A Weyl-Type Theorem for Geometrized Newtonian Gravity†

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ABSTRACT

I state and prove, in the context of a space having only the metrical and affine structure imposed by the geometrized version of Newtonian gravitational theory, a theorem analagous to that of Weyl's for a Lorentz manifold. The theorem says that a projective structure and a suitably defined compatible conformal structure jointly suffice for fixing the metric structure of a Newtonian spacetime model up to constant factors, and for fixing its affine structure as well. The theorem allows one to give a natural, physically compelling interpretation of the spatiotemporal geometry of a geometrized Newtonian gravity spacetime manifold, in close analogy with the way Weyl's Theorem allows one to do in general relativity.

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1 Weyl's Theorem

Soon after Einstein first proposed the theory of general relativity, Weyl (1918) formulated and proved a theorem that has since served as the foundation for one of the most influential and compelling ways to give a physical interpretation to the theory's mathematical machinery, Lorentzian geometry:[1]

Theorem 1.1 (Weyl 1918) *Given a conformal structure on a differential manifold, and a projective structure agreeing with it on (images) of its null geodesics, there is a Lorentz metric, fixed up to a constant factor, having each as its associated structure of that kind.*

In essence, to know the conformal structure is to know which curves are timelike (the possible paths of massive bodies), which are null geodesics (the possible paths of light-rays *in vacuo*), and which are spacelike (the possible paths of no physical system); to know the projective structure allows one further to say which timelike curves are (images of) geodesics, *i.e.*, the possible paths of freely falling massive bodies. To know both, then, the theorem tells us, is to know the spacetime metric up to a fixed constant, *i.e.*, up to the choosing of a system of units of measurement, thus giving us the physical significance of the metric: it is determined by the behavior of light rays and freely falling bodies.[2] In this paper, I construct the necessary machinery, a Newtonian conformal structure, for a formulation and proof of a natural analogue of this theorem in the context of geometrized Newtonian gravity. It provides in the same way the basis for a compelling physical interpretation of the spatiotemporal structures of that theory.

In §2, I characterize the analogue of a conformal structure in Newtonian spaces. I conclude in §3 with a statement and proof of the theorem, and an explanation of the way it grounds a physical interpretation of the theory. In appendix A, I sketch the machinery of geometrized Newtonian gravity required for the paper's definitions and proofs, following the treatment of Malament (2012, ch. 4, §2) (with minor emendations and simplifications).

2 Newtonian Conformal Structure

The following definition encapsulates almost the entirety of the formal structure of geometrized Newtonian gravity as a physical theory, the remainder being elaboration of and derivations from its elements.

[1]Researchers as varied in their backgrounds, aims and temperaments as Eddington (1923), Møller (1952), Trautman (1965), Hawking and Ellis (1973) and Malament (2012) have used it so, to mention only a small sample. This form of interpretation can be enlighteningly contrasted with one based, *e.g.*, on chronometry, as in Einstein (2001) or Synge (1960a).

[2]See, *e.g.*, Ehlers, Pirani, and Schild (1972) for a thorough exposition of the mathematics behind the theorem and its intended physical significance, and Malament (2012, ch. 2, §1, pp. 120–121) for a lapidary account of such a physical interpretation, including a discussion of its virtues and possible problems.

Definition 2.1 *A* Newtonian spacetime model *is an ordered quintuplet* $(\mathcal{M}, \rho, h^{ab}, t_{ab}, \nabla_a)$ *such that:*

1. $\mathcal{M}$ *is a four-dimensional, paracompact, connected, smooth, differential manifold*
2. ρ *is a smooth, non-negative, scalar function on* $\mathcal{M}$
3. h^{ab} *is a smooth, symmetric tensor field on* $\mathcal{M}$ *of signature* $(0, 1, 1, 1)$[3]
4. t_{ab} *is a smooth, symmetric tensor field on* $\mathcal{M}$ *of signature* $(1, 0, 0, 0)$[4]
5. t_{ab} *and* h^{ab} *are orthogonal,* i.e., $t_{an}h^{nb} = 0$
6. ∇_a *is a smooth derivative operator on* $\mathcal{M}$*, compatible with* t_{ab} *and* h^{ab} *in the sense that* $\nabla_a t_{bc} = 0$ *and* $\nabla_a h^{bc} = 0$

$\mathcal{M}$ represents spacetime, the "totality of all point-events", and ∇_a the physical affine structure, *i.e.*, the one whose geodesics represent unaccelerated paths in the spacetime. ρ represents the mass-density distribution of matter. h^{ab} and t_{ab} represent, respectively, the closest we come to having spatial and temporal metric structures on $\mathcal{M}$. (See appendix A for an explanation of the sense in which these tensors represent such structure.) We refer to the ordered pair (h^{ab}, t_{ab}) as a *Newtonian metrical structure*.

From hereon, we assume all Newtonian spacetime models to be spatially flat (*i.e.*, R^{abcd}, the spatialized Riemann tensor, vanishes; see appendix A, especially proposition A.6, for an explanation of the significance of this condition). It follows from the results and discussion of Malament (1986) that this represents no real loss of generality, as it is those models that best capture the idea of the possible spaces of classical Newtonian gravitational theory. We also assume in what follows that $\mathcal{M}$ is simply connected and the spacetime model is temporally orientable (*i.e.*, that there exists a globally defined temporal function t such that $t_{ab} = t_a t_b := \nabla_a t \nabla_b t$). Again, this is no real loss of generality, for all arguments and conclusions would still go through without the assumption, at the cost of constant hedging about which results are local and which global, and hence much technical work of a nit-picky sort without any counter-balancing gain in physical insight.

From hereon, we will need to keep track of the difference between a curve considered, on the one hand, as a smooth, injective mapping from a real interval to $\mathcal{M}$, and, on the other, as the point-set image in $\mathcal{M}$ of such a mapping. I will use 'curve' when I mean the former, and 'image of a curve' when I mean the latter.

[3]This use of 'signature' differs from the standard usage. In this case, the signature indicates that there exists at every point of the manifold a basis $\{\overset{i}{\xi}{}^a\}_{i\in\{0,1,2,3\}}$ for the tangent plane such that $h_{mn}\overset{i}{\xi}{}^m\overset{j}{\xi}{}^n = 0$ for $i \neq j$, and

$$h_{mn}\overset{i}{\xi}{}^m\overset{i}{\xi}{}^n = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1, 2, 3 \end{cases}$$

[4]Again, this use of signature differs from the standard one, and is the same as explained in footnote 3.

