A few points on gunky space

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1 Introduction

In set-theoretic models of space, one begins with a dehydrated dust of points. This dust is reconstituted using measure theory, and the result is glued together using topology. This is all quite standard, but there's something dissatisfying about it. It requires taking a number of philosophical stands about the nature of space, choices which presumably could have gone another way. One major assumption is that there are, in fact, points—unextended and indivisible regions of space. One might think, like Brentano and Whitehead (Zimmerman, 1996), that space has no atomic parts. At the very least this seems to be a *possibility*. And this possibility has a long philosophical pedigree, appearing in Zeno's paradoxes, Kant's antinomies, and contemporary metaphysics (Sider, 1993).

Call a spatial region that is not comprised of points 'gunky' (after Lewis, 1991). The past few years have seen a number of mathematical models of gunky space (Russell, 2008), but I take Arntzenius's (2012) "measure-theoretic gunk" to be the most promising among these. This model is of particular interest because Arntzenius is motivated in part by considerations from quantum mechanics (QM), giving philosophers of physics a reason to care about mereology. After briefly reviewing the measure-theoretic gunk model (§2.1), I tie up a few loose ends (§2.2). Arntzenius gives an axiomatic characterization of a topology on such a model, but does not have a representation theorem on offer. I provide a characterization of these structures in terms of locales, a generalization of topological spaces, and discuss how these relate to pointy topological spaces.

After establishing the gunky model, I turn to physics on this space. In particular, I consider whether it can support QM, one of the motivating cases for Arntzenius's model. I argue that it cannot. Though parts of the QM formalism suggest pointlessness, this apparent suggestion is the result of not taking the whole formalism into account. I use an example from Halvorson (2004) to illustrate these claims (§3.1) and to show that QM does not motivate a theory of gunky space after all (§3.2). A final verdict on gunk ultimately relies, however, on an account of unitarily inequivalent representations of the algebra of observables. This is perhaps not too surprising, because addressing the small-scale structure of space brings us into the realm of QM_∞ (Ruetsche, 2011), which is already beleaguered by these problems. Nevertheless, our investigation will show that in fact there is nothing essentially quantum about Arntzenius's motivations for gunk, and similar considerations apply in classical theories. I close with some questions that arise as a result.

2 Measure-theoretic gunk

Since I will only be concerned with the measure-theoretic approach to gunk, I hereby drop the qualifier. Arntzenius describes his model as the real number line seen through "blurry glasses". One begins with the real number line, and then identifies two sets if they differ only by a set of Lebesgue measure zero. The purpose of this section is to reconstruct Arntzenius's model in a way that makes clear the role of his assumptions. In brief, the argument is as follows: assuming that the regions of space behave according to standard mereology, and that every region has a determinate, non-zero size, there are very few models available. They can each be labeled by two cardinals, $(\mathfrak{m}, \mathfrak{n})$, where $\mathfrak{n}$ is either 0 or infinite. $\mathfrak{m}$ represents the number of spatial points, so for gunk we have $\mathfrak{m} = 0$. $\mathfrak{n}$ is fixed by the further assumption that the space must be separable, giving $\mathfrak{n} = \aleph_0$. Thus there is one model of gunky space.

2.1 The gunky measurable locale

A suitably general framework in which to model regions should not presuppose that a space is comprised of points, as a topology does. Looking ahead, it should also provide the tools to express Arntzenius's gunk. Locales provide such a framework.[1] Mathematically, a *locale* is a lattice with all finite meets and all joins, and in which these are compatible. One can interpret this in physical terms as follows. A locale X is a collection of spatial regions, denoted $\mathcal{O}(X)$, and one can talk about region B being part of region A, written $B \subseteq A$. For any two regions A and B, there is some region $A \wedge B$, the meet of A and B, which is the largest region part of both A and B. For any family of regions $\mathcal{F}$, there is a region $\bigvee \mathcal{F}$, the join of $\mathcal{F}$, which is the fusion of every region in $\mathcal{F}$. Finally, we have the identity

$$B \wedge \bigvee_i A_i = \bigvee_i (B \wedge A_i)$$

Physically, this means that the largest part shared by B and the fusion of the As is the fusion of the largest parts B shares with each A.

