon}(\mathbf{M}(-)), \mathcal{B}) \longrightarrow \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\}$$

where the category of elements of $\Upsilon(\mathbf{M}(-))$ is denoted by $\int(\boldsymbol{\Upsilon}(\mathbf{M}(-)), \mathcal{B})$. Its objects are all pairs (B, λ), where λ is a subobject of $\mathbf{M}(B)$, that is a monic quantum homomorphism in $\mathbf{M}(B)$. The morphisms of the category of elements of $\Upsilon(\mathbf{M}(-))$ are given by the arrows $(\acute{B}, \acute{\lambda}) \longrightarrow (B, \lambda)$, namely they are those morphisms $v : \acute{B} \longrightarrow B$ of $\mathcal{B}$ for which $\lambda * v = \acute{\lambda}$, where $\lambda * v$ denotes the pullback of the subobject λ of $\mathbf{M}(B)$ along v and $\acute{\lambda}$ is a subobject of $\mathbf{M}(\acute{B})$.

The colimit in the category of elements of the Boolean frames modeled subobject functor can be equivalently represented as a coequalizer of coproduct using standard category-theoretic arguments (Appendix A.2):

$$\coprod_{v:\acute{B}\to B} \mathbf{M}(\acute{B}) \quad \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \quad \coprod_{(B,\lambda)} \mathbf{M}(B) \xrightarrow{\chi} \mathbf{L}\Upsilon(\mathbf{M}(-)) = \Omega$$

where the second coproduct is over all the pairs (B, λ) with $\lambda \in \Upsilon(\mathbf{M}(B))$ of the category of elements, while the first coproduct is over all the maps $v : (\acute{B}, \acute{\lambda}) \longrightarrow (B, \lambda)$ of that category, so that $v : \acute{B} \longrightarrow B$ and the condition $\lambda * u = \acute{\lambda}$ is satisfied.

First, we may interpret the above representation of the colimit in the category of elements of the Boolean frames modeled subobject functor in the category $\mathbf{Sets}$. In this case, the coproduct $\coprod_{(B,\lambda)} \mathbf{M}(B)$ is a coproduct of sets, which is equivalent to the product $\Upsilon(\mathbf{M}(B)) \times \mathbf{M}(B)$ for B in $\mathcal{B}$. The coequalizer is thus the definition of the tensor product $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ of the set valued functors:

$$\Upsilon(\mathbf{M}(-)) : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}, \qquad \mathbf{M} : \mathcal{B} \longrightarrow \mathbf{Sets}$$

$$\coprod_{B,\acute{B}} \Upsilon(\mathbf{M}(B)) \times Hom(\acute{B}, B) \times \mathbf{M}(\acute{B}) \quad \overset{\zeta}{\underset{\eta}{\rightrightarrows}}$$

$$\overset{\zeta}{\underset{\eta}{\rightrightarrows}} \quad \coprod_{B} \Upsilon(\mathbf{M}(B)) \times \mathbf{M}(B) \xrightarrow{\chi} \quad \Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$$

where the functor $\Upsilon(\mathbf{M}(-))$ is considered as a right $\mathcal{B}$-module and the functor $\mathbf{M}$ as a left $\mathcal{B}$-module, in complete analogy with the definition of the tensor product of a right $\mathcal{B}$-module with a left $\mathcal{B}$-module over a ring of coefficients $\mathcal{B}$. We call this the *functorial tensor product decomposition of the colimit in the category of elements of the Boolean frames modeled subobject functor* and we make use of the tensor notation in the sequel.

According to the preceding diagram, for elements $\lambda \in \Upsilon(\mathbf{M}(B))$, $v : \acute{B} \to B$ and $\acute{q} \in \mathbf{M}(\acute{B})$ the following equations hold:

$$\zeta(\lambda, v, \acute{q}) = (\lambda * v, \acute{q}), \qquad \eta(\lambda, v, \acute{q}) = (\lambda, v(\acute{q}))$$

symmetric in $\Upsilon(\mathbf{M}(-))$ and $\mathbf{M}$. Hence the elements of the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(\lambda, q)$. This element can be written in tensor product form as follows:

$$\chi(\lambda, q) = \lambda \otimes q, \quad \lambda \in \Upsilon(\mathbf{M}(B)), q \in \mathbf{M}(B).$$

Thus, if we take into account the definitions of ζ and η above, we obtain:

$$[\lambda * v] \otimes \acute{q} = \lambda \otimes v(\acute{q}), \quad \lambda \in \Upsilon(\mathbf{M}(B)), \acute{q} \in \mathbf{M}(\acute{B}), v : \acute{B} \longrightarrow B.$$

We conclude that the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ is actually the quotient of the set $\amalg_B \Upsilon(\mathbf{M}(B)) \times \mathbf{M}(B)$ by the equivalence relation generated by the above equations. Furthermore, if we define $\lambda * v = \acute{\lambda}$, $v(\acute{q}) = q$, where $\acute{\lambda}$ is a subobject of $\mathbf{M}(\acute{B})$ and $q \in \mathbf{M}(B)$, we obtain the equations:

$$\acute{\lambda} \otimes \acute{q} = \lambda \otimes q.$$

Moreover, since pullbacks exist in $\mathcal{L}$, we may consider the arrows $h : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\acute{h} : \mathbf{M}(D) \to \mathbf{M}(\acute{B})$ and the following pullback diagram in $\mathcal{L}$:

$$\begin{tikzcd}
\mathbf{M}(D) \arrow[r, "h"] \arrow[d, "\acute{h}"] & \mathbf{M}(B) \arrow[d] \\
\mathbf{M}(\acute{B}) \arrow[r] & L
\end{tikzcd}$$

such that the following relations are satisfied: $h(d) = q$, $\acute{h}(d) = \acute{q}$ and $\lambda * h = \acute{\lambda} * \acute{h}$. Then, we obtain:

$$\lambda \otimes q = \lambda \otimes h(d) = [\lambda * h] \otimes d = [\acute{\lambda} * \acute{h}] \otimes d = \acute{\lambda} \otimes \acute{h}(d) = \acute{\lambda} \otimes \acute{q}.$$

We may further define,

$$\lambda * h = \acute{\lambda} * \acute{h} = \tau,$$

where τ is a subobject of $\mathbf{M}(D)$. Then, it is obvious that:

$$\lambda \otimes q = \tau \otimes d$$

$$\acute{\lambda} \otimes \acute{q} = \tau \otimes d.$$

It is then evident that we may define a partial order on the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ as follows:

$$\lambda \otimes b \leq \rho \otimes c$$

if and only if there exist quantum algebraic homomorphisms $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$, and some d_1, d_2 in $\mathbf{M}(D)$, such that: $\beta(d_1) = b$, $\gamma(d_2) = c$, and $\lambda * \beta = \rho * \gamma = \tau$. Thus, we obtain:

$$\lambda \otimes b = \tau \otimes d_1$$

$$\rho \otimes c = \tau \otimes d_2.$$

We conclude that:

$$\lambda \otimes b \leq \rho \otimes c$$

if and only if

$$\tau \otimes d_1 \leq \tau \otimes d_2 \Longleftrightarrow d_1 \leq d_2.$$

The set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ may be further endowed with a maximal element which admits the following presentations:

