Maxwell-Cartan gravitation

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Abstract

In this paper, I build upon and extend [Weatherall, 2015b]'s answer to the question posed in [Saunders, 2013]: "What is the relation between a theory of gravity (and other forces) formulated in Maxwell space-time and one based on Newton-Cartan space-time?" I do so by showing how one can give an explicit dynamics for Newtonian gravitation, without presupposing any more structure than that found in Maxwell spacetime. Doing so helps to further illuminate the relationship between geometrised and non-geometrised formulations of Newtonian gravitation.

1. Introduction

The following two observations are well-known to philosophers of physics:

1. Newtonian gravitation admits, in addition to the well-known velocity-boost and potential-shift symmetries, a "gravitational gauge symmetry" in which the gravitational field is altered.

2. Newtonian gravitation may be presented in a "geometrised" form,[1] in which the dynamically allowed trajectories are the geodesics of a non-flat connection.

Moreover, it is widely held that these two observations are intimately related. However, aspects of this relationship remain somewhat obscure. In particular, there is widespread disagreement over the sense in which the symmetry of observation 1 motivates the move from a non-geometrised formulation to the geometrised formulation of observation 2; and over the extent to which such motivation ought to be regarded as analogous to the use of the velocity-boost symmetry to motivate the move from Newtonian to Galilean spacetime, or to the use of the potential-shift symmetry to motivate the move from a formulation in terms of gravitational potentials to a formulation in terms of gravitational fields.

In this paper, I seek to clarify this relationship. In §2, I introduce some preliminary mathematical notions. §3 introduces Newtonian gravitation, set on Galilean spacetime and in terms of gravitational fields, and presents the gravitational gauge symmetry referred to in observation 1. §4 presents Newton-Cartan theory, the geometrised formulation of Newtonian gravitation referred to by observation 2, and discusses how it relates to the gravitational gauge symmetry. In §5, I consider some reasons (highlighted in [Saunders, 2013]) for being confused about the way in which Newton-Cartan theory relates to the gravitational gauge symmetry, and the desirability of a gravitational theory set on Maxwellian spacetime. I then discuss [Weatherall, 2015b]'s suggestion for how to construct such a theory, and indicate some ways in which it is not as perspicuous as we might wish. §6 contains the main contribution of this paper: the specification of a gravitational dynamics, set upon Maxwellian spacetime, which bears a particularly

[1] Due originally to [Trautman, 1965].

perspicuous relationship to Newton-Cartan theory and to the gravitational gauge symmetry. §7 concludes.[2]

2. Leibnizian spacetime

All of the spacetime structures we will be considering contain at least as much structure as *Leibnizian spacetime*.[3] Such a spacetime may be defined as a structure comprising the following data:[4]

- A differential manifold M, which we take to be diffeomorphic to $\mathbb{R}^4$
- A smooth, curl-free 1-form t_a on M
- A smooth, symmetric $(0,2)$-rank tensor h^{ab} on M; this is required to be *flat*

subject to the *orthogonality condition*

$$t_a h^{ab} = 0 \tag{1}$$

We will denote such a structure by either $\mathfrak{L}$ or $\langle M, t_a, h^{ab} \rangle$. A Leibnizian spacetime contains enough structure to permit judgments regarding the continuity and smoothness of spatiotemporal paths or regions; the temporal distance between any two events; and the spatial distance between any two simultaneous events. However, it does not contain much structure beyond that.

In particular, it does not permit judgments regarding the straightness of spatiotemporal paths. That kind of structure is represented by an *affine connection* ∇ on the base manifold M. Such a connection provides (roughly speaking) a way of differentiating tensor fields on the manifold, taking any tensor field $T^{a_1 \ldots a_m}_{b_1 \ldots b_n}$ to a tensor field $\nabla_c T^{a_1 \ldots a_m}_{b_1 \ldots b_n}$. The main result we will need from the general theory of affine connections is the following:

Proposition 1. [Malament, 2012, Proposition 1.7.3] Let ∇ and ∇' be derivative operators on the manifold M. Then there exists a smooth symmetric tensor field C^a_{bc} on M that satisfies the following condition for all smooth tensor fields $T^{a_1 \ldots a_r}_{b_1 \ldots b_s}$ on M:

$$\begin{aligned} \nabla'_c T^{a_1 \ldots a_r}_{b_1 \ldots b_s} = & \nabla_c T^{a_1 \ldots a_r}_{b_1 \ldots b_s} \\ & - C^{a_1}_{cn} T^{n a_2 \ldots a_r}_{b_1 \ldots b_s} - \cdots - C^{a_r}_{cn} T^{a_1 \ldots a_{n-1} n}_{b_1 \ldots b_s} \\ & + C^n_{cb_1} T^{a_1 \ldots a_r}_{n b_2 \ldots b_s} + \cdots + C^n_{cb_s} T^{a_1 \ldots a_r}_{b_1 \ldots b_{s-1} n} \end{aligned} \tag{2}$$

Conversely, given any derivative operator ∇ on M and any smooth symmetric tensor field C^a_{bc} on M, if ∇' is defined by equation (2), then ∇' is also a derivative operator on M.

Proof. See [Malament, 2012, pp. 51–52]. □

The field C^a_{bc} plays a role precisely analogous to that of the vector potential A_μ in the theory of connections on fibre bundles. For this reason, we will refer to it as the *affine vector potential* (of ∇' relative to ∇). It is closely related to the Christoffel symbols. Given a coordinate chart $\phi : U \subseteq M \to \mathbb{R}^4$, let ∂ be the affine connection naturally induced on U by ϕ (i.e., the pullback to U, by ϕ, of the canonical affine connection on $\mathbb{R}^4$). In the coordinate chart ϕ, the Christoffel symbols for an arbitrary connection ∇ on U are then the components, in ϕ, of the affine vector

[2] See also [Wallace, 2015] for an alternative analysis of these issues.
[3] [Earman, 1989, chap. 2]
[4] Note that I'm using "a spacetime" here to mean a mathematical structure which is apt to represent some kind of physical spacetime. Although this terminology is a little unfortunate, it is sufficiently convenient and standard to be worth cooperating with. We just have to be careful to avoid assuming that all and only the structure in a spacetime is best interpreted as representing spatiotemporal structure (this will come up at p. 9 below).

potential of ∇ relative to ∂.[5] The advantage of working with affine vector potentials, rather than Christoffel symbols, is that they are coordinate-independent. When C^a_{bc} is the affine vector potential of ∇' relative to ∇, we will write $\nabla' = (\nabla, C^a_{bc})$.

We will only consider affine connections on $\mathfrak{L}$ which are *compatible* with the temporal and spatial metric fields, i.e., which satisfy the compatibility conditions

$$\nabla_a t_b = 0 \tag{3a}$$

$$\nabla_a h^{bc} = 0 \tag{3b}$$

Note that these equations parallel the defining equation $\nabla_a g_{bc} = 0$ for the Levi-Civita connection on a metric space $\langle M, g_{ab} \rangle$; however, unlike the Levi-Civita connection, there is not a *unique* connection satisfying (3) for a given Leibnizian spacetime $\mathfrak{L}$. Given our earlier condition that h^{ab} be flat, it turns out that every compatible connection is *spatially flat*: that is, for any compatible connection ∇ with curvature tensor $R^a{}_{bcd}$, $R^{abcd} = 0$. (Indeed, the converse also holds—if every compatible connection is spatially flat, then h^{ab} is flat—so we could have chosen to define a Leibnizian spacetime

A connection not only characterises which curves are straight and which are not: it precisely quantifies the deviation from straightness. More precisely, if θ^a is a unit timelike vector field, then the *acceleration field* of θ^a (representing the acceleration of particles whose worldlines are the integral curves of θ^a) is

$$\theta^n \nabla_n \theta^a \tag{4}$$

θ^a is a *geodesic field* (its integral curves are *geodesics* just if its acceleration field vanishes.

Furthermore, a connection characterises the rotation of the integral curves of a given vector field. Without going into full details, I will remark only that a unit timelike field θ^a is said to be *non-rotating* or *twist-free*, relative to a (compatible) connection ∇, if

$$\nabla^{[a}\theta^{b]} = 0 \tag{5}$$

3. Galilean gravitation

As indicated above, Leibnizian spacetime lacks sufficient spacetime structure to be the backdrop for a gravitational dynamics. The first dynamics we consider supplements Leibnizian spacetime with an affine structure, to differentiate between straight and curved spacetime paths. To that end, define a *Galilean spacetime*[6] to comprise

- A Leibnizian spacetime $\mathfrak{L}$
- A flat affine connection $\overline{\nabla}$[7] which is compatible with $\mathfrak{L}$

We can now define our first gravitational theory.

Definition 1. A model of *Galilean gravitation* comprises

- A Galilean spacetime $\langle \mathfrak{L}, \overline{\nabla} \rangle$
- A scalar field $\mu : \mathfrak{L} \to \mathbb{R}$

[5] This may sound puzzling: how can they be the components of the affine vector potential (a tensor field), when every schoolchild knows that the Christoffel symbols are not the components of a tensor field? The answer is that the Christoffel symbols of ∇ in a different chart, ϕ', are the components in ϕ' of the affine vector potential of ∇ relative to ∂' (*not* ∂), where ∂' is the affine connection induced on U by ϕ'. In other words, the Christoffel symbols are (as it were) coordinate-dependent twice over: the choice of chart affects both which tensor field is the affine vector potential of interest, and (as ever) what the components of that field are.

[6] [Earman, 1989, chap. 2]

[7] As a notational mnemonic, I will use an overline to indicate a flat connection.

- A spacelike vector field G^a
- A maximal class Ξ of unit timelike vector fields

satisfying the following equations:

$$\overline{\nabla}_a G^a = -4\pi\mu \tag{6a}$$

$$\overline{\nabla}^{[c} G^{a]} = 0 \tag{6b}$$

$$\xi^n \overline{\nabla}_n \xi^a = G^a \tag{6c}$$

for every $\xi^a \in \Xi$.

The scalar field μ represents the mass density, the vector field G^a represents the gravitational field, and the set Ξ represents all possible tangents of possible test particles. Given a model of Galilean gravitation, we will refer to curves through $\mathfrak{L}$ which are integral curves of some member of Ξ as *dynamically allowed trajectories*. When I say that the class Ξ is *maximal*, I mean that it contains every field satisfying condition (6c) (so that we get all the possible curves that a test particle could take). The gravitational field is related to the mass density by equation (6a) (the source equation for this theory), whilst the dynamically allowed trajectories are fixed by equation (6c). Note that I have chosen to work with a gravitational field, rather than the gravitational potential. This is simply in order to remove the gauge symmetries of the potential, so that we can focus on those symmetries that alter the field itself. The condition (6b) ensures that this decision is harmless: it holds of G^a if and only if there is a scalar field φ such that $G^a = \overline{\nabla}^a \varphi$.[8] It will be helpful to have a term for a structure $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a, \Xi \rangle$ which does not necessarily satisfy equations (6). We will refer to such a structure as a *kinematical structure* for Galilean gravitation. In intuitive terms, we may think of kinematical structures as representing worlds which are metaphysically possible according to Galilean gravitation (they contain the right ontological ingredients) and the models as representing worlds which are physically possible according to Galilean gravitation (they contain the right ontological ingredients, arranged in the right way).