Two affinities ∇_a and $\widetilde{\nabla}_a$ are projectively equivalent if they agree on images of geodesics, which is to say, if they agree on geodesics up to arbitrary (smooth, monotonic) reparametrization: $\xi^n \nabla_n \xi^a = 0$ if and only if $\xi^n \widetilde{\nabla}_n \xi^a = \lambda \xi^a$ for λ a smooth function on the image ξ. A projective structure, then, is a maximal collection of affine structures on a Newtonian spacetime model all of which agree on images of all geodesics ("maximal" in the sense that we throw in every affine structure that meets the criterion). One can equally well define the projective structure as the complete family of images of geodesics on which the affine structures agree. A member of such a family of images of geodesics is a projective geodesic.

Now, to characterize the analogue of conformal structure in geometrized Newtonian gravity. In general relativity, one can define a conformal structure to be an assignment of a smoothly varying field of quadratic cones at every point of the spacetime manifold, the null-cones. Two Lorentz metrics g_{ab} and $\tilde{g}_{ab}$ are conformally equivalent if they have the same null-cone structure. This holds if and only if the two metrics agree on their null geodesics, which holds if and only if $g_{ab} = \Omega^2 \tilde{g}_{ab}$ for some smooth, non-zero scalar field Ω. Because we have no such metric structure in geometrized Newtonian gravity, and correlatively no cone-structure (except the degenerate one, which will not do) or non-trivially null type of vector, one cannot apply such a characterization of conformal structure here.

The following observation provides the required clue for moving forward: two Lorentz metrics are conformally equivalent if and only if they agree on orthogonality relations for all pairs of vectors. Equivalently, g_{ab} and $\tilde{g}_{ab}$ agree in assignments of ratios of lengths to any pair of (non-null) vectors. The conformal structure so characterized then allows one to distinguish among timelike, null and spacelike vectors, to distinguish null geodesics, and to reconstruct the null-cone structure. Null-vectors are those non-zero vectors orthogonal to themselves, which picks out the null-cones; and a null geodesic is one contained in a null-cone (in an appropriate sense—see Ehlers, Pirani, and Schild 1972). Timelike vectors are those pointing into the interior of the null cones, and spacelike vectors are all the rest.

Because, given two spacelike vectors, there is always a third that, added to the first, makes it orthogonal to the second, the trigonometric functions allow one to define angles among arbitrary pairs of spacelike vectors at a point. No two (non-zero) timelike vectors are orthogonal to each other, however, so one cannot define ordinary trigonemetric angles between them. In the same way as is done in special relativity, however, one can define hyperbolic trigonometric angles between timelike vectors by considering a transformation from one to another as signifying a "difference in velocity between inertial observers", which can be represented by a rotation through a hyperbolic angle (Synge 1960b, ch. IV, §13). This then allows one to define ratios of lengths for timelike vectors. This all suggests that we attempt to characterize conformal structure in the context of geometrized Newtonian gravity by making use of orthogonality relations and more general angle relations among (appropriate) pairs of vectors.

Now, the structure of orthogonality relations among all pairs of spacelike vectors at all points suffices for defining a family of symmetric tensor fields characterized by a given h_{ab} with signature (0, 1, 1, 1), fixed up to multiplication by a positive scalar field Ω^2: α^a, β^a are orthogonal if and only if $h_{mn}\alpha^m\beta^n = 0$. This gives us too much structure, however. As explained in Appendix A, spacelike vectors in geometrized Newtonian gravity do not admit univocal attributions of length: if they did, then the spatial separation of two non-simultaneous events could be absolutely determined, but that is inconsistent with Galileian relativity. The orthogonality relations among spacelike vectors, therefore, must be determined derivatively from primitive orthogonality relations imposed on 1-forms. This gives us a family of symmetric tensor fields characterized by a given h^{ab} with signature (0, 1, 1, 1), fixed up to multiplication by a positive scalar field Ω^2: α_a, β_a are orthogonal if and only if $h^{mn}\alpha_m\beta_n = 0$. This family represents the conformal spatial metric.

The set of 1-forms α_a such that $h^{mn}\alpha_m\alpha_n \neq 0$ becomes a vector space if we include the zero vector. To know h^{ab}, therefore, allows one to distinguish timelike from spacelike vectors: (non-zero) spacelike vectors are those that result from raising an index of a (non-zero) 1-form; timelike are those that cannot be so derived. Thus, the family of spacelike vectors at each point forms a three-dimensional subspace of the tangent plane. If $\xi^a = h^{an}\alpha_n$, we say α_a is a spacelike representative 1-form of ξ^a, and ξ^a and η^a are orthogonal if and only if they have representative 1-forms that are so. (It is easy to check that this is independent of choice of representative 1-form.) We can now also determine a second family of symmetric tensor fields $\{\overset{\lambda}{t}_{ab}\}_{\lambda\in\Lambda}$ (Λ some indexing set), each with signature (1, 0, 0, 0): a vector ξ^a is timelike if and only if $\overset{\lambda}{t}_{mn}\xi^m\xi^n \geq 0$, for any λ. The family of timelike vectors thus forms an open, convex four-dimensional subset of the tangent plane. Clearly, any two representative tensor fields h^{ab} and $\overset{\lambda}{t}_{ab}$ from the families are orthogonal, in the sense that $\overset{\lambda}{t}_{an}h^{nb} = 0$.

What we have in hand so far cannot capture all the properties we require of a true conformal structure for a Newtonian spacetime model (as characterized by definition 2.1 and the further conditions we required in the discussion after that). The orthogonality relations among spacelike 1-forms (as encoded in the family of the h^{ab}) allow us to define angles and ratios of lengths between spacelike vectors, but the temporal structure we have so far, encoded in the family of $\overset{\lambda}{t}_{ab}$, does not allow unambiguous attribution of angles and ratios of length to pairs of timelike vectors, and so the family of $\overset{\lambda}{t}_{ab}$ does not yet represent a true conformal temporal metric—call it a temporal pre-conformal structure. Neither does the structure we have so far permit us to characterize the required conditions of spatial flatness and temporal orientability for Newtonian spacetime models. Indeed, at this point nothing guarantees that the distribution of spacelike three-dimensional subspaces of the tangent planes even be integrable (in the sense of Fröbenius), and so be compatible with the existence of global simultaneity slices (as the existence of a global time function guarantees in the standard case). It is perhaps surprising that all requried remaining structure can be determined by fixing only more spatial structure.