$$\mathbf{1} = \tau \otimes 1 := true \qquad \forall \tau \in \Upsilon(\mathbf{M}(D))$$

$$\mathbf{1} = id_{\mathbf{M}(B)} \otimes b := true \qquad \forall b \in \mathbf{M}(B),$$

and an orthocomplementation operator,

$$[\tau \otimes d]^{\star} = \tau \otimes d^{\star}.$$

Then, it is easy to verify that the set $\Omega = \Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ endowed with the prescribed operations is actually a quantum event algebra, for every Boolean event algebra B in $\mathcal{B}$. Consequently, the elements of the quantum truth values algebra $\Omega = \Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ are equivalence classes represented in tensor product form as follows:

$$[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda}(b) := \Delta^{\lambda}(b) = \lambda \otimes b, \quad \text{where,}$$

$$[\lambda * v] \otimes \acute{b} = \lambda \otimes v(\acute{b}), \quad \lambda \in \Upsilon(\mathbf{M}(B)), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b,$$

and $[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda} := \Delta^{\lambda}$ denotes a local Boolean cover of Ω in the Boolean localization functor $[\delta_{\Upsilon(\mathbf{M}(-))}]^{(-)}$ of Ω using the unit isomorphism.

Corollary - Criterion of Truth : The criterion of truth for the category of quantum event algebras $\mathcal{L}$ with respect to a functor of Boolean localizations is the following:

$$[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda}(b) := \Delta^{\lambda}(b) = \lambda \otimes b = \mathit{true} \qquad \textbf{iff} \qquad b \in \mathit{Image}(\lambda),$$

where b may be thought of as representing the element (e.g., projection operator) that identifies a corresponding quantum proposition p in the context of $\mathbf{M}(B)$.

3 Physical Interpretation

We have shown explicitly that the category of quantum event algebras $\mathcal{L}$ has a quantum truth values object Ω, which is defined by the colimit (inductive limit) in the category of elements of the Boolean frames modeled subobject functor, viz. $\Omega = \Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$. To recapitulate, the truth values are equivalence classes represented in tensor product form as follows:

$$\Delta^{\lambda}(b) = \lambda \otimes b, \quad \text{where,}$$

$$[\lambda * v] \otimes \acute{b} = \lambda \otimes v(\acute{b}), \quad \lambda \in \Upsilon(\mathbf{M}(B)), \acute{b} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B, v(\acute{b}) = b,$$

and Δ^{λ} denotes a local Boolean cover of Ω in the Boolean localization functor $\Delta^{(-)}$ of Ω using the unit isomorphism. Thus, we have proved that truth-value assignment in quantum mechanics is localized with respect to equivalence classes of compatible Boolean frames belonging to a Boolean localization functor of a quantum event algebra. We require that a Boolean localization functor of L in $\mathcal{L}$ is closed with respect to Boolean covers, viz. it contains all Boolean frames covering L and satisfy the compatibility conditions. In this way, the quantum value *true* (equivalence class) used for the evaluation of quantum propositions is characterized by:

$$\tau \otimes 1 := \mathit{true} \qquad \forall \tau \in \Upsilon(\mathbf{M}(D))$$

$$id_{\mathbf{M}(B)} \otimes b := \mathit{true} \qquad \forall b \in \mathbf{M}(B)$$

$$\Delta^{\lambda}(b) = \lambda \otimes b = \mathit{true} \qquad \textbf{iff} \qquad b \in \mathit{Image}(\lambda), \quad \forall \lambda \in \Upsilon(\mathbf{M}(B)).$$

The classifying quantum event algebra Ω in $\mathcal{L}$ plays a role similar to the role played by the two-valued Boolean algebra $\mathbf{2} := \{0, 1\}$ in characterizing the logic of propositions referring to the behavior of classical systems. Thus, in the quantum case, subobjects of a quantum event algebra should be characterized in terms of characteristic functions, which take values not in $\mathbf{2}$, but precisely in the truth values object Ω in $\mathcal{L}$. Let us explain the functionality of the quantum truth values object Ω according to the following diagram:

$$\begin{tikzcd}
Dom(l * e) \arrow[r] \arrow[d, "l * e = \lambda"] & K \arrow[d, "l"] \\
\mathbf{M}(B) \arrow[r, "e"] & L
\end{tikzcd}$$

where $l : K \hookrightarrow L$ is a subobject of a quantum event algebra L, $e : \mathbf{M}(B) \to L$ is a Boolean cover of L in a Boolean localization functor of L, and $l * e = \lambda$ is the pullback of l along e, that is the subobject λ of $\mathbf{M}(B)$. According to the truth-value criterion, the characteristic function of the subobject $l : K \hookrightarrow L$ of L is specified as an equivalence class of pullbacks of the subobject l along its restrictions on all Boolean covers in a Boolean localization functor of L. For each Boolean cover $\mathbf{M}(B)$ of L we have that $(l * e) \otimes b = \lambda \otimes b = true$ if and only if b belongs to the $Image(\lambda)$, for all λ in $\Upsilon(\mathbf{M}(B))$. Thus, for each Boolean cover $\mathbf{M}(B)$ of L, the value $\mathbf{1} = true$ in Ω is assigned to all those b in $\mathbf{M}(B)$ belonging to the restriction of the subobject $l : K \hookrightarrow L$ of L with respect to the subobject λ of $\mathbf{M}(B)$, for all λ. In particular, if the Boolean covers are monic morphisms, each pullback is expressed as the intersection of the subobject l with the corresponding cover in the Boolean localization functor.

Conceptually, every quantum event or proposition of a quantum event algebra L is contextualized with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of L by means of pulling back or restricting. In this way, a quantum proposition refers to an equivalence class of all its restricted propositions with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of L. Note that all restricted propositions are logically compatible since they belong to a Boolean localization functor of L. With respect to each such contextualization we obtain a contextual truth valuation of the restricted proposition associated with the corresponding frame $\mathbf{M}(B)$ specified by the truth rule $\lambda \otimes b = true$ if and only if b belongs to the $Image(\lambda)$, holding for every subobject λ of $\mathbf{M}(B)$, where b represents the restricted quantum proposition with respect to the Boolean frame $\mathbf{M}(B)$. The important thing to emphasize is that all these contextual truth valuations of some quantum proposition are appropriately related to each other by the formation of equivalence classes gluing together all its restrictions with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor. This, in effect, is established by the quantum truth value *true* specification constraints in Ω determining a complete description of states of affairs and defined by: $\tau \otimes 1 := true$ for all $\tau \in \Upsilon(\mathbf{M}(B))$ and $id_{\mathbf{M}(B)} \otimes b := true$ for all $b \in \mathbf{M}(B)$ and for all Boolean frame domains $\mathbf{M}(B)$ in a Boolean localization functor.

Let us now apply the above truth valuation scheme to a typical measurement situation referring to a quantum system prepared to pass through a slit, where a counter has been set to record by clicking or not the passage through the slit. If we denote a Boolean domain preparation context by

$\mathbf{M}(B)$, containing both the measuring apparatus and the system observed, then we may form the propositions:

$\langle c \rangle$:=[*counter clicks*]

$\langle d \rangle$:=[*system passes through the slit*]

$\langle\ c \Rightarrow d\ \rangle := \langle b \rangle :=$ [*counter clicks* $\Rightarrow$ *system passes through the slit*].