A locale is a generalization of a topological space. Any topological space X naturally gives rise to a locale in which the regions are the open sets of X, $\wedge$ is set intersection, and $\bigvee$ is set union. A locale that can be obtained by this process is called a *spatial* locale. This relationship, combined with the fact that the definition of a locale makes no reference to points, means that locale theory is sometimes referred to as "pointless topology." However, this nomenclature can be misleading, because some locales do have points. Spelling this out requires some attention to the way in which locales can be related to one another. If A and B are locales, then a continuous function $f\colon A \to B$ is a function $f^*\colon \mathcal{O}(B) \to \mathcal{O}(A)$ such that $f^*(A \wedge B) = f^*(A) \wedge f^*(B)$ and $f^*\left(\bigvee_i A_i\right) = \bigvee_i f^*(A_i)$. In the topological case, where $\wedge$ and $\bigvee$ have their set-theoretic meanings, $f^*(A)$ is the preimage of A, and continuity has its usual topological meaning: f is continuous just in case the preimage of any open is open.

The notion of a point of a locale can be defined in a number of equivalent ways. The simplest uses the abstract point 1, which is the locale obtained from the one-element set considered as a topological space. A point of a locale X is then some region that looks like the abstract point; i.e., it is a continuous map $f : 1 \to X$. An equivalent definition, solely in terms of $\mathcal{O}(X)$, uses a filter $\mathcal{F}$ on $\mathcal{O}(X)$. A filter is a collection of elements of $\mathcal{O}(X)$ that contains X and such that if $A \in \mathcal{F}$, then $A \vee B \in \mathcal{F}$ when $B \in \mathcal{O}(X)$ and $A \wedge B \in \mathcal{F}$ when $B \in \mathcal{F}$. Call a filter completely prime if it does not contain the empty region and, whenever $\bigvee_i A_i \in \mathcal{F}$, one of the A_i is in $\mathcal{F}$. A point $f : 1 \to X$ is equivalent to a completely prime filter on X. While $f : 1 \to X$ picks out a particular region of X that looks like a point, a completely prime filter on $\mathcal{O}(X)$ is a nested collection of regions which "zeroes in" on a point.

This notion of point is the one with which Arntzenius is concerned. The idea underlying his discussion is that the points of space which appear in our mathematical representation of the wavefunction are artifacts of this representation. In other words, they won't appear in every representation; or at least, facts about them are not guaranteed to be consistent across representations. Those points which *do* appear in every model, and which do real representational work in Arntzenius's sense, are those which are picked out by maps $1 \to X$. Such points are really part of space, and are therefore (indivisible) regions in their own right. So any point-sized region must be a member of $\mathcal{O}(X)$, and must stand in the above relations to the other regions.

Because the pointy structure of a locale X is determined by $\mathcal{O}(X)$ via completely prime filters, there is a well-defined notion of the collection of points of X, denoted $\mathrm{pt}(X)$. One can then construct a topology on $\mathrm{pt}(X)$ using $\mathcal{O}(X)$ as the frame of opens. This topological space is not necessarily a faithful representation of the locale, because it might be the case that $A, B \in \mathcal{O}(X)$ have the same points, even though $A \neq B$. So if we were to try reconstructing the locale by forgetting about the underlying points, we would not generally end

[1] For a textbook introductions to locales, see Mac Lane and Moerdijk (1992, Ch. IX) or Vickers (2007).

up back where we started. Say that a locale X has enough points if for any two regions $A, B \in \mathcal{O}(X)$, there is a point that belongs to one but not the other. A locale has enough points if and only if it is spatial. Since a gunky locale won't have *any* points, it can't be spatial, hence can't be faithfully equipped with a topology. In §3, I argue that a topological description of space is necessary in the version of QM under consideration, hence gunk is inadmissible.

Arntzenius is concerned with a particular class of locales; namely, those which can support the standard mereology of gunky regions (2012, p. 138). These locales must be complete Boolean algebras; that is, any family $\mathcal{F}$ must have an infinite meet $\bigwedge \mathcal{F}$, in addition to all the other structure. Furthermore, it must be possible to equip the algebra of regions with a measurable structure, to verify that no region has zero extension. So we need a way to talk about measurable structure on locales. This is obtained by generalizing measurable spaces.