First, require that the distribution of spacelike three-dimensional subspaces of the tangent planes be integrable. The leaves of the resulting foliation of $\mathcal{M}$ are to be the simultaneity slices of the Newtonian spacetime model. Because these leaves are three-dimensional hypersurfaces, they allow the fixing of a global non-zero 1-form t_a, one that annihilates all vectors tangent to the hypersurfaces (*i.e.*, spacelike vectors). By Fröbenius's Theorem, this 1-form is at least locally exact, *i.e.*, there exists a function t such that $t_a = \nabla_a t$, again at least locally. Because we have assumed $\mathcal{M}$ to be simply connected, this can be extended to a global relation; the simultaneity slices are then the surfaces of constant t. Clearly, $t_a t_b$ belongs to the family of temporal metrics our construction has already fixed. Now, this allows us to single out a family of conformally related temporal metrics. To see this, let s_a be another non-zero 1-form that annihilates all spacelike vectors. The family of 1-forms that annihilate spacelike vectors forms a 1-dimensional vector space, so there is a non-zero function χ such that $t_a = \chi s_a$, and so $\chi^2 s_a s_b$ is also in the family of metrics. The family of temporal metrics has been fixed up to multiplication by a conformal factor. A Newtonian spacetime model having any of those as its temporal metric, moreover, is automatically temporally orientable, and each member of the family determines a unique time function (up to the addition of a constant). A temporal pre-conformal structure may therefore be divided into equivalence classes, each of which constitutes a true temporal conformal structure, and fixing the spacelike distribution to be integrable picks out a unique equivalence class. This family of temporal metrics, moreover, suffices for the unambiguous fixing of hyperbolic angles between timelike vectors, and so represents the conformal temporal metric. Conversely, it is clear that to fix the hyperbolic angles between all pairs of timelike vectors at all points characterizes the conformal temporal metric.

Thus,

Definition 2.2 *A* Newtonian Spatial Conformal Structure $\mathfrak{SC}$ *on a candidate Newtonian spacetime manifold* $\mathcal{M}$ *consists of:*

1. *at each point* $p \in \mathcal{M}$*, a three-dimensional vector subspace* $\mathfrak{SC}_p$ *of* $T^*_p M$ *(the tensor space of 1-forms over* p*), smoothly varying from point to point*
2. *orthogonality relations fixed for all pairs of elements of* $\mathfrak{SC}_p$*, for all* p*, such that the distribution of three-dimensional subspaces of tangent vectors determined by the associated family of* h^{ab} *is integrable*
3. *the leaves of the foliation determined by the integrable distribution admit flat derivative operators, all jointly defined by restriction of a global derivative operator on* $\mathcal{M}$ *to each leaf respectively*

A spatial conformal derivative operator is the family of derivative operators each of which is compatible with some spatial metric in the conformal families. A representative of the spatial conformal structure is an ordered pair (h^{ab}, ∇_a) compatible with each other.

Definition 2.3 *A* Newtonian Temporal Conformal Structure $\mathfrak{TC}$ *on a candidate Newtonian spacetime manifold* $\mathcal{M}$ *consists of a four-dimensional open, convex set of tangent vectors* $\mathfrak{TC}_p$ *at each point* $p \in \mathcal{M}$ *and a fixing of hyperbolic angles among all pairs of vectors in each* $\mathfrak{TC}_p$*, all such that:*

1. *the associated family of* t_{ab} *(fixed up to multiplication by a positive function) are all smooth, and of the form* $t_{ab} = t_a t_b$
2. *the distribution of three-dimensional vector subspaces of the tangent planes determined by a representative 1-form* t_a *is integrable*

A temporal conformal derivative operator is the family of derivative operators each of which is compatible with some temporal metric in the conformal families. A representative of the temporal conformal structure is an ordered pair $(t_{ab},\, \nabla_a)$ compatible with each other.

Definition 2.4 *A* Newtonian Conformal Structure $\mathfrak{C}$ *on a candidate Newtonian spacetime manifold* $\mathcal{M}$ *consists of a spatial and a temporal conformal structure, compatible in the sense that they agree on their associated distribution of three-dimensional vector subspaces,* i.e.*, on the leaves of the induced foliations.*

It follows automatically that every h^{ab} in a Newtonian conformal structure is compatible with every t_{ab} in it, *i.e.*, $h^{an}t_{nb} = 0$. To fix a Newtonian spatial conformal structure fixes the entire conformal structure, as there is only one possible Newtonian temporal conformal structure compatible with it. To fix a Newtonian temporal conformal structure also fixes the entire conformal structure, so long as the integral hypersurfaces it determines admit flat derivative operators.

A conformal derivative operator is the family of derivative operators each of which is compatible with some pair of spatial and temporal metrics in the conformal families. A conformal representative is a triplet $(h^{ab},\, t_{ab},\, \nabla_a)$ all compatible with each other. By construction, we are guaranteed that at least one conformal representative has a spatially flat derivative operator.[5]

It will give some insight into the character of a Newtonian conformal structure, as well as being useful in the proof of theorem 3.1, to show that being spatially twist-free for a vector field ($\nabla^{[a}\xi^{b]} = 0$) is invariant with respect to spatial conformal structure, *i.e.*, a spatial conformal derivative operator allows one to determine whether a given vector field is twist-free or not. This makes intuitive sense, as being twist-free essentially means that "nearby" vectors in the vector field have no "angular velocity" with respect to each other, but angular velocity is a conformal notion. More precisely, fix two spatial conformal representatives $(h^{ab},\, \nabla_a)$ and $(\tilde{h}^{ab},\, \widetilde{\nabla}_a)$, with $\tilde{h}^{ab} = \Omega^2 h^{ab}$, and the difference vector $C^a{}_{bc}$ between the two derivative operators. First,

$$\begin{aligned} 0 &= \widetilde{\nabla}_a \tilde{h}^{bc} \\ &= \widetilde{\nabla}_a (\Omega^2 h^{bc}) \end{aligned}$$

so

$$\Omega^2 \widetilde{\nabla}_a h^{bc} + h^{bc} \widetilde{\nabla}_a \Omega^2 = 0$$

Using $C^a{}_{bc}$ to re-express this, and noting that ∇_a and $\widetilde{\nabla}_a$ agree in their action on Ω^2, we get

$$\begin{aligned} 0 &= \Omega^2 \nabla_a h^{bc} + \Omega^2 C^b{}_{an} h^{nc} + \Omega^2 C^c{}_{an} h^{bn} + h^{bc} \nabla_a \Omega^2 \\ &= 2\Omega^2 h^{cn} C^b{}_{an} + h^{bc} \nabla_a \Omega^2 \end{aligned} \tag{2.1}$$

[5] In general, the derivative operator of a given conformal representative will not be spatially flat, and so cannot represent the affine structure of a Newtonian spacetime model according to the constraints of this paper, but that is not a problem. The conformal structure I sketch here is strictly that of the metric structure of a candidate Newtonian spacetime model. A conformal structure in this sense need not embody all the geometry of a complete Newtonian spacetime model, in particular not the affine structure. Although spatial flatness of a complete Newtonian spacetime model may fix the spatial metric up to a constant, that does not imply that spatial conformal factors for a Newtonian conformal structure can be only constants. Any $\mathbb{S}^2$-preserving transformation of R^3 will preserve orthogonality, and there are far more of those than just multiplication by a constant—the Möbius transformations, *e.g.*

