Quine's Conjecture on Many-Sorted Logic

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Abstract

In this paper we settle a conjecture suggested by Quine (1937, 1938, 1951, 1956, 1960, 1963). Our theorem makes precise the relationship between many-sorted logic and single-sorted logic and yields a remark about a criterion for theoretical equivalence proposed by Glymour (1970, 1977, 1980).

Introduction

Quine expressed the following thought about the relationship between many-sorted logic and single-sorted logic:

> Every many-sorted theory "is equivalent to" a single-sorted theory.

We will call this claim *Quine's conjecture.*[1]

In this paper we aim to capture the sense in which Quine's conjecture is true. Although the basic idea behind the conjecture is clear, before proving it one needs to make precise what it means for two theories to be "equivalent." We consider three ways to make Quine's conjecture precise. We show that the first two versions of the conjecture are false, but we conclude by proving the third.

Preliminaries

We begin with some preliminaries about many-sorted logic.[2] A **signature** Σ is a set of sort symbols, predicate symbols, function symbols, and constant

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[1]For example, see Quine (1951, 69–71), Quine (1960, 209–10), and Quine (1963, 267–8). He explains the conjecture as follows: "[...] we can always reduce multiple sorts of variables to one sort if we adopt appropriate predicates. Wherever we might have used a special sort of variable we may use instead a general variable and *restrict* it to the appropriate predicate" (Quine, 1963, 268). Quine (1937, 1938, 1956) provides support for the conjecture by describing a method of "translating" between many-sorted and single-sorted logic and applying it to von Neumann-Bernays set theory and Russell's theory of types.

[2]The reader is encouraged to consult Hodges (2008) and Barrett and Halvorson (2015b) for details.

symbols. Every signature is required to contain at least one sort symbol. The predicate, function, and constant symbols in Σ are assigned **arities** constructed from sorts in Σ. The arity of a symbol specifies which sorts the symbol "applies to." The Σ-terms, Σ-formulas, and Σ-sentences are recursively defined in the standard way. The only difference from the syntax of single-sorted logic is that the quantifiers $\forall_\sigma$ and $\exists_\sigma$ that appear in Σ-formulas must be indexed by sorts $\sigma \in \Sigma$. We will use the notation $\exists_{\sigma=1} x\phi(x)$ to abbreviate the sentence "there exists a unique x of sort σ such that $\phi(x)$."

A **Σ-structure** A is a family of nonempty and pairwise disjoint sets A_σ, one for each sort symbol $\sigma \in \Sigma$, in which the predicates, functions, and constant symbols in Σ have been interpreted. One recursively defines when elements $a_1, \ldots, a_n \in A$ **satisfy** a Σ-formula $\phi(x_1, \ldots, x_n)$ in the Σ-structure A, written $A \vDash \phi[a_1, \ldots, a_n]$.

A **Σ-theory** T is a set of Σ-sentences. The sentences $\phi \in T$ are called the axioms of T. If the signature Σ has only one sort symbol, then the Σ-theory T is called a **single-sorted theory**, while if Σ has more than one sort symbol, then T is called a **many-sorted theory**. A Σ-structure M is a **model** of a Σ-theory T if $M \vDash \phi$ for all $\phi \in T$. We will use the notation $\mathrm{Mod}(T)$ to denote the class of models of a theory T. A theory T **entails** a sentence ϕ, written $T \vDash \phi$, if $M \vDash \phi$ for every model M of T.

We begin with the following preliminary criterion for theoretical equivalence.

Definition. Theories T_1 and T_2 are **logically equivalent** if they have the same class of models, i.e. if $\mathrm{Mod}(T_1) = \mathrm{Mod}(T_2)$.

One can verify that theories T_1 and T_2 are logically equivalent if and only if $\{\phi : T_1 \vDash \phi\} = \{\psi : T_2 \vDash \psi\}$. It is easy to see that logical equivalence is too strict to capture the sense in which Quine's conjecture is true. Theories can only be logically equivalent if they are formulated in the same signature, so no many-sorted theory is logically equivalent to a single-sorted theory.

Since logical equivalence is such a strict criterion for theoretical equivalence, logicians and philosophers of science have proposed other criteria for theoretical equivalence.[3] We will begin by considering a criterion called *definitional equivalence* that was introduced into philosophy of science by Glymour (1970, 1977, 1980).

The first version of Quine's conjecture is the following.

[3]See Quine (1975), Sklar (1982), Halvorson (2012, 2013, 2015), Glymour (2013), van Fraassen (2014), and Coffey (2014) for general discussion of theoretical equivalence in philosophy of science. See Glymour (1977), North (2009), Swanson and Halvorson (2012), Curiel (2014), Knox (2013), Barrett (2014), Weatherall (2015a,b,c), and Rosenstock et al. (2015) for discussion of whether or not particular physical theories should be considered theoretically equivalent. Finally, see de Bouvére (1965), Kanger (1968), Pinter (1978), Pelletier and Urquhart (2003), Andréka et al. (2005), Friedman and Visser (2014), and Barrett and Halvorson (2015a,b) for some results that have been proven about varieties of theoretical equivalence.

Quine's conjecture 1

Every theory is definitionally equivalent to a single-sorted theory. ⌟

In order to understand Quine's conjecture 1, we need to describe definitional equivalence. We begin by formalizing the concept of a definition.