Notice that the proposition $\langle b \rangle$ is a compound proposition referring to the coupling of the measuring apparatus with the quantum system in the Boolean context $\mathbf{M}(B)$. The proposition $\langle\ c \Rightarrow d\ \rangle := \langle b \rangle$ is assigned the value *true* in Ω, expressing a complete description of the state of affairs. More precisely with respect to the logical frame of the Boolean context $\mathbf{M}(B)$, or more concretely with respect to $id_{\mathbf{M}(B)}$, we have that $id_{\mathbf{M}(B)} \otimes b = true$. We notice that the above truth-value assignment does not suffice in order to infer that the proposition $\langle d \rangle$ is true. In order to infer the above, we need to use the Boolean context $\mathbf{M}(C)$ which contains only the measuring apparatus, which is a subobject of the preparatory Boolean context $\mathbf{M}(B)$, viz. $\lambda : \mathbf{M}(C) \hookrightarrow \mathbf{M}(B)$. Then, we obtain that: $id_{\mathbf{M}(C)} \otimes c = true$ and $\lambda \otimes \xi = true$ if and only if ξ belongs to the $Image(\lambda)$ in $\mathbf{M}(B)$. The Boolean contexts $\mathbf{M}(C)$ and $\mathbf{M}(B)$ induce the formation of a Boolean localization functor for the evaluation of $\langle d \rangle$ and thus they are glued compatibly together. Hence, we deduce that $\lambda \otimes b = true$ because b belongs to the $Image(\lambda)$ in $\mathbf{M}(B)$. In particular b in $\mathbf{M}(B)$ is the image of c in $\mathbf{M}(C) \hookrightarrow \mathbf{M}(B)$ and the truth of $\langle d \rangle$ is indirectly inferred by the equivalence class, which is induced by the ideal of compatible Boolean covers $\mathbf{M}(C)$ and $\mathbf{M}(B)$, and defined by:

$$id_{\mathbf{M}(C)} \otimes c = \lambda \otimes b = id_{\mathbf{M}(B)} \otimes b = true.$$

4 Conclusions

In the present study we considered a category-theoretic framework for the interpretation of quantum event structures and their logical semantics. The scheme of interpretation is based on the existence of the Boolean frames-quantum adjunction, namely, a categorical adjunction between presheaves of Boolean event algebras and quantum event algebras. On the basis of this adjoint correspondence, we proved the existence of an object of truth values in the category of quantum event algebras, characterized as subobject classifier. The conceptual essence of this classifying object Ω is associated with the fact that Ω constitutes the appropriate tool for the valuation of propositions describing the behavior of a quantum system, in analogous correspondence with the classical case, where the two-valued Boolean object is used. We explicitly constructed the quantum object of truth values in tensor product form, and furthermore, demonstrated its functioning in a typical measurement situation. In addition, we provided a criterion of truth valuation that corresponds to the truth-value *true* in the quantum domain.

We would like to conclude our paper by remarking on the conceptual merits of the suggested approach. The attribution of truth values to quantum mechanical propositions arising out of the preceding category-theoretic scheme bears the following consequences that seem to be intuitively satisfactory. Firstly, it avoids the semantic ambiguity with respect to truth-value assignment to propositions that is inherent in conventional quantum mechanics, in the following sense: all propositions that are certainly true or certainly false (assigned probability value 1 or 0) according to conventional quantum mechanics are also certainly true or certainly false according to the category-theoretic approach. The remaining propositions (assigned probability value different from 1 and 0) are semantically undecidable according to the former interpretation (they are neither true nor false), while they have determinate truth values according to the latter. These values, however, depend not only on the state of the physical system that is considered but also on the context through which the system is investigated, thus capturing the endemic feature of quantum contextuality. Indeed, the existence of the subobject classifier Ω leads naturally to contextual truth-value assignments to quantum mechanical propositions, where each proposition pertaining to a physical system under investigation acquires a determinate truth value with respect to the context defined by the corresponding observable to be measured.

Secondly, the quantum truth values object Ω enables not only a determinate truth valuation in each fixed experimental context, but in addition, it amalgamates internally all compatible truth valuations with respect to all Boolean frames belonging to some Boolean localization functor of a quantum event algebra. The amalgamation is expressed by the formation of equivalence classes, which are represented in tensor product form, via the truth value *true* in Ω determining a complete description of states of affairs with respect to the considered Boolean localization functor. In this way, truth-value assignment in quantum mechanics is topologically localized and consequently contextualized with respect to tensor product equivalence classes formed among compatible Boolean frames belonging to a Boolean localization functor of a quantum event algebra. Conceptually, every quantum event or proposition of a quantum event algebra L is contextualized with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of L. Thus, the truth of a quantum proposition is specified by the tensor product equivalence class of the truth value *true* in Ω interconnecting together all contextual truth valuations of all its restricted propositions with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of L.

A particularly interesting application of the proposed scheme refers to the following case involving partially overlapping, incompatible, Boolean experimental contexts: Let A and E be two incompatible observables of a quantum system in a given state sharing one or more projection operators in their corresponding spectral decompositions. Let $\mathbf{M}(B_A)$ and $\mathbf{M}(B_E)$ be the corresponding Boolean subalgebras in the system's Hilbert space quantum event structure associated to the observables A and E, respectively. From a physical perspective, the quantum truth values object Ω takes into account the whole set of possible ways of assigning truth values to the propositions associated with the projectors of the spectral decomposition of a given observable. Then, the subobject classifier Ω makes it possible to refer, at least partially, to the truth valuation of propositions represented by projectors pertaining to incompatible observables with respect to the initially chosen, without facing a Kochen-Specker contradiction, in the following sense: once an observable is selected to be measured, say A, and thus the associated context of measurement is fixed, we may consistently refer to Boolean truth valuations of observable E, as far as its common projectors with A are concerned, by taking into account the Boolean information that $\mathbf{M}(B_A) \cap \mathbf{M}(B_E)$ has about $\mathbf{M}(B_E)$. It is important to realize that in this framework no Kochen-Specker contradiction arises, since these truth valuations are considered from a fixed context. Furthermore, the sheaf theoretical representation of a quantum algebra of events, in terms of Boolean localization functors, takes precisely into account the compatibility conditions of these Boolean subalgebras with respect to their intersection in such a way as to leave invariant the amount of information contained in a quantum system. As indicated in Section 2.2, this underlying invariance property is satisfied if and only if the counit of the adjunction, restricted to those Boolean localization functors, is an isomorphism, that is, structure-preserving, 1-1 and onto. Inevitably, this state of affairs allows one to formalize the extent to which we can consider as objective properties of a physical system, and hence, attribute well-defined truth values to their corresponding propositions, those properties represented by projectors pertaining to the overlaps of different Boolean covers without facing no-go theorems. Still, one may ask, what about probabilities? What may be the probabilistic relations between events pertaining to different Boolean covers? Importantly, by analogous reasoning, within this framework one is able to refer consistently to the conditional probability of one quantum event, say e_j, given another, a_i, when working in two different, partially overlapping, Boolean contexts. On this approach, to ask what the probability is that a measurement of observable E will yield result e_j, given that an event a_i has occurred, is to ask for what value of ω the statement $p(e_j \mid a_i) = \omega$ is true, provided that a measurement of A has already taken place. Since p is a probability measure defined on the system's quantum event structure, obeying the requirements of Gleason's theorem, and hence representable by a quantum probabilistic state (a density operator) D on the system's Hilbert space, the conditional probability $p(e_j \mid a_i)$ is given by Lüders' conditionalization rule, $p(e_j \mid a_i) = Tr(P_A D P_A P_E)/Tr(DP_A)$, where P_A and P_E are the projection operators onto the associated one-dimensional subspaces of the system's Hilbert space corresponding to events a_i and e_j, respectively. It may be worth remarking in passing that on the so-called Copenhagen approach of Bohr and Heisenberg there are no available means of dealing with sequences of events; $p_n(e_j \mid a_i)$ will always be zero if A and E are incompatible, even if they share certain projections as in the previous case, since there is no single experimental arrangement that may correspond to the two Boolean subalgebras $\mathbf{M}(B_A)$ and $\mathbf{M}(B_E)$ generated by the associated magnitudes.