Thus $2\Omega^2 h^{an} C^b{}_{nc} = -h^{ab}\nabla_c \Omega^2$ and so $h^{n[a} C^{b]}{}_{nc} = 0$. Thus

$$\widetilde{\nabla}^{[a}\xi^{b]} = 0 \quad \text{if and only if} \quad \nabla^{[a}\xi^{b]} = 0$$

Now, for a Newtonian conformal structure to be a physically meaningful analogue of conformal structure in general relativity, it should allow one to distinguish a preferred family of images of curves, the conformal spacelike geodesics, *i.e.*, images of curves that can be reparametrized so as to be geodesics for each representative derivative operator of the conformal derivative operator associated with the conformal structure.[6] And it does. They are the spacelike curves that, in an appropriate sense, preserve orthogonality relations. Indeed, only the Newtonian spatial conformal structure is required. Fix two spacelike vector-fields ξ^a and η^a everywhere orthogonal to each other. One can always do this. Pick a spatially flat representative of the spatial conformal derivative operator, ∇_a; fix spacelike ξ^a and η^a orthogonal at a point p; then construct a curve (unique up to parametrization) that has ξ^a as its tangent vector at p, and that parallel-transports η^a along it with respect to ∇_a. Then do the same with the roles of ξ^a and η^a reversed. Do the same along every point of the constructed curves. Pick a spacelike vector θ^a orthogonal to both ξ^a and η^a at p, and parallel-transport the constructed curves along θ^a. And so on. Because we are working with a spatially flat derivative operator, we are guaranteed that parallel-transport of all these vectors is path-independent, so the construction is consistent.

The integral curves of ξ^a and η^a are conformal spacelike geodesics. To see this, pick again a representative ∇_a of the spatial conformal derivative operator (not necessarily spatially flat), with its associated h^{ab}. We will use h^{ab} to effectively lower indices in the following calculations, by using it to arbitrarily fix spacelike representative 1-forms of ξ^a and η^a respectively. (It is a simple calculation to show that the following argument is independent of the initial choice of spacelike representative 1-forms.) First, it is clear by construction that the two vector-fields ξ^a and η^a Lie-derive each other. Thus $\xi^n\nabla_n\eta^a = \eta^n\nabla_n\xi^a$, and, by orthogonality, $\nabla_a\xi^n\eta_n = 0$. Now the result follows by playing these off each other in the standard way.

$$\begin{aligned}\eta_m\xi^n\nabla_n\xi^m &= \xi^m\xi^n\nabla_n\eta_m \\ &= \xi^n\nabla_n\xi^m\eta_m - \eta_m\xi^n\nabla_n\xi^m\end{aligned}$$

The first term in the last line is zero by orthogonality, and so we are left with $\eta_m\xi^n\nabla_n\xi^m = -\eta_m\xi^n\nabla_n\xi^m$, *i.e.*, $\eta_m\xi^n\nabla_n\xi^m = 0$. Since this is true for all vectors orthogonal to ξ^a, we conclude that $\xi^n\nabla_n\xi^a = \lambda\xi^a$, for some non-negative λ. Thus, the integral curves of ξ^a are spacelike geodesics of ∇_a up to reparametrization.

Before moving on to the main result, it will be useful to show that the mass density ρ of a Newtonian spacetime model is a spatially conformal invariant. ρ, of course, is not responsible only

[6] Although a conformal structure in general relativity fixes the null geodesics as curves, not just images of curves, we cannot expect that here: that happens in the null case only because all vectors tangent to all curves formed by parametrizing the images have the same length (*viz.*, 0), and so the images remain true geodesics under arbitrary parametrizations. That will not be the case for the spacelike curves here, which will be true geodesics (not just re-parametrizable so as to be geodesics) only under a preferred family of parametrizations.

for gravitational effects in geometrized Newtonian gravity, but also represents inertial mass. As such, by Newton's Second Law, to know the ratio of the magnitude of a (non-gravitational) force acting on a given mass distribution to the magnitude of its resulting acceleration determines ρ. A simple calculation, however, shows that any timelike curve ξ^a parametrized by arc-length ("proper time") will satisfy $\xi^n \nabla_n \xi^a = \eta^a$, where η^a is spacelike. This says that for any ponderable body, the worldine of which by definition instantiates a timelike curve, its acceleration is everywhere spacelike.[7] Because forces are directly proportional to accelerations, they must also be spacelike. To know the spatial conformal structure, therefore, suffices for knowing the mass distribution everywhere, as it is given by the ratios of the magnitudes of force and acceleration. Thus, it makes sense to speak of a conformal structure for an entire Newtonian spacetime model (including ρ), not just for the Newtonian metric and affine structures.

3 The Theorem

A projective structure and a Newtonian conformal structure are compatible with each other if the conformal spacelike geodesics determined by the Newtonian conformal structure are also projective geodesics. We can now state the main result of the paper.

Theorem 3.1 *A Newtonian spatial conformal structure and a compatible projective structure suffice to fix the complete geometry of a Newtonian spacetime model,* i.e.*, to fix the temporal and spatial metrics up to a constant factor and to fix the affine structure.*

PROOF:
The spacelike conformal structure fixes the temporal conformal structure, and so the entire Newtonian conformal structure. Assume, then, that we have two pairs of conformal representatives, $(h^{ab}, t_{ab}, \nabla_a)$ and $(\tilde{h}^{ab}, \tilde{t}_{ab}, \widetilde{\nabla}_a)$, where $\tilde{h}^{ab} = \Omega^2 h^{ab}$ and $\tilde{t}_{ab} = \chi^2 t_{ab}$, or, rewriting the latter using the temporal 1-forms t_a and $\tilde{t}_a$ that, respectively, define the temporal metrics, $\tilde{t}_a = \chi t_a$. Because we know the Newtonian spacetime model must be spatially flat, without loss of generality we can assume that both derivative operators are spatially flat.

Now, because their respective compatible derivative operators, ∇_a and $\widetilde{\nabla}_a$, agree on images of geodesics, their difference tensor $C^a{}_{bc}$ is of the form $\delta^a{}_{(b}\phi_{c)}$ for some 1-form ϕ_a. In particular, because they are conformally related, from equation (2.1) we know that

$$h^{nc} C^a{}_{bn} = -\frac{1}{2} h^{ac} \nabla_b \ln \Omega^2 \tag{3.2}$$

and the analogous calculation shows that

$$t_n C^n{}_{ab} = t_a \nabla_b \ln \chi \tag{3.3}$$

[7] If this were not the case, then one could have ponderable bodies accelerating in a timelike direction, with attendant timelike forces, and I see no possible way to make physical sense of such a conceit, except perhaps in a 1950's B-movie "Great Scot! He's beginning to mutate!"-sort of way, since this would seem to allow for material bodies following spacelike trajectories.