Let $\Sigma \subset \Sigma^+$ be signatures and let $p \in \Sigma^+ - \Sigma$ be a predicate symbol of arity $\sigma_1 \times \ldots \times \sigma_n$. An **explicit definition of p in terms of $\boldsymbol{\Sigma}$** is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \ldots \forall_{\sigma_n} x_n \big(p(x_1, \ldots, x_n) \leftrightarrow \phi(x_1, \ldots, x_n) \big)$$

where $\phi(x_1, \ldots, x_n)$ is a Σ-formula. Similarly, an explicit definition of a function symbol $f \in \Sigma^+ - \Sigma$ of arity $\sigma_1 \times \ldots \times \sigma_n \to \sigma$ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \ldots \forall_{\sigma_n} x_n \forall_\sigma y \big(f(x_1, \ldots, x_n) = y \leftrightarrow \phi(x_1, \ldots, x_n, y) \big) \tag{1}$$

and an explicit definition of a constant symbol $c \in \Sigma^+ - \Sigma$ of sort σ is a Σ^+-sentence of the form

$$\forall_\sigma x \big(x = c \leftrightarrow \psi(x) \big) \tag{2}$$

where $\phi(x_1, \ldots, x_n, y)$ and $\psi(x)$ are both Σ-formulas. Note that in all of these cases it must be that the sorts $\sigma_1, \ldots, \sigma_n, \sigma \in \Sigma$.

Although they are Σ^+-sentences, (1) and (2) have consequences in the signature Σ. In particular, (1) and (2) imply the following sentences, respectively:

$$\forall_{\sigma_1} x_1 \ldots \forall_{\sigma_n} x_n \exists_{\sigma=1} y \phi(x_1, \ldots, x_n, y)$$
$$\exists_{\sigma=1} x \psi(x)$$

These two sentences are called the **admissibility conditions** for the explicit definitions (1) and (2).

We now have the resources necessary to describe the concept of a definitional extension. A **definitional extension** of a Σ-theory T to the signature Σ^+ is a theory

$$T^+ = T \cup \{\delta_s : s \in \Sigma^+ - \Sigma\}$$

that satisfies the following two conditions. First, for each symbol $s \in \Sigma^+ - \Sigma$ the sentence δ_s is an explicit definition of s in terms of Σ, and second, if s is a constant symbol or a function symbol and α_s is the admissibility condition for δ_s, then $T \vDash \alpha_s$.

One can think of a definitional extension of a theory as "saying no more" than the original theory (Barrett and Halvorson, 2015b). It simply allows one to add "abbreviations" of old formulas to the theory. With this thought in mind, we can describe definitional equivalence.

Definition. Let T_1 be a Σ_1-theory and T_2 be a Σ_2-theory. T_1 and T_2 are **definitionally equivalent** if there are theories T_1^+ and T_2^+ that satisfy the following three conditions:

- T_1^+ is a definitional extension of T_1,

- T_2^+ is a definitional extension of T_2,
- T_1^+ and T_2^+ are logically equivalent $\Sigma_1 \cup \Sigma_2$-theories.

One often says that T_1 and T_2 are definitionally equivalent if they have a "common definitional extension." Definitional equivalence captures a sense in which two theories are "intertranslatable" (Barrett and Halvorson, 2015a).

One can easily verify that definitional equivalence is a strictly weaker criterion than logical equivalence. Unlike logical equivalence, theories in different signatures can be definitionally equivalent. But definitional equivalence is still incapable of substantiating Quine's conjecture. As we have described it, a definitional extension does not allow one to define new sorts. If T_1 and T_2 are definitionally equivalent, therefore, they must be formulated in signatures with the same sort symbols. So no many-sorted theory is definitionally equivalent to a single-sorted theory. Quine's conjecture 1 is therefore **false**.

Fortunately, there are criteria for theoretical equivalence that are more general than definitional equivalence. The one that will be of particular interest to us is called *Morita equivalence*.[4] Morita equivalence is a natural generalization of definitional equivalence. Indeed, it is essentially the same as definitional equivalence, but it allows one to define new sort symbols in addition to new predicate, function, and constant symbols.

Our second version of Quine's conjecture is the following.

Quine's conjecture 2

Every theory is Morita equivalent to a single-sorted theory. ⌟

In order to understand Quine's conjecture 2, we need to describe Morita equivalence. We begin by discussing how to define new sort symbols. Let $\Sigma \subset \Sigma^+$ be signatures and consider a sort symbol $\sigma \in \Sigma^+ - \Sigma$. One can define the sort σ as a product sort, a coproduct sort, a subsort, or a quotient sort. In each case one defines σ using old sorts from Σ and new function symbols from $\Sigma^+ - \Sigma$. These new function symbols specify how the new sort σ is related to the old sorts in Σ. We describe in detail these four ways to define new sorts.

In order to define σ as a product sort, one needs two function symbols $\pi_1, \pi_2 \in \Sigma^+ - \Sigma$ with π_1 of arity $\sigma \to \sigma_1$, π_2 of arity $\sigma \to \sigma_2$, and $\sigma_1, \sigma_2 \in \Sigma$. The function symbols π_1 and π_2 serve as the "canonical projections" associated with the product sort σ. An explicit definition of the symbols σ, π_1, and π_2 as a **product sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x \forall_{\sigma_2} y \exists_{\sigma=1} z (\pi_1(z) = x \wedge \pi_2(z) = y)$$

One should think of a product sort σ as the sort whose elements are ordered pairs, where the first element of each pair is of sort σ_1 and the second is of sort σ_2.

[4]See Barrett and Halvorson (2015b) for an introduction to Morita equivalence and Andréka et al. (2008) for a presentation of closely related ideas.

One can also define σ as a coproduct sort. In this case, one needs two function symbols $\rho_1, \rho_2 \in \Sigma^+ - \Sigma$ with ρ_1 of arity $\sigma_1 \to \sigma$, ρ_2 of arity $\sigma_2 \to \sigma$, and $\sigma_1, \sigma_2 \in \Sigma$. The function symbols ρ_1 and ρ_2 are the "canonical injections" associated with the coproduct sort σ. An explicit definition of the symbols σ, ρ_1, and ρ_2 as a **coproduct sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_\sigma z\big(\exists_{\sigma_1=1} x(\rho_1(x) = z) \vee \exists_{\sigma_2=1} y(\rho_2(y) = z)\big) \wedge \forall_{\sigma_1} x \forall_{\sigma_2} y \neg\big(\rho_1(x) = \rho_2(y)\big)$$

One should think of a coproduct sort σ as the disjoint union of the elements of sorts σ_1 and σ_2.