A *measurable space* is a triple $\langle X, \mathcal{M}, \mathcal{N} \rangle$, where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and $\mathcal{N}$ is a σ-ideal of $\mathcal{M}$. Given a set X equipped with a measure, we can obtain its underlying measurable space by forgetting the measures of all but the null sets. Then $\mathcal{M}$ is the collection of regions which have a size, and $\mathcal{N}$ is the collection of regions that have size zero. A measurable space is called *localizable* if $\mathcal{M}/\mathcal{N}$ is a complete Boolean algebra. Every "reasonable" measurable space is localizable, in the sense that localizability is a necessary and sufficient condition for the applicability of every major theorem of measure theory. In particular, both the Riesz representation theorem, used throughout QM, and Maharam's theorem, used below, apply to all and only localizable measurable spaces (Segal, 1951).[2] Call the collection of all localizable measurable spaces LocMeas. Each of these gives a complete Boolean algebra $\mathcal{M}/\mathcal{N}$, which represents a collection of regions, all of which have a determinate size, so they form a locale. Let MeasLoc denote the collection of locales which are isomorphic to $\mathcal{M}/\mathcal{N}$ for some localizable measurable space $\langle X, \mathcal{M}, \mathcal{N} \rangle$.[3]

MeasLoc is the collection of all the locales which can be given a measurable structure. In other words, it is the menu of the possible mereological algebras of sizable regions. So these are the options from which Arntzenius's gunky models will have to draw. As it turns out, there are very few: by Maharam's (1942) theorem, any measurable locale can be decomposed into the disjoint union of $\mathfrak{m}$ points and $\mathfrak{n}$ copies of the real line with Lebesgue measure. So the possibilities are classified by a pair of cardinals $(\mathfrak{m}, \mathfrak{n})$, with $\mathfrak{n}$ zero or infinite. But if space is gunky, then it has no points, hence $\mathfrak{m} = 0$. So all that is left is to determine $\mathfrak{n}$.

If $\mathfrak{n} = 0$ as well, then we are left with the empty locale, which has no regions at all. The disjoint union of a non-zero finite number of copies of $\mathbb{R}$ is isomorphic to the disjoint union of countably many copies, so we may take $\mathfrak{n}$ to be infinite. A bound on $\mathfrak{n}$ arises from Arntzenius's demand that the measurable space be separable, a requirement which deserves some attention. A topological space T is separable if it has a countable subset $\{x_i\}$ that is dense in T. The fact that this subset is dense means that every point of T can be approximated by the x_i, so we can do a great deal of our reasoning about the space in terms of only countably many of its points. This is a familiar feature of QM in separable Hilbert spaces, where states can be decomposed into superpositions of countably many eigenvectors, each with an associated eigenvalue. In heuristic terms, a separable topological space is one whose features are determined by countably many of its pieces.

If we want to reason in terms of countable superpositions, then we must require that our states form a separable space, which we may do in a number of equivalent ways. If $\langle X, \mathcal{M}, \mathcal{N} \rangle$ is the measurable space underlying our state space, then the states are given by elements of $L^2(X)$; so we can require that $L^2(X)$ be separable. This requirement is equivalent to the requirement that $L^p(X)$ be separable for $1 \leq p < \infty$. This is not too surprising; the L^p spaces are determined by the measurable space, so if one of them is separable—that is, if one of them is determined by what happens in countably many places—then this indicates that the measurable space itself is determined by what happens in countably many places. And indeed, one can define separability for measurable spaces directly by constructing a natural topology on the algebra of regions, and this notion of separability coincides with separability of the L^p spaces on X. A much more direct definition, however, is in terms of the Maharam classification. As should be expected, a measurable

[2] The relevant form of the Riesz representation theorem says that for any measure μ on $\langle X, \mathcal{M}, \mathcal{N} \rangle$ and any continuous linear functional ϕ on $L^1(M)$, there is a bounded measurable function k on X such that $\phi(f) = \int_X kf \, d\mu$ for all $f \in L^1(M)$. This holds if and only if $\langle X, \mathcal{M}, \mathcal{N} \rangle$ is localizable.