The first remark to make about this theory is that it is *grossly* unphysical. The mass density μ is just represented as a phenomenological background, in the sense that there is nothing constraining the motion of the matter whose density μ allegedly represents—in particular, nothing requiring that that it follow a dynamically allowed trajectory. (In fact, there isn't even anything in the models which can be identified as representing the motion of the matter comprising μ.) This has a number of counter-intuitive consequences. One is that if a model contains some region with vanishing mass density, it will nevertheless be threaded by dynamically allowed trajectories—the test particles do not contribute to the local mass density. Indeed, note that (because gravitational coupling is universal) we have not even had to assign the test particles any mass at all!

Moreover, it means that there is a substantial amount of underdetermination in the dynamics. In particular, fixing a mass density μ on a Galilean spacetime $\langle \mathfrak{L}, \overline{\nabla} \rangle$ does not uniquely determine the allowed trajectories, as the following proposition demonstrates.

Proposition 2. Let $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a, \Xi \rangle$ be a model of Galilean gravitation, and consider any spacelike field η^a such that $\nabla^a \eta^b = 0$. Then $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a + \eta^a, \Xi' \rangle$ is a model of Galilean gravitation, where Ξ' is defined by the condition that for any unit timelike field ξ'^a on $\mathfrak{L}$,

$$\xi'^a \in \Xi' \text{ iff } (\xi'^n \overline{\nabla}_n \xi'^a - \eta^a) \in \Xi \tag{7}$$

Proof. Since $\nabla^a \eta^b = 0$, $\nabla_a \eta^b = t_a \theta^n \nabla_n \eta^b$, where θ^n is any future-directed unit timelike field; it follows that $\nabla_a \eta^a = 0$.[9] The proposition immediately follows. □

[8] See [Malament, 2012, Proposition 4.1.6]. Note that this is precisely analogous to the role played by the equation $\nabla \times \mathbf{E} = 0$ in electrostatics.

[9] This observation is adapted from [Malament, 2012, p. 277].

This also means that the theory is indeterministic: by letting $\eta^a = 0$ prior to some arbitrary time, and then smoothly increasing thereafter, we can make the two models agree up to that time but disagree thereafter. (Such an η^a is permissible, for the condition $\nabla^a \eta^b =$ requires only that η^a be spatially constant at each time, not that it be constant over time.) One small remark, on why this should be considered genuine underdetermination (and hence, genuine indeterminism): note that the change from $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a, \Xi \rangle$ to $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a + \eta^a, \Xi' \rangle$ is *not* a symmetry transformation. A symmetry transformation ought to be specificiable in terms of systematic transformations of the structures in the theory, and there is no way—at least, no obvious way—to characterise Ξ' as arising from the application of a systematic transformation to the members of Ξ.[10] This also points up the benefits of representing the test-particle trajectories explicitly in the model, even though they play a very minimal dynamical role: had they not been there, then we would have a (formal) symmetry transformation to hand, and might have been tempted to interpret the underdetermination as merely apparent.[11] However, although the underdetermination is a genuine feature of the theory, it is important to recognise that it is an artefact of decoupling the mass density μ from the dynamically allowed trajectories. We can block it if we require that the mass density "flows" along some particular dynamically allowed tangent field, ζ^a: if $\zeta^a \in \Xi$, then $\zeta^a \notin \Xi'$, and so the above move no longer lets us construct a second model consistent with the same background data.

Clearly, working with a theory in which we do subject the mass density to appropriate dynamics would have significant conceptual advantages. However, I do not intend to do so in this paper. In my defence, I offer two considerations. One is just that this more realistic dynamics involves more complicated mathematics; so there is some advantage to working in the simpler case, bearing its defects in mind. The other is that formal presentations of Newtonian gravitation in the literature standardly do not relate the mass density to dynamically allowed trajectories (indeed, many presentations omit the dynamically allowed trajectories altogether).[12] By not doing so either, I make comparison easier—and in particular, can highlight some consequences of making this choice, which might otherwise go unremarked.

Let us now turn to our main topic. The jumping-off point is the presence of a certain kind of symmetry in the theory of Galilean gravitation. However, we have to be a little careful here, for one finds two presentations of the symmetry in the literature. Ultimately this is harmless, since the two ways of presenting it turn out to be more or less equivalent—but showing that to be the case is somewhat non-trivial, and illuminating to work through.

First, some discussions[13] present the relevant symmetry as involving a transformation of the *connection* and the *gravitational field*. More specifically, the symmetry is presented as follows:

$$\overline{\nabla} \mapsto (\overline{\nabla}, \eta^a t_b t_c) \tag{8a}$$

$$G^a \mapsto G^a + \eta^a \tag{8b}$$

where η^a is any spacelike vector field such that $\overline{\nabla}^a \eta^b = 0$. It is straightforward to show that this is, indeed, a symmetry of the above: if $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$ and $G'^a = G^a + \eta^a$ are substituted into the above equations, we get the same equations out again. It will be helpful to introduce some terminology for when two connections are related in the manner (8a) above:

[10] For example, one might be tempted to try the map $\xi^a \mapsto \sigma^a$, for some vector field σ^a with appropriate properties, and hope to identify η^a with some construction out of σ^a. But the accelerations transform as $\xi^n \nabla_n \mapsto \xi^n$-$\nabla_n \xi^a + \sigma^n \nabla_n \xi^a + \xi^n \nabla_n \sigma^a + \sigma^n \nabla_n \sigma^a$: so this will only work if $\sigma^n \nabla_n \xi^a + \xi^n \nabla_n \sigma^a = 0$, and I am sceptical that there are conditions that could be imposed on σ^a that will make this so for all $\xi^a \in \Xi$.

[11] [Belot, 2013] uses essentially this phenomenon, but in the case of electromagnetism, to question the claim that symmetry transformations should, in general, be taken to reveal surplus structure. Again, taking the trajectories of matter guided by the fields into account means that the transformations in question cease to be symmetries.

[12] e.g. [Friedman, 1983], [Knox, 2014], [Weatherall, 2015b].

[13] e.g. [Knox, 2014].

Definition 2. Let $\mathfrak{L}$ be a Leibnizian spacetime structure, and suppose that ∇ and ∇' are two connections compatible with $\mathfrak{L}$. We say that ∇' is *rigidly linearly accelerated* relative to ∇ if $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^a \eta^b = 0$.

Sometimes, however,[14] the symmetry is instead presented as acting upon the gravitational field, the mass density, and the test-particle tangents. More precisely, let a *swerve* be a diffeomorphism $s : \mathfrak{L} \to \mathfrak{L}$ which, in any coordinate system adapted to $\mathfrak{L}$, takes the form

$$t \mapsto t \tag{9a}$$

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}(t) \tag{9b}$$

where $\mathbf{a}(t)$ is an arbitrary time-dependent translation. Furthermore, this spacetime transformation is conjoined with a certain kind of internal transformation. Again in terms of coordinates, we characterise it as follows: if the components of G^a in the same adapted coordinates as above are g,[15] then define $\widetilde{G}^a$ as the vector field with components (in that selfsame coordinate system)

$$\widetilde{\mathbf{g}} = \mathbf{g} + \ddot{\mathbf{a}}(t) \tag{10}$$

where $\ddot{\mathbf{a}}(t)$ is, as one would expect, the second temporal derivative of $\mathbf{a}(t)$.

In order to assess whether this transformation is a symmetry, we need to specify how the models of the theory are affected by such "external" transformations (i.e., by maps on the manifold representing spacetime). In general discussions of spacetime theories, such a transformation is traditionally taken to act by pushing forward some of the structures in each model, whilst leaving the other structures alone; call the structures which get pushed forward "foreground" structures, and those which get left alone "background" structures.[16] For the purposes of this symmetry, the right way to make the split into background and foreground is to take $\mathfrak{L}$ and $\overline{\nabla}$ as background, and the remainder as foreground. So (on this characterisation), the effect of the combined transformation is

$$G^a \mapsto s^*(\widetilde{G}^a) \tag{11a}$$

$$\mu \mapsto s^*\mu \tag{11b}$$

$$\Xi \mapsto s^*\Xi \tag{11c}$$

where $s^*\Xi = \{s^*\xi^a : \xi^a \in \Xi\}$. Note that G^a undergoes a "double transformation": it is both acted on by the diffeomorphism, and by the internal transformation (10). Showing that this is a symmetry is not very easy in coordinate-free terms—but by translating the dynamical equations (6) into coordinates, we can do so.

Thus, it might look at first glance as though we have two symmetry transformations on the table: one of which turns a model $\langle \mathfrak{L}, \overline{\nabla}, G^a, \mu, \Xi \rangle$ into $\langle \mathfrak{L}, \overline{\nabla}', G'^a, \mu, \Xi \rangle$, and the other of which turns that same model into $\langle \mathfrak{L}, \overline{\nabla}, s^*(\widetilde{G}^a), s^*\mu, \{s^*\xi^a\} \rangle$. However, appearances can be deceptive: in an important sense, the effects of these two transformations are equivalent to one another. First, observe that $s^*\mathfrak{L} = \mathfrak{L}$ (that is, $s^*t_a = t_a$ and $s^*h^{ab} = h^{ab}$). As a result,

$$\langle \mathfrak{L}, (s^{-1})^*\overline{\nabla}, \widetilde{G}^a, \mu, \Xi \rangle \cong \langle \mathfrak{L}, \overline{\nabla}, s^*(\widetilde{G}^a), s^*\mu, \{s^*\xi^a\} \rangle \tag{12}$$

[14] e.g. [Saunders, 2013], [Pooley, 2013].

[15] Note that there are only three components, since $G^0 = 0$ in any adapted coordinate system.

