Brussels-Austin Nonequilibrium Statistical Mechanics in the Later Years: Large Poincaré Systems and Rigged Hilbert Space

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Abstract

This second part of a two-part essay discusses recent developments in the Brussels-Austin Group after the mid 1980s. The fundamental concerns are the same as in their similarity transformation approach (see Part I), but the contemporary approach utilizes rigged Hilbert space (whereas the older approach used Hilbert space). While the emphasis on nonequilibrium statistical mechanics remains the same, the use of similarity transformations shifts to the background. In its place arose an interest in the physical features of large Poincaré systems, nonlinear dynamics and the mathematical tools necessary to analyze them.

Keywords: Thermodynamics, Statistical Mechanics, Integrable Systems, Nonlinear Dynamics, Probability, Arrow of Time
Word count (including notes): 9,547

1 Introduction

Part I of this essay discussed the earlier similarity transformation approach to nonequilibrium statistical mechanics of Ilya Prigogine and his coworkers. This approach, along with that of subdynamics, is perhaps somewhat familiar as it has received some attention in philosophical literature and was the subject of Prigogine's well-known book, *From Being to Becoming: Time & Complexity in the Physical Sciences* (1980). Part II of this essay focuses on their more recent and less familiar work on nonequilibrium statistical mechanics in rigged Hilbert spaces.

It has been argued that no current approaches to microscopic dynamics can explain or derive the second law of thermodynamics, since it is both necessary and sufficient for the derivation of the second law from microscopic dynamics that the dynamics be exact (e.g. Mackey 1992, pp. 98-100; 2002).[1] Although it can be shown that the coarse-grained projection operator arising from the earlier Brussels-Austin approach yields an exact dynamics, whether their similarity transformation yields exact dynamics is unknown (Antoniou and Gustafson 1993; Antoniou, Gustafson and Suchanecki 1998, p. 119). Nevertheless, one of the crucial claims of the earlier approach was that trajectory descriptions at the microscopic level and probabilistic descriptions at the macroscopic level of thermodynamic behavior are related via a transformation (Part I).

This way of viewing the relationship be trajectory and probabilistic descriptions is de-emphasized in their more recent work. So the core point is no longer to derive irreversible thermodynamic behavior from reversible microscopic descriptions, so much as to argue for the priority of irreversible macroscopic descriptions for a particular class of systems known as large Poincaré systems. However, the core intuitions of the new approach remain continuous with their earlier work; namely, that irreversibility is fundamentally dynamical in character and that distributions are ontologically fundamental explanatory elements for complex statistical systems.

The Brussels-Austin Group's recent work develops a method for constructing a complete set of eigenvectors for the model equations describing the thermodynamic approach to equilibrium for Large Poincaré systems as well as nonlinear dynamics more generally. This approach reformulates the question of how to relate reversible trajectory and irreversible probabilistic descriptions as follows: How can the trajectory dynamics of a large Poincaré system (LPS) yield necessary conditions for the thermodynamics approach to equilibrium and what further mechanisms account for the sufficient conditions for such behavior?

Large Poincaré systems are defined and illustrated in §2 using nonintegrable Hamiltonians and classical perturbation theory as a way of motivating some of the key physical and mathematical problems for such systems. The rigged Hilbert space approach to these systems is outlined in §3, and the corresponding time-ordering rule and semigroup operators governing the dynamics are introduced. Particular details of the approach are discussed in §4, where an alternative interpretation of Prigogine's treatment of trajectories and their relationship to the dynamics of distributions is developed. Some remarks on probabilistic vs. deterministic dynamics closes the essay (§5).

[1] A dynamics on a state space Ω with a transfer operator P_t is *exact* if and only if $\lim_{t\to\infty} |P_t\rho - \rho_{eq}|_{L^1} = 0$ for every initial density ρ, where ρ_{eq} is the unique stationary density (i.e. equilibrium density), P_t governs the dynamics (e.g. Liouville or the Frobenius-Perron operators), and the norm is in the sense of Lebesgue integrable functions. Among other properties, exact dynamics are noninvertible and always yield a unique stationary density.

2 Large Poincaré Systems and Integrability

Toward the end of the 19th century, Poincaré was investigating planetary motion, among other things. Solving the equations of motion for the solar system is extremely difficult because all the planets interact with each other through gravitational forces. One of the questions Poincaré pursued was whether there was a suitable way to transform these equations of motion into a system of equations where the gravitational interaction would vanish and one could solve the evolution equations for the angle variables of each planet independently of the others. What Poincaré showed was that in general such a transformation was impossible for systems of N mutually interacting bodies. If a canonical transformation for a system of equations describing a set of interacting particles that carries the equations into a set where the interactions vanish exists, then the system is classified as *integrable*. This means that the original system of equations can be transformed into one where each particle's angle variable is fully described by an equation that is independent of any other particle's angle variable.

Poincaré showed that systems of equations were nonintegrable when they contained *resonances* between various degrees of freedom. In essence a resonance is a transient metastable state establishing a narrow, precise frequency gateway through which energy can be efficiently transferred from one element of a physical system to another. Physical examples of resonances include transient bound states produced in particle collisions and transient intermediates in chemical reactions.

2.1 Integrable Systems and Classical Perturbation Theory

In order to make these notions of resonances and nonintegrability more precise, consider Hamiltonian systems in classical mechanics. While models with completely integrable Hamiltonians are rare, they are still very useful in the study of physical systems. For many systems can be modeled using Hamiltonians of the form

$$H = H_0(\vec{J}) + \lambda V(\vec{J}, \vec{\alpha}), \tag{1}$$

where H_0 is assumed to be completely integrable, $\vec{J}$ represents the action variables (e.g. generalized momentum vectors), $\vec{\alpha}$ the angle variables (e.g. generalized coordinate vectors) and λ (assumed $\ll 1$) is the coupling coefficient roughly describing the strength of the interactions through the potential V. The question of whether or not a Hamiltonian system is integrable is equivalent to being able to find a canonical transformation from the old state space coordinates $(\vec{J}, \vec{\alpha})$ to the new coordinates $(\vec{I}, \vec{\beta})$ corresponding to a transformation operator of the form

$$e^{iF(\vec{I},\vec{\beta})} \tag{2}$$

decoupling all the equations for the angle variables (in essence turning off all the interactions by making λ zero). When such a transformation can be found, the Hamiltonian is said to be completely integrable and I will refer to this type of integrability as complete integrability (to be distinguished from the Brussels-Austin sense of integrability below).