Finally, given the preceding conceptual and technical advantages, it would be wrong to assume that they are achievable at the expense of resorting to anti-realist approaches with respect to truth-semantics, as, for instance, identifying truth with a positivist verificationist position. On the contrary, the proposed account of truth conforms to a realist conception of truth, which, moreover, is compatible with contemporary physics. The account of truth-value assignment suggested here essentially denies that there can be a "God's-eye view" or an absolute Archimedean standpoint from which to state the totality of facts of nature. For, an elementary quantum mechanical proposition is not true or false *simpliciter*, independently of a particular context of reference, as in the case of classical mechanics. On account of the Kochen-Specker theorem, there simply does not exist, within a quantum mechanical discourse, a consistent binary assignment of determinately true or determinately false propositions independent of the appeal to a context. Propositional content seems to be linked to a context. This connection between referential context and propositional content means that a descriptive elementary proposition in the domain of quantum mechanics is, in a sense, incomplete unless it is accompanied by the specified conditions of an experimental context under which the proposition becomes effectively truth-valued (see, in addition, Karakostas 2012). Hence, from the category theoretical perspective of the present paper, the reference to a Boolean preparatory experimental context should not be viewed primarily as offering the evidential or verificationist basis for the truth of a proposition; it does not aim to equate truth to verification. Nor should it be associated with practices of instrumentalism, operationalism, and the like; it does not aim to reduce theoretical terms to products of operational procedures. It rather provides the appropriate conditions under which it is possible for a proposition to receive consistently a truth value. Whereas in classical mechanics the conditions under which elementary propositions are claimed to be true or false are determinate independently of the context in which they are expressed, in contradistinction, the truth-conditions of quantum mechanical propositions are determinate within a context. In other words, the specification of the context is part and parcel of the truth-conditions that should obtain for a proposition in order the latter to be invested with a determinate (albeit unknown) truth value. Otherwise, the proposition is, in general, semantically undecidable. Thus, the specification of the context provides the necessary conditions whereby bivalent assignment of truth values to quantum mechanical propositions is in principle applicable. This marks the fundamental difference between conditions for well-defined attribution of truth values to propositions and mere verification conditions. In the quantum description, therefore, the specification of the experimental context forms a pre-condition of quantum physical experience, which is necessary if quantum mechanics is to grasp empirical reality at all. In this respect, the specification of the context constitutes a methodological act preceding any empirical truth in the quantum domain and making it possible.

A APPENDIX

A.1 Comparison With Other Categorial Approaches

It is instructive to attempt a brief comparison of our topos-theoretic representation scheme with other categorial and topos-theoretic approaches. The current interest in applying methods of topos theory in the logical foundations of quantum physics was initiated by the work of Isham and Butterfield (1998, 1999), who provided a topos-theoretic reformulation of the Kochen-Specker theorem. For this purpose, they considered the partially ordered set of commutative von Neumann subalgebras of the non-commutative algebra of all bounded operators on a quantum Hilbert space as a "category of contexts" where the only arrows are inclusions. This "category of contexts" served as a base category for defining the topos of presheaves of sets over the poset of commutative subalgebras. The reformulation of the Kochen-Specker theorem took place by defining a special presheaf, called the spectral presheaf, and showing that the latter has no global sections. We note that the action of the spectral presheaf on each commutative von Neumann subalgebra gives its maximal ideal spectrum (Gelfand spectrum). Alternatively, the former "category of contexts" may be replaced by the poset of all Boolean subalgebras of the non-Boolean lattice of projection operators on a quantum Hilbert space. Similarly, the action of the spectral presheaf in this case (called the dual presheaf) on each Boolean subalgebra gives its Stone spectrum (that is the set of all its homomorphisms to the 2-valued Boolean algebra $\{0, 1\}$). In this case, the statement of the Kochen-Specker theorem is equivalent to the assertion that the dual presheaf has no global sections.

This topos-theoretic research initiative has been extended, elaborated and developed further by Döring and Isham (DI) (e.g., 2011 and references therein). The central principle of their work is that the construction of a theory of physics is equivalent to finding a representation in "a topos of a certain formal language" that is attached to the system. In particular, regarding quantum theory, their proposal is to use the formal language associated with the topos of presheaves of sets over the poset of commutative von Neumann subalgebras (or the poset of Boolean subalgebras) and mimic the classical topos, set-theoretic formulation of physical theories. This analogy is pursued up to the point of constructing a topos-theoretic framework of quantum kinematics (dynamical ideas have not yet been addressed in their framework). This difficult task required the following: [i] The association of physical quantities with morphisms in the topos of presheaves from a "state-object" to a "quantity-value" object; [ii] the definition of an appropriate "object of truth-values" in this topos, and [iii], the construction of the so called "daseinisation map" for projections (and self-adjoint operators), which is used as a translation mechanism from the Hilbert space formalism to the topos formalism. Regarding [i] the "state object" is identified as the spectral presheaf, while regarding [ii] the "object of truth-values" in the topos of presheaves is identified with the subobject classifier, which assigns to each context the Heyting algebra of all sieves on that context. Regarding [iii] the "daseinisation map" transforms a projection operator to a clopen subobject of the spectral presheaf by approximating it, for each context of the base poset, by the smallest projection greater than or equal to it. In this way, propositions are represented by the clopen subobjects of the spectral presheaf (a Heyting algebra representation). At a next stage, the procedure of daseinisation is extended to self-adjoint operators by considering their spectral families and approximating with respect to the spectral order. This method comes in two versions depending on the procedure of approximating self-adjoint operators from above or from below in the spectral order. If for each context, the best approximations to a self-adjoint operator from above and below become evaluated at a state, they define an interval of real numbers, which is interpreted as the unsharp value of this operator at the selected context in that state. By essentially building on this insight, (DI) construct a "quantity-value" object, which is a presheaf different from the real number object of the (DI) topos of presheaves. In this way, the daseinisation of a self-adjoint operator is described as a natural transformation from the "state-object" to the "quantity-value" object. In classical physics, for every state of a system a proposition acquires a definite truth value (true/false) or equivalently each state defines a homomorphism from the Boolean algebra of propositions to the two-valued Boolean algebra. In the (DI) case, a state is not represented by a global section of the spectral presheaf ("state object") due to the topos version of Kochen-Specker's theorem, but by a probability measure on the spectral presheaf. This forces (DI) to define the truth value of a proposition at a state as the sieve (downwards closed set) of contexts for which the probability of its daseinisation at each such context is 1.