[16] Terminology here is a nightmare. [Earman, 1989, p. 45] gives the same prescription here for how external transformations act on the models of a spacetime theory, although he uses the terms "absolute objects" and "dynamical objects" to refer to what I have called background and foreground structues. [Friedman, 1983] does the same, but also argues that the split into absolute and dynamical objects should always be done by reference to which objects are constant across different models of the theory; by contrast, I want to allow that the split might be something we simply stipulate. Thus, the foreground/background distinction, as I am using the terminology, is closer to the distinction [Pooley, 2015] draws between "dynamical" and "fixed" fields.

since the latter results from the former by applying the diffeomorphism s to all the structures comprising it. We can then show that for any swerve s, there is a spacelike field η^a satisfying $\overline{\nabla}^a\eta^b = 0$, such that

$$(s^{-1})^*(\overline{\nabla}) = (\overline{\nabla}, \eta^a t_b t_c) \tag{13}$$

and

$$\widetilde{G}^a = G^a + \eta^a = G'^a \tag{14}$$

Thus, we can indeed think of both of these transformations as capturing the same symmetry. Moreover, it is a bona fide symmetry (unlike the case above): we are engaged in systematically transforming the constituent fields.

So, using either characterisation, we can conclude that the models of Galilean gravitation are not invariant under symmetry transformations: more precisely, there is a symmetry transformation which relates non-isomorphic models to one another. But there is a longstanding tradition of thinking that structure which is variant under a symmetry is "surplus structure": that the differences between symmetry-related models of a theory are (in some sense) not differences that should be taken seriously, and which should motivate us either to interpret the theory in such a way that it is not committed to that structure, or to replace theory by a more parsimonious one. It is not my intention in this paper to assess the merits or demerits of this proposal in general,[17] but merely to note that it could plausibly be applied to this case. (After all, the kinds of considerations that motivate the application of this principle, such as that symmetry-related models will be empirically indistinguishable, look like they are going to apply here.) If it is so applied, then there appears to be something defective about the formalism of Galilean gravitation: it fails to most perspicuously represent the structure to which the best interpretation of the theory is committed. Fortunately, however, there is (or was) a reasonably broad consensus about the formalism which this symmetry motivates us to move to: that of *Newton-Cartan theory*.

4. Newton-Cartan gravitation

In Newton-Cartan theory, we still enrich a Leibnizian spacetime structure with an affine connection. However, this connection is no longer required to be flat. To that end, define a *Newton-Cartan spacetime* to comprise

- A Leibnizian spacetime, $\mathfrak{L}$
- An affine connection $\widetilde{\nabla}$ which is compatible with $\mathfrak{L}$, and which satisfies the *homogeneous Trautman conditions*:

$$\widetilde{R}^{ab}{}_{cd} = 0 \tag{15a}$$

$$\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b} \tag{15b}$$

The conditions (15) can be given geometrical interpretations. Equation (15a) holds iff parallel transport of spacelike vectors is path independent.[18] As a result, any Newton-Cartan spacetime comes naturally equipped with a standard of rotation (a notion about which more will be said below): to find out if a pair of objects are rotating, for example, just parallel-transport their displacement vector along the path of one of them, and see if it coincides with the displacement vector at a second time. Equation (15b) holds iff it is possible to find a smooth, unit timelike

[17] On which there already exists a substantial literature: see, for example, [Saunders, 2003], [Brading and Castellani, 2003], [Baker, 2010], [Dasgupta, 2014], [Dewar, 2015], [Caulton, 2015], and references therein.

[18] [Malament, 2012, Proposition 4.3.1]

field θ^a which is both geodesic ($\theta^n \widetilde{\nabla}_n \theta^a = 0$) and twist-free ($\widetilde{\nabla}^{[a}\theta^{b]} = 0$).[19] It follows from Equation (15b) that a vector field χ^a which is geodesic relative to $\widetilde{\nabla}$ will not start to "spontaneously rotate": along any integral curve of χ^a, if χ^a is twist-free at some point on the curve then it is twist-free at all points on the curve.[20] Thus, Equation (15a) ensures that the connection admits the comparison of spatial directions at different times (in a path-independent fashion), whilst Equation (15b) ensures that geodesics do not exhibit abberrant rotational behaviour.

A dynamically possible model of *Newton-Cartan gravitation* then comprises

- A Newton-Cartan spacetime structure $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$
- A scalar field $\mu : \mathfrak{L} \to \mathbb{R}$
- A maximal class Ξ of unit timelike vector fields

satisfying the following equations:

$$\widetilde{R}_{bd} = 4\pi\mu t_b t_d \tag{16a}$$

$$\xi^n \widetilde{\nabla}_n \xi^a = 0 \tag{16b}$$

What is the relationship between Galilean and Newton-Cartan gravitation? Mathematically, their relationship is given by the following pair of theorems:[21]

Theorem 1. Geometrisation Theorem. Let $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a, \Xi \rangle$ be a model of Galilean gravitation. Then there is a unique derivative operator $\widetilde{\nabla}$ on $\mathfrak{L}$ given by

$$\widetilde{\nabla} = (\overline{\nabla}, G^a t_b t_c) \tag{17}$$

such that $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation.

Theorem 2. Recovery Theorem. Let $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ be a model of Newton-Cartan gravitation. Then any flat derivative operator $\overline{\nabla}$ on $\mathfrak{L}$ which is related to $\widetilde{\nabla}$ by equation (17), is such that $\langle \mathfrak{L}, \overline{\nabla}, \mu, G^a, \Xi \rangle$ is a model of Galilean gravitation (and, there exists at least one such flat derivative operator).

For proofs, see [Malament, 2012, §4.2]. Theorems 1 and 2 indicate the sense in which Newton-Cartan theory may be considered to include only those structures which are invariant under the gravitational gauge symmetry discussed above. First, consider the case where we characterise that symmetry in terms of a transformation of the connection and the gravitational field. For suppose that $\langle \mathfrak{L}, \overline{\nabla}, G^a, \mu, \Xi \rangle$ and $\langle \mathfrak{L}, \overline{\nabla}', G'^a, \mu, \Xi \rangle$ are models related by the application of the symmetry transformation (8). It is straightforward to show that

$$(\overline{\nabla}', G'^a t_b t_c) = (\overline{\nabla}, G^a t_b t_c) \tag{18}$$

and so that both models of Galilean gravitation give rise to the same model of Newton-Cartan gravitation. This bears a natural comparison to the shift from a potentials-based formulation of electromagnetism to a fields-based formulation: the sense in which the fields-based formulation contains only the invariants of the other formulation is that two models of the latter, related by a local potential symmetry, give rise to the same fields-based model.

Second, consider the case where we characterise it in terms of a "swerve" diffeomorphism, combined with a certain internal transformation; suppose that $\langle \mathfrak{L}, \overline{\nabla}, G^a, \mu, \Xi \rangle$ and $\langle \mathfrak{L}, \overline{\nabla}, s^* G'^a, s^*\mu, s^*\Xi \rangle$ are indeed related by such a diffeomorphism s accompanied by the internal transformation (10).

[19] [Malament, 2012, Propositions 4.3.3 and 4.3.7]
[20] [Malament, 2012, Proposition 4.3.6]
[21] Due to [Trautman, 1965].

Let $\widetilde{\nabla} = (\overline{\nabla}, G^a t_b t_c)$, so that the former model gets turned into $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ by (17). Then observe that

$$\begin{aligned}(\overline{\nabla}, s^*G'a) &= s^*((s^{-1})^*\overline{\nabla}, G'^a) \\ &= s^*(\overline{\nabla}', G'^a) \\ &= s^*\widetilde{\nabla}\end{aligned}$$

So the latter model gets turned into $\langle \mathfrak{L}, s^*\widetilde{\nabla}, s^*\mu, s^*\Xi \rangle$, which is isomorphic to $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$. The fact that symmetry-related models are rendered isomorphic rather than identical is interesting, but in keeping with other external symmetries: the shift from Newtonian to Galilean spacetime, for example, renders boost-related models isomorphic but not identical.

It is controversial what we should conclude from this about the interpretative relationship between the two theories. On the one hand, if the theories are interpreted so that their commitments about the structure of spacetime are simply read off the formalism, then they come out inequivalent: for in Galilean gravitation, spacetime is flat, whilst in Newton-Cartan gravitation, spacetime is (typically) curved. On the other, if they are interpreted so that their commitments about the structure of spacetime are not to be so straightforwardly read off—perhaps because of a general anti-realism about spatiotemporal structure, or perhaps because of a view about the way theories express commitments about spatiotemporal structure[22]—then one could deem them equivalent.

This is not an issue I wish to get involved in here. I will discuss judgments of equivalence in the below, but for the most part, we will appeal only to a weaker characterisation of equivalence: namely, one in which two theories are equivalent if there exists some kind of "translation" between the structures used in one theory and those used in the other, which allow us to convert any model of the one theory into a unique model of the other, and back again. Unfortunately , I don't have a general account of what gets to count as a "translation" in this context; we shall have to trust that we can know it when we see it. I stress that this characterisation of equivalence is a reasonably weak one. It is the kind of equivalence that holds, for example, between electromagnetism formulated in terms of a pair of vector fields, $\mathbf{E}$ and $\mathbf{B}$, and electromagnetism formulated as an anti-symmetric tensor field $F_{\mu\nu}$. It may be controversial whether electromagnetism formulated in terms of fields ought to be regarded as equivalent to electromagnetism formulated in terms of potentials, but it surely should not be controversial that electromagnetism formulated in terms of $\mathbf{E}$ and $\mathbf{B}$ is equivalent to electromagnetism formulated in terms of $F_{\mu\nu}$.[23]

5. Maxwell-Weatherall gravitation

Recently, however, Saunders[24] has queried whether we really should regard Newton-Cartan theory as the spacetime theory that properly encodes the lessons of the symmetry canvassed above. Roughly speaking, Saunders' concern might be paraphrased as follows: the symmetry above, at least on the characterisation in terms of swerves, looked like it ought to lead us to the repudiation of absolute accelerations, analogously to the way that boost symmetries lead us to repudiate absolute velocities. Yet any Newton-Cartan spacetime comes with a perfectly well-defined notion of acceleration: after all, each such spacetime carries a connection, albeit one which is not flat. So what gives?

Here is another way of getting at the same kind of worry. Look again at the sense in which

[22]See [Knox, 2014].

[23]I choose these examples deliberately, since [Weatherall, 2015a] argues that the relationship between field- and potentials-based formulations of electromagnetism is analogous to that between Newton-Cartan and Galilean gravitation. (In the terms of that paper, I am appealing only to the claim that theories with isomorphic categories of models are equivalent.)