In general one then must proceed using a perturbation method where the strategy is to find approximate solutions of (1) in terms of $H_o(\vec{I})$ plus small perturbations due to $V(\vec{I},\vec{\beta})$. In the course of standard perturbation analysis of such a model (e.g. Tabor 1989, 89-108), terms of the form

$$\frac{V_{n_i,n_j,n_k\ldots}}{n_i\beta_i + n_j\beta_j + n_k\beta_k + \ldots} \tag{3}$$

emerge where i, j, and k are integers labeling the particles, $V_{n_i n_j n_k}$ represents the Fourier transformed potential, the n_l indicate the (discrete) degrees of freedom of the particles in the Fourier expansion, and the β_l can be negative and are often interpreted as generalized frequencies. Clearly terms like (3) increase without bounds when the denominator approaches zero. The denominator being zero represents a resonance. It is the presence of a sufficient number of these resonances that prevents us from using the standard canonical transformation techniques to turn the model into a completely integrable system of equations. For an N-body problem, the resonance condition takes the form that the finite sum $n_i\beta_i + n_j\beta_j + n_k\beta_k + \ldots + n_N\beta_N = 0$. In general there are several combinations of n_l's and β_l's satisfying this condition.

2.2 Large Poincaré Systems

First consider an integrable Hamiltonian for a system with two degrees of freedom. The state space trajectories will then be confined to the surfaces of nested tori, where each surface corresponds to a different combination of the values of the two constants of the motion. Now add perturbations λV to this Hamiltonian where $\lambda \ll 1$. If the perturbations leave the Hamiltonian integrable, then the model dynamics are not appreciably affected. In contrast, if the perturbations render the Hamiltonian nonintegrable (e.g. resonance phenomena), then these periodic orbits will be disrupted because such perturbations are as physically important as the unperturbed orbits of the integrable part of the model, due to the transfer of energy involved. The KAM theorem specifies the conditions under which tori associated with quasi-periodic trajectories survive and constitute the majority of motions realized in state space, so that most regions in state space for nonintegrable models close to integrable models show stable nonperiodic orbits (e.g. Hilborn 1994, 337-9).

There are two types of fixed points for the state space trajectories in Hamiltonians of the form (1): elliptic and hyperbolic (saddle points). Elliptic fixed points correspond to stable periodic orbits which are disrupted by resonances. Hyperbolic fixed points present complex behavior: trajectories exhibiting sensitive dependence on initial conditions and which wander erratically over large regions of state space. These structures also exhibit self-similarity. The chaotic behavior in Hamiltonian systems is similar to chaotic behavior in dissipative systems. However, since Hamiltonian systems do not contract to some fixed point as do dissipative systems, orbits near hyperbolic fixed points will become unstable leading to exponentially diverging trajectories. It should be pointed out that stable and chaotic orbits can coexist simultaneously in state space.

Large Poincaré systems are of interest to Prigogine and coworkers. Consider a typical SM Hamiltonian of the form

$$H(p,q) = \sum_{i=1}^{N} \frac{\vec{P}_i^2}{2m_i} + \lambda \sum_{j>i}^{N} V(|\vec{q}_i - \vec{q}_j|), \tag{4}$$

where $\vec{q}$ and $\vec{p}$ are N-component vectors representing generalized coordinates and momenta respectively, and the system is in a large box with volume L^3. The Brussels-Austin group is interested in "large" systems, meaning they work in the limit $L^3 \to \infty$ (the number of particles N may be finite or infinite). A LPS is obtained when the system is large and the number of degrees of freedom of the system tends to infinity. An example of a LPS with a finite number of particles would be a finite number of charges interacting with an electromagnetic field, while an example with an infinite number of particles would be the *thermodynamic limit* ($L^3 \to \infty$, $N \to \infty$, N/L^3 finite). Such systems possess "continuous sets of resonances". By continuous sets of resonances, the Brussels-Austin Group means that in the Fourier transformed representation, the eigenfrequencies are continuous functions of the wave vector k, so that the summation operations over terms like (3) must be replaced by integrals and the denominators of such terms can be arbitrarily close to zero.

The resonance condition for a continuous set of resonances for a LPS in the context of perturbation theory takes the form

$$\int \int b\beta db d\beta = 0, \tag{5}$$

where b (representing degrees of freedom) and β are continuous functions defined over the real numbers. Under condition (5) motion will not even be quasiperiodic so that variables have a continuous spectrum.[2] No canonical transformation exists that can turn these LPS models into completely integrable models (Prigogine et al. 1991, pp. 6-7). Such models exhibit the type of randomness associated with mixing, K-flows and Bernoulli systems, but are usually interpreted as deterministic.[3]

As an example of a LPS, imagine a gas containing an infinite number of particles continually undergoing collisions, where the collision processes never

[2] As Koopman and von Neumann first pointed out, for dynamical systems with continuous spectra, 'the states of motion corresponding to any set become more and more spread out into an amorphous everywhere dense *chaos*. Periodic orbits, and such like, appear only as very special possibilities of negligible probability' (Koopman and von Neumann 1932, p. 261). This is generally acknowledged to be the first reference to the term "chaos" in the context of dynamics.

[3] The Baker's transformation is a favorite model of a deterministic random system for the Brussels-Austin Group (Part I). The equations are reversible, deterministic and conservative, yet the mapping turns out to have the Bernoulli property (randomness of a coin toss).

cease. A more realistic example is an electromagnetic oscillator with frequency ω_{osc} interacting with an electromagnetic field. The field has an infinite number of degrees of freedom and the frequency ω_k of the field varies continuously with k, giving rise to an infinite number of resonances. Continuous resonances like those in LPS are involved in fundamental phenomena such as absorption and emission of light, decay of unstable particles and the scattering of electromagnetic waves off of fluids or other forms of matter, and are found in both classical mechanics (CM) and quantum mechanics (QM).

The rigged Hilbert space (RHS) approach of the Prigogine school is a method for solving the equations of a LPS (both CM and QM) consisting in constructing a complete set of eigenvalues and eigenvectors for the Liouville operator acting on distribution functions ρ.[4] The construction of such eigenvalues and eigenfunctions is what Prigogine and colleagues call the 'generalized problem of integration' (Prigogine et al. 1991, p. 4). To be clear about terminology, finding a transformation that decouples the Hamiltonian in (1) is what is required to show that the system is completely integrable in the sense described earlier. Constructing the complete set of eigenvalues and eigenvectors for a set of equations derived from (1) is what Prigogine and colleagues refer to as 'integrating' or solving the equations of motion. Although initially motivated in the context of perturbation theory (as sketched here), the rigged Hilbert space approach is more general in nature and applicable to any LPS (e.g. most systems in SM, systems involving interacting fields).