Let us now attempt a brief comparison of our approach with the (DI) approach. Initially, it is useful to focus on the different conceptual aspects involved in the utilization of topos-theoretic ideas in the foundations of quantum physics. (DI) use the notion of topos as a semantical framework of intuitionistic propositional or predicate logic in its function to serve as a linguistic representation (that is the topos is "a topos of a certain formal language") attached to a system. Precise criteria of this attachment are not provided, which would justify the reasons of adherence to an intuitionistic framework. Rather, the scheme is built on the strong analogy provided by the notion of an "elementary topos" (that is the logical embodiment of the topos concept) as a generalized model of set theory being equipped with a subobject classifier (that is a distributive Heyting algebra classifying object, which forces the intuitionistic semantics) generalizing the classifying function of the Boolean two-valued object in the universe of sets. Behind this analogy there is the philosophical claim of "neo-realism". This is also conceived in a purely logical manner by (DI), on the basis of the claim that a new form of realism in physics is restored if both the propositional structure and the truth values structure of the "linguistic representation" of a physical system are distributive and "almost Boolean". In comparison, our approach uses the notion of topos in the sense of a generalized geometric environment, which makes it possible to constitute the structural information content of "complex objects" (like quantum event algebras) from the non-trivial localization properties of observables, which are used in order to probe (or technically cover) these "complex objects". More precisely, the proposed crucial notion of topos in physics is the one associated with the conceptual framework of Grothendieck topoi. Every Grothendieck topos can be represented as a category of sheaves for some Grothendieck topology on a base category of "contexts". Moreover, every Grothendieck topos is also an elementary topos, and thus equipped with an internal classifying object of truth values. Thus, in our perspective the "linguistic representation" is a consequence of the above mathematical fact and not the ultimate aim of formulating a physical theory in elementary topos logical terms in order to restore some form of traditional realism. (DI) avoid any reference to the notion of observables, mainly because of the possible instrumentalist connotations of this term, and use instead the term "physical quantity". Still observables denote physical quantities that, in principle, can be measured and the constitution of quantum observables from interconnected families of local Boolean observables (with respect to an appropriate Grothendieck topology) reveals the non-trivial (unsharp) localization properties in the quantum realm. Thus, it is precisely these non-trivial localization properties that necessitate the constitution of quantum objects via factorization through a Grothendieck topos (a "superstructure of measurement", viz. a "category of sheaves" in Grothendieck's words) over a base category of Boolean localizing measurement contexts. In the topos scheme of (DI), which follows an inverse conceptual direction by attempting to reduce "quantum objects" (for instance self-adjoint operators) to "objects or arrows in a topos" (for instance a topos-conceived physical quantity), instead of constituting or inducing "quantum objects" by factorization through "objects or arrows in a topos" reflecting Boolean localization properties, the localization problem is not avoided but appears in another guise in the elaborate construction of the "quantity-value" object. It is important to stress that our conception of the functional role of topos in quantum mechanics is still realist (although in a different sense in comparison to "neo-realism") since the consideration of Boolean localizing contexts forms a pre-condition of quantum physical experience, as we have explained previously.

The above brings into focus two other important issues in the attempted comparison between these two topos approaches to the foundations of quantum physics. The first refers to the role of "Boolean contexts" or "commutative contexts" as the objects of the base category and the other refers to the idea of translation between "quantum objects" and "topos objects". Let us start with the comparison referring to the issue of "contexts". The idea of a "context" describes an algebra of commuting physical quantities, or equivalently, a complete Boolean algebra of commuting projection operators (the idempotent elements of a "commutative context"). In the framework of (DI) the contexts are partially ordered by inclusion forming a poset which serves as the base category of presheaves. The contexts are called heuristically "local" since no topology is defined on the base category. Note that since the base category is a poset the consideration of the Alexandroff topology of upper or lower sets in the order does not make any difference at the topos level since every presheaf is a sheaf for this topology on a poset. In any case, since they consider a topos as "a topos of a certain formal language" attached to a quantum system, the consideration of the topos of presheaves over this partial order, being naturally equipped with a subobject classifier (the Heyting algebra of all sieves at each context), is adequate for their purpose to provide truth values of propositions (after the procedure of daseinisation) in an "almost Boolean" truth values object in this topos. Their intention is to use all these partially ordered "local" contexts simultaneously in order to capture the information of "quantum objects" (not homomorphically) in terms of truth valuations in the subobject classifier. A natural question arising in this setting is if the orthomodular lattice of all projections in a global non-commutative von Neumann algebra is determined by the partially ordered set of its Boolean subalgebras of projections, that is, by the poset of its "Boolean contexts". This is not the case since at least the inclusions of the "Boolean contexts" together with the order relation should be taken into account. Still, it seems that this does not appear as a problem in the topos approach of (DI), because they are only interested in a non-homomorphic translation of projections into their daseinised approximations with respect to the partially ordered "contexts", followed by another non-homomorphic mapping (of Heyting algebras) into the subobject classifier. In comparison, in our approach the specification of the base category of "Boolean contexts" plays a major role and is different from a poset. Initially, we define as a base category the category of complete Boolean algebras with morphisms all the corresponding homomorphisms (the technicality of considering σ-Boolean algebras is forced upon the requirement of having a well defined theory of observables according to standard measure-theoretic arguments). The choice of the category of complete Boolean algebras as a base category is justified by the fact that given any set of pairwise commuting self-adjoint operators, there exists a complete Boolean algebra which contains all the projection operators generating the spectral decomposition of these operators. Thus, complete Boolean algebras play the functional role of logical frames relative to which we are able to coordinatize the measurements of the observables corresponding to these self-adjoint operators. The semantic connotation of "Boolean contexts" as "Boolean logical frames" for covering the global non-Boolean lattice of projections poses the necessity to make precise the meaning of what is "local" in the base category. For this purpose, we define an appropriate Grothendieck topology on the (opposite) category of complete Boolean algebras (the sub-canonical topology of epimorphic families of Boolean covers), which boils down to the notion of Boolean localization functors forming a partially ordered set by inclusion. The notion of Boolean covers as probing frames of a quantum event algebra requires further explanation for the aims of the comparison. For this reason we point out that the spectral presheaf, the so called "state-object" of (DI) is different from our corresponding spectral presheaf, which is called functor of Boolean frames of a quantum event algebra. The (DI) spectral presheaf, at each "Boolean context" gives the set of Boolean homomorphisms from that context to the two-valued context (the Stone spectrum of the "Boolean context"). In our case, the functor of Boolean frames, at each "Boolean context" gives the set of quantum homomorphisms from the "modeled Boolean context" (that is the quantum event algebra image of the "Boolean context" under the action of the modeling functor) to a fixed quantum event algebra. These "modeled Boolean contexts" are the generators of covering families of a quantum event algebra, that is families of "Boolean covers" or "Boolean logical frames" of a quantum event algebra localizing it. Thus, it is convenient to think of these "Boolean covers" in terms of covering Boolean coordinate patches of a global quantum event algebra, so that there might be many with the same image. Notice also that, in contradistinction to the case where they are related only by inclusion, there may be many homomorphisms between each pair of them. Finally, instead of pairwise intersections we have to look at their fibered products (which define the pullback compatibility conditions for Boolean covers in some Boolean localization functor of a quantum event algebra). The upshot of this difference boils down to the following consequences: First, the homomorphism from a "modeled Boolean context" to some fixed quantum event algebra always factors in a homomorphic way through the inductive limit (colimit) in the category of elements of the functor of Boolean frames of the quantum event algebra. Second, the functor of Boolean frames becomes a sheaf with respect to compatible Boolean covering families in the defined topology (Boolean localization functors). Third, by restriction to such Boolean localization functors, a quantum event algebra can be represented isomorphically by the inductive limit in the category of elements of its functor (sheaf) of Boolean frames. Fourth, the whole structural information of a global quantum event algebra is constituted sheaf-theoretically (up to isomorphism) and inversely preserved by this inductive limit construction (restricted to Boolean covers in the topology). Fifth, the same idea can be implemented in an analogous way for the categories of quantum observables and quantum probabilities by passage to the corresponding slice categories of the base category of quantum event algebras. Hence, there is no need to introduce separately notions of "quantity-value" objects and "quasi-states". Sixth, the Grothendieck topos of sheaves on the defined site is the geometric localization environment via which it becomes possible to constitute "quantum objects" contextually (from the local to the global level) by probing them through interconnected families of Boolean frames. Seventh, by reflection of the localization topos the category of quantum event algebras itself becomes equipped with a classifying object, which can be used for truth valuations of quantum propositions in analogy to the classical case. Eighth, the exact analogue of the spectral logical object in the localization topos assigns to each "modeled Boolean context" the set of quantum homomorphisms from this context to the quantum classifying algebra (instead of the Stone spectrum).