[3] I am not aware of any published material on measurable locales. Unpublished material is available at the *n*Lab (2012).

space is separable just in case $\mathfrak{m}$ and $\mathfrak{n}$ are each countable.

These two requirements uniquely specify the measurable locale tagged $(0, \aleph_0)$ as the algebra of gunky regions. For ease of reference, call this measurable locale $\mathfrak{G}$. By construction, it does not have any points. Moreover, it is isomorphic to the Lebesgue measurable locale; i.e., the measurable locale underlying all second-countable smooth manifolds with nonzero dimension. I will return to this point at the end; for now, I will wrap up a couple of the loose ends of Arntzenius's (2012) discussion. Then I will address the argument that we ought to take $\mathfrak{G}$ to represent the structure of space in QM. As far as Arntzenius's mathematical program is concerned, the prognosis is good. Since $\mathfrak{G}$ isn't spatial, there is no good way to represent it as a set equipped with extra structure.[4] Rather, it must be represented as an equivalence class of objects in LocMeas. That is, one represents it as the Lebesgue measure space while taking care to avoid accepting anything that isn't true in every equivalent representation. So as long as we are careful to ignore the artifacts of our representation (i.e., the measure-zero sets), there's no harm in using the standard Lebesgue measure space as a stand-in for $\mathfrak{G}$.

2.2 Loose ends

The language of measurable locales makes the answers to some of Arntzenius's open questions easily accessible. There is first the question of putting a topology on the gunky space. Arntzenius proposes that this be a "connectedness" and "limitedness" structure on $\mathfrak{G}$ satisfying certain axioms, which are motivated by the topological case. Connectedness is a binary relation on regions that says whether the two regions share a part. Limitedness, meanwhile, is a generalization of compactness from the topological situation. These relations are inspired by certain topological relations, and these generalize directly to locales. Two regions A and B share a part if their overlap $A \wedge B$ is not the null region. As for compactness, the topological definition makes no reference to points and may be defined exactly as in the topological setting. Call a family U_i a cover of X if $\bigvee_i U_i = X$. A locale X is compact if every cover has a finite subfamily that also covers X. So any measurable locale, $\mathfrak{G}$ included, naturally has a connectedness and limitedness structure, given by $\wedge$ and compactness.

But recall that since $\mathfrak{G}$ doesn't have any points, it isn't spatial. So a gunky space can't have a nontrivial topology on it. We can, however, do something analogous to the pointy case. There, one starts with a metric and obtains a topology and a measurable space from that. Similarly, to any locale there is associated a measurable locale, obtained in the same manner as the topological case. So if one could construct a pointless locale that represents connectedness and limitedness information, one could also obtain an underlying measurable locale. These would stand in relation to one another in a way analogous to topological spaces and measurable spaces in the pointy case. The problem is that there's no obvious choice of locale to capture the topological information. $\mathfrak{G}$ won't do, because it lacks crucial information like dimension. Any smooth manifold naturally has the structure of $\mathfrak{G}$ as a measurable locale, but not every smooth manifold is the same space. So, as expected, the measurable structure does not capture facts about locality. If other words, the regions of a measurable space can be scrambled around without changing the measurable structure, as long as their sizes are preserved.

3 Quantum mechanics isn't pointless

In addition to the philosophical motivations from the opening, Arntzenius takes QM to motivate a gunky model of space. His argument goes like this: the Lebesgue integral $\int_A f$ of f over the region A is defined by adding up the measure of each subregion of A, weighted by the value of f there. So measure-zero regions make no difference to the Lebesgue integral. A particle in space is represented by a function ψ in $L^2(\mathbb{R})$, and the probability of finding it in a region A is given by $\int_A |\psi|^2$. If ψ and ψ' differ only on one point, then $\int_A |\psi|^2 = \int_A |\psi'|^2$ for all regions A, so there is no observable difference between ψ and ψ'. Applying

[4] What counts as a "good" representation can be made precise in a number of ways. For one, Loc isn't well-powered (Johnstone, 1982, p. 57), i.e., it has a proper class of subobjects. So there is no set of subregions.

a principle of parsimony, we should eliminate the freeloading points of space, which do no representative work.