3 Mathematical Details of the Rigged Hilbert Space Approach

There are three key elements in the Brussels-Austin method to solving LPS equations. First, they utilize distribution functions to describe the dynamics. Second, they adopt extended spaces such as RHS as a mathematical framework for solving the equations. Third, they introduce an "appropriate" time ordering of the dynamical states of the system.

3.1 The Need for Distributions

When solutions of the generalized integration problem sketched at the end of §2.2 exist, they reduce to classical trajectories for most CM systems and to state vectors for most QM systems. In the context of a LPS, however, Prigogine and colleagues argue solutions are not reducible beyond distributions for CM systems. Examples include systems in kinetic theory, radiation damping and interacting fields. One important feature of such physical contexts is that they are characterized by *persistent interactions*. According to Petrosky and Prigogine,

[4] These distribution functions may be understood in terms of the probability density $\rho(\vec{q}_1, \vec{q}_2, \vec{q}_3, ..., \vec{p}_1, \vec{p}_2, \vec{p}_3..., t)$ of finding a set of molecules (say) with coordinates $\vec{q}_1, \vec{q}_2, \vec{q}_3, ...$ and momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3...$ at time t on the relevant energy surface and are analogous to the microcanonical distribution.

a system's interactions are persistent if there are no asymptotic states such that the interactions finally cease (1997, pp. 33 and 35). For example in kinetic theory, the molecules of a gas are in constant interaction with one another because they are undergoing continuous collisions. This physical situation should be contrasted with the idealized case of a single neutral particle scattering off a fixed target. In the latter situation, there is a *transitory interaction* because the particle undergoes an interaction only in a finite region near the target over a very short time interval, while the particle spends the majority of its life in the so-called asymptotic in and out states free of any interactions with the target. Since interactions never cease for systems with persistent interactions, the model equations typically will not be completely integrable.

The presence of persistent interactions is one of the features giving rise to the continuous set of resonances in a LPS. In a gas containing a large number of particles, these resonances allow for energy to be transferred and leveled throughout the system. Through persistent interactions and the resulting resonances, the particles will loose energy and any ordered patterns are destroyed through *diffusion* (see §4.2 below).

A further consequence is that the physical dynamics are no longer localized, but are spread throughout the space occupied by the LPS. For the gas example, these nonlocal dynamics will take the form of correlations as described in §4.2 below. In addition if the number of particles is large enough, then the degrees of freedom for such a gas of particles will have a continuous spectrum qualifying it as a LPS. This implies that we should expect the dynamical description of such systems to be in terms of distributions of particles rather than in terms of individual particles, because the effects of long-range and higher-order correlations due to such interactions become at least as important as the trajectory dynamics. The particles remain coupled to one another through their interactions resulting in collective effects (§4.2 below). This type of long-range coupling at least implies that the global or collective dynamics of the system cannot be accurately represented by trajectory dynamics alone (see §4.3 below). As a consequence, Prigogine and colleagues believe we must view irreversibility as a property of a system that emerges at the global level which is not derivable from the trajectory description, meaning that distributions are the natural elements for representing statistical phenomena rather than trajectories.[5]

3.2 The Need for RHS

A RHS is an extended mathematical space first introduced by the Russian mathematician Gel'fand and his collaborators (Gel'fand and Vilenkin 1964).[6] Briefly a RHS can be understood in the following way. Let Ψ be an abstract linear

[5] To avoid a simple confusion (e.g. Bricmont 1995, pp. 165-6), note that singular distributions such as delta functions *are not* used to represent probability distributions in the rigged Hilbert space approach.

[6] In more recent work Petrosky and Prigogine (1997) have explored rigging "Liouville space"–the space of density functions or density operators–for dynamics. Ordóñez (1998) has demonstrated that these Liouville spaces can be rigged as a Gel'fand triplet, yielding semi-group operators and generalized eigenvectors.

scalar product space and complete it with respect to two topologies. The first topology is the standard Hilbert space (HS) topology $\tau_{\mathcal{H}}$

$$|h| = \sqrt{(h,h)}, \tag{6}$$

where h is an element of Ψ resulting a HS $\mathcal{H}$. The second topology τ_Φ is defined by a countable set of norms

$$|\phi_n| = \sqrt{(\phi,\phi)_n},\ n = 0, 1, 2... \tag{7}$$

where ϕ is also an element of Ψ and the scalar product in (7) is given by

$$(\phi,\phi')_n = (\phi, (\Delta+1)^n\phi'), n = 0, 1, 2... \tag{8}$$

and

$$\phi_\gamma \to \phi \text{ in } \tau_\Phi \text{ iff } |\phi_\gamma - \phi|_n \to 0 \text{ for every } n, \tag{9}$$

where Δ is the Nelson operator $\Delta = \sum X_i^2$ (Nelson 1959, 587). The χ_i are the generators of an enveloping algebra of observables for the system in question and they form a basis for a Lie algebra (Nelson 1959; Bohm *et al.* 1999). For example if we are modeling the harmonic oscillator, the X_i could be the raising and lowering operators (Bohm 1978, 7-9). Furthermore if the operator $\Delta+1$ is a nuclear operator then this ensures that Φ is a nuclear space (Treves 1967, 509-34; Bohm 1967, 276-7). An operator A is nuclear if it is linear, essentially self-adjoint (Roman 1975, pp. 540-3) and its inverse is *Hilbert-Schmidt*. The operator A^{-1} is Hilbert Schmidt if $A^{-1} = \sum X_i P_i$, where the P_i are mutually orthogonal projection operators on a finite dimensional vector space and $\sum a_i^2 < \infty$, a_i denoting the eigenvalues of A^{-1} (Bohm 1967, 273-6). Notice that the norm (6) is a special case of (7) where $n = 0$.[7]

We obtain a Gel'fand triplet if we complete Ψ with respect to τ_Φ to obtain Φ and with respect to $\tau_{\mathcal{H}}$ to obtain $\mathcal{H}$. In addition we consider the dual spaces of continuous linear functionals $\Phi^\times$ and $\mathcal{H}^\times$ respectively. Since $\mathcal{H}$ is self dual, we obtain

$$\Phi \subset \mathcal{H} \subset \Phi^\times, \tag{10}$$

where $\Phi^\times$ is characterized by the induced topology $\tau_\times$. The meaning of the symbol $\subset$ in relation (10) is that every space to the left of $\subset$ a is a subspace of every space to the right of $\subset$ and every space to the left of $\subset$ is dense in the space to the right of $\subset$ with respect to the topology of the space to the right of $\subset$ (see Gel'fand and Vilenkin 1964 for more details).