The final issue of our comparison refers to the idea of translation between "quantum objects" and "topos objects". In the (DI) framework the translation is implemented from the "quantum side" to the "topos side" through the procedure of daseinisation of projectors (and self-adjoint operators). This is a procedure of order-theoretic approximation of each projector in the global non-Boolean lattice (representing a proposition about the value of a physical quantity) by some projector in each "classical context" of the base poset, such that all the "classical contexts" are taken into account simultaneously. The order-theoretic approximation procedure may be conducted either from above (outer daseinisation) or from below (inner daseinisation) with distinct physical interpretations. For example, in the outer case each approximating projector (with respect to a "classical context") denotes the strongest consequence in that context of the original projector. In a nutshell, (outer) daseinisation produces an order embedding of the global non-distributive lattice of projections into a distributive lattice (complete Heyting algebra of clopen subobjects of the spectral presheaf), which does not preserve the conjunction and the negation operations of the quantum lattice as well as the law of excluded middle. Conceptually, daseinisation in its functional role as a translation from "quantum objects" to "topos objects" is interpreted as a means to "bring-a-quantum-property-into-existence" (inspired from Heidegger's Dasein) by "hurling it into the collection of all possible classical snapshots of the world" in the words of (DI). In comparison, we think of the process of translation between "quantum objects" and "topos objects" in a different way. The key idea is the existence of a categorical adjunction (pair of adjoint functors) between the topos of presheaves (over the base category of complete Boolean algebras) and the category of quantum event algebras. The adjunction provides a bidirectional functorial correlation between this topos and the category of quantum event algebras, where the right adjoint is the functor of Boolean frames (of a quantum event algebra) and the left adjoint is the inductive limit of an object in the topos (taken in the category of its elements). Thus, in comparison to daseinisation, which translates (not homomorphically) a "quantum object" to a "topos object", the adjunction is a bidirectional and functorial translation mechanism of encoding and decoding information from "topos objects" to "quantum objects" and inversely, by preserving the structural form of the correlated categories. The crucial part of the adjunction is the construction of the left adjoint, by means of which we obtain a homomorphism from the inductive limit of a "topos object" to a "quantum object". In particular, the counit of the adjunction, evaluated at a quantum event algebra, is a quantum homomorphism from the inductive limit in the category of elements of the functor of Boolean frames to a quantum event algebra, which can be made into a quantum isomorphism by restriction to a Boolean localization functor. In this way, the global structural information of a quantum event algebra can be approximated homomorphically or (in the latter case) completely constituted (up to isomorphism) by means of gluing together the observable information collected in all compatible Boolean frames in the form of appropriate equivalence classes (by the inductive limit construction).

Moreover, the "Boolean frames-quantum adjunction" provides the key conceptual and technical device to show that the category of quantum event algebras is equipped with a classifying object, which should be used for the valuation of quantum propositions by analogy to the classical case, where the two-valued Boolean algebra plays this role. For this purpose we use the unit of the adjunction evaluated at the subobject functor (a "topos object") and show that it becomes representable in the category of quantum event algebras by a classifying object in this category (a "quantum object"), which is again constructed by an inductive limit operation (in the category of elements of the subobject functor). Intuitively, this quantum classifying object contains the information of equivalence classes of truth valuations with respect to all compatible Boolean frames belonging to a Boolean localization functor of a quantum event algebra. In comparison, the truth value object of (DI) is the subobject classifier in their topos of presheaves over the poset of "classical contexts" (a "topos object"). In their case there does not exist a homomorphism (of Heyting algebras) from the (clopen) subobjects of their spectral presheaf to the subobject classifier of this topos, which would provide the analogy with the classical case. This is so because a state is not represented by a global element of their spectral presheaf due to the Kochen-Specker theorem, but by a probability measure on the spectral presheaf. Thus, the truth value of a proposition (at a state) is identified with the downwards closed set of "classical contexts" for which the probability of its daseinisation at each such context is 1. Nevertheless, from an inverse viewpoint, the truth of a "daseinized proposition" in a "classical context" does not convey any information about the truth of the original quantum proposition. In comparison, in our approach the truth of a proposition in a Boolean frame makes it equivalent to all other propositions in all Boolean frames being compatible with it with respect to a Boolean localization functor of a quantum event algebra by the explicit truth-value criterion.

Conclusively, in our approach the *"Boolean frames-quantum adjunction"* is a theoretical platform for probing the quantum domain of discourse via a *localization* *topos* by: [I] *Decoding* the global information contained in quantum event structures *inductively* via *equivalence classes* of partially compatible processes of localization in *Boolean logical frames* realized as physical contexts for measurement of observables, and [II] *classifying* quantum information in terms of *contextual* truth valuations with respect to these Boolean logical frames. We claim that the functioning of this *bidirectional translation* platform is fundamental philosophically for a novel *realist* understanding of the part-whole relation and the corresponding contextualist account of truth suited to the quantum domain.