When defining a new sort σ as a product sort or a coproduct sort, one uses two sort symbols in Σ and two function symbols in $\Sigma^+ - \Sigma$. The next two ways of defining a new sort σ only require one sort symbol in Σ and one function symbol in $\Sigma^+ - \Sigma$.

In order to define σ as a subsort, one needs a function symbol $i \in \Sigma^+ - \Sigma$ of arity $\sigma \to \sigma_1$ with $\sigma_1 \in \Sigma$. The function symbol i is the "canonical inclusion" associated with the subsort σ. An explicit definition of the symbols σ and i as a **subsort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x\big(\phi(x) \leftrightarrow \exists_\sigma z(i(z) = x)\big) \wedge \forall_\sigma z_1 \forall_\sigma z_2\big(i(z_1) = i(z_2) \to z_1 = z_2\big) \qquad (3)$$

where $\phi(x)$ is a Σ-formula. One should think of σ as "the things of sort σ_1 that are ϕ." The sentence (3) entails the following Σ-sentence:

$$\exists_{\sigma_1} x \phi(x)$$

As above, we will call this Σ-sentence the **admissibility condition** for the definition (3).

Lastly, in order to define σ as a quotient sort one needs a function symbol $\epsilon \in \Sigma^+ - \Sigma$ of arity $\sigma_1 \to \sigma$ with $\sigma_1 \in \Sigma$. An explicit definition of the symbols σ and ϵ as a **quotient sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2\big(\epsilon(x_1) = \epsilon(x_2) \leftrightarrow \phi(x_1, x_2)\big) \wedge \forall_\sigma z \exists_{\sigma_1} x(\epsilon(x) = z) \qquad (4)$$

where $\phi(x_1, x_2)$ is a Σ-formula. This sentence defines σ as a quotient sort that is obtained by "quotienting out" the sort σ_1 with respect to the formula $\phi(x_1, x_2)$. The sort σ should be thought of as the set of "equivalence classes of elements of σ_1 with respect to the relation $\phi(x_1, x_2)$," and the function symbol ϵ is the "canonical projection" that maps an element to its equivalence class. And indeed, one can verify that the sentence (4) implies that $\phi(x_1, x_2)$ is an equivalence relation. In particular, (4) entails the following Σ-sentences:

$$\forall_{\sigma_1} x(\phi(x, x))$$
$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2(\phi(x_1, x_2) \to \phi(x_2, x_1))$$
$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2 \forall_{\sigma_1} x_3\big((\phi(x_1, x_2) \wedge \phi(x_2, x_3)) \to \phi(x_1, x_3)\big)$$

These Σ-sentences are the **admissibility conditions** for the definition (4).

Now that we have described the four ways of defining new sort symbols, we can define the concept of a Morita extension. A Morita extension is a natural generalization of a definitional extension. The only difference is that now one is allowed to define new sort symbols. Let $\Sigma \subset \Sigma^+$ be signatures and T a Σ-theory. A **Morita extension** of T to the signature Σ^+ is a Σ^+-theory

$$T^+ = T \cup \{\delta_s : s \in \Sigma^+ - \Sigma\}$$

that satisfies the following three conditions. First, for each symbol $s \in \Sigma^+ - \Sigma$ the sentence δ_s is an explicit definition of s in terms of Σ. Second, if $\sigma \in \Sigma^+ - \Sigma$ is a sort symbol and $f \in \Sigma^+ - \Sigma$ is a function symbol that is used in the explicit definition of σ, then $\delta_f = \delta_\sigma$. (For example, if σ is defined as a product sort with projections π_1 and π_2, then $\delta_\sigma = \delta_{\pi_1} = \delta_{\pi_2}$.) And third, if α_s is an admissibility condition for a definition δ_s, then $T \vDash \alpha_s$.

A Morita extension of a theory again "says no more" than the original theory (Barrett and Halvorson, 2015b). Our definition of Morita equivalence is perfectly analogous to definitional equivalence.

Definition. Let T_1 be a Σ_1-theory and T_2 a Σ_2-theory. T_1 and T_2 are **Morita equivalent** if there are theories $T_1^1, \ldots, T_1^n$ and $T_2^1, \ldots, T_2^m$ that satisfy the following three conditions:

- Each theory T_1^{i+1} is a Morita extension of T_1^i,
- Each theory T_2^{i+1} is a Morita extension of T_2^i,
- T_1^n and T_2^m are logically equivalent Σ-theories with $\Sigma_1 \cup \Sigma_2 \subset \Sigma$.

Two theories are Morita equivalent if they have a "common Morita extension." One can easily verify that Morita equivalence is a strictly weaker criterion than definitional equivalence. If two theories are definitionally equivalent, then they are Morita equivalent. But in general the converse does not hold. Theories can be Morita equivalent even if they are formulated in signatures with different sort symbols.

We can now consider Quine's conjecture 2. Although this version of the conjecture is not trivially refuted like Quine's conjecture 1, it too is false. The following theorem provides an example of a theory that is not Morita equivalent to any single-sorted theory.

Theorem 1. *Let $\Sigma_1 = \{\sigma_1, \sigma_2, \ldots\}$ be a signature containing a countable infinity of sort symbols. The Σ_1-theory $T_1 = \emptyset$ is not Morita equivalent to any single-sorted theory.*

The idea behind Theorem 1 is simple. One can think of the theory T_1 as saying the following: "Every element is either of kind$_1$ or of kind$_2$ or of kind$_3$ or..., no element is of more than one kind, and there is at least one element of every kind." A single-sorted theory in first-order logic simply does not have the expressive power to say this. In particular, it cannot express the first conjunct. The theory T_1 should therefore not be Morita equivalent to any single-sorted theory. Before proving this we need a simple lemma.