In this section I argue that points do some representative work in QM. There is an easy argument for this conclusion: physics textbooks and journal articles are replete with δ distributions, which correspond to precise points in space. As Arntzenius notes, these cannot be captured in the standard separable Hilbert space formulation of QM, requiring instead something like a rigged Hilbert space treatment (de la Madrid, 2005). Despite the ubiquity of δ functions in practice, one might deny their theoretical necessity, so I will put them to the side. Still, I think, points of space can be found in QM. Though they can be eliminated from the Hilbert space in some cases, they remain in the algebra of observables.

3.1 Pointy quantum systems

The previous section was essentially a discussion of a particular kind of duality between algebra and geometry called Stone duality. This section addresses another duality, Gelfand duality. This is a duality between spaces and the algebras of functions defined on them. Measurable locales, like $\mathfrak{G}$, correspond to commutative von Neumann algebras. In particular, the collection of bounded continuous functions on a measurable locale form a von Neumann algebra, and any commutative von Neumann algebra can be written as the collection of bounded continuous functions on a measurable locale. However, QM requires a larger class of algebras, C^*-algebras. Since commutative C^*-algebras correspond to spatial locales, this forces QM to involve points.

The high level picture is this (Wolters, 2013). Butterfield and Isham (1998) have shown how to formulate the operator algebra of a QM system inside a particular mathematical universe (a topos). When formulated in this universe, the operator algebra becomes commutative, making it more tractable. The cost is that this universe does not obey classical logic. But this is not too heavy a burden, especially if we are already thinking of spaces as locales, through Stone duality. When working with this logic, Gelfand duality says that commutative C^*-algebras are dual to locales, and these locales are not necessarily spatial. So the state space of the system is represented inside the universe as a locale, possibly without points.

When making predictions using this system, one uses an external description of the universe. That is, in measurement contexts there is a way to represent the state space of the system as a pointy topological space outside of the mathematical universe. However, constructing this representation requires a choice of which operators will be represented. Since this external state space is a topological space, the operators it supports must all commute. So, for example, it cannot represent both position and momentum states. This is a formal expression of Bohr's principle of complementarity, in that modeling a measurement requires a choice of context, and in a particular context not all operators will have a non-trivial representation. It is also a topological interpretation of the Kochen–Specker theorem, as Heunen et al. (2011) show.

Halvorson (2004) has provided a pertinent illustration of this. Let $l_2(\mathbb{R})$ denote the vector space of complex functions supported on a countable subset $S_f \subseteq \mathbb{R}$ with $\sum_{S_f} |f|^2 < \infty$. This space can be made into a Hilbert space by equipping it with the inner product $\langle f|g\rangle = \sum_{S_f \cap S_g} \bar{f}g$. Then the collection of characteristic functions φ_λ for $\lambda \in \mathbb{R}$ form an orthonormal basis of $l_2(\mathbb{R})$, and one can come up with a representation of the Weyl form of the canonical commutation relations (CCRs) such that there is an operator Q with $Q\varphi_\lambda = \lambda\varphi_\lambda$. So Q is an operator on a Hilbert space that has an uncountable spectrum, as a position operator in a continuum ought.

Though one can define a position operator Q, there is no way to define a momentum operator P that satisfies the CCRs with Q. A parallel construction allows the definition of P, but similarly rules out the definition of Q. But this is not to say that position or momentum is completely meaningless in either representation. Rather, in both representations there is a pair of projection-valued measures E^Q and E^P which represent the position and momentum observables. In the Q representation, $E^P(\mathbb{R})$ is the 0 operator. More generally, if either $E^Q(\{\lambda\})$ or $E^P(\{\lambda\})$ is nonzero for some $\lambda \in \mathbb{R}$, then the other will vanish on all of $\mathbb{R}$. This is an instance of the foregoing general picture: in a particular measurement context, such as a position measurement, we only look at the relevant commutative subalgebra of the system's C^*-algebra. This means that operators which don't commute with position—such as momentum or the Hamiltonian—have no spectrum of eigenstates. However, their localic structure remains: E^P is a map from the Borel locale on $\mathbb{R}$ to the one-point locale $\{0\}$. On the other hand, the commutative subalgebra of the position operator Q and those that commute with it (*inter alia*, the values of E^Q) has a spectrum that is a spatial locale. In the Halvorson construction, this is the topological space $\mathbb{R}$. Note that there are two spatial locales here: there is the topological space $\mathbb{R}$ of positions, and there is *also* the Borel locale that acts as the domain of E^P. And this latter space is the space of momentum values. So there is a real sense in which one can speak of points in momentum space, even when a momentum operator is undefined.