[7]There are many different inequivalent irreducible representations of an enveloping algebra of a group characterizing a physical system (e.g. the rotation group has an inequivalent irreducible representation for each value of j). They can be combined in many ways by taking direct products describing combinations of physical systems. These representations are characterized by the values of the invariant or Casimir operators of the group. So although the Nelson operator fully determines the topology of Φ, there is freedom in choosing the enveloping algebra describing elementary physical systems. Further restrictions on the choice of function space for a realization of Φ are due to the particular characteristics of the physical system being modeled. This is analogous to the situation for W^*-algebras in the algebraic approach to QM (Primas 1981 pp. 161-249; Amann and Atmanspacher 1999).

For the Brussels-Austin Group, the chief reason to work in a RHS is the ability to naturally model unstable physical phenomena such as decay, scattering and the irreversible approach to equilibrium which is lacking in HS (e.g., Bishop 2003a). These kinds of time-dependent processes require complex eigenvalues and generalized eigenfunctions (Gel'fand and Shilov 1967). Such mathematical quantities are not well-defined in a HS, but are given rigorous justification in a suitable RHS. In particular the Liouville operator, which characterizes a LPS's approach to equilibrium, does not have a complete set of eigenvalues and eigenfunctions in a HS. Recently the Brussels-Austin Group has demonstrated that a complete set of eigenvalues and eigenvectors for this important operator can be defined and calculated for several chaotic models in extended spaces (Antoniou and Tasaki 1992 and 1993; Hasegawa and Shapir 1992; Hasegawa and Driebe 1993). An additional motivation for switching to a RHS is that the equations of motion defined on a HS are time-symmetric. Time-asymmetric equations may be defined and solved in a RHS making the latter type of space a natural choice for modeling intrinsic irreversible processes (irreversibility without explicit reference to an environment; see Part I). Intrinsic irreversibility is of prime interest to the Brussels-Austin Group because these types of irreversible processes are related to intrinsic arrows of time in physics (i.e. arrows of time which are independent of human intervention or approximation).

3.3 Semigroup Operators in RHS and Irreversibility

One of the important features of RHS is that evolution operators are often elements of semigroups rather than groups, so that irreversible behavior can be appropriately modeled. The case of simple scattering is a good example for illustrating the concepts. An idealized version of a scattering experiment is sketched in Figure 1. There is a preparation apparatus which prepares particles in a particular state (energy, angular momentum, etc.). The particles are emitted at a target (assumed to be fixed in this analysis). The free particle Hamiltonian in (1) is H_0 while the potential in the interaction region surrounding the scattering center is given by V. After the interaction with the target, the detector registers the particle measuring quantities such as the angle of scattering relative to the initial direction of the particle as emitted from the accelerator or the energy of the particle after the scattering event.

Each interaction involves a resonance which can be described as

$$|E^{\pm}>=\left(1+\frac{1}{E-H\pm i\varepsilon}V\right)|E>\,, \tag{11}$$

a Lippmann-Schwinger-type equation for the evolution of the energy eigenstates as they pass through the scattering region. Whenever the operator on the right hand side of (11) applied to the energy eigenstate $|E>$ goes to infinity, we have a resonance. According to the Brussels-Austin Group, if, given a sufficiently large number of interacting particles, the number of resonances in a system is sufficiently large, then the system will evolve from a highly ordered state to

Figure 1: An idealized scattering experiment.

a completely randomized or equilibrium state. This evolution is intrinsically irreversible, due to the internal dynamics of the system.

The intrinsic irreversibility of LPS models must be described by semigroups. This necessitates leaving the HS framework and working in a broader mathematical space such as a RHS which Antoniou and Prigogine (1993) adopt in their analysis of the Friedrich's model for scattering. In the Gel'fand triplet $\Phi \subset \mathcal{H} \subset \Phi^{\times}$, $\Phi^{\times}$ is the space of particle distribution functions. Furthermore Antoniou and Prigogine adopt the following time ordering condition: any excitations or preparations are to be interpreted as events taking place before $t = 0$ while any de-excitations or detections are to be interpreted as events taking place after $t = 0$ (1993, pp. 445 and 455).

At the point in the analysis of the scattering experiment where choices have to be made regarding how to interpret the directions of integration for the analytic functions involved in the upper and lower complex half-planes, they choose the following interpretations (1993, pp. 454-5): excitations are identified as taking place *before* $t = 0$ (taken to be represented as extensions from the lower to the upper half-plane), while de-excitations are identified as taking place after $t = 0$ (taken to be represented as extensions from the upper to the lower half-plane). So the time-ordering rule is applied with respect to the choice of how to deform the contours in the complex plane with respect to the choice of direction of integration along the contours. Proceeding in this fashion Antoniou and Prigogine derive concrete realizations for the space Φ involving Hardy class function spaces (1993 pp. 457-9; see also Bishop 2003a and 2003b).

Antoniou and Prigogine discuss two semigroups of evolution operators. The first is $U^{\dagger}(t) = e^{-iHt}$, initially defined on $\mathcal{H}$ for $-\infty < t < \infty$, extended to $\Phi^{\times}$. It is continuous and complete in the topology $\tau_{\times}$ of $\Phi^{\times}$, valid for $t \geq 0$ and they identify its temporal direction as carrying states into the forward direction of time. This operator describes evolution reaching equilibrium in the future. The second operator is $U^\dagger(t)$ extended to $\Phi^\times$, continuous and complete in the topology $\tau_\times$, but valid for $t \leq 0$.[8] They identify the temporal direction of this latter operator as carrying states into the *backward direction* of time ($-t$ increasing), so this operator describes evolution reaching equilibrium in the past. Since no physical systems are ever observed evolving to equilibrium from the future into the past, they *select* $U^\dagger(t)$ extended to $\Phi^\times$ for $t \geq 0$ as the physically relevant semigroup of evolution operators for modeling statistical mechanical systems. This selection is taken to be an expression of the second law of thermodynamics based on our empirical observations (Antoniou and Prigogine 1993, p. 461).

The approach sketched in this section for the case of transient scattering can be extended to the case where the interactions are continuous and persistent, yielding similar results (Petrosky and Prigogine 1996 and 1997).

4 Discussion of the RHS Approach

The Brussels-Austin Group's RHS approach has yielded solutions (mostly numerical) to nonequilibrium statistical mechanical system equations. Based on these solutions and the insights gained from the new approach, Prigogine and coworkers make a number of important claims needing detailed discussion.

4.1 Thermodynamic Arrow of Time

One of the claimed virtues of the approach is that it provides an explanation for the thermodynamic arrow of time (the law of increasing entropy defined entropy close to equilibrium). This has been one of the central goals of Prigogine since he began his work in SM. One feature that both the earlier similarity transformation approach (discussed in Part I) and the RHS approach share in this quest is a kind of vacillation between seeking an explanation of the thermodynamic arrow in the dynamics of the physical system, and taking the empirically observed direction of the arrow as a fundamental principle.