Lemma. *Let $\Sigma \subset \Sigma^+$ be signatures and T a Σ-theory. Suppose that A is a model of T with A_σ a finite set for every sort $\sigma \in \Sigma$. Let T^+ be a Morita extension of T to a signature Σ^+ and A^+ a model of T^+ such that $A^+|_\Sigma = A$.*[5] *Then A^+_σ is a finite set for every $\sigma \in \Sigma^+$.*

Proof. Let $\sigma \in \Sigma^+ - \Sigma$ be a sort symbol. We show that A^+_σ is a finite set in the cases where σ is defined as a product sort or a subsort. If T^+ defines σ as a product sort of σ_1 and σ_2, then A^+_σ has exactly as many elements as $A_{\sigma_1} \times A_{\sigma_2}$, which is finite by assumption. If T^+ defines σ as a subsort of $\sigma_1 \in \Sigma$, then the cardinality of A^+_σ is less than or equal to the cardinality of A_{σ_1}, which is also finite. The coproduct and quotient cases follow analogously. □

We now turn to the proof of Theorem 1.[6]

Proof of Theorem 1. Suppose for contradiction that there is a single-sorted theory T_2 that is Morita equivalent to T_1. This means that T_2 is a Σ_2-theory with $\sigma \in \Sigma_2$ the unique sort symbol. Let T be the "common Morita extension" of T_1 and T_2 to a signature $\Sigma \supset \Sigma_1 \cup \Sigma_2$. We consider the model A of T_1 defined by $A_{\sigma_i} = \{i, i'\}$ for each $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ there is an isomorphism $f_i : A \to A$ that is the identity on A_{σ_j} for $j \neq i$, but on A_{σ_i} maps $f_i : i \mapsto i'$ and $f_i : i' \mapsto i$. The $f_i : A \to A$ are isomorphisms, and so are elementary embeddings. This implies that there are infinitely many arrows $f : A \to A$ in the category $\mathrm{Mod}(T_1)$.

There is an equivalence of categories $F : \mathrm{Mod}(T_1) \to \mathrm{Mod}(T_2)$ such that for every model M of T_1

$$F(M) = M^+|_{\Sigma_2}$$

for some model M^+ of T that is isomorphic to an expansion of M (Barrett and Halvorson, 2015b, Theorem 5.1). We consider the model A^+ of T. The Lemma implies that A^+_σ is a finite set. This implies that $F(A)_\sigma$ is a finite set. Since Σ_2 contains only the sort σ and $F(A)_\sigma$ is finite, there can be at most finitely many arrows $g : F(A) \to F(A)$ in the category $\mathrm{Mod}(T_2)$. But since F is an equivalence (and therefore full and faithful), this cannot be the case. □

Theorem 1 immediately implies that Quine's conjecture 2 is **false**. It is not the case that every many-sorted theory is Morita equivalent to a single-sorted theory. This disproof of Quine's conjecture 2, however, suggests the following slight modification of the conjecture.

Quine's conjecture 3

If Σ is a signature with finitely many sorts, then every Σ-theory is Morita equivalent to a single-sorted theory. ⌟

[5] One can show that such a model A^+ exists and is unique up to isomorphism (Barrett and Halvorson, 2015b, Theorem 4.1).

[6] The reader is encouraged to consult Barrett and Halvorson (2015b) for notation.

This final version of Quine's conjecture is true. One proves Quine's conjecture 3 by explicitly constructing a "corresponding" single-sorted theory $\widehat{T}$ for every many-sorted theory T. The basic idea behind the construction is intuitive. The theory $\widehat{T}$ simply replaces the sort symbols that the theory T uses with predicate symbols.[7] It takes some work, however, to make this idea precise.

Let Σ be a signature with finitely many sort symbols $\sigma_1, \ldots, \sigma_n$. We begin by constructing a corresponding signature $\widehat{\Sigma}$ that contains one sort symbol σ. The symbols in $\widehat{\Sigma}$ are defined as follows. For every sort symbol $\sigma_j \in \Sigma$ we let q_{σ_j} be a predicate symbol of sort σ. For every predicate symbol $p \in \Sigma$ of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m}$ we let q_p be a predicate symbol of arity σ^m (the m-fold product of σ). Likewise, for every function symbol $f \in \Sigma$ of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m} \to \sigma_j$ we let q_f be a predicate symbol of arity σ^{m+1}. And lastly, for every constant symbol $c \in \Sigma$ we let d_c be a constant symbol of sort σ. The single-sorted signature $\widehat{\Sigma}$ corresponding to Σ is then defined to be

$$\widehat{\Sigma} = \{\sigma\} \cup \{q_{\sigma_1}, \ldots, q_{\sigma_n}\} \cup \{q_p : p \in \Sigma\} \cup \{q_f : f \in \Sigma\} \cup \{d_c : c \in \Sigma\}$$

We can now describe a method of "translating" Σ-theories into $\widehat{\Sigma}$-theories. Let T be an arbitrary Σ-theory. We define a corresponding $\widehat{\Sigma}$-theory $\widehat{T}$, and then show that $\widehat{T}$ is Morita equivalent to T.

We begin by translating the axioms of T into the signature $\widehat{\Sigma}$. This will take two steps. First, we describe a way to translate the Σ-terms into $\widehat{\Sigma}$-formulas. Given a Σ-term $t(x_1, \ldots, x_n)$, we define the $\widehat{\Sigma}$-formula $\widehat{\psi}_t(y_1, \ldots, y_n, y)$ recursively as follows.