We continue our comparison by commenting briefly on a similar topos-theoretic approach to that of Döring and Isham (DI), which has been developed by Heunen, Landsman and Spitters (HLS) (e.g., 2009 and references therein). The similarity is based on the following facts: [I] They also use the notion of topos as a semantical framework of intuitionistic predicate logic in its function to serve as a linguistic representation (that is the topos is "a topos of a certain formal language") attached to a quantum system. [II] The choice of the base category of their topos scheme is closely related to the one by (DI), meaning that it is also a partially ordered set of "classical contexts", the essential difference being that they are not commutative von Neumann algebras but more general star algebras over the complex numbers. Regarding these structural similarities, our comparison comments referring to the (DI) scheme pertain to this scheme as well. Repeating concisely, the difference pertains to the following: [i] The distinct notions of an elementary topos in comparison to a Grothendieck localization topos (realized as a category of sheaves for an appropriate Grothendieck topology) as a foundation to probe the content of a physical theory, and [ii] the choice of the partial order relation among "classical contexts" as an adequate base category to capture the complexity of quantum logic, in contradistinction to the category of complete Boolean algebras and homomorphisms together with their function as Boolean logical frames in quantum logic.

Notwithstanding the above similarities there are considerable differences between the topos approaches of these two groups. They can be very concisely summarized as follows: [i] The (HLS) topos approach uses a covariant functorial perspective, which is based on the topos of co-presheaves on the partial order of "classical contexts". [ii] The conceptual and philosophical underpinning of the topos scheme serves different purposes and is interpreted in distinctively different ways: in the (DI) case it is interpreted as a framework of "neo-realism" in the sense of resembling classical physics in an "almost Boolean" way, whereas in the (HLS) case it is interpreted as a framework making precise Bohr's "doctrine of classical concepts" invoking explicitly the notions of experiments, measurement and observables. This is also reflected in the terminology (for example, (DI) speak of physical quantities, whereas (HLS) speak of observables), and it is somehow strange to us that (HLS) also use the term "daseinisation" of (DI) in order to describe the approximation procedure, although the meaning of this term is at odds with Bohr's doctrine. [iii] The essential point of the (HLS) topos approach is that there exists an internal commutative star algebra (or an internal Boolean algebra) within the topos of co-presheaves over the poset of "classical contexts", so their topos comes equipped together with an "internal commutative algebraic object", which is not the case in the (DI) approach.

For our comparison purposes, we focus on the aspect [iii] above, marking the basic technical difference between the (HLS) and (DI) approaches in relation to ours. An initial remark is that the "internal commutative algebraic object" is introduced in the topos by means of a tautological covariant functor, which assigns to each object in the poset of "classical contexts" itself, seen as a set. So, it is this tautological covariant functor which serves as an "internal commutative algebraic object" in the topos of (HLS). Then, the use of the constructive version of the Gelfand duality theorem of Banaschewski and Mulvey (2006), generalizing Gelfand duality internally in topoi, allows (HLS) to define the internal Gelfand spectrum of this "internal commutative algebraic object" in their topos, which is a frame (and thus a Heyting algebra in the topos) to act as the topos intuitionistic logical surrogate of quantum logic. The process of passing from a non-commutative star algebra to an internal commutative star algebra via a tautological functor in the topos of covariant functors over the poset of "classical contexts" is called "Bohrification" by (HLS). Now, the internal observables are given by the self-adjoint elements in the "internal commutative algebraic object" and the internal states by the linear functionals to the constant functor of complex numbers in the topos. Moreover, there exists an internal complete Boolean algebra in the topos formed by the idempotent internal observables. In a nutshell, (HLS) using these "internal objects" define embeddings of the standard "quantum objects" into "topos objects", set up an analogous approximation procedure (inner daseinisation) for projections (and self-adjoint elements), and manage to embed the standard quantum logic into a "topos object" analogous to the "clopen subfunctors of the spectral presheaf" of the (DI) approach, which is not an "internal Boolean algebra" "topos object". This is, similarly to the (DI) case, an order embedding to a Heyting algebra object in a topos, which does not preserve disjunctions and the negation operator. In comparison, in our approach we have not considered the existence of any analogous "internal Boolean algebra object" in our topos (which is different from the topos of (HLS) both in terms of the base category and the fact that we use a topos of (pre)sheaves and not a topos of co-presheaves). It is not clear if the existence of such an "internal commutative object" has been somehow forced by the employment of a tautological functor (together with the choice of the base category as a poset) or is a more general phenomenon. At least (HLS) do not provide any other instance, except the tautological case, and do not make any further remark concerning this issue. In the physical state of affairs, apart from the functionality of the "internal commutative object" in order to define observables and states internally in their topos - thus bypassing the issues with the "quantity value object" and "quasi-states" in the (DI) approach - they do not use an appropriate "internal Boolean algebra" "topos object" for valuations of quantum propositions, and therefore such an internal object is not relevant for logical classification internally in their topos. In our case, focusing on the viewpoint of a topos as a Grothendieck localization topos of sheaves, we may further make use of the notion of "Boolean localization" implied by a result of Barr and Diaconescu (Mac Lane and Moerdijk 1992), according to which for any Grothendieck topos of sheaves there exists a Boolean cover, that is a geometric morphism from the topos of sheaves over a complete Boolean algebra to this topos. This theorem, applied in our case, provides also an adjunction between the topos of sheaves over a complete Boolean algebra and the category of quantum event algebras. This notion of "Boolean localization" as pertaining to our approach will be explored in detail elsewhere.

An interesting further development in the "Bohrification program" of (HLS) is the work of van der Berg and Heunen (BH) (2012), who make the claim that this program is most naturally developed in the context of partial algebras, a concept introduced in quantum mechanical considerations by Kochen and Specker. They show that every partial Boolean algebra is the inductive limit of its total subalgebras, viz. the commeasurable Boolean subalgebras. Note that in the proof of this result they use a partial Boolean algebra together with a prescribed poset of total subalgebras as well as the inclusions of the total subalgebras into the partial algebra. This is, in fact, another form of a well-known theorem in the theory of orthomodular lattices, called "Kalmbach's Bundle Lemma" (1983), as (BH) also point out. As we have also stressed previously in our remarks to the (DI) approach, this result shows that the partial order relation of "classical contexts" is not adequate to capture the structural information of quantum logic, and at least, the inclusion functions of the "Boolean contexts" to the quantum lattice should be also taken into account. In comparison, our approach to the specification of a quantum event algebra via the left adjoint functor of the "Boolean frames-quantum adjunction" is more general. In our case, the Boolean algebras of the base category do not form a poset and actually they are not even required to be subalgebras of a quantum event algebra. Moreover, the inductive limit is taken in the category of elements of the functor of Boolean frames of a quantum event algebra. It is instructive to remark that the partial order relation of a quantum event algebra in this way is induced by lifting morphisms from the base category of Boolean algebras to the fibers of the category of elements subject to the pullback compatibility conditions.