In the RHS approach, the types of mechanisms to which the Brussels-Austin Group appeals for explaining the thermodynamic arrow are diffusion, the growth of correlations and collective effects, all of which are generated by Poincaré resonances (Antoniou and Prigogine 1993; Petrosky and Prigogine 1996 and 1997). The extension of the description of a LPS with their Poincaré resonances, persistent interactions and chaotic dynamics to Gel'fand triplet spaces allows the eigenvector equations to be solved. In the course of analyzing these solutions, characteristically there are two semigroups that emerge as sketched in §3.3. At this point in the analysis, one semigroup is selected because it represents systems approaching equilibrium in the temporal direction of the future, while the other semigroup is disregarded because it describes systems approaching equilibrium in the temporal direction of the past which is never observed and, therefore,

[8] The requirements of continuity and completeness force the unitary group extended to $\Phi^\times$ to be restricted to the separate time ranges $t \leq 0$ and $t \geq 0$ (Bohm and Gadella 1989, pp. 35-119).

deemed to be unphysical (Antoniou and Prigogine 1993, p. 461; Petrosky and Prigogine 1996, p. 453 and 1997, p. 13). By making this latter appeal to observations, the Brussels-Austin Group is appealing to the very facts they seek to explain *via* the dynamics of the physical system.

The model equations alone do not uniquely determine which semigroup is the appropriate one, so some kind of appeal to physical considerations is needed. As discussed in §3.3 above, the Brussels-Austin Group does make an appeal to a criterion for choosing a temporal ordering: any excitations are to be interpreted as events taking place before $t = 0$ while any de-excitations are to be interpreted as events taking place after $t = 0$. While there is a clear ordering of time from excitation to de-excitation, the criterion invoked still ultimately rests upon our observations that a system is excited before it undergoes de-excitation. The *physical* reason why the thermodynamic arrow runs from the past toward the future is still undiscovered in the RHS approach, though the approach gives us the mathematical tools to explore and describe the arrow precisely.

4.2 Correlation Dynamics

The RHS approach highlights the role of nonlocal and collective effects due to long-range correlations that introduce new dynamics in the probabilistic description that are typically ignored in the trajectory description of a LPS.[9] The term "collective effects" is used to describe the behavior of an aggregate of particles coupled together in some fashion that is distinct from the behavior of individual particles. Collective effects can arise from long-range forces such as electromagnetism, gravity or from spatial correlations caused by interactions.

Spatial correlations play an important role in the temporal ordering of the dynamics of SM systems. In atomic or molecular gases, collective effects are due to collisions. Consider the idealized textbook situation, where we start with an isolated gas of N particles in a volume V that have yet to interact with one another. If the initial distribution of the particles is homogeneous and isotropic, then the particles are equally likely to be at any point $\vec{r}$ in V.[10] This result holds for each individual particle under the condition that the positions of the other particles are arbitrary. In a typical gas or liquid, this latter condition is not fulfilled in general, however. Consider two particles at a time in our gas. Given the position of one particle, different positions of the second particle are not equally likely to obtain; namely, the second particle cannot occupy the position of the first particle. Due to interparticle interactions and the symmetry properties of the state vectors, different values of the relative position $(\vec{r}_2 - \vec{r}_1)$ between our two test particles in the entire gas do not appear with equal likelihood. This feature is known as a *spatial correlation* between the simultaneous positions $\vec{r}_1$ and $\vec{r}_2$ of the two particles.

[9] Prigogine (1962, 138-95) introduced a simplified version of correlation dynamics and George (1973a) developed the idea in the direction indicated in this section.

[10] Of course, in this idealized example the assumption of equiprobability of states is reasonable. In a LPS, by contrast, interactions are persistent, so this assumption cannot be made.

In a plasma, for example, where the gas is composed of charged particles, spatial correlations are the tendencies of unlike charges to cluster together and the tendencies of like charges to repel each other. The simultaneous positions of the particles in the plasma are not all equally likely. It turns out that there is a simple relationship between the spatial integral of the correlation function representing spatial correlation and the mean square fluctuation of the density of the gas particles (Pathria 1972, 447-50), meaning the spatial distribution of the particles is influenced by the presence of such correlations. In addition these correlations are directly dependent on the density of particles in the gas. As the density decreases, such collective effects disappear because the *mean free path* of the particle, a measure of the likelihood of a collision during a given distance traveled, becomes comparable to V. This means collision events will be very rare and correlations will be kept to a minimum when the mean free path is large.

Collisions are frequent in dense gases and the spatial correlations induced by collisions couple each particle with many other particles in the gas. It is this coupling due to correlations that leads to collective behavior responsible for gas particles being collected into coherent structures rather than being uniformly spread throughout the volume. Examples would be turbulence and shock waves.

To see how these correlations develop, start with the particles in the gas before they have interacted with each other. As they begin colliding, the first interactions set up binary correlations between particles. As the interactions persist, ternary correlations begin to appear. The process will continue by establishing quaternary correlations and so on through N-ary correlations as more and more particles become involved others through collisions. The progression from lower order correlations (which appear first) to higher order correlations (which appear later) corresponds to a natural temporal ordering for the evolution of the states of the gas. Correlations and other collective effects can rival or exceed the role of individual particle trajectories and be masked by a dynamical description that treats trajectories as its basic explanatory element.

For example the electromagnetic force is a long-range force. It is the dominant force in many situations in a plasma, so the behavior of a plasma is not reducible to the dynamics of the trajectories of the individual particles alone. In the case of a plasma, the energy of the plasma is affected by the presence of correlations, such that one of the differences between the energy of a plasma and that of an ideal gas (noninteracting particles) is given by a correction term due to correlation effects (Krall and Trivelpiece 1986, 63-5). Not only do these effects interact with the electromagnetic fields of the plasma itself, but they also generate new electromagnetic fields that react back on the plasma leading to very complex dynamics.

Among the physical mechanisms playing a role in LPS, correlations appear to play a crucial role in irreversibility. As was apparent in the earlier similarity transformation approach, the progression of correlations suggests a natural direction for the thermodynamic arrow (George 1973a). But this is not simply another way of saying that entropy increases for such systems because in an open system the order of correlations may continue to grow while the measure of disorder in the system may remain constant or decrease. So correlations are not the complete explanation for the thermodynamic arrow of time.