- If $t(x_1, \ldots, x_n)$ is the variable x_i, then $\widehat{\psi}_t$ is the $\widehat{\Sigma}$-formula $y_i = y$.
- If $t(x_1, \ldots, x_n)$ is the constant c, then $\widehat{\psi}_t$ is the $\widehat{\Sigma}$-formula $d_c = y$.
- Suppose that $t(x_1, \ldots, x_n)$ is the term $f(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n))$ and that each of the $\widehat{\Sigma}$-formulas $\widehat{\psi}_{t_i}(y_1, \ldots, y_n, y)$ have been defined. Then $\widehat{\psi}_t(y_1, \ldots, y_n, y)$ is the $\widehat{\Sigma}$-formula

  $$\exists_\sigma z_1 \ldots \exists_\sigma z_k \big(\widehat{\psi}_{t_1}(y_1, \ldots, y_n, z_1) \wedge \ldots \wedge \widehat{\psi}_{t_k}(y_1, \ldots, y_n, z_k) \wedge q_f(z_1, \ldots, z_k, y)\big)$$

One can think of the formula $\psi_t(y_1, \ldots, y_n, y)$ as the translation of the expression "$t(x_1, \ldots, x_n) = x$" into the signature $\widehat{\Sigma}$.

Second, we use this map from Σ-terms to $\widehat{\Sigma}$-formulas to describe a map from Σ-formulas to $\widehat{\Sigma}$-formulas. Given a Σ-formula $\psi(x_1, \ldots, x_n)$, we define the $\widehat{\Sigma}$-formula $\widehat{\psi}(y_1, \ldots, y_n)$ recursively as follows.

[7] This construction recalls the proof that every theory is definitionally equivalent to a theory that uses only predicate symbols (Barrett and Halvorson, 2015a, Prop. 2). Quine (1937, 1938, 1956, 1963) suggests the basic idea behind our proof, as does Burgess (2005, 12). The theorem that we prove here is much more general than Quine's results because we make no assumption about what the theory T is.

- If $\psi(x_1, \ldots, x_n)$ is $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$, where s and t are Σ-terms of sort σ_i, then $\widehat{\psi}(y_1, \ldots, y_n)$ is the $\widehat{\Sigma}$-formula

$$\exists_\sigma z\big(\widehat{\psi}_t(y_1, \ldots, y_n, z) \wedge \widehat{\psi}_s(y_1, \ldots, y_n, z) \wedge q_{\sigma_i}(z)\big)$$

- If $\psi(x_1, \ldots, x_n)$ is $p(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n))$, where $p \in \Sigma$ is a predicate symbol, then $\widehat{\psi}(y_1, \ldots, y_n)$ is the $\widehat{\Sigma}$-formula

$$\exists_\sigma z_1 \ldots \exists_\sigma z_k\big(\widehat{\psi}_{t_1}(y_1, \ldots, y_n, z_1) \wedge \ldots \wedge \widehat{\psi}_{t_k}(y_1, \ldots, y_n, z_k) \wedge q_p(z_1, \ldots, z_k)\big)$$

- This definition extends to all Σ-formulas in the standard way. We define the $\widehat{\Sigma}$-formulas $\widehat{\neg\psi} := \neg\widehat{\psi}$, $\widehat{\psi_1 \wedge \psi_2} := \widehat{\psi_1} \wedge \widehat{\psi_2}$, $\widehat{\psi_1 \vee \psi_2} := \widehat{\psi_1} \vee \widehat{\psi_2}$, and $\widehat{\psi_1 \to \psi_2} := \widehat{\psi_1} \to \widehat{\psi_2}$. Furthermore, if $\psi(x_1, \ldots, x_n, x)$ is a Σ-formula, then we define both of the following:

$$\widehat{\forall_{\sigma_i} x\psi} := \forall_\sigma y(q_{\sigma_i}(y) \to \widehat{\psi}(y_1, \ldots, y_n, y))$$
$$\widehat{\exists_{\sigma_i} x\psi} := \exists_\sigma y(q_{\sigma_i}(y) \wedge \widehat{\psi}(y_1, \ldots, y_n, y))$$

One should think of the formula $\widehat{\psi}$ as the translation of the Σ-formula ψ into the signature $\widehat{\Sigma}$.

This allows us to consider the translations $\widehat{\alpha}$ of the axioms $\alpha \in T$. The single-sorted theory $\widehat{T}$ will have the $\widehat{\Sigma}$-sentences $\widehat{\alpha}$ as some of its axioms. But $\widehat{T}$ will have more axioms than just the sentences $\widehat{\alpha}$. It will also have some **auxiliary axioms**. These auxiliary axioms will guarantee that the symbols in $\widehat{\Sigma}$ "behave like" their counterparts in Σ. We define auxiliary axioms for the predicate symbols $q_{\sigma_1}, \ldots, q_{\sigma_n} \in \widehat{\Sigma}$, $q_p \in \widehat{\Sigma}$, and $q_f \in \widehat{\Sigma}$, and for the constant symbols $d_c \in \widehat{\Sigma}$. We discuss each of these four cases in detail.

We first define auxiliary axioms to guarantee that the symbols $q_{\sigma_1}, \ldots, q_{\sigma_n}$ behave like sort symbols. The $\widehat{\Sigma}$-sentence ϕ is defined to be $\forall_\sigma y(q_{\sigma_1}(y) \vee \ldots \vee q_{\sigma_n}(y))$.[8] Furthermore, for each sort symbol $\sigma_j \in \Sigma$ we define the $\widehat{\Sigma}$-sentence ϕ_{σ_j} to be

$$\exists_\sigma y(q_{\sigma_j}(y)) \wedge \forall_\sigma y\big(q_{\sigma_j}(y) \to (\neg q_{\sigma_1}(y) \wedge \ldots \wedge \neg q_{\sigma_{j-1}}(y) \wedge \neg q_{\sigma_{j+1}}(y) \wedge \ldots \wedge \neg q_{\sigma_n}(y))\big)$$

One can think of the sentences $\phi_{\sigma_1}, \ldots, \phi_{\sigma_n}$, and ϕ as saying that "everything is of some sort, nothing is of more than one sort, and every sort is nonempty."