[24][Saunders, 2013]

Newton-Cartan theory is invariant under swerves: above, I suggested that it was because swerves relate the models $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ and $\langle \mathfrak{L}, s^*\widetilde{\nabla}, s^*\mu, s^*\Xi \rangle$, which we can see to be isomorphic to one another. This suggests that we have gotten this invariance by taking the Leibnizian spacetime $\mathfrak{L}$ as the background, and everything else (i.e., $\widetilde{\nabla}$, μ and Ξ) as foreground. But if so, then we lose the notion that there is something distinctive about swerves. For Leibnizian spacetime is invariant under a *much* larger class of transformations than swerves: for example, any transformation which (in adapated coordinates) takes the form

$$\mathbf{x} \mapsto \mathbf{R}(t)\mathbf{x} \tag{19a}$$

$$t \mapsto t \tag{19b}$$

will also be an automorphism of $\mathfrak{L}$, where $\mathbf{R}(t)$ is any time-dependent rotation matrix. So if r is a diffeomorphism of this kind (call it a "twist"), then $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle \cong \langle \mathfrak{L}, r^*\widetilde{\nabla}, r^*\mu, r^*\Xi \rangle$, and it looks as though swerves and twists are on an equal footing. But this isn't true! For twists are not symmetries of Galilean gravitation, no matter what kind of modifications we try to make to the gravitational field.

Nor are we any better off by taking Newton-Cartan spacetime $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$ to be the background, for a Newton-Cartan spacetime is not invariant under swerves: $\langle \mathfrak{L}, \widetilde{\nabla} \rangle \ncong \langle \mathfrak{L}, s^*\widetilde{\nabla} \rangle$, in general. Hence (in general), $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle \ncong \langle \mathfrak{L}, \widetilde{\nabla}, s^*\mu, s^*\Xi \rangle$. This isn't an idle concern, either: it is precisely this observation that has led various authors[25] to argue that—even though Galilean gravitation admits a symmetry involving arbitrary linear accelerations—it is nevertheless folly to claim that acceleration is relative.

The answer is that neither $\mathfrak{L}$ nor $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$ are appropriate candidates for being regarded as the "background structure" in gravitational theory. For under the application of a swerve s to a Galilean spacetime $\langle \mathfrak{L}, \overline{\nabla} \rangle$, more structure than just $\mathfrak{L}$ is invariant. The structure that is invariant goes by the moniker of *Maxwell spacetime*.[26] Intuitively, the idea is that a Maxwell spacetime contains a "standard of rotation", but no "standard of acceleration". First, I define what it is for two connections to agree on their standard of rotation,[27] and then define a Maxwell spacetime in terms of that.

Definition 3. Let $\mathfrak{L}$ be a Leibnizian spacetime, and let ∇ and ∇' be two connections compatible with $\mathfrak{L}$. ∇ and ∇' are *rotationally equivalent* if, for any unit timelike field θ^a on $\mathfrak{L}$, $\nabla^{[a}\theta^{b]} = 0$ iff $\nabla'^{[a}\theta^{b]} = 0$.

Definition 4. A *Maxwell spacetime* comprises

- A Leibnizian spacetime $\mathfrak{L}$
- A *standard of rotation* W: an equivalence class of rotationally equivalent flat affine connections (compatible with $\mathfrak{L}$)

The different connections in a given standard of rotation are related to one another in a fairly straightforward way, as the following proposition demonstrates.

Proposition 3. Let $\langle \mathfrak{L}, W \rangle$ be a Maxwell spacetime, and consider any $\overline{\nabla} \in W$. For any other connection $\overline{\nabla}'$, $\overline{\nabla}' \in W$ iff $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$, for some spacelike field η^a such that $\overline{\nabla}^a \eta^b = 0$.[28]

Proof. See Appendix A. □

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[25] e.g. [Friedman, 1983, §V.4]
[26] [Earman, 1989, chap. 2]
[27] This definition follows [Weatherall, 2015b].
[28] cf. [Earman, 1977, p. 99]

In other words, all of the connections in a Maxwell spacetime's standard of rotation are rigidly linearly accelerated relative to one another. This proposition demonstrates the invariance of Maxwell spacetime under swerves: defining $s^*W = \{s^*\overline{\nabla} : \overline{\nabla} \in W\}$, we get that for any Maxwell spacetime $\langle \mathfrak{L}, W \rangle$, $\langle s^*\mathfrak{L}, s^*W \rangle = \langle \mathfrak{L}, W \rangle$.

Maxwell spacetime lacks absolute acceleration. However, the sense in which we are led to a theory repudiating absolute acceleration—or in which the role of swerve symmetries is made manifest—depends on our being able to construct a gravitational dynamics that is, in some appropriate sense, "set" upon Maxwell spacetime. The challenge, as [Weatherall, 2015b] observes, is that without a connection, it is not clear how to do so: without a connection, one cannot characterise the dynamically allowed trajectories as those which deviate from inertial motion just insofar as they are acted on by forces. However, he argues that there is a natural indirect solution. In the terminology and notation used here, his core claim is that given a Maxwell spacetime $\langle \mathfrak{L}, W \rangle$, for any $\overline{\nabla} \in W$, there exists some twist-free vector field G^a such that (1) $\overline{\nabla}_a G^a = -4\pi\mu$, where μ is the mass density distribution of spacetime, and (2) the allowed trajectories of bodies are curves, with tangents ξ^a, whose acceleration (relative to $\overline{\nabla}$) is given by $\xi^n \nabla_n \xi^a = G^a$.[29] Here is a way of cashing this out in precise terms.

The idea is that we relativise the gravitational field to an arbitrary choice of derivative operator from W. Of course, the way in which gravitational fields are assigned to operators will have to be constrained, so that (roughly speaking) two fields are related to one another in the same way the operators are. More precisely:

Definition 5. Given a Maxwell spacetime $\langle \mathfrak{L}, W \rangle$, a *relative gravitational field* on $\langle \mathfrak{L}, W \rangle$ is a map $G^a_* : \overline{\nabla} \in W \mapsto G^a_{\overline{\nabla}} \in \mathfrak{X}(\mathfrak{L})$,[30] such that if $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$, then

$$G^a_{\overline{\nabla}'} = G^a_{\overline{\nabla}} - \eta^a \tag{20}$$

Definition 6. A model of *Maxwell-Weatherall gravitation* comprises

- A Maxwell spacetime $\langle \mathfrak{L}, W \rangle$
- A relative gravitational field $G^a_* : W \to \mathfrak{X}(M)$
- A scalar field $\mu : \mathfrak{L} \to \mathbb{R}$
- A maximal class Ξ of unit timelike vector fields

satisfying the following equations:

$$\overline{\nabla}_a G^a_{\overline{\nabla}} = -4\pi\mu \tag{21a}$$

$$\overline{\nabla}^{[a} G^{c]}_{\overline{\nabla}} = 0 \tag{21b}$$

$$\xi^n \overline{\nabla}_n \xi^a = G^a_{\overline{\nabla}} \tag{21c}$$

for any $\overline{\nabla} \in W$ and every $\xi^a \in \Xi$.

One might object: in what sense is this a dynamics set upon Maxwell spacetime? After all, the equations (21) clearly make use of a particular derivative operator from W! However, that need not express a commitment to the structure of that derivative operator in particular, *provided* that whether or not $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ satisfies the equations (21) is independent of our choice of $\overline{\nabla}$ from W. This is, indeed, the case.

[29] The above is intended as a verbatim transcription of the following: "given a Maxwell-Huygens spacetime $(M, t_a, h^{ab}, [\nabla])$, for any $\nabla \in [\nabla]$, there exists some scalar field φ such that (1) $\nabla_a \nabla^a \varphi = 4\pi\rho$, where ρ is the mass density distribution of spacetime, and (2) the allowed trajectories of bodies are curves γ whose acceleration (relative to ∇) is given by $\xi^n \nabla_n \xi^a = \nabla^a \varphi$." [Weatherall, 2015b, p. 8].

[30] Where $\mathfrak{X}(\mathfrak{L})$ is the space of vector fields on $\mathfrak{L}$.

Proposition 4. Let $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ be a possible model of Maxwell-Weatherall gravitation, and consider any $\overline{\nabla}, \overline{\nabla}' \in W$. Then (for any $\xi^a \in \Xi$) the equations (21) hold with respect to $\overline{\nabla}$ iff they hold with respect to $\overline{\nabla}'$.

Proof. This follows straightforwardly from the Trautman Recovery Theorem (Theorem 2). □

What is the relationship between Maxwell-Weatherall gravitation and the theories of gravitation we saw above? Simply this: Maxwell-Weatherall gravitation is (plausibly) equivalent to Newton-Cartan gravitation. There are generic translations which, given any model of Maxwell-Weatherall-gravitation, let us construct a model of Newton-Cartan gravitation; and which, given any model of Newton-Cartan gravitation, let us construct a model of Maxwell-Weatherall gravitation.

Proposition 5. Let $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ be a model of Newton-Cartan gravitation. Consider all pairs $\langle \overline{\nabla}, G^a_{\overline{\nabla}} \rangle$ where $\overline{\nabla}$ is a flat affine connection, $G^a_{\overline{\nabla}}$ is a spacelike twist-free vector field, and $\widetilde{\nabla} = (\overline{\nabla}, G^a_{\overline{\nabla}} t_b t_c)$. Define W as consisting of all the flat connections $\overline{\nabla}$ that feature in some such pair; and define G^a_* as the map $W \to \mathfrak{X}(\mathfrak{L})$ which assigns $\overline{\nabla}$ to $G^a_{\overline{\nabla}}$. Then $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ is a model of Maxwell-Weatherall gravitation.

Proof. See Appendix A. □

Proposition 6. Let $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ be a model of Maxwell-Weatherall gravitation. Let $\overline{\nabla}$ be an arbitrary element of W. Define $\widetilde{\nabla} = (\overline{\nabla}, G^a_{\overline{\nabla}} t_b t_c)$. So defined, $\widetilde{\nabla}$ is independent of the choice of $\overline{\nabla}$; and $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation.

Proof. See Appendix A. □

Saunders asks the question, "What is the relation between a theory of gravity (and other forces) formulated in Maxwell space-time and one based on Newton-Cartan space-time?"[31] We now have a partial answer: at least for Maxwell-Weatherall gravitation (which could presumably be supplemented with dynamics for other forces as appropriate), the relation is one of equivalence. However, Maxwell-Weatherall gravitation is a rather inelegant theory. It would be nice to have a dynamics which did not have to traffic in "relative gravitational fields", whatever those are. Moreover, the relationship to Newton-Cartan theory is—as Weatherall acknowledges—rather indirect. Apart from anything else, there is still a mismatch between the explicitly geometrical structures of Maxwell-Weatherall theory and those of Newton-Cartan theory: the Newton-Cartan connection defined in Proposition 6 is not a member of W. Finally, one might feel a little unhappy about the prominence that standards of acceleration still play in this theory. It remains the case that the equation of motion (21c) is presented as a requirement that the acceleration of a test particle be such-and-such—it's just that the such-and-such is relativised to the standard used to measure the acceleration. It would be preferable to have a condition that is phrased, so far as possible, in terms of relative acceleration.