Long-range correlations are another effect in the dynamics of correlations that become apparent in RHS (discussed in its earliest form in Prigogine 1962 and George 1973a). As gas particles begin to interact, correlations develop among the particles due to interactions (recall that in a LPS these interactions are associated with resonances). Along with the growing order of correlations, long-range correlations develop as particles interact with one another and then separate over long distances while carrying the "memory" of their prior interactions (correlations) with them to other parts of the gas. Over short time scales, the growing order of correlations appears to be the more dominant of the two effects. As time goes on, the long-range correlations due to resonances are built up so that collective effects become influential. These long-range correlations are associated with nonequilibrium modes of energy transfer (Petrosky and Prigogine 1996, p. 468).

Over longer time-scales, another very interesting phenomenon occurs. Equilibrium short-range binary correlations remain finite, but nonzero around each particle. In turn ternary nonequilibrium correlations are built up among particles in a small region. These correlations diffuse throughout the system, leaving the equilibrium correlations, while quartinary nonequilibrium correlations are built up among the local particles. These correlations diffuse throughout the system while quintinary nonequilibrium correlations build up and so forth. As time continues the variously ordered nonequilibrium correlations can propagate over large distances due to diffusion so that the corresponding information is transferred globally among the particles of the gas. The end result is a "sea" of multiple incoherent correlations (Petrosky and Prigogine 1996, p. 468). This effect provides a natural temporal direction for the flow of entropy and is revealed in the types of complex spectral representations of the statistical evolution operators made possible by working in an RHS.

In this sense one might argue that as the order of correlations increases, as long-range correlations grow and as higher-order nonequilibrium correlations propagate throughout the gas, the effects of individual trajectories on the global dynamics of the gas become less important relative to the effects of the dynamics of correlations. This does not mean that particles lack trajectories and positions in state space as these types of interaction events are parasitic on these concepts (e.g. mean free path between collisions). In my view correlations and collective effects make the significant contributions to the global dynamics while the effects of trajectories play a role only locally (see below).[11]

One might object that the dynamics of correlations can somehow be reversed even though the probability of the right kinds of reversals to run the whole evolution backwards (like a film in reverse) is extremely small. If true, then the situation is still the same as in standard thermodynamics where the increase

[11] Of course I have used idealized examples in this section in the sense that we imagined starting with a gas of noninteracting particles and then "turning on" the interactions. Recall that interactions are persistent in a LPS so there is never a time in such systems when the microscopic dynamics can be characterized by smooth, smooth trajectories.

in entropy in systems is viewed as being reversible though the probability is vanishingly small.

The Brussels-Austin response to such an objection for open systems has been given in §4 of Part I. For closed systems they have shown that as the dynamics of correlations continue, an "entropy barrier" against inversion develops. This barrier can be defined as the value of the H-function–a thermodynamic function related to the entropy, which does not require coarse graining or the invocation of an environment in the Brussels-Austin approach–after such an inversion minus its value before such an inversion. This difference increases exponentially with time, so the longer the LPS evolves, the higher the barrier to inversion. Essentially this means that the energy requirements to invert the system of particles increases very rapidly with time. As the model approaches equilibrium, this energy barrier diverges, hence, there is no physical way of "going back" in the anti-thermodynamic direction (Petrosky and Prigogine 1996, pp. 468-9 and 494-5).

4.3 "Collapse of Trajectories"

In the similarity transformation approach (Part I), Prigogine and collaborators put forward several arguments to the effect that smooth (i.e., everywhere differentiable), deterministic trajectories do not exist for unstable statistical mechanical systems. These arguments were fundamentally flawed in similar ways in that epistemological claims were treated as ontological claims. In the new approach, this bias against such smooth trajectories and the dynamics derivable from trajectories resurfaces in a different form that clarifies the Brussels-Austin attitude toward trajectories.

It is well known that in the traditional description, the trajectory of a point particle free of any external forces can be represented mathematically as a superposition of "plane waves" by taking the position of the particle and applying a Fourier transform from (q, p) space to (k, p) space. In this latter space, a trajectory is a coherent superposition of plane waves and this superposition is modeled by a Dirac delta function. For a particle undergoing free motion, this distribution function is a solution to the equation of motion, has unchanging width and is everywhere differentiable throughout its deterministic evolution ("smooth" trajectory).

For a finite number of particles with normalizable distributions, the trajectory description in (k, p) space and the Brussels-Austin probabilistic description agree.[12] In the thermodynamic limit, however, Prigogine and coworkers argue that resonances destroy smooth trajectories in the following way. In the thermodynamic limit, the Dirac delta function describing the trajectories of particles at $t = 0$, once evolution begins, immediately begins spreading throughout a subspace of (k, p) space under the action of resonances, though maintaining a delta

[12] Some critics, such as Bricmont (1995, pp. 165 and 175), have overlooked the way in which the RHS approach reduces to standard SM approaches for small numbers of particles when LPS conditions are not fulfilled.

function singularity[13] (Petrosky and Prigogine 1996, pp. 479-481 and 1997, pp. 35-37). The trajectories are no longer representable as delta functions, but by broader kinds of distribution functions. Petrosky and Prigogine unfortunately described this phenomenon as the "collapse of trajectories", but all they really mean is that a different notion of trajectory is required in a LPS.

In (q,p) space, this implies that there are no longer any smooth (everywhere differentiable) trajectories, but, rather, trajectories exhibiting *Brownian motion*. A simple way to see this is to return to our idealized gas example. As before, assume initially that the particles have not interacted with each other. Prior to any collisions, the motion of the particles can be characterized by smooth trajectories. As they begin interacting, the particle trajectories become piece-wise continuous as instantaneous discontinuities arise associated with each collision. Continuous interactions of this type would then prevent trajectories from being everywhere differentiable, resulting in particles exhibiting Brownian trajectories rather than smooth ones, but this in no way implies that there are no trajectories whatsoever.

Consider the special case of a single smooth trajectory represented as

$$\gamma(p,q) = \prod_{i=1}^{N} \delta(\vec{q}_i - \vec{q}_i^{\,0})\delta(\vec{p}_i - \vec{p}_i^{\,0}) \tag{12}$$

in a LPS model where the superscript 0 indicates the contribution from the unperturbed Hamiltonian. To first order the time evolution of the momentum for the component $i = 1$ is giving by

$$\vec{p}_1(t) = \vec{p}_1^{\,0} + \lim_{\Omega\to\infty} \frac{\lambda}{\Omega} \sum_k \sum_{n=2}^{N} (-\vec{k}) \frac{V_k}{\vec{k}\cdot(\vec{v}_1^{\,0} - \vec{v}_n^{\,0}) - i\varepsilon} \left(e^{-i\vec{k}\cdot(\vec{v}_1^{\,0} - \vec{v}_n^{\,0})t} - 1 \right) e^{-i\vec{k}\cdot(\vec{q}_1^{\,0} - \vec{q}_n^{\,0})}, \tag{13}$$