Next we define auxiliary axioms to guarantee that the symbols q_p, q_f, and d_c behave like their counterparts p, f, and c in Σ. For each predicate symbol $p \in \Sigma$ of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m}$, we define the $\widehat{\Sigma}$-sentence ϕ_p to be

$$\forall_\sigma y_1 \ldots \forall_\sigma y_m \left(q_p(y_1, \ldots, y_m) \to \left(q_{\sigma_{j_1}}(y_1) \wedge \ldots \wedge q_{\sigma_{j_m}}(y_m)\right)\right)$$

[8] Note that if there were infinitely many sort symbols in Σ, then we could not define the $\widehat{\Sigma}$-sentence ϕ in this way.

This sentence guarantees that the predicate q_p has "the appropriate arity." Likewise, for each function symbol $f \in \Sigma$ of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m} \to \sigma_j$ we define the $\widehat{\Sigma}$-sentence ϕ_f to be the conjunction

$$\begin{aligned}&\forall_\sigma y_1 \ldots \forall_\sigma y_m \forall_\sigma y \big(q_f(y_1, \ldots, y_m, y) \to (q_{\sigma_{j_1}}(y_1) \wedge \ldots \wedge q_{\sigma_{j_m}}(y_m) \wedge q_{\sigma_j}(y))\big)\\ &\wedge \forall_\sigma y_1 \ldots \forall_\sigma y_m \big((q_{\sigma_{j_1}}(y_1) \wedge \ldots \wedge q_{\sigma_{j_m}}(y_m)) \to \exists_{\sigma=1} y (q_f(y_1, \ldots, y_m, y))\big)\end{aligned}$$

The first conjunct guarantees that the symbol q_f has "the appropriate arity," and the second conjunct guarantees that q_f behaves like a function. Lastly, if $c \in \Sigma$ is a constant symbol of arity σ_j, then we define the $\widehat{\Sigma}$-sentence ϕ_c to be $q_{\sigma_j}(d_c)$. This sentence guarantees that the constant symbol d_c also has "the appropriate arity."

We now have the resources to define a $\widehat{\Sigma}$-theory $\widehat{T}$ that is Morita equivalent to T.

$$\begin{aligned}\widehat{T} = \{\widehat{\alpha} : \alpha \in T\} &\cup \{\phi, \phi_{\sigma_1}, \ldots, \phi_{\sigma_n}\}\\ &\cup \{\phi_p : p \in \Sigma\}\\ &\cup \{\phi_f : f \in \Sigma\}\\ &\cup \{\phi_c : c \in \Sigma\}\end{aligned}$$

The theory $\widehat{T}$ has two kinds of axioms, the translated axioms of T and the auxiliary axioms. These axioms allow $\widehat{T}$ to imitate the theory T in the signature $\widehat{\Sigma}$. Indeed, one can prove the following result.

Theorem 2. *The theories T and $\widehat{T}$ are Morita equivalent.*

The proof of Theorem 2 requires some work, and has therefore been placed in an appendix. But the idea behind the proof is simple. The theory T needs to define symbols in $\widehat{\Sigma}$. It defines the sort symbol σ as a "universal sort," by taking the coproduct of the sorts $\sigma_1, \ldots, \sigma_n \in \Sigma$. The theory T then defines the symbols q_p, q_f, and d_c in $\widehat{\Sigma}$ simply by using the corresponding symbols p, f, and c in Σ. Likewise, the theory $\widehat{T}$ needs to define the symbols in Σ. It defines the sort symbol σ_j as the subsort of "things that are q_{σ_j}" for each $j = 1, \ldots, n$. And $\widehat{T}$ defines the symbols p, f, and c again by using the corresponding symbols q_p, q_f, and d_c.

Since the theory T was arbitrary, Theorem 2 immediately implies that Quine's conjecture 3 is **true**.

Conclusion

Our proof of Quine's conjecture 3 substantiates Quine's original thought about the relationship between many-sorted logic and single-sorted logic. It captures a precise sense in which single-sorted logic has exactly the same expressive power as (finitely) many-sorted logic. And in addition to making this relationship precise, our results provide a reason to prefer Morita equivalence over definitional equivalence. If one follows Glymour (1970, 1977, 1980) in taking definitional equivalence as the standard for theoretical equivalence, then one cannot recover any sense in which Quine's conjecture is true. If one wants to substantiate Quine's conjecture, one needs to move to Morita equivalence.

One might be tempted to draw a further philosophical conclusion from our proof of Quine's conjecture 3. In several places, Quine himself concludes that this fact implies that many-sorted logic is dispensable. In particular, he suggests that explicitly naming sorts and specifying the sorts of the vocabulary in the signature of a theory is an "artificial device" (Quine and Carnap, 1990, 409), and that we are licensed to ignore it.[9]

We urge one to resist this temptation. Quine's suggestion here is highly misleading, and it is not supported by our result. We have shown that for each theory T in a signature Σ with finitely many sorts, there is a Morita equivalent single-sorted theory $\widehat{T}$. This result does *not* show that we can ignore sorts altogether. Indeed, two single-sorted theories might have different sorts, in which case there is a non-trivial question about whether these sorts can be defined from each other. One theory's single sort might, for example, be definable as a product of another theory's single sort. In particular, this means that two single-sorted theories can be Morita equivalent even if they are not definitionally equivalent.[10] We therefore ignore sorts at our own peril. By doing so, we blind ourselves to the variety of ways in which theories in different signatures can be equivalent.[⋆]

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[9]Quine (1960, 229) makes this attitude particularly clear when he says: "All in all, I find an overwhelming case for a single unpartitioned universe of values of bound variables, and a simple grammar of predication which admits general terms all on an equal footing. Subsidiary distinctions can still be drawn as one pleases, both on methodological considerations and on considerations of natural kind; but we may think of them as distinctions special to the sciences and unreflected in the structure of our notation." For other expressions of this same attitude, see Quine (1951, 69–71), Quine (1963, 267–8), and Quine and Carnap (1990, 409).