6. Maxwell-Cartan gravitation

In this section, I present a theory that meets these desiderata. First, I define the kind of spacetime structure which we will be using, which will let us make the relationship between our theory and Newton-Cartan theory more explicit. The starting observation is that the connections which figure in a given model of Newton-Cartan gravitation and its various "de-geometrisations" (i.e. the models of Galilean gravitation related to it by equation 17) are all rotationally equivalent. (Both in intuitive terms, and in the precise sense canvassed above.) This suggests an obvious

[31] [Saunders, 2013, p. 46]

way to liberalise the notion of a "standard of rotation" that was used in Maxwell spacetime: simply expand it to include non-flat connections. However, we don't want to expand it too far. After all, as we saw above, we know that whatever connection we wind up with after the geometrisation will still obey the homogeneous Trautman conditions (15). This observation motivates the following definition.

Definition 7. A *Maxwell-Cartan* spacetime, $\mathfrak{M}$, consists of data $\langle \mathfrak{L}, J \rangle$, where

- $\mathfrak{L}$ is a Leibnizian spacetime
- J is a non-empty equivalence class of Newton-Cartan connections (i.e. $\mathfrak{L}$-compatible connections satisfying the homogeneous Trautman conditions), under the equivalence relation of rotational equivalence.

We saw above that all the connections in a Maxwell spacetime's standard of rotation are related by rigid linear accelerations. An analogous result holds for the connections in a Maxwell-Cartan spacetime's standard of rotation.

Proposition 7. Let $\langle \mathfrak{L}, J \rangle$ be a Maxwell-Cartan spacetime, and consider any $\nabla \in J$. For any other connection ∇', $\nabla' \in J$ iff $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^{[a}\eta^{b]} = 0$.

Proof. See Appendix A. □

Again, it is helpful to have some specific terminology for the kind of relationship that holds between the members of J.

Definition 8. Let $\mathfrak{L}$ be a Leibnizian spacetime, and suppose that ∇ and ∇' are two connections compatible with $\mathfrak{L}$. We say that ∇' is *linearly accelerated* relative to ∇ if $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^{[a}\eta^{b]} = 0$.

As one would expect, this is a strictly weaker condition that that of being rigidly linearly accelerated: that requires that $\nabla^a \eta^b = 0$.

How does Maxwell-Cartan spacetime relate to Maxwell spacetime? It turns out to be a little difficult to say. On the one hand, given any Maxwell spacetime, one can construct a unique Maxwell-Cartan spacetime, and vice versa. The following two propositions make this precise.

Proposition 8. Let $\langle \mathfrak{L}, W \rangle$ be a Maxwell spacetime. Define the class of connections J by taking the closure of W under linear acceleration (so $\nabla \in J$ iff $\nabla = (\overline{\nabla}, \eta^a t_b t_c)$, for some $\overline{\nabla} \in W$ and some spacelike twist-free η^a). Then $\langle \mathfrak{L}, J \rangle$ is a Maxwell-Cartan spacetime.

Proof. See Appendix A. □

Proposition 9. Let $\langle \mathfrak{L}, J \rangle$ be a Maxwell-Cartan spacetime. Define the class of connections W to consist of just those members of J which are flat. Then $\langle \mathfrak{L}, W \rangle$ is a Maxwell spacetime.

Proof. See Appendix A. □

This suggests a sense in which Maxwell-Cartan and Maxwell spacetime are equivalent: it seems that a structure $\langle \mathfrak{L}, W, J \rangle$, where $W \subset J$, could be regarded as a "definitional expansion" of both the Maxwell spacetime $\langle \mathfrak{L}, W \rangle$ and the Maxwell-Cartan spacetime $\langle \mathfrak{L}, J \rangle$ (since each of W and J can be defined in terms of the other). Against this, however, there is a concern that the automorphisms of $\langle \mathfrak{L}, W \rangle$ will turn out to be a strict subset of the automorphisms of $\langle \mathfrak{L}, J \rangle$: for, as just discussed, Maxwell-Cartan spacetime is invariant under linear acceleration, whereas Maxwell spacetime is not. This isn't a rigorous argument, since I haven't explicitly linked my stipulative terminology of "(rigid) linear acceleration" to theses about the behaviour of the connections under pullback. Doing so isn't very straightforward, but it seems plausible that such links will hold; if they do, then the automorphism groups will not coincide. But if so, then this provides a sense in which Maxwell spacetime and Maxwell-Cartan spacetime are not equivalent—if the automorphisms of the former are a strict subset of those of the latter, then that suggests taking Maxwell spacetime to have strictly more structure than Maxwell-Cartan spacetime.[32] In support of this claim of inequivalence, note that Maxwell spacetime canonically induces an affine holonomy map for each point $p \in \mathfrak{L}$, namely the trivial holonomy map: all the connections in W agree that a tangent vector is invariant when parallel-transported round a loop. By contrast, the connections in J do not agree on what happens to such a tangent vector, and so one cannot canonically associate holonomy maps to the points in a Maxwell-Cartan spacetime in the same way.

I am genuinely unsure, given these competing considerations, whether it is better to regard Maxwell spacetime and Maxwell-Cartan spacetime as equivalent or not. Note, however, that if we decide they are inequivalent, then Maxwell-Cartan spacetime is the weaker structure of the two; so showing that Newtonian gravitation may be set on Maxwell-Cartan spacetime serves, *a fortiori*, to show that it may be set on Maxwell spacetime. I now proceed to show just this. That is, I now specify a gravitational dynamics, against the backdrop of Maxwell-Cartan spacetime, which makes no use of gravitational fields (even relative ones), nor of absolute accelerations. As with Maxwell-Weatherall gravitation, the equations will be expressed in terms of a derivative operator; but we will show that the equations' holding is independent of the choice of derivative operator from the background standard of rotation, J, and therefore presupposes no more structure than that specified by J.

We begin with the following preliminary definition.

Definition 9. Let ξ^a and ξ'^a be unit timelike fields on a Maxwell-Cartan spacetime $\langle \mathfrak{L}, J \rangle$. ξ^a and ξ'^a are *acceleratively equivalent* just in case they have the same acceleration at every point in $\mathfrak{L}$:

$$\xi^n \nabla_n \xi^a = \xi'^n \nabla_n \xi'^a \tag{22}$$

Note that this definition is independent of the choice of derivative operator (from J) used to measure the acceleration of ξ^a and ξ'^a. An equivalence class of acceleratively equivalent fields is *maximal* if there is no equivalence class of which it is a proper subclass.

We can now state the dynamics. A dynamically possible model of *Maxwell-Cartan gravitation* comprises

- A Maxwell-Cartan spacetime $\langle \mathfrak{L}, J \rangle$
- A scalar field $\mu : \mathfrak{L} \to \mathbb{R}$
- A maximal equivalence class Ξ of acceleratively equivalent unit timelike fields

obeying the following equations:

$$R_{bd}\xi^b\xi^d - \nabla_a(\xi^n \nabla_n \xi^a) = 4\pi\mu \tag{23a}$$

$$\nabla^c(\xi^n \nabla_n \xi^a) - \nabla^a(\xi^n \nabla_n \xi^c) = 0 \tag{23b}$$

for any $\nabla \in J$ (with Ricci tensor R_{ab}) and $\xi^a \in \Xi$.

As with Maxwell-Weatherall gravitation, in order to justify the claim that we are presupposing only the structure of Maxwell-Cartan spacetime, we need to verify that these equations are invariant under choice of $\nabla \in J$. The following proposition does so.

Proposition 10. Let $\langle \mathfrak{L}, J, \mu, \Xi \rangle$ be a possible model of Maxwell-Cartan gravitation, and consider any $\nabla, \nabla' \in J$. Then (for any $\xi^a \in \Xi$) the equations (23) hold with respect to ∇ iff they hold with respect to ∇'.

[32] This is the principle SYM* in [Barrett, 2014].

Proof. See Appendix A. □

Note that, as promised, neither of the equations (23) involve the assertion that the absolute acceleration of the test-particle trajectories takes a particular value—not even where that value is relativised to the standard used to measure acceleration. The second equation, equation (23b), asserts that all dynamically allowed accelerations are twist-free; i.e. that the four-acceleration field associated with any field in Ξ is non-rotating. The first equation, equation (23a), asserts that the "average radial relative acceleration" of any field in Ξ is given by $-4/3\pi\mu$.[33] That is, let λ^a be a *connecting field* for ξ^a: a spacelike vector field such that $\mathcal{L}_\xi \lambda^a = 0$ (where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ^a). Intuitively, we think of λ^a as joining integral curves of ξ^a to "neighbouring" integral curves. The relative acceleration of such neighbouring curves is then given by

$$\xi^n \nabla_n (\xi^m \nabla_m \lambda^a) \tag{24}$$

and has radial component (magnitude in the direction of λ^a)

$$\lambda_a \xi^n \nabla_n (\xi^m \nabla_m \lambda^a) \tag{25}$$

where $\lambda_a = \hat{h}_{ab}\lambda^b$, for $\hat{h}_{ab}$ the spatial metric associated to ξ^a.[34] This depends on λ^a. But if we introduce three connecting fields $\overset{1}{\lambda}{}^a, \overset{2}{\lambda}{}^a, \overset{3}{\lambda}{}^a$ which are orthonormal to one another, then we can introduce the *average radial acceleration* of ξ^a as the average of the three radial components,

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_a \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^a) \tag{26}$$

It can then be shown that the average radial acceleration is independent of the choice of connecting fields $\overset{i}{\lambda}{}^a$; indeed, we have

Proposition 11. Let ξ^a be a unit timelike field, and suppose that $\{\overset{i}{\lambda}{}^a\}_i$ are three orthonormal spacelike fields such that $\mathcal{L}_\xi \overset{i}{\lambda}{}^a = 0$. Then

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_a \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^a) = \frac{1}{3}\left(\nabla_a(\xi^n \nabla_n \xi^a) - R_{bd}\xi^b \xi^d\right) \tag{27}$$

Proof. See Appendix A. □

Thus, the average radial acceleration of ξ^a is $-4/3\pi\mu$ iff ξ^a obeys equation (23a).

Now, I claim that Maxwell-Cartan gravitation is equivalent to Newton-Cartan gravitation. The following two propositions support this claim.

Proposition 12. Let $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ be a model of Newton-Cartan gravitation. Define

$$J = \{\nabla : \nabla = (\widetilde{\nabla}, \eta^a t_b t_c)\} \tag{28}$$

for any spacelike η^a such that $\widetilde{\nabla}^{[a}\eta^{b]} = 0$. Then $\langle \mathfrak{L}, J, \mu, \Xi \rangle$ is a model of Maxwell-Cartan gravitation.