The point of reviewing this construction is twofold. First, it gives an illustration of how the topological information encoded in the algebra of observables "fits together" with the measure-theoretic information encoded in the Hilbert space of states. More precisely, it gives a concrete example of how a measurement context gives a commutative C^*-algebra, which gives a spatial locale. The second reason to address this construction in particular is that Arntzenius addresses it directly as a competitor to gunk, rejecting it. His objections, addressed presently, help draw out the consequences for gunk in QM.

3.2 Traces of points in quantum mechanics

Arntzenius (2003, p. 1454) finds this construction unsatisfying. One reason is that he finds it less mathematically attractive than denying points and sticking to separable Hilbert spaces. To this one is tempted to say "*de gustibus mathematicis non est disputandum*." For my part, $l_2(\mathbb{R})$ is just what I'd expect, looking at the $(\mathfrak{m}, \mathfrak{n})$ menu; there are continuum-many points, and for each point there is a state representing the particle's occupying that point. However, Arntzenius points out four specific problematic features of this model. In this subsection I argue that two of them are not problems at all. The third feature is the crux of the relationship between gunk and QM, and I will argue that a gunky picture does not obviously square with continuous quantities in QM. The fourth, unfortunately, extends beyond the scope of this paper.

The first problem with the $l_2(\mathbb{R})$ model is that it isn't separable. Recalling the menu of measurable spaces, this should not be at all surprising. The measurable space $(\mathfrak{m}, \mathfrak{n})$ is separable only if both $\mathfrak{m}$ and $\mathfrak{n}$ are countable. If space is comprised of points, then there are more than countably of them, so $\mathfrak{m} > \aleph_0$ prevents the Hilbert space from being separable. The lack of separability is no reason to think that this Hilbert space is pathological unless there are independent grounds supporting separability. Arntzenius suggests that separability is nice because it supports "reasoning in terms of finite or countable superpositions," whereas a nonseparable space does not. But this is just a restatement of what separability means, not a reason in its favor. Reasoning in terms of countable superpositions shouldn't be expected if there are continuum-many points.

The second problem is that any Hilbert space containing continuum-many points as well as a diffuse part (i.e., the measurable space $(\mathfrak{c}, \aleph_0)$) can be written as the disjoint union of these two parts. Again, it is not obvious how to get from this mathematical fact to a problem with the formalism. Perhaps the idea is that $(\mathfrak{c}, \aleph_0)$ is somehow artificial, because we created it by gluing on enough points to represent the points of space. This artificiality claim requires some further support, however. As discussed above, a measurable space represents no information about the locations of different regions. So to say that the space is decomposable like this is to say that it has $\mathfrak{c}$ parts that are points and one part that is a diffuse region. It says nothing about the location of these parts. So this fact is not a problem for the $l_2(\mathbb{R})$ model, it is simply a reflection of how little structure is captured by a measurable locale.

Arntzenius's third problem is that, as mentioned above, when a system is in a position eigenstate, it has no well-defined momentum. The argument taking us from this mathematical fact to a problem must rest on some interpretational assumptions. The first decision must be about what the various bits and pieces of the formalism represent, physically. I take it that an element of a Hilbert space represents a possible state of the system modeled by the Hilbert space. When we are thinking about the position operator, such a state can be interpreted as a field on space, possibly up to measure-zero differences. This interpretation seems to be behind Arntzenius's contention that QM suggests space is gunky. But this is too quick, because filling in the details matters. For one example, Bohmian mechanics further specifies the state of the system using point particles, which are not obviously compatible with gunk. For another, a realist about configuration space will take space to be emergent, the result of particular symmetries of configuration space. A more cautious approach takes the properties of a quantum system to be probative of the structure of space, rather than determinative of it.