[10]For example, the theory T_1 (in a signature containing one sort symbol σ_1) with the axiom "there is exactly one thing" is Morita equivalent to the theory T_2 (in a signature containing one sort symbol σ_2 and a unary predicate p) with the axiom "there are exactly two things, the first is p, and the second is $\neg p$."

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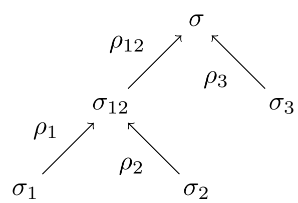
Appendix

The objective of this appendix is to prove Theorem 2. For convenience we prove a special case of the result. We will assume that Σ has only three sort symbols $\sigma_1, \sigma_2, \sigma_3$ and that Σ does not contain function or constant symbols. A perfectly analogous (though more tedious) proof goes through in the general case.

We prove the result by explicitly constructing a "common Morita extension" $T_4 \cong \widehat{T}_4$ of T and $\widehat{T}$ to the following signature.

$$\Sigma^+ = \Sigma \cup \widehat{\Sigma} \cup \{\sigma_{12}\} \cup \{\rho_1, \rho_2, \rho_{12}, \rho_3\} \cup \{i_1, i_2, i_3\}$$

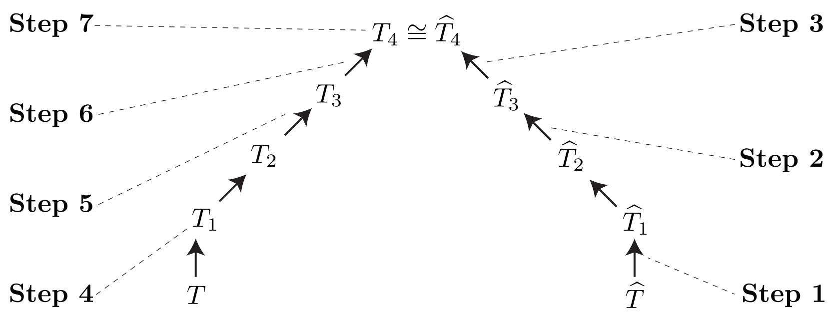
The symbol $\sigma_{12} \in \Sigma^+$ is a sort symbol. The symbols denoted by subscripted ρ are function symbols. Their arities are expressed in the following figure.



The symbols i_1, i_2, and i_3 are function symbols with arity $\sigma_1 \to \sigma$, $\sigma_2 \to \sigma$, and $\sigma_3 \to \sigma$, respectively.

We now turn to the proof.

Proof of Theorem 2. The following figure illustrates how our proof will be organized.



Steps 1–3 define the theories $\widehat{T}_1, \ldots, \widehat{T}_4$, steps 4–6 define $T_1, \ldots, T_4$, and step 7 shows that T_4 and $\widehat{T}_4$ are logically equivalent.

Step 1. We begin by defining the theory $\widehat{T}_1$. For each sort $\sigma_j \in \Sigma$ we consider the following sentence.

$$\begin{aligned} &\forall_\sigma y \left(q_{\sigma_j}(y) \leftrightarrow \exists_{\sigma_j} x(i_j(x) = y)\right) \\ &\quad \wedge \forall_{\sigma_j} x_1 \forall_{\sigma_j} x_2 (i_j(x_1) = i_j(x_2) \to x_1 = x_2) \end{aligned} \tag{θ_{σ_j}}$$

The sentence θ_{σ_j} defines the symbols σ_j and i_j as the subsort of "things that are q_{σ_j}." The auxiliary axioms ϕ_{σ_j} of $\widehat{T}$ guarantee that the admissibility conditions for these definitions are satisfied. The theory $\widehat{T}_1 = \widehat{T} \cup \{\theta_{\sigma_1}, \theta_{\sigma_2}, \theta_{\sigma_3}\}$ is therefore a Morita extension of $\widehat{T}$ to the signature $\widehat{\Sigma} \cup \{\sigma_1, \sigma_2, \sigma_3, i_1, i_2, i_3\}$.

Step 2. We now define the theories $\widehat{T}_2$ and $\widehat{T}_3$. Let $\theta_{\sigma_{12}}$ be a sentence that defines the symbols $\sigma_{12}, \rho_1, \rho_2$ as a coproduct sort. The theory $\widehat{T}_2 = \widehat{T}_1 \cup \{\theta_{\sigma_{12}}\}$

is clearly a Morita extension of $\widehat{T}_1$. We have yet to define the function symbols ρ_{12} and ρ_3. The following two sentences define these symbols.

$$\forall_{\sigma_3} x \forall_\sigma y(\rho_3(x) = y \leftrightarrow i_3(x) = y) \qquad (\theta_{\rho_3})$$

$$\forall_{\sigma_{12}} x \forall_\sigma y(\rho_{12}(x) = y \leftrightarrow \psi(x, y)) \qquad (\theta_{\rho_{12}})$$

The sentence θ_{ρ_3} simply defines ρ_3 to be equal to the function i_3. For the sentence $\theta_{\rho_{12}}$, we define the formula $\psi(x, y)$ to be

$$\exists_{\sigma_1} z_1\big(\rho_1(z_1) = x \wedge i_1(z_1) = y\big) \vee \exists_{\sigma_2} z_2\big(\rho_2(z_2) = x \wedge i_2(z_2) = y\big)$$