Proof. See Appendix A. □

[33] The below is modelled on [Malament, 2012, Propositions 2.7.2 and 4.3.2].

[34] In fact, given that λ^a is spacelike, we could have used the spatial metric associated to any unit timelike field; but since we have a particular such field knocking around, it is helpful to fix on it.

Proposition 13. Let $\langle \mathfrak{L}, J, \mu, \Xi \rangle$ be a model of Maxwell-Cartan gravitation. Let ∇ be an arbitrary element of J, and let ξ^a be an arbitrary element of Ξ. Define a derivative operator $\widetilde{\nabla}$ by

$$\widetilde{\nabla} = (\nabla, t_b t_c \xi^n \nabla_n \xi^a) \tag{29}$$

So defined, $\widetilde{\nabla}$ is independent of the choice of ξ^a and ∇; and $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation.

Proof. See Appendix A. □

There are two comments to make about the above—both of them regarding choices I have made in constructing the theory, and some of the consequences of those choices. The first regards the role of requiring accelerative equivalence among the dynamically allowed trajectories. One might be concerned that this sneaks in a little extra structure "under the rug", as it were. Note, in particular, that the reason why no such requirement was involved in our earlier theories is that there was some object which coordinated the four-accelerations with one another: the gravitational field in Galilean gravitation, the Newton-Cartan connection in Newton-Cartan gravitation, and the relative gravitational field in Maxwell-Weatherall gravitation. So it might be thought that Maxwell-Cartan gravitation is secretly committed to something like a (relative) gravitational field, even though it doesn't appear explicitly on the list of ingredients of a model.

I think that this claim is essentially right. Indeed, it had better be right, if Maxwell-Cartan gravitation is meant to be equivalent to Newton-Cartan or Maxwell-Weatherall gravitation: that equivalence holds precisely because one can construct a (unique) Newton-Cartan connection, or a (unique) relative gravitational field, within any model of Maxwell-Cartan gravitation. Now, if the worry is just that this commitment isn't obvious from the formalism of the theory, then it doesn't strike me as compelling: that is, I don't think we should be worried about the fact that this commitment is less "visible" in Maxwell-Cartan theory than in these other cases. After all, plenty of theories are committed to structure other than that which shows up on the list of ingredients: as we have already seen, for instance, models of Galilean and Newton-Cartan gravitation are committed to a standard of rotation (in the sense that they pick out such a standard), even though no such object shows up in between the angle-brackets of any model.

Alternatively, the worry might be that there is something objectionably "arbitrary" about the fact that it is one equivalence class of trajectories that is picked out as the class of dynamically allowed trajectories, rather than any other. For, note that there will exist multiple models of Maxwell-Cartan gravitation with the same Maxwell-Cartan spacetime and mass density, but with distinct classes of dynamically allowed trajectories. I think that it is correct to recognise this as a problem, but incorrect to diagnose it as a problem attending specifically to Maxwell-Cartan gravitation: rather, it is just the underdetermination we first saw in section 3, illustrated by Proposition 2. And just as was the case there, this underdetermination is (one hopes) an artefact of the decision to treat the mass density as merely phenomenological, rather than as comprised of matter flowing along a dynamically allowed trajectory. It seems plausible to conjecture that if we reversed that decision, then the problem would go away: there would only be one accelerative equivalence class of trajectories consistent with the flow field ζ^a. It would be of value to show this explicitly; but unfortunately, I lack the space to do so here.

The second remark concerns how the above relates to the work of Weatherall and Saunders. Weatherall makes the following claim, regarding the theory that was presented above as "Maxwell-Weatherall gravitation":

> What is the invariant physical structure in this theory? For one, as we have seen, there is the standard of rotation shared between the derivative operators. This gives the sense in which this is a theory in Maxwell-Huygens [[i.e., Maxwell]] spacetime. The other invariant structure, however, is the collection of allowed trajectories for bodies. These are calculated in different ways depending on which representative

one chooses from [[W]], and the accelerations associated with each such curve varies similarly. So we do not have the structure to say that these curves are accelerating or not. But however they are described, i.e., whatever acceleration (if any) is attributed to them, the curves themselves are fixed. Indeed, given some distribution of matter is spacetime, it is these curves that form the empirical content of Newtonian gravitational theory.

> Now suppose that we are given some such collection of curves, $\{\gamma\}_\rho$, relativized to a matter distribution ρ, in Maxwell-Huygens spacetime. Suppose, too, that however these curves are determined—whether by the calculational procedure just mentioned or some other method—they agree with the possible trajectories allowed by ordinary Newtonian gravitation. It turns out that with this information, one can uniquely reconstruct a [[model of Newton-Cartan gravitation]].[35]

We are now in a position to clarify some points about this. First, as just discussed, Weatherall is not quite correct to say that the curves are fixed: the underdetermination means that there are several available equivalences classes of dynamically allowed curves. Moreover, if we are working in the context of Maxwell-Weatherall gravitation, then we do not simply get dynamically allowed collections of curves on Maxwell spacetime: each collection is accompanied by a relative gravitational field, pairing the connections in the standard of rotation to gravitational fields.

However, it is natural to see Maxwell-Cartan gravitation as capturing the intuitions expressed in the above passage. In Maxwell-Cartan gravitation, we can indeed make sense of just taking a Maxwell-Cartan spacetime, laying down a mass density upon it, and then considering some particular equivalence class of dynamically allowed trajectories consistent with that mass density. And having done so, we can indeed then construct a unique model of Newton-Cartan gravitation. This construction—i.e., Proposition 13—provides a precise analogue, in the context of Maxwell-Cartan gravitation, to Weatherall's proposition 4, which (translated into the notation used here) states that

> Let $\{\gamma\}_\mu$ be the collection of allowed trajectories for a given mass distribution μ in Maxwell-Huygen spacetime $\langle \mathfrak{L}, W \rangle$, as described above [i.e. in the passage just quoted]. Then there exists a unique derivative operator $\widetilde{\nabla}$ such that (1) $\{\gamma\}_\rho$ consists in the timelike geodesics of $\widetilde{\nabla}$ and (2) $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$ is a model of Newton-Cartan theory for mass density μ.[36]

Weatherall goes on to conclude that

> this result—at least as I interpret it here—reveals a certain inadequacy in Saunders' account. Saunders insists that there is no privileged standard of acceleration in Maxwell-Huygens spacetime. And there are a few senses in which that is right: (1) before accounting for gravitational influences, Maxwell-Huygens spacetime does not have enough structure to make sense of acceleration; and (2) even in the presence of dynamical considerations, there is in general no privileged flat derivative operator, and thus no privileged collection of inertial frames in the standard sense, relative to which acceleration may be defined. Nonetheless, it turns out that once one takes the dynamically allowed trajectories into account, one can define a standard of acceleration, namely, the unique one relative to which the allowed trajectories are geodesics.[37]

Understood as a claim about Maxwell-Weatherall or Maxwell-Cartan gravitation, this is quite right. However, it is not clear that either of these really captures what Saunders had in mind. Consider the following passage:

[35] [Weatherall, 2015b, p. 9]
[36] [Weatherall, 2015b, p. 9]
[37] [Weatherall, 2015b, p. 10]

> Take possible worlds each with only a single structureless particle. Depending on the connection, there will be infinitely many distinct trajectories, infinitely many distinct worlds of this kind. But in Newton-Huygens terms, as in Barbour-Bertotti theory, there is only one such world—a trivial one in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn.[38]

This suggests that Saunders would object to the fact that models of Maxwell-Cartan theory cheerfully include all the dynamically allowed trajectories: for it is precisely by keeping track of all such trajectories that we are able to reconstruct the Newton-Cartan connection from any model of Maxwell-Cartan theory. (This is made clear in the proof of Proposition 13.)

Now, when working with a theory of the kinds discussed so far—in which the mass density is not subject to any kind of non-trivial dynamics—keeping those trajectories in play is pretty crucial (since otherwise, as Proposition 2 illustrates, we will overestimate the symmetry group of the theory). But in the context of a theory in which the mass distribution is *not* treated phenomenologically, Saunders' remark suggests a natural alternative: a version of Maxwell-Cartan theory whose models include only the dynamically *realised* trajectories, i.e., include only the matter flow field ζ^a—and *not* the full collection Ξ of dynamically allowed trajectories. Again, although examining such a theory would be interesting, I lack the space to do so here. I do want to note, however, that the empiricist warrant for such a theory is a little more strained than might at first appear. Empiricist scruples classically require us to permit only structure in our theory which is empirically accessible. A natural way to cash out the idea of empirical accessibility is as " that which can be directly observed by experiment". But the trajectories of possible test particles *can* be directly observed: *if* a particle were to be released at a given point, then observing its path would reveal the trajectory! In other words, the fact that these trajectories do not *in fact* contain any matter (in a given model), and so are not in fact *observed*, does not mean that they are *unobservable*—had they contained matter, they would have been observed.

7. Conclusion

To conclude, I wish to observe that Proposition 13 is illustrating something quite striking. We begin with a theory containing a relatively minimal amount of geometrical structure; this structure is nevertheless sufficient to permit the characterisation of a non-trivial dynamics; that dynamics then permits us to introduce further geometrical structure (to wit, the connection) as a codification of the behaviour of the dynamical objects.[39] In other words, Proposition 13 provides an (extremely partial) illustration of the so-called *dynamical approach to spacetime geometry*,[40] in which one seeks to characterise spacetime geometry as a codification of the behaviour of dynamical structures.[41] A natural question is whether this can be extended: can we do the same trick, but starting from a more minimal geometrical basis yet? A starting-point for an answer would be to try and replicate the analysis here, but using the Künzle-Ehlers Recovery Theorem[42] rather than the Trautman Recovery Theorem. It would also be of interest to know to what extent, if any, such techniques could be extended to relativistic rather than Newtonian spacetimes. I postpone these questions to another time.

[38] [Saunders, 2013, pp. 46–47]
[39] cf. [Knox, 2014]
[40] [Brown, 2005], [Stevens, 2015]
[41] [Wallace, 2015] discusses these issues in more depth.
[42] [Malament, 2012, Proposition 4.5.2]

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A. Proofs of propositions

Proposition 3. Let $\langle \mathfrak{L}, W\rangle$ be a Maxwell spacetime, and consider any $\overline{\nabla} \in W$. For any other connection $\overline{\nabla}'$, $\overline{\nabla}' \in W$ iff $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$, for some spacelike field η^a such that $\overline{\nabla}^a \eta^b = 0$.[43]

Proof. First, suppose that $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$ for some spacelike rigid η^a. To show that ∇ and ∇' are rotationally equivalent, let θ^a be any unit timelike field, and observe that

$$\begin{aligned}\nabla'^{[a}\theta^{b]} &= \nabla^{[a}\theta^{b]} - h^{c[a}\eta^{b]}t_c t_n \theta^n \\ &= \nabla^{[a}\theta^{b]}\end{aligned} \tag{30}$$

It remains to show that ∇' is flat. Using the standard expression relating two Riemann tensors, we can obtain

$$R'^a{}_{bcd} = R^a{}_{bcd} + 2t_b t_{[d}\nabla_{c]}\eta^a \tag{31}$$

But $R^a{}_{bcd} = 0$, and as discussed in the proof of proposition 2, if $\nabla^a \eta^b = 0$ then $t_{[d}\nabla_{c]}\eta^a = 0$. So $R'^a{}_{bcd} = 0$.