Finally, we would like to comment briefly on a currently emerging research program by Abramsky and Brandenburger (AB) (2011), who have proposed the modeling of contextuality and non-locality using the framework of sheaf theory. Their setting is quite general by using weaker assumptions than standard quantum theory, and their aim is to explicate the introduced sheaf-theoretic notions by applying them on empirical models in a clear and simple way. An interesting aspect of this approach is that the phenomena of contextuality and non-locality are detached from their quantum-theoretic origins since they become applicable in a much wider spectrum through their association with sheaf-theoretic notions. In particular, the central claim of (AB) is that the phenomena of contextuality and non-locality should be modeled in sheaf-theoretic terms as giving rise to obstructions to the existence of global sections. More precisely, they show that the existence of a global section, gluing together uniquely a compatible family of elements in a presheaf pertaining to the empirical modeling of a system, is equivalent to the realization of this system by a factorizable hidden-variable model. Their empirical model of a system involves a measurement space (a finite and discrete set), a finite covering of the measurement space (called a measurement covering consisting of a family of subsets, corresponding to measurement contexts, where a measurement context is a set of measurements that can be performed jointly), a finite set of outcomes, a presheaf of events assigning to each measurement context its set of outcomes (being trivially a sheaf over a discrete space), and a presheaf of distributions assigning to each measurement context its set of distributions on the sections defined over this context (such that the operation of restriction in the presheaf corresponds to taking the marginal of a distribution). Then, for a measurement covering, a compatible family of elements of the presheaf of distributions (thus a sheaf of distributions with respect to this measurement covering) defines a no-signalling empirical model corresponding to this measurement covering. Of particular interest for our purposes is the quantum representation of these empirical models. In this case, a measurement covering consists of measurement contexts, which are identified as sets of maximal commuting subsets of the set of all observables on a fixed Hilbert space (i.e., the set of all observables on a fixed Hilbert space define the measurement space of a quantum empirical model according to (AB)). In comparison to our sheaf-theoretic model, we notice the following: Instead of the set of all observables on a fixed Hilbert space, we take into account the global quantum event and observable structure explicitly, thus our measurement space at the level of events is a quantum event algebra (a quantum logic) and at the level of observables is a partial commutative algebra (a quantum observable algebra). The measurement covering of (AB) by sets of "maximal commutative contexts" corresponds to a Boolean covering consisting of maximal complete Boolean algebras of projections, where each one of them generates the spectral resolution of each "maximal commutative context". Now, in this setting of a quantum empirical model, (AB) define a quantum representation by a state (density operator) on the fixed Hilbert space. Then, for each "maximal commutative context" in the measurement covering, the state defines a probability distribution on the set of commuting observables belonging to this context, by the standard "trace rule", and thus defines a presheaf of probability distributions on the measurement space with respect to the measurement covering. This is analogous in our case to the presheaf functor of Boolean measure theoretic (probabilistic) frames of a quantum state with respect to a Boolean covering of a quantum event algebra, which we have shown that it is a sheaf (Zafiris 2006b). The pertinent question in the setting of (AB) is if their presheaf of probability distributions is a sheaf with respect to the considered measurement covering. (AB) show that this is actually the case, namely, families of distributions are compatible on overlaps of measurement contexts in the covering, and thus can be glued together. The important conceptual insight of (AB) is that this result implies a "generalized no-signalling theorem" in quantum mechanics, which incorporates the standard no-signalling theorem of Bell-type scenarios corresponding to special cases of measurement coverings.

A.2 The Left Adjoint Colimit Construction

The left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ of the Boolean realization functor of $\mathcal{L}$ is defined for each presheaf $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit (inductive limit)

$$\mathbf{L}(\mathbf{P}) = Colim\{\int(\mathbf{P}, \mathcal{B}) \xrightarrow{\int_{\mathbf{P}}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\}.$$

We can provide an explicit form of the left adjoint functor by expressing the above colimit as a *coequalizer of a coproduct* using standard category-theoretic arguments. For this purpose, if we consider the category of elements of the presheaf of Boolean algebras $\mathbf{P}$, that is $\int(\mathbf{P}, \mathcal{B})$, as an index category $\mathcal{I}$, then the colimit of the functor $\mathbf{M} \circ \int_{\mathbf{P}} : \mathcal{I} \to \mathcal{L}$ is expressed as follows:

$$\coprod_{v:\acute{B}\to B} \mathbf{M}(\acute{B}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(B,p)} \mathbf{M}(B) \xrightarrow{\chi} \mathbf{L_M}(\mathbf{P})$$

where χ is the coequalizer of the arrows ζ and η. In the diagram above the second coproduct is over all the objects (B, p) with $p \in \mathbf{P}(B)$ of the category of elements, while the first coproduct is over all the maps $v : (\acute{B}, \acute{p}) \to (B, p)$ of that category, so that $v : \acute{B} \to B$ and the condition $p \cdot v = \acute{p}$ is satisfied.

In order to analyze in more detail the colimit in the category of elements of $\mathbf{P}$ induced by the functor of local Boolean frames $\mathbf{M}$, and because of the fact that $\mathcal{L}$ is a concrete category, we may consider the forgetful functor from $\mathcal{L}$ to **Sets**. Then, the coproduct $\coprod_{(B,p)} \mathbf{M}(B)$ is a coproduct of sets, which is equivalent to the product $\mathbf{P}(B) \times \mathbf{M}(B)$ for $B \in \mathcal{B}$. The coequalizer is thus equivalent to the definition of the tensor product $\mathbf{P} \otimes_{\mathcal{B}} \mathbf{M}$ of the set valued functors $\mathbf{P} : \mathcal{B}^{op} \to \mathbf{Sets}$ and $\mathbf{M} : \mathcal{B} \to \mathbf{Sets}$. We call this construction the *functorial tensor product decomposition of the colimit in the category of elements of* $\mathbf{P}$ *induced by the functor of local Boolean frames* $\mathbf{M}$:

$$\coprod_{B,\acute{B}} \mathbf{P}(B) \times Hom(\acute{B}, B) \times \mathbf{M}(\acute{B}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{B} \mathbf{P}(B) \times \mathbf{M}(B) \xrightarrow{\chi} \mathbf{P} \otimes_{\mathcal{B}} \mathbf{M}$$

According to the above diagram, for elements $p \in \mathbf{P}(B)$, $v : \acute{B} \to B$ and $\acute{q} \in \mathbf{M}(\acute{B})$ the following equations hold:

$$\zeta(p, v, \acute{q}) = (p \cdot v, \acute{q}), \qquad \eta(p, v, \acute{q}) = (p, v(\acute{q}))$$

symmetric in $\mathbf{P}$ and $\mathbf{M}$. Hence the elements of the set $\mathbf{P} \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(p, q)$. This element can be written as:

$$\chi(p, q) = p \otimes q, \quad p \in \mathbf{P}(B), q \in \mathbf{M}(B).$$

Thus, if we take into account the definitions of ζ and η above, we obtain:

$$p \cdot v \otimes \acute{q} = p \otimes v(\acute{q}), \quad p \in \mathbf{P}(B), \acute{q} \in \mathbf{M}(\acute{B}), v : \acute{B} \to B.$$

We conclude that the set $\mathbf{P} \otimes_{\mathcal{B}} \mathbf{M}$ is actually the quotient of the set $\coprod_{B} \mathbf{P}(B) \times \mathbf{M}(B)$ by the smallest equivalence relation generated by the above equations. The equivalence classes of this relation can be further endowed with the structure of a quantum event algebra, thus completing the construction of the left adjoint colimit in $\mathcal{L}$ via the category of **Sets**.

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