Now,

$$\begin{aligned}\nabla_a \tilde{t}_b &= \widetilde{\nabla}_a \tilde{t}_b + 2\tilde{t}_n \delta^n{}_{(a}\phi_{b)} \\ &= 2\tilde{t}_{(a}\phi_{b)}\end{aligned} \tag{3.4}$$

Also, $\nabla_a \tilde{t}_b = \nabla_a(\chi t_b) = t_b \nabla_a \chi$. For that last expression to be symmetric, as implied by equation (3.4), $\nabla_a \chi$ must be proportional to t_a; therefore, ϕ_a is proportional to t_a as well, say $\phi_a = \lambda t_a$. Thus $h^{an}\nabla_n \chi = 0$, so χ is spatially constant. We also have

$$C^n{}_{na} = \lambda \delta^n{}_{(n} t_{a)} = 5\lambda t_a$$

and so

$$0 = h^{am} C^n{}_{nm} = -\frac{1}{2} h^{na} \nabla_n \ln \Omega^2$$

Thus Ω is also spatially constant. This fixes the spacelike geodesics up to affine parametrization: if σ is the image of a spacelike geodesic, then any parametrization σ^a such that $h^{mn}\sigma_m \sigma_n = 1$, for any spacelike representative σ_a, makes it a geodesic with respect to both ∇_a and $\widetilde{\nabla}_a$. An explicit expression for $C^a{}_{bc}$ now follows from equations (3.2) and (3.3):

$$C^a{}_{bc} = \alpha^a t_b t_c - \frac{1}{2} \overset{s}{\delta}{}^a{}_{(b} \nabla_{c)} \ln \Omega^2 \tag{3.5}$$

where α^a is unit timelike (with respect to t_a), and $\overset{s}{\delta}{}^a{}_b$ is the spacelike identity tensor (*i.e.*, $\delta^a{}_b$ restricted to simultaneity slices).

Now, by the Trautman Recovery Theorem (A.5) we know that there is a flat derivative operator $\overset{f}{\nabla}_a$ and a scalar field ϕ such that

1. $\overset{f}{\nabla}_a$ and ϕ jointly satisfy the Poisson equation for the ρ of the Newtonian spacetime model
2. ξ^a is a timelike geodesic with respect to ∇_a if and only if $\xi^n \overset{f}{\nabla}_n \xi^a = -h^{an}\overset{f}{\nabla}_n \phi$, *i.e.*,
$$\xi^n \nabla_n \xi^a = 0 \quad \text{if and only if} \quad \xi^n \overset{f}{\nabla}_n \xi^a = -h^{an}\overset{f}{\nabla}_n \phi$$
3. $\overset{f}{\nabla}_a t_b = \overset{f}{\nabla}_a h^{bc} = 0$

We also know the analogous statements holds for $\widetilde{\nabla}_a$, some flat $\overset{\tilde{f}}{\nabla}_a$ and some $\tilde{\phi}$. From the proof of the Trautman Theorem (Malament 2012, pp. 274*ff.*), we know explicit expressions for the two flat derivative operators. Let θ^a be rigid (with respect to h^{ab}), twist-free, unit and timelike, *i.e.*, $h^{an}\nabla_n \theta^b = 0$ and $t_n \theta^n = 1$. Then the difference vector $\overset{f}{C}{}^a{}_{bc}$ of ∇_a and $\overset{f}{\nabla}_a$ equals $\theta^a t_b t_c$. For η^a the same for $\tilde{h}_{ab}$, the difference vector $\overset{\tilde{f}}{C}{}^a{}_{bc}$ of $\widetilde{\nabla}_a$ and $\overset{\tilde{f}}{\nabla}_a$ equals $\eta^a \tilde{t}_b \tilde{t}_c$.

We now want to determine the difference tensor $D^a{}_{bc}$ of $\overset{f}{\nabla}_a$ and $\overset{\tilde{f}}{\nabla}_a$.

$$\begin{aligned}0 &= \overset{f}{\nabla}_a t_b \\ &= \overset{\tilde{f}}{\nabla}_a t_b + t_n D^n{}_{ab}\end{aligned}$$

but also

$$0 = \overset{\tilde{f}}{\nabla}_a \chi t_b$$
$$= t_b \overset{\tilde{f}}{\nabla}_a \chi + \chi \overset{\tilde{f}}{\nabla}_a t_b$$

so $t_n D^n{}_{ab} = t_b \overset{\tilde{f}}{\nabla}_a \ln\chi = t_b \nabla_a \ln\chi$. The analogous calculation for h^{ab} yields $h^{nc} C^a{}_{bn} = -\frac{1}{2} h^{ac} \nabla_b \ln\Omega^2$, from which the explicit form of the difference tensor follows,

$$D^a{}_{bc} = \beta^a t_b t_c - \frac{1}{2} \overset{s}{\delta}{}^a{}_{(b} \nabla_{c)} \ln\Omega^2 \tag{3.6}$$

where β^a is unit timelike (with respect to t_a).

Now, by considering the relations between the behavior of timelike geodesics of ∇_a with respect to $\overset{f}{\nabla}_a$ and that of those of $\widetilde{\nabla}_a$ with respect to $\overset{\tilde{f}}{\nabla}_a$, we will be able to show that χ is a constant, and thence that Ω is as well.

$$\begin{aligned} \tilde{\xi}^n \overset{\tilde{f}}{\nabla}_n \tilde{\xi}^a &= \chi^{-1} \xi^n \left(\overset{f}{\nabla}_n \chi^{-1} \xi^a - \chi^{-1} D^a{}_{nr} \xi^r \right) \\ &= \chi^{-1} \xi^a \xi^n \overset{f}{\nabla}_n \chi^{-1} + \chi^{-2} \xi^n \overset{f}{\nabla}_n \xi^a - \chi^{-2} \xi^n \xi^r (\beta^a t_n t_r - \frac{1}{2} \overset{s}{\delta}{}^a{}_{(n} \nabla_{r)} \ln\Omega^2) \\ &= \chi^{-1} \xi^a \xi^n \overset{f}{\nabla}_n \chi^{-1} - \chi^{-2} h^{an} \overset{f}{\nabla}_n \phi - \chi^{-2} \beta^a \end{aligned} \tag{3.7}$$