□

Proposition 5. Let $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi\rangle$ be a model of Newton-Cartan gravitation. Consider all pairs $\langle \overline{\nabla}, G^a_{\overline{\nabla}}\rangle$ where $\overline{\nabla}$ is a flat affine connection, $G^a_{\overline{\nabla}}$ is a spacelike twist-free vector field, and $\widetilde{\nabla} = (\overline{\nabla}, G^a_{\overline{\nabla}} t_b t_c)$. Define W as consisting of all the flat connections $\overline{\nabla}$ that feature in some such pair; and define G^a_* as the map $W \to \mathfrak{X}(\mathfrak{L})$ which assigns $\overline{\nabla}$ to $G^a_{\overline{\nabla}}$. Then $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi\rangle$ is a model of Maxwell-Weatherall gravitation.

Proof. Consider any pair $\overline{\nabla}, \overline{\nabla}' \in W$. Equation (30) indicates that they are both rotationally equivalent to $\widetilde{\nabla}$, and hence to one another; so $\langle \mathfrak{L}, W\rangle$ is a Maxwell spacetime. Moreover, if $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$, then since

$$(\overline{\nabla}, G^a_{\overline{\nabla}} t_b t_c) = \widetilde{\nabla} = (\overline{\nabla}', G^a_{\overline{\nabla}'} t_b t_c) \tag{32}$$

we obtain that $\eta^a = G^a_{\overline{\nabla}} - G^a_{\overline{\nabla}'}$; so equation (20) is satisfied, and G^a_* is a relative gravitational field.

[43] cf. [Earman, 1977, p. 99]

We now show that $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ satisfies the equations (21). That equation (21b) is satisfied is given by the definition of G^a_*. As for the others, let $\overline{\nabla} = (\widetilde{\nabla}, -G^a_{\overline{\nabla}} t_b t_c)$ be an arbitrary member of W. Then:

$$\begin{aligned}\xi^n \overline{\nabla}_n \xi^a &= \xi^n \widetilde{\nabla}_n \xi^a + G^a_{\overline{\nabla}} \\ &= G^a_{\overline{\nabla}}\end{aligned}$$

So equation (21c) is satisfied. Next, expressing the Riemann tensors of $\widetilde{\nabla}$ and $\overline{\nabla}$ in terms of one another, and using the fact that the latter's Riemann tensor vanishes:

$$\widetilde{R}^a{}_{bcd} = 2 t_b t_{[d} \overline{\nabla}_{c]} G^a_{\overline{\nabla}} \tag{33}$$

Then, using equation (16a),

$$\begin{aligned}4\pi\mu t_b t_d &= \widetilde{R}_{bd} \\ &= -\widetilde{R}^a{}_{bad} \\ &= -t_b t_d \overline{\nabla}_a G^a_{\overline{\nabla}}\end{aligned}$$

So equation (21a) is satisfied.

□

Proposition 6. Let $\langle \mathfrak{L}, W, G^a_*, \mu, \Xi \rangle$ be a model of Maxwell-Weatherall gravitation. Let $\overline{\nabla}$ be an arbitrary element of W. Define $\widetilde{\nabla} = (\overline{\nabla}, G^a_{\overline{\nabla}} t_b t_c)$. So defined, $\widetilde{\nabla}$ is independent of the choice of $\overline{\nabla}$; and $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation.

Proof. First, suppose (having defined $\widetilde{\nabla}$ with respect to $\overline{\nabla}$) that $\overline{\nabla}'$ is any other member of W. By Proposition 3, $\overline{\nabla}' = (\overline{\nabla}, \eta^a t_b t_c)$, for some η^a such that $\overline{\nabla}^a \eta^b = 0$. Then, using equation (20),

$$\begin{aligned}(\overline{\nabla}', G^a_{\overline{\nabla}'}) &= ((\overline{\nabla}, \eta^a), G^a_{\overline{\nabla}'}) \\ &= (\overline{\nabla}, \eta^a + G^a_{\overline{\nabla}'}) \\ &= (\overline{\nabla}, G^a_{\overline{\nabla}}) \\ &= \widetilde{\nabla}\end{aligned}$$

So the definition is independent of the choice of connection in W. Next, using equation (33), we immediately get that

$$\widetilde{R}^{ab}{}_{cd} = 0 \tag{34}$$

and using equation 21b, that

$$\begin{aligned}\widetilde{R}^a{}_b{}^c{}_d &= t_b t_d \overline{\nabla}^c G^a_{\overline{\nabla}} \\ &= t_d t_b \overline{\nabla}^a G^c_{\overline{\nabla}} \\ &= \widetilde{R}^c{}_d{}^a{}_b\end{aligned}$$

So $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$ is a Newton-Cartan spacetime. Using equation (33) in conjunction with equation (23a) yields

$$\begin{aligned}\widetilde{R}_{bd} &= -t_b t_d \overline{\nabla}_a G^a_{\overline{\nabla}} \\ &= 4\pi\mu t_b t_d\end{aligned}$$

Thus, equation (16a) is satisfied. Finally, for any $\xi^a \in \Xi$, equation (21c) ensures that

$$\begin{aligned}\xi^n \widetilde{\nabla}_n \xi^a &= \xi^n \overline{\nabla}_n \xi^a - G^a_{\overline{\nabla}} \\ &= 0\end{aligned}$$

□

So equation (16b) is satisfied.

Proposition 7. Let $\langle \mathfrak{L}, J \rangle$ be a Maxwell-Cartan spacetime, and consider any $\nabla \in J$. For any other connection ∇', $\nabla' \in J$ iff $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^{[a}\eta^{b]} = 0$.

Proof. First, suppose that $\nabla' = (\nabla, \eta^a t_b t_c)$ for some spacelike twist-free field η^a. Equation (30) shows that ∇ and ∇' are rotationally equivalent. So all we need to show is that ∇' satisfies the homogeneous Trautman conditions (15). Using (31),

$$\begin{aligned}R'^{ab}{}_{cd} &= R^{ab}{}_{cd} + 2h^{bn} t_n t_{[d} \nabla_{c]} \eta^a \\ &= R^{ab}{}_{cd}\end{aligned}$$

So clearly, $R'^{ab}{}_{cd} = 0$ iff $R^{ab}{}_{cd} = 0$.

Next, suppose that $R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b$. Then

$$\begin{aligned}R'^a{}_b{}^c{}_d &= R^a{}_b{}^c{}_d + 2h^{cn} t_b t_{[d} \nabla_{n]} \eta^a \\ &= R^a{}_b{}^c{}_d + t_b t_d \nabla^c \eta^a \\ &= R^c{}_d{}^a{}_b + t_d t_b \nabla^a \eta^c \\ &= R'^c{}_d{}^a{}_b\end{aligned}$$

where the third equality uses our supposition, and the twist-freedom of η^a. Showing that if $R'^a{}_b{}^c{}_d = R'^c{}_d{}^a{}_b$ then $R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b$ proceeds similarly.

Conversely, suppose that $\nabla' \in J$.[44] Since ∇ and ∇' are both compatible with $\mathfrak{L}$, there is some antisymmetric tensor field κ_{ab} such that $\nabla' = (\nabla, 2h^{an} t_{(b} \kappa_{c)n})$.[45] Now let θ^a be some unit timelike field such that $\nabla^{[a}\theta^{b]} = 0$ (some such field is guaranteed to exist, since ∇' obeys the homogeneous Trautman conditions). It follows that

$$\begin{aligned}0 &= \nabla'^{[a}\theta^{b]} \\ &= \nabla^{[a}\theta^{b]} + 2h^{d[b}h^{a]n} t_{(n}\kappa_{m)d}\theta^m \\ &= h^{d[b}h^{a]n} t_n \kappa_{md}\theta^m + h^{d[b}h^{a]n} t_m \kappa_{nd}\theta^m \\ &= \tfrac{1}{2}(\kappa^{ab} - \kappa^{ba}) \\ &= \kappa^{ab} (= h^{ac}h^{bd}\kappa_{cd})\end{aligned}$$

So $\kappa_{ab} = t_{[a}\sigma_{b]}$, for some 1-form σ_b; and so $2h^{an} t_{(b}\kappa_{c)n} = \eta^a t_b t_c$ for the spacelike field $\eta^a = 2h^{an}\sigma_n$.

It remains to show that η^a is twist-free. By using equation (31), we obtain

$$R'^a{}_b{}^c{}_d = R^a{}_b{}^c{}_d + 2t_b t_d \nabla^c \eta^a \tag{35}$$

So by exchange of indices, and applying the second homogeneous Trautman condition,

$$t_b t_d \nabla^c \eta^a = t_b t_d \nabla^a \eta^c \tag{36}$$

[44] This half of the proof is adapted from a proof in [Weatherall, 2015b].
[45] [Malament, 2012, Proposition 4.1.3]

Since $t_a \neq 0$, $\nabla^{[c}\eta^{a]} = 0$. □

Proposition 8. Let $\langle \mathfrak{L}, W \rangle$ be a Maxwell spacetime. Define the class of connections J by taking the closure of W under linear acceleration (so $\nabla \in J$ iff $\nabla = (\overline{\nabla}, \eta^a t_b t_c)$, for some $\overline{\nabla} \in W$ and some spacelike twist-free η^a). Then $\langle \mathfrak{L}, J \rangle$ is a Maxwell-Cartan spacetime.

Proof. Since any flat connection satisfies the homogeneous Trautman equations, this is a straightforward consequence of Proposition 7. □

Proposition 9. Let $\langle \mathfrak{L}, J \rangle$ be a Maxwell-Cartan spacetime. Define the class of connections W to consist of just those members of J which are flat. Then $\langle \mathfrak{L}, W \rangle$ is a Maxwell spacetime.