More explicitly, suppose that it were impossible to represent a system as being in a position eigenstate. This suggests that, for empirical purposes, systems cannot be in precise locations. It doesn't say anything directly about the structure of space. If one further supposes some sort of parsimony principle, then perhaps this impossibility suggests that space has no points. Note, however, that this approach also demands that the E^Q and E^P functions be modified to take arguments that are gunky regions, rather than the pointy Borel regions that they standardly take. Even in the standard Hilbert space $L^2(\mathbb{R})$, one is dealing with spatial locales in the form of the Borel regions of $\mathbb{R}$. So one must modify the standard algebra of projections given by E^Q and E^P, even when working on $L^2(\mathbb{R})$.

All of these considerations are moot, because it *isn't* impossible to represent a system as being in a position eigenstate. I take it that Arntzenius's objection to such a representation is that this fails to represent the quantum system as having a well-defined momentum, and that this is problematic. If it is problematic, it's not obviously so. To motivate a problem here, one would have to argue for the incoherence of the position and momentum representations. It's not enough to simply rule out the physical possibility of these $l_2(\mathbb{R})$ representations. It might well be that $L^2(\mathbb{R})$ is the one true state space of a system with continuously many degrees of freedom, but the straightforward physical interpretation of this fact is that QM systems can't be measured to have precise values of position and momentum. And the phrase "precise values" is given content by reference to the points of the Borel locale that is the domain of E^Q and E^P. So an argument that QM doesn't take place in a pointy space requires first an argument that rules out $l_2(\mathbb{R})$ on stronger than empirical grounds, followed by justification for replacing E^Q and E^P with their gunky cousins.

Arntzenius's fourth complaint starts to provide such an argument for the first of these tasks. The Hilbert space $l_2(\mathbb{R})$ is unitarily inequivalent to the space $L^2(\mathbb{R})$. On the view that Ruetsche (2011) terms "Hilbert space conservatism," when we are faced with inequivalent Hilbert spaces, we ought to pick just one (presumably $L^2(\mathbb{R})$) and stick with it. Assuming this position, one can then build in gunk by hand, by modifying the standard projection-valued measures E^Q and E^P. I lack the space to enter into this debate, hence the space to say anything definitive about the viability of gunk in QM. However, the conclusion that gunky models of space bump up against interpretive issues in the foundations of quantum field theory is somewhat surprising in itself.

4 Conclusion

My primary aims were to make explicit (i) the commitments of Arntzenius's model of gunk, and (ii) the pointy role of the algebra of observables. As for (i), I presented a framework that captures any mereological approach to regions of space and which includes size data from measure theory. This left a possibility space indexed by $(\mathfrak{m}, \mathfrak{n})$, which provides an easy way to determine how many points each has ($\mathfrak{m}$), as well as making Arntzenius's restrictions easily expressible: gunk requires $\mathfrak{m} = 0$ and separability requires $\mathfrak{n} = \aleph_0$. Turning to (ii), I evaluated the motivation for gunk from QM and found it wanting. Even if the state space for systems with continuous degrees of freedom is the gunky $L^2(\mathbb{R})$, the algebra of position observables is on a spatial locale, hence is composed of points. This is but one instance of a more general relationship between measurable spaces and topological spaces: some measurable spaces are gunky, but in order to equip one with a topology, points must be introduced.

From where we now stand, it looks like quantum mechanics as such has little to do with Arntzenius's motivations for introducing gunk. As we have seen, a space can be said to have the structure of gunk if it has the Lebesgue measurable space underlying it. But this structure is ubiquitous: it underlies any second-countable smooth manifold of positive dimension. So gunk appears as soon as we begin to deal with smooth spaces. And indeed, the space which Arntzenius identifies as being gunky is configuration space, which is a smooth manifold. So despite the connections to QM_∞ that appear, what is really at issue is our understanding of smooth geometry. So we seem to be left with two main questions. First: what *is* the small-scale structure of smooth geometry, and how does it correspond to our physical theories of space? And second: what, if anything, does this tell us about the structure of space in QM and its relativistic cousins?

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