$\tilde{\xi}^n \overset{\tilde{f}}{\nabla}_n \tilde{\xi}^a$ is spacelike. The righthand side of the final line, however, is the sum of a fixed timelike vector and spacelike vector, and an arbitrary timelike vector (proportional to ξ^a). The sum of a timelike and a spacelike vector is always timelike, so, in order for $\chi^{-2} h^{an} \overset{f}{\nabla}_n \phi - \chi^{-2} \beta^a$ added to $\chi^{-1} \xi^a \xi^n \overset{f}{\nabla}_n \chi^{-1}$ to always be spacelike, for arbitrary timelike ξ^a, it must be the case that the entire timelike part of the righthand side of the final line must vanish,

$$\chi^{-1} \xi^a \xi^n \overset{f}{\nabla}_n \chi^{-1} - \chi^{-2} \beta^a = 0$$

Again, ξ^a is arbitrary, however, so we must have $\xi^n \overset{f}{\nabla}_n \chi^{-1} = 0$ and $\beta^a = 0$. Since χ is constant in timelike directions, it must be a global constant (since we already know it is spatially constant).

To show that Ω is a constant, let us first sum up the relations we now know among all the derivative operators in play.

1. $\nabla_a - \widetilde{\nabla}_a := C^a{}_{bc} = \alpha^a t_b t_c - \frac{1}{2} \overset{s}{\delta}{}^a{}_{(b} \nabla_{c)} \ln\Omega^2$
2. $\nabla_a - \overset{f}{\nabla}_a := \overset{f}{C}{}^a{}_{bc} = \theta^a t_b t_c$
3. $\widetilde{\nabla}_a - \overset{\tilde{f}}{\nabla}_a := \overset{\tilde{f}}{C}{}^a{}_{bc} = \eta^a t_b t_c$
4. $\overset{f}{\nabla}_a - \overset{\tilde{f}}{\nabla}_a := D^a{}_{bc} = -\frac{1}{2} \overset{s}{\delta}{}^a{}_{(b} \nabla_{c)} \ln\Omega^2$

So

$$(\nabla_a - \overset{f}{\nabla}_a) - (\widetilde{\nabla}_a - \overset{\tilde{f}}{\nabla}_a) = (\theta^a - \chi^2 \eta^a) t_b t_c$$

but also

$$\begin{aligned}(\nabla_a - \overset{f}{\nabla}_a) - (\widetilde{\nabla}_a - \overset{\tilde{f}}{\nabla}_a) &= (\nabla_a - \widetilde{\nabla}_a) - (\overset{\tilde{f}}{\nabla}_a - \overset{f}{\nabla}_a) \\ &= \alpha^a t_b t_c - \overset{s}{\delta}{}^a{}_{(b} \nabla_{c)} \ln \Omega^2\end{aligned}$$

Equating these two expressions, it follows that $\nabla_a \Omega = 0$, since $\overset{s}{\delta}{}^a{}_b$ is spacelike in both a and b. Thus $C^a{}_{bc} = \alpha^a t_b t_c$.

Finally, parametrize the images of the timelike geodesics ξ so that they are actual geodesics with respect to t_{ab}, *viz.*, $\xi^n \nabla_n \xi^a = 0$. It suffices for this to fix a parametrization such that $t_n \xi^n = 1$ everywhere along ξ. $\tilde{t}_n \xi^n = \chi$, however, a constant, so ξ^a is also a parametrization that makes ξ a geodesic with respect to $\widetilde{\nabla}_a$. Thus,

$$0 = \xi^n \widetilde{\nabla}_n \xi^a = \xi^n \nabla_n \xi^a - \alpha^a t_m t_n \xi^m \xi^n$$

so $\alpha^a = 0$. Thus, the affine structure ∇_a is fixed, and t_a and h^{ab} are fixed up to global constants.

■

The converse of the theorem is trivial.

One can now use the theorem to render a physical interpretation to the Newtonian metric structure of a Newtonian spacetime model, in the following way.[8] We will use the following interpretive principles.

C1 Timelike curves represent the possible paths of massive bodies.

C2 Segments of images of spacelike geodesics represent the spatial position and extent of (rigid) yard-sticks.

P1 Images of timelike geodesics represent the paths of freely falling massive bodies.

Now, if we can determine when two coincident rigid rods are orthogonal at an instant of time, then, by **C2**, we can determine the Newtonian conformal structure. This can be operationalized in any of a number of ways, such as by the use of a compass. The projective structure is then fixed by determining the images of paths of freely falling bodies (**P1**). Thus, by the theorem, to know when rigid rods are orthogonal to each other at a given moment of time and to know the images of the paths of freely falling bodies is to fix the entirety of the metrical and affine structure of a Newtonian spacetime model. (A Newtonian metrical structure does not by itself fix a unique affine structure, in contradistinction to the situation in general relativity.)

[8] I do not claim this is the unique or even just a canonical way of doing so, or the most elegant or concise or what have you, only that it is a perspicuous and natural way, with clear physical significance.

A Appendix: Geometrized Newtonian Gravity

A Newtonian spacetime model is as defined in definition 2.1.

Definition A.1 *The* temporal length *of a vector* ξ^a *is* $(\xi^m \xi^n t_{mn})^{\frac{1}{2}}$*.* ξ^a *is* timelike *if its temporal length is non-zero,* spacelike *if the vector is non-zero and has a temporal length of zero.*

It is natural to think of the zero-vector as both timelike and spacelike, in so far as we know what it means for two events to occur at the same time as well as at the same place. The addition of any spacelike vector to a timelike one is always timelike. It follows that the family of timelike vectors forms an open, convex four-dimensional subset of the tangent plane. (It cannot form a vector-space, as the sum of two timelike vectors may be spacelike.) It also follows that the family of spacelike vectors at a point forms a three-dimensional vector subspace of the tangent plane.