Proof. First, we show that J contains at least one flat connection (so the set W is nonempty).[46] Let ∇ be some member of J, with curvature tensor $R^a{}_{bcd}$. Since $R^{ab}{}_{cd} = 0$, there is some unit timelike field χ^a which is rigid and twist-free ($\nabla^a \chi^b = 0$).[47] Let $\overline{\nabla} = (\nabla, t_b t_c \chi^n \nabla_n \chi^a)$; that is, let the acceleration field of $\overline{\nabla}$ relative to ∇ be the four-acceleration of χ^a. By Proposition 7, $\overline{\nabla} \in J$; one can also show that it is flat.

Now consider the closure of $\overline{\nabla}$ under the relation of rigid linear acceleration, $[\overline{\nabla}]$. By Proposition 3, $\langle \mathfrak{L}, [\overline{\nabla}] \rangle$ is a Maxwell spacetime. So all we need to show is that $W = [\overline{\nabla}]$. So consider any other flat operator $\overline{\nabla}' \in J$. Using equation (31), we obtain that $t_b t_{[d} \nabla_{c]} \eta^a = 0$. Acting on both sides with h^{cn} yields $t_b t_d \nabla^c \eta^a = 0$; since $t_a \neq 0$, $\nabla^c \eta^a = 0$. So all the flat operators in J are related to one another by rigid accelerations, from which it follows that $W = [\overline{\nabla}]$, and hence that $\langle \mathfrak{L}, W \rangle$ is a Maxwell spacetime. □

Proposition 10. Let $\langle \mathfrak{L}, J, \mu, \Xi \rangle$ be a possible model of Maxwell-Cartan gravitation, and consider any $\nabla, \nabla' \in J$. Then (for any $\xi^a \in \Xi$) the equations (23) hold with respect to ∇ iff they hold with respect to ∇'.

Proof. By Proposition 7, $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^{[a}\eta^{b]} = 0$. Now, using equation (31),

$$\begin{aligned} R'_{bd} &= R_{bd} + 2 t_b t_{[d} \nabla_{a]} \eta^a \\ &= R_{bd} + t_b t_d \nabla_a \eta^a \end{aligned}$$

and so for any timelike ξ^a,

$$R'_{bd} \xi^b \xi^d = R_{bd} \xi^b \xi^d + \nabla_a \eta^a \tag{37}$$

On the other hand,

$$\begin{aligned} \nabla'_a(\xi^n \nabla'_n \xi^a) &= \nabla'_a(\xi^n \nabla_n \xi^a + \eta^a) \\ &= \nabla_a(\xi^n \nabla_n \xi^a + \eta^a) + \eta^a t_a t_r (\xi^n \nabla_n \xi^r + \eta^r) \\ &= \nabla_a(\xi^n \nabla_n \xi^a) + \nabla_a \eta^a \end{aligned}$$

Hence,

$$R_{bd} \xi^b \xi^d - \nabla_a(\xi^n \nabla_n \xi^a) = R'_{bd} \xi^b \xi^d - \nabla'_a(\xi^n \nabla'_n \xi^a) \tag{38}$$

So equation (23a) holds with respect to ∇ iff it holds with respect to ∇'. Second,

$$\begin{aligned} \nabla'^c(\xi^n \nabla'_n \xi^a) &= \nabla'^c(\xi^n \nabla_n \xi^a + \eta^a) \\ &= \nabla^c(\xi^n \nabla_n \xi^a + \eta^a) + h^{dc} \eta^a t_d t_e (\xi^n \nabla_n \xi^e + \eta^e) \\ &= \nabla^c(\xi^n \nabla_n \xi^a) + \nabla^c \eta^a \end{aligned}$$

[46] This part of the proof is essentially just the first part of the Trautman recovery theorem; I follow Malament's treatment.

[47] [Malament, 2012, Proposition 4.2.4 (1)]

Similarly,

$$\nabla'^a(\xi^n \nabla'_n \xi^c) = \nabla^a(\xi^n \nabla_n \xi^c) + \nabla^a \eta^c$$

Since $\nabla^c \eta^a = \nabla^a \eta^c$ (i.e., since η^a is twist-free), equation (23b) also holds with respect to ∇ iff it holds with respect to ∇'. □

Proposition 11. Let ξ^a be a unit timelike field, and suppose that $\{\overset{i}{\lambda}{}^a\}_i$ are three orthonormal spacelike fields such that $\mathcal{L}_\xi \overset{i}{\lambda}{}^a = 0$. Then

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_a \xi^n \nabla_n(\xi^m \nabla_m \overset{i}{\lambda}{}^a) = \frac{1}{3}\big(\nabla_a(\xi^n \nabla_n \xi^a) - R_{bd}\xi^b \xi^d\big) \tag{27}$$

Proof. First, note that for any connecting field λ^a, $\xi^n \nabla_n \lambda^a = \lambda^n \nabla_n \xi^a$ (since $\mathcal{L}_\xi \lambda^a = 0$). Hence,[48]

$$\begin{aligned}
\xi^n \nabla_n(\xi^m \nabla_m \lambda^a) &= \xi^n \nabla_n(\lambda^m \nabla_m \xi^a) \\
&= (\xi^n \nabla_n \lambda^m)\nabla_m \xi^a + \xi^n \lambda^m \nabla_n \nabla_m \xi^a \\
&= (\lambda^n \nabla_n \xi^m)\nabla_m \xi^a + \xi^n \lambda^m \nabla_m \nabla_n \xi^a + \xi^n \lambda^m R^a{}_{rmn}\xi^r \\
&= (\lambda^m \nabla_m \xi^n)\nabla_n \xi^a + \lambda^m \nabla_m(\xi^n \nabla_n \xi^a) - \lambda^m(\nabla_m \xi^n)(\nabla_n \xi^a) + \xi^n \lambda^m R^a{}_{rmn}\xi^r \\
&= \lambda^m(\nabla_m(\xi^n \nabla_n \xi^a) + R^a{}_{rmn}\xi^r \xi^n)
\end{aligned}$$

Since the connecting fields are orthonormal,

$$\hat{h}_{ab} = \sum_i \overset{i}{\lambda}_a \overset{i}{\lambda}_b \tag{39}$$

and so[49]

$$\sum_i \overset{i}{\lambda}_a \overset{i}{\lambda}{}^c = \delta_a{}^c - t_a \xi^c \tag{40}$$

Therefore,

$$\begin{aligned}
\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_a \xi^n \nabla_n(\xi^m \nabla_m \overset{i}{\lambda}{}^a) &= \frac{1}{3}\Big(\sum_i \overset{i}{\lambda}_a \overset{i}{\lambda}{}^c\Big)(\nabla_c(\xi^n \nabla_n \xi^a) + R^a{}_{bcd}\xi^b \xi^d) \\
&= \frac{1}{3}(\delta_a{}^c - t_a \xi^c)(\nabla_c(\xi^n \nabla_n \xi^a) + R^a{}_{bcd}\xi^b \xi^d) \\
&= \frac{1}{3}(\nabla_a(\xi^n \nabla_n \xi^a) - R_{bd}\xi^b \xi^d)
\end{aligned}$$

□

Proposition 12. Let $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi\rangle$ be a model of Newton-Cartan gravitation. Define

$$J = \{\nabla : \nabla = (\widetilde{\nabla}, \eta^a t_b t_c)\} \tag{28}$$

for any spacelike η^a such that $\widetilde{\nabla}^{[a}\eta^{b]} = 0$. Then $\langle \mathfrak{L}, J, \mu, \Xi\rangle$ is a model of Maxwell-Cartan gravitation.

Proof. By Proposition 7, $\langle \mathfrak{L}, J\rangle$ is a Maxwell-Cartan spacetime. The equations (16) guarantee that the equations (23) hold with respect to $\widetilde{\nabla}$; by Proposition 10, they therefore hold with respect to any $\nabla \in J$. Thus, $\langle \mathfrak{L}, J, \mu, \Xi\rangle$ is a model of Maxwell-Cartan gravitation. □

[48] This calculation is just an extension of the proof of [Malament, 2012, Proposition 1.8.5], for the case where ξ^a is not a geodesic.

[49] [Malament, 2012, Equation 4.1.12]

Proposition 13. Let $\langle \mathfrak{L}, J, \mu, \Xi \rangle$ be a model of Maxwell-Cartan gravitation. Let ∇ be an arbitrary element of J, and let ξ^a be an arbitrary element of Ξ. Define a derivative operator $\widetilde{\nabla}$ by

$$\widetilde{\nabla} = (\nabla, t_b t_c \xi^n \nabla_n \xi^a) \tag{29}$$

So defined, $\widetilde{\nabla}$ is independent of the choice of ξ^a and ∇; and $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation.

Proof. First, suppose (having defined $\widetilde{\nabla}$ with respect to ξ^a and ∇) that ξ'^a is any other member of Ξ. Since all members of Ξ are acceleratively equivalent to one another,

$$(\nabla, t_b t_c \xi'^n \nabla_n \xi'^a) = (\nabla, t_b t_c \xi^n \nabla_n \xi^a) \tag{41}$$

so the definition of $\widetilde{\nabla}$ is independent of the choice of ξ^a. Now suppose that ∇' is any other member of J. By Proposition 7, $\nabla' = (\nabla, \eta^a t_b t_c)$ for some η^a such that $\nabla^{[a}\eta^{b]} = 0$. Then

$$\begin{aligned}(\nabla', t_b t_c \xi^n \nabla'_n \xi^a) &= ((\nabla, \eta^a t_b t_c), t_b t_c(\xi^n \nabla_n \xi^a - \eta^a)) \\ &= (\nabla, t_b t_c \xi^n \nabla_n \xi^a) \\ &= \widetilde{\nabla}\end{aligned}$$

So the definition is independent of the choice of ∇.

Since $\xi^n \nabla_n \xi^a$ is spacelike, and given (23b) (for ∇), we get (by Proposition 7) that $\widetilde{\nabla} \in J$, and so obeys the homogeneous Trautman conditions. So $\langle \mathfrak{L}, \widetilde{\nabla} \rangle$ is a Newton-Cartan spacetime. Moreover, for any $\xi^a \in \Xi$,

$$\xi^n \widetilde{\nabla}_n \xi^a = \xi^n \nabla_n \xi^a - \xi^n \nabla_n \xi^a = 0 \tag{42}$$

So we immediately have equation (16b): that is, the dynamically allowed trajectories are precisely the geodesics of $\widetilde{\nabla}$. Since $\widetilde{\nabla} \in J$, we also have that it satisfies (23a) for all such geodesic vector fields, i.e., for every $\xi^a \in \Xi$,

$$\widetilde{R}_{bd} \xi^b \xi^d = 4\pi\mu \tag{43}$$

But if this holds for *every* geodesic field only if (16a) holds. So $\langle \mathfrak{L}, \widetilde{\nabla}, \mu, \Xi \rangle$ is a model of Newton-Cartan gravitation. □