where Ω is the volume, $\vec{k}$ is the wave vector, $\vec{v}_1$ is the velocity vector of particle 1, $\vec{v}_n$ is the velocity vector of particle n, and ε is an infinitesimal positive constant. The first term represents the contribution from the unperturbed Hamiltonian and the second term represents contributions from the interactions. If N is finite, (13) becomes

$$\vec{p}_1(t) = \vec{p}_1^{\,0} + \lambda \sum_{n=2}^{N} \int d\vec{k} \frac{V_k}{\vec{k}\cdot(\vec{v}_1^{\,0} - \vec{v}_n^{\,0}) - i\varepsilon} \vec{k} e^{-i\vec{k}\cdot(\vec{q}_1^{\,0} - \vec{q}_n^{\,0})} + O(\lambda^2), \tag{14}$$

in the limit $t \to \infty$ because the pole at $\vec{k}\cdot(\vec{v}_1^{\,0} - \vec{v}_n^{\,0}) = i\varepsilon$ vanishes as $\Omega \to \infty$, the LPS condition. According to (14) the value of the momentum to first order asymptotically approaches a constant and the time dependence drops out. Note

[13] The significance of the delta function singularity appears to be more mathematical than physical. Mathematically it means that so-called reduced distribution functions–where the distribution function refers to a subset s of the total number of particles in the system–exists in the thermodynamic limit, but such distribution functions almost always exist for molecules under most realistic forces. Reduced distributions were introduced into nonequilibrium contexts by (Brout and Prigogine 1956; Prigogine and Balescu 1959).

that in the limit $|\vec{q}_1^0 - \vec{q}_n^0| \to \infty$, the interactions from particles n remains finite even if such interactions are short-ranged due to resonances, so that long-range correlations are built up. In the thermodynamic limit, (13) generally diverges and Petrosky and Prigogine conclude that point distributions such as (12) representing trajectories are not physically admissible and, therefore, smooth trajectories are inconsistent with the thermodynamic limit in a LPS (1996, p. 480). Only singular nonlocal distributions appear to be consistent with the thermodynamic limit and such distributions lie outside of HS (Petrosky and Prigogine 1996, pp. 479-81).

These results are related to the nonlocal nature of the collective effects of the entire distribution described in §4.2 above. If any arbitrary finite number of particles were selected within the system and treated in isolation, all nonlocal diffusion and correlation effects become negligible and we are left with the standard description and results in terms of trajectories (however, these trajectories would not necessarily be everywhere differentiable).

In more realistic situations, the nonexistence of smooth trajectories leads directly to the Brussels-Austin claim that a LPS exhibits behavior that *cannot be derived from trajectory dynamics*. Such effects include the breaking of time symmetry (i.e., the appearance of semigroups of operators governing the evolution instead of groups), diffusion and nonlocal correlations. Prigogine and coworkers refer to these effects as "non-Newtonian" to emphasize the fact that the trajectory description is inadequate to account for them. The existence of collision operators such as the Fokker-Planck operator is only a necessary condition for irreversibility and other "non-Newtonian" effects. Particular types of distributions (namely singular distributions) must also be present in order to have sufficient conditions for such behavior. The class of singular distribution functions is quite broad and applicable to many ordinary situations in SM (the canonical distribution is an example; see also Prigogine 1962 and 1997). Petrosky and Prigogine have carried out algebraic and computer modeling to demonstrate that the trajectory and distribution descriptions yield different results for LPS (e.g. 1993, 1994 and 1996).

I believe the appropriate way to understand this new approach with its "non-Newtonian" effects is to agree with them that distribution descriptions cannot be reduced to point-wise descriptions. However, both descriptions should be viewed as valid within their domains. The trajectory description is valid for local regions of a LPS, where there are relatively few particles, so that trajectory dynamics is the dominant feature (the trajectories may be either smooth and exact, or exhibit random walks). Interactions take place among particles at this local level and to the extent that we can ignore higher-order and long-range correlations, trajectory and distribution descriptions agree in their account of physical behavior as was noted earlier.

Where my interpretation of the Brussels-Austin work differs from their own is when the conditions for a LPS are met (large number of particles, continuous frequencies, etc.). I agree that in examining the global evolution of LPS, higher-order correlations and collective effects due to long-range, persistent interactions are the dominant features, which are not reducible to trajectory dynamics alone.

Trajectories are not irrelevant, however, because such features as correlations and collective effects presuppose particle positions and trajectories. For example, collective effects in ordinary gases do not disconfirm the existence of trajectories, though the effects of correlations can rival or exceed the effects of individual particle trajectories and be masked by a dynamical description that treats trajectories as the sole explanatory element. Note that (14) does not imply smooth point trajectories are immediately expunged from a LPS. Physically smooth trajectories are converted into random walks due to the persistent interactions and the long-range higher-order correlations that diffuse throughout the system over time. As described above, resonances, collisions and correlations are closely related to long-range correlations and collective effects, behavioral features of unstable systems for which the trajectory description alone cannot adequately account. For LPS models the whole is more than the sum of its parts. Particle trajectories are necessary for global distributions to exist, but are insufficient for determining how such global distributions evolve in time. The thermodynamic paradox might be dissolved because (1) the time-symmetric behavior of the trajectory dynamics contributes nothing more to the global evolution of the SM system than the necessary conditions for the existence of such a system and (2) in a LPS trajectories exhibit Brownian motion and correlation dynamics dominate the macroscopic dynamics. Thermodynamic behavior is, then, an emergent global phenomenon possessing a temporal direction.

My interpretation suggests a way to reduce the tension in their view between operationalism with respect to trajectories and realism with respect to distributions (see Part I), where the Brownian trajectories of the system give the *necessary* conditions for the existence of the distribution ρ, but *not sufficient* conditions for its evolution. In my judgement the new approach the Brussels-Austin Group has been exploring illuminates some of the underlying physical mechanisms of thermodynamic behavior. Focusing on the growth and dynamics of correlations and collective effects are important physical insights which have advanced our understanding of thermodynamics processes. And by employing extended mathematical structures such as RHS, they have developed powerful tools for describing such processes which will doubtless lead to further insights.