The signature of t_{ab} implies that at every point there exists a neighborhood and smooth 1-form t_a defined on that neighborhood such that $t_{ab} = t_a t_b$ on that neighborhood. If there is a globally defined such 1-form, then the spacetime model is *temporally orientable* and t_a is a *temporal orientation* on it; a timelike vector ξ^a is future-directed if $t_n \xi^n > 0$. Item 6 in definition 2.1 implies that t_a is closed, and so, at least locally, there exists a smooth function t such that $\nabla_a t = t_a$, the (local) *time-function*. If $\mathcal{M}$ is simply connected and the spacetime model temporally orientable, t is defined globally. In this case, its constant surfaces represent the Newtonian idea of "all of space at a given moment of time", *i.e.*, a maximal collection of events all (absolutely) simultaneous with each other, a *simultaneity slice*, and so $\mathcal{M}$ has the topology $\mathbb{R} \times \Sigma$, where Σ is a three-dimensional manifold diffeomorphic to a simultaneity slice. This t is unique up to the addition of a constant, which may be thought of as a change of temporal origin. Its scale, however, is fixed by the temporal length of vectors: to multiply t_{ab} by a constant factor changes nothing physically, but rather represents only a change in units of temporal measurement.

t_{ab}, then, determines an absolute temporal structure—the temporal separation of any two events in the spacetime is fixed once and for all, independent of any other state of affairs in the world; the equivalence classes of spacetime points under the relation "having a temporal separation of zero with" are exactly the simultaneity slices. A *timelike curve* is one whose tangent vectors are everywhere timelike. Such curves represent the possible worldlines of ponderable bodies. The signature of t_{ab} has another consequence of note: it does not allow one to define orthogonality between two timelike vectors. This makes physical sense: two orthogonal, timelike vector-fields would define time-functions different in the sense that they would not share surfaces of constancy, *i.e.*, they would define incommensurable temporal structures.[9]

h^{ab} defines the spatial metric structure in a more indirect way.

[9] One can see the physical sense this makes in another, more indirect way: in so far as one can consider all the spacelike vectors at a point in Newtonian spacetime to be the result, in the limit, of "flattening" the null-cones in a relativistic spacetime—"letting the upper bound of possible velocities go to infinity"—the timelike Newtonian vectors at that point encode essentially the same information as the timelike vectors in the interior of the original null-cones, and no two timelike vectors can be orthogonal to each other with respect to a fixed Lorentz metric. See Malament (1986) for a precise characterization and analysis of the process of flattening the null-cones in a relativistic spacetime.

Proposition A.2

1. *A vector ξ^a is spacelike at a point p if and only if there is a covector α_a at p such that $\xi^a = h^{an}\alpha_n$.*

2. *For all covectors α_a and β_a at a point, if $h^{an}\alpha_n = h^{an}\beta_n$, then $h^{mn}\alpha_m\alpha_n = h^{mn}\beta_m\beta_n$.*

In virtue of this proposition, the following is well formulated.

Definition A.3 *The* spatial length *of a spacelike vector ξ^a is $(h^{nm}\alpha_n\alpha_m)^{\frac{1}{2}}$, where α_a is any covector satisfying $h^{an}\alpha_n = \xi^a$.*

If a spacelike η^a equals $h^{an}\alpha_n$, then we say α_a is a *representative spacelike 1-form* of η^a. In so far as h^{ab} determines lengths only for spacelike vectors, but not at all for timelike vectors (whereas t_{ab} determines lengths for all spacelike vectors: 0), it defines a spatial metric in only a Pickwickian sense, as it ought to according to Galileian relativity. We know what it means in Newtonian theory to assign a definite distance between two simultaneous events, by employing yard-sticks and the like, independent of any other state of affairs in the world. We do not know how to do so for non-simultaneous events.[10]

The following construction captures the content of this observation. Specify at a point p a constant timelike vector ξ^a of unit temporal length, and define $\hat{h}_{ab}$ to be the unique symmetric tensor at p satisfying

$$\hat{h}_{an}h^{nb} = \delta_a{}^b - t_{an}\xi^b\xi^n$$
$$\hat{h}_{an}\xi^n = 0$$

$\hat{h}_{ab}$ represents the covariant spatial metric determined by projection relative to ξ^a. Then the spatial length of any vector assigned indirectly by h^{ab} will agree with that assigned directly by $\hat{h}_{ab}$. The spatial separation of two non-simultaneous events, however, as determined by such a $\hat{h}_{ab}$, depends on the choice of ξ^a. Again, this is as it should be. If we knew how to assign a spatial length to timelike vectors, and so a fixed spatial separation between non-simultaneous events, then we could define a notion of absolute rest: "a particle is at absolute rest if the timelike curve representing its worldline has everywhere tangent vectors of zero spatial length". This, however, we cannot do.

In contradistinction to t_{ab}, h^{ab} allows one to define a relation of orthogonality among spacelike vectors: two spacelike vectors η^a and θ^a are orthogonal if $h^{mn}\alpha_m\beta_n = 0$, where α_m and β_n are any two representative spacelike 1-forms of η^a and θ^a respectively. This makes physical sense as well: we know in Newtonian spacetime how to determine whether two yardsticks at a single moment of time are at a right angle to each other.

A simple calculation shows that any timelike curve parametrized by arc-length ("proper time") will satisfy $\xi^n\nabla_n\xi^a = \eta^a$, where ξ^a is the tangent to the curve and η^a is spacelike. This says that for any ponderable body, the worldine of which by definition instantiates a timelike curve, its acceleration is everywhere spacelike, and thus a viable candidate for entering Newton's Second Law

[10] See Stein (1967) for a thorough discussion.

on the righthand side. Thus, we can assign determinate magnitudes, fixed once and for all, to the acceleration of bodies, as Galileian relativity allows us to do, and Newtonian mechanics demands we do (Stein 1967). In sum, h^{ab} encodes the right amount of structure of the right kind, nothing more, nothing less. (One may think of this as physics practiced in accord with the Goldilocks Principle.)

In what follows, we assume that $\mathcal{M}$ is simply connected and the spacetime model is temporally orientable; this is no real loss of generality, for all arguments and conclusions would still go through without the assumption, at the cost of constant hedging and much technical work of a nit-picky sort without any counter-balancing gain in physical insight. In particular, we do not need to worry about which results hold only locally and which globally.

Before moving on to the definition of the appropriate analogue of a conformal structure for a Newtonian metric structure, we rehearse a few of its properties and state the most important two theorems in geometrized Newtonian gravity, which will be of use later. When there is no chance for ambiguity, we will use raised indices to represent the action of h^{ab} on a tensorial object, *e.g.*, $\nabla^a := h^{an}\nabla_n$.

Theorem A.4 (The Geometrization Lemma) *Fix a Newtonian spacetime model* ($\mathcal{M}$, ρ, h^{ab}, t_{ab}, ∇_a)*, such that* ∇_a *is flat,* i.e., *its associated Riemann tensor* $R^a{}_{bcd} = 0$*, and a smooth scalar field* ϕ *such that* $\nabla^n \nabla_n \phi = 4\pi\rho$ *(the Poisson equation). Define another derivative operator* $\tilde{\nabla}_a = (\nabla_a,\, C^a{}_{bc})$*, where* $C^a{}_{bc} = -t_{cb}\nabla^a\phi$*. Then:*

1. ($\mathcal{M}$, ρ, h^{ab}, $t_{ab}, \tilde{\nabla}_a$) *is a Newtonian spacetime model*
2. $\tilde{\nabla}_a$ *is the unique derivative operator such that*

$$\xi^n \tilde{\nabla}_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \nabla_n \xi^a = -\nabla^a \phi$$

3. *the Riemann tensor* $\tilde{R}^a{}_{bcd}$ *associated with* $\tilde{\nabla}_a$ *satisfies*
   a. $\tilde{R}_{ab} = 4\pi\rho t_{ab}$
   b. $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$
   c. $\tilde{R}^{ab}{}_{cd} = 0$

A few remarks are in order to explain the physical significance of the theorem. In essence it says that, given a Newtonian spacetime model with a flat affine structure, we can always construct one in which the geodesics (possible paths of "freely falling bodies") of a curved affine structure are the same paths as those representing bodies moving under the force of the ambient gravitational field associated with ρ in the original model. The constructed model "geometrizes" gravity: it incorporates the effects of gravity into the metric structure of the new model, in analogy with general relativity.