We should take a moment here to understand the definition $\theta_{\rho_{12}}$. We want to define what the function ρ_{12} does to an element a of sort σ_{12}. Since the sort σ_{12} is the coproduct of the sorts σ_1 and σ_2, the element a must "actually be" of one of the sorts σ_1 or σ_2. (The disjuncts in the formula $\psi(x, y)$ correspond to these possibilities.) The definition $\theta_{\rho_{12}}$ stipulates that if a is "actually" of sort σ_j, then the value of ρ_{12} at a is the same as the value of i_j at a. One can verify that $\widehat{T}_2$ satisfies the admissibility conditions for θ_{ρ_3} and $\theta_{\rho_{12}}$, so the theory $\widehat{T}_3 = \widehat{T}_2 \cup \{\theta_{\rho_3}, \theta_{\rho_{12}}\}$ is a Morita extension of $\widehat{T}_2$ to the signature

$$\widehat{\Sigma} \cup \{\sigma_1, \sigma_2, \sigma_3, \sigma_{12}, i_1, i_2, i_3, \rho_1, \rho_2, \rho_3, \rho_{12}\}$$

Step 3. We now describe the Σ^+-theory $\widehat{T}_4$. This theory defines the predicates in the signature Σ. Let $p \in \Sigma$ be a predicate symbol of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m}$. We consider the following sentence.

$$\forall_{\sigma_{j_1}} x_1 \ldots \forall_{\sigma_{j_m}} x_m \, (p(x_1, \ldots, x_m) \leftrightarrow q_p(i_{j_1}(x_1), \ldots, i_{j_m}(x_m))) \qquad (\theta_p)$$

The theory $\widehat{T}_4 = \widehat{T}_3 \cup \{\theta_p : p \in \Sigma\}$ is therefore a Morita extension of $\widehat{T}_3$ to the signature Σ^+.

Step 4. We turn to the left-hand side of our organizational figure and define the theories T_1 and T_2. We proceed in an analogous manner to the first part of Step 2. The theory $T_1 = T \cup \{\theta_{\sigma_{12}}\}$ is a Morita extension of T to the signature $\Sigma \cup \{\sigma_{12}, \rho_1, \rho_2\}$. Now let θ_σ be the sentence that defines the symbols $\sigma, \rho_{12}, \rho_3$ as a coproduct sort. The theory $T_2 = T_1 \cup \{\theta_\sigma\}$ is a Morita extension of T_1 to the signature $\Sigma \cup \{\sigma_{12}, \sigma, \rho_1, \rho_2, \rho_3, \rho_{12}\}$.

Step 5. This step defines the function symbols i_1, i_2, and i_3. We consider the following sentences.

$$\forall_{\sigma_3} x_3 \forall_\sigma y(i_3(x_3) = y \leftrightarrow \rho_3(x_3) = y) \qquad (\theta_{i_3})$$

$$\forall_{\sigma_2} x_2 \forall_\sigma y\big(i_2(x_2) = y \leftrightarrow \exists_{\sigma_{12}} z(\rho_2(x_2) = z \wedge \rho_{12}(z) = y)\big) \qquad (\theta_{i_2})$$

$$\forall_{\sigma_1} x_1 \forall_\sigma y\big(i_1(x_1) = y \leftrightarrow \exists_{\sigma_{12}} z(\rho_1(x_1) = z \wedge \rho_{12}(z) = y)\big) \qquad (\theta_{i_1})$$

The sentence θ_{i_3} defines the function symbol i_3 to be equal to ρ_3. The sentence θ_{i_2} defines the function symbol i_2 to be equal to the composition "$\rho_{12} \circ \rho_2$." Likewise, the sentence θ_{i_1} defines the function symbol i_1 to be "$\rho_{12} \circ \rho_1$." The theory $T_3 = T_2 \cup \{\theta_{i_1}, \theta_{i_2}, \theta_{i_3}\}$ is a Morita extension of T_2 to the signature $\Sigma \cup \{\sigma_{12}, \sigma, \rho_1, \rho_2, \rho_3, \rho_{12}, i_1, i_2, i_3\}$.

Step 6. We still need to define the predicate symbols in $\widehat{\Sigma}$. Let $\sigma_j \in \Sigma$ be a sort symbol and $p \in \Sigma$ a predicate symbol of arity $\sigma_{j_1} \times \ldots \times \sigma_{j_m}$. We consider the following sentences.

$$\forall_\sigma y(q_{\sigma_j}(y) \leftrightarrow \exists_{\sigma_j} x(i_j(x) = y)) \qquad (\theta_{q_{\sigma_j}})$$

$$\forall_\sigma y_1 \ldots \forall_\sigma y_m \big(q_p(y_1, \ldots, y_m) \leftrightarrow \exists_{\sigma_{j_1}} x_1 \ldots \exists_{\sigma_{j_m}} x_m(i_{j_1}(x_1) = y_1 \wedge \ldots \wedge i_{j_m}(x_m) = y_m \wedge p(x_1, \ldots, x_m))\big) \qquad (\theta_{q_p})$$

These sentences define the predicates $q_{\sigma_j} \in \widehat{\Sigma}$ and $q_p \in \widehat{\Sigma}$. One can verify that T_3 satisfies the admissibility conditions for the definitions $\theta_{q_{\sigma_j}}$. And therefore the theory $T_4 = T_3 \cup \{\theta_{q_{\sigma_1}}, \theta_{q_{\sigma_2}}, \theta_{q_{\sigma_3}}\} \cup \{\theta_{q_p} : p \in \Sigma\}$ is a Morita extension of T_3 to the signature Σ^+.

Step 7. It only remains to show that the Σ^+-theories T_4 and $\widehat{T}_4$ are logically equivalent. One can verify by induction on the complexity of ψ that

$$T_4 \vDash \psi \leftrightarrow \widehat{\psi} \text{ and } \widehat{T}_4 \vDash \psi \leftrightarrow \widehat{\psi}. \qquad (5)$$

for every Σ-sentence ψ. One then uses (5) to show that $\mathrm{Mod}(T_4) = \mathrm{Mod}(\widehat{T}_4)$. The argument involves a number of cases, but since each case is straightforward we leave them to the reader to verify. The theories T_4 and $\widehat{T}_4$ are logically equivalent, which implies that T and $\widehat{T}$ are Morita equivalent. □