As a last comment, I should point out that this RHS approach does not represent a kind of coarse-graining approach, at least as normally understood. Emphasis shifts away from trajectories because they are only a part of the story of the behavior of a LPS (coarse-grained accounts typically assume that trajectory dynamics is the whole story, but that complete descriptions at the trajectory level are computationally intractable). And, as in the similarity transformation approach, the RHS approach distinguishes between manifolds of stable and unstable motions (in contrast to typical coarse-grained accounts). Furthermore, if the global behavior of a LPS is not only emergent, but also constrains the motion of individual particles (say by restricting the modes of energy transfer), then an appropriate mathematical description should be able to describe this kind of feedback between levels in a system. The RHS approach can describe such feedback effects, whereas coarse-grained accounts cannot because they deal with only one level of a given system. Finally, whether trajectories that are not everywhere continuous nor everywhere differentiable are deterministic or not is an open question in the RHS approach, as I discuss in the next section (coarse-grained accounts typically assume trajectories are deterministic, though usually no explicit assumptions are made regarding the trajectories' continuity and differentiablility).

5 Possibility Rather than Certainty?

Prigogine's provocatively titled book, *The End of Certainty* (1997), sums up one of arguably the most important and far reaching consequences of the Brussels-Austin Group's work: Namely, that the certainty of the deterministic, time-symmetric trajectory description is not applicable to the global dynamics of a LPS. Instead only a statistical description of probability densities remains. In conventional CM and SM models, particle positions and trajectories are treated as the fundamental ontological entities determining the dynamical evolution of the system. In the Brussels-Austin view this is no longer the case for LPS models. The fundamental ontological feature for these models are the probability distributions, i.e., the large-scale arrangements of the particles themselves. To reformulate the laws of classical dynamics along the statistical lines suggested by Prigogine and co-workers leads to the conclusion that such laws now 'express "possibilities" and no more "certainties"' (Petrosky and Prigogine 1997, p. 1).

Where there are relatively few numbers of particles, the Brussels-Austin Group's approach to dynamics reduces to the standard results of CM, so the trajectory picture with its deterministic and time-reversible character is preserved as a limiting case. In non-LPS cases, the RHS approach recovers the usual results of SM (e.g. Fokker-Planck equation, Boltzmann equation, non-Markovian master equations). It is in cases where the LPS criteria apply that probability becomes the fundamental notion, irreducible to the trajectory description. Systems must be treated as wholes. If any subset of the total number of particles N is treated by itself all the "non-Newtonian" effects disappear and the conventional descriptions are recovered. It is in this sense that Prigogine believes, 'What is now emerging is an "intermediate" description that lies somewhere between the two alienating images of a deterministic world and an arbitrary world of pure chance...[T]he new laws of nature deal with the possibility of events, but do not reduce these events to deductible, predictable consequences' (Prigogine 1997, p. 189).

The nature of this possibility supposedly represents a new conception which remains to be clarified, however. It is clearly not the kind of irreducible indeterminism described in von Neumann collapse, where some sort of collapse from multiple possibilities to a single actuality is envisioned. As Prigogine and colleagues describe it, their probabilistic formulation of physics is also to be distinguished from the type of chaotic dynamics, where the underlying dynamics is deterministic, but the outcomes of the system are not predictable. The latter is *epistemically* indeterminable but not *ontically* indeterministic.[14] Instead

[14]Understanding what it means for a system or a description to be ontically indeterministic

the dynamics envisioned by Prigogine and his colleagues involve an interplay between unitary reversible processes and irreversible processes. The LPS are important examples of dynamical systems which show this kind of interplay and are, therefore, intrinsically probabilistic.

But the relationship of this probabilistic evolution to deterministic dynamics remains unclear and requires attention because under some conditions the dynamics of probability distributions can be "embedded" into completely deterministic dynamics and Markov processes can almost always be "embedded" into deterministic Kolmogorov processes (Antoniou and Gustafson 1993; Gustafson 1997, pp. 55-76). This leaves open the possibility that there is no *significant fundamental difference between this new conception of probabilistic evolution and the conventional conception of deterministic evolution*, or so one could plausibly argue.[15]

Though more needs to be said regarding the notion of probabilistic dynamics they are working out, it must be internally generated by the dynamics of the system (e.g. *via* correlation dynamics) rather than imposed from the outside by observers, measuring apparatuses or the environment. I do not take it that this need for more clarification is a serious weakness of their program. On the contrary, I look at the situation as analogous to the early days of quantum theory where many concepts (indeterminacy being one of them) were very hazy at the start inviting serious reflection and exploration.

The RHS formalism gives us a unified description of dynamics and thermodynamics within a statistical framework and a consistent, rigorous description of irreversible processes. The mathematical developments are indeed impressive, including new results regarding the theory of complex spectral representations of operators. Furthermore this framework is powerful enough to allow a unification between CM and QM (Prigogine et al. 1991; Petrosky, Prigogine and Tasaki 1991; Petrosky and Prigogine 1994). However, the promise of the recent Brussels-Austin work must be balanced against two important open questions: (1) What is the physical and mathematical status of the past-directed $t \leq 0$ semigroup (§3.3) and (2) What is the precise nature of the probability lying at the heart of an LPS? Answering these two questions holds the key to their being able to offer an explanation for the thermodynamic arrow of time and for their developing a notion of indeterminism that is different in kind from that discussed in conventional QM developments that would be truly revolutionary.

As things stand, the Brussels-Austin Group has given us a powerful descriptive tool for irreversible processes, and nonlinear dynamics more generally, but they have not given us an explanation for the origination of the irreversibility we observe in our world. One might object that the RHS approach is ultimately

is by no means straightforward (e.g. Bishop 2002).

[15] I should point out that although there may exist theorems showing that given any Markov process, that process can be embedded in a larger deterministic Kolmogorov process, the general result does not necessarily mean that the given Markov process is deterministic. Whether or not a given Markov process is deterministic or not is an ontological rather than a mathematical question. It should also be clear, however, that simply characterizing the probability densities via Kolmogorov measures is insufficient because this cannot settle the ontological nature of the probability.

only of mathematical interest since there is nothing philosophically interesting given the current state of the above open questions. This response is too quick, however. These open questions can also be viewed as opportunities for exploration of the underlying concepts of the approach in order to attempt to answer these questions. For example, by adopting a different arrow of time in the context of scattering in a RHS formulation of QM, one can show that the $t \leq 0$ semigroup is also future oriented (this time arrow is, however, highly operational in character and not generally applicable outside of laboratory contexts; for discussion, see Bishop 2003a and 2003b). So interesting conceptual questions are raised by the Brussels-Austin work. Besides, even if questions (1) and (2) should ultimately be answered in a way that closes off this avenue for nonequilibrium SM, that information is also valuable to philosophers.

Acknowledgments–This essay was considerably strengthened from discussions with and comments by Ioannis Antoniou, Harald Atmanspacher, Jeremy Butterfield, Dean Driebe, Fred Kronz, Tomio Petrosky, Michael Redhead, Michael Silberstein and numerous anonymous referees. I take full responsibility for remaining weaknesses.

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