The effective converse of the Geometrization Lemma holds as well.

Theorem A.5 (Trautman Recovery Theorem) *Fix a Newtonian spacetime model* ($\mathcal{M}$, ρ, h^{ab}, t_{ab}, ∇_a) *satisfying*

1. $R_{ab} = 4\pi\rho t_{ab}$

2. $R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b$

3. $R^{ab}{}_{cd} = 0$

Then there is a derivative operator $\widetilde{\nabla}^a$ *and a scalar field* $\tilde{\phi}$ *such that*

1. $\widetilde{\nabla}^n \widetilde{\nabla}_n \tilde{\phi} = 4\pi\rho$ *(the Poisson equation)*

2. $\widetilde{\nabla}^a$ *is compatible with* t_{ab} *and* h^{ab}

3. $\widetilde{\nabla}^a$ *is flat*

4. for all timelike curves with tangent vector ξ^a

$$\xi^n \nabla_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \widetilde{\nabla}_n \xi^a = -\widetilde{\nabla}^a \tilde{\phi}$$

Moreover, $(\widetilde{\nabla}_a, \tilde{\phi})$ *is not unique. Any other such pair* $(\hat{\nabla}_a, \hat{\phi})$ *will satisfy the stated conditions if and only if*

a. $\nabla^a \nabla^b (\tilde{\phi} - \hat{\phi}) = 0$

b. $\hat{\nabla}_a = (\widetilde{\nabla}_a, C^a{}_{bc})$*, where* $C^a{}_{bc} = t_{bc} \nabla^a (\tilde{\phi} - \hat{\phi})$

Given a Newtonian spacetime model in which the gravitational effects of ρ are geometrized (incorporated into the curved affine structure), the theorem tells us that we can recover one in which the gravitational effects of ρ are rather represented by the acceleration of the paths of freely-falling bodies, as defined by a flat affine structure.

In disanalogy with general relativity, and crucially for our purposes, in this context $R^{ab}{}_{cd} = 0$ is *not* equivalent to $R^a{}_{bcd} = 0$. The interpretation of the condition $R^{ab}{}_{cd} = 0$ and its relative strength as compared to $R^a{}_{bcd} = 0$ follow from a proposition whose proof is straightforward, albeit tedious (Malament 2012, §§4.2–4.3). To state it, we need a few more definitions. A Newtonian spacetime model is *spatially flat* if $R^{abcd} = 0$, *i.e.*, if the affine structure on the simultaneity slices derived by restricting to them the action of the global affine structure is flat. A unit, future-directed timelike vector-field ξ^a is *rigid* if $\pounds_\xi h^{ab} = 0$, *i.e.*, if $\nabla^{(a}\xi^{b)} = 0$. (One may think of a rigid vector field as the analogue of a Killing field in general relativity.) ξ^a is *twist-free* if $\nabla^{[a}\xi^{b]} = 0$, and is *acceleration-free* if $\nabla_a \xi^b$.

Proposition A.6 *Fix a spatially flat Newtonian spacetime model. Then:*

1. $R^{ab}{}_{cd} = 0$ *if and only if there exists a rigid and twist-free vector field*

2. $R^{abcd} = 0$ *if and only if there exists a rigid, twist-free and acceleration-free vector field*

Another illuminating characterization of $R^{ab}{}_{cd} = 0$ follows from this proposition: the condition holds if and only if parallel-transport of spacelike vectors is path-independent.

References

Barbour, J. and B. Bertotti (1982, August). Mach's principle and the structure of dynamical theories. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 382*(1783), 295–306. doi:10.1098/rspa.1982.0102.

Earman, J. (1989). *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time*. Cambridge, MA: MIT Press.

Eddington, A. (1923). *Mathematical Theory of Relativity* (Second ed.). Cambridge: Cambridge University Press.

Ehlers, J., F. Pirani, and A. Schild (1972). The geometry of free fall and light propagation. In L. O'Raifeartaigh (Ed.), *General Relativity: Papers in Honour of J. L. Synge*, Chapter 4, pp. 63–84. Oxford: Clarendon Press. Republished in *General Relativity and Gravitation*, 44(2012, 6):1587–1609, doi:10.1007/s10714-012-1353-4.

Einstein, A. (2001). *Relativity: The Special and General Theory* (Fifteenth, enlarged ed.). London: Routledge. Translated by R. W. Lawson. The original German was published in 1916. The first English edition was published in 1920 by Methuen & Co., London. The Fifteenth edition was first published in 1954.

Hawking, S. and G. Ellis (1973). *The Large Scale Structure of Space-Time*. Cambridge: Cambridge University Press.

Malament, D. (1986). Newtonian gravity, limits, and the geometry of space. In R. Colodny (Ed.), *From Quarks to Quasars: Philosophical Problems of Modern Physics*, pp. 181–201. Pittsburgh: Pittsburgh University Press.

Malament, D. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitational Theory*. Chicago: University of Chicago Press. Uncorrected final proofs for the book are available for download at `http://strangebeautiful.com/other-texts/malament-founds-gr-ngt.pdf`.

Møller, C. (1952). *The Theory of Relativity* (First ed.). Oxford: The Clarendon Press.

Stein, H. (1967). Newtonian space-time. *Texas Quarterly 10*, 174–200.

Synge, J. (1960a). *Relativity: The General Theory*. Amsterdam: North-Holland Publishing Co.

Synge, J. (1960b). *Relativity: The Special Theory*. Amsterdam: North-Holland Publishing Co.

Trautman, A. (1965). Foundations and current problems of general relativity. In S. Deser. and K. Ford (Eds.), *Lectures on General Relativity*, Volume 1 of *Brandeis Summer Institute in Theoretical Physics*, pp. 1–248. Englewood Cliffs, NJ: Prentice-Hall, Inc.

Weyl, H. (1918). *Raum-Zeit-Materie: Vorlesungen über Allgeimene Relativitätstheorie*. Berlin: Springer-Verlag.