FREGE'S BEGRIFFSSCHRIFT IS FIRST-ORDER COMPLETE

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1. It is widely taken that the first-order part of Frege's *Begriffsschrift* is complete.[1] However, there does not seem to be a formal verification of this received claim. The general concern is that Frege's system is one axiom short in first-order predicate calculus compared to, by now, the standard theory. Yet Frege has one extra inference rule in his system. Then the question is whether Frege's first-order calculus is still deductively sufficient as far as the first-order completeness is concerned. In this short note we confirm that the missing axiom is derivable from his stated axioms and inference rules, and hence the logic system in the *Begriffsschrift* is indeed first-order complete.

2. We provide a list of Frege's axioms and inference rules for first-order calculus. We use modern notations instead of Frege's two dimensional notations.[2]

Logical Axioms

Sentential calculus

A1. $A \to (B \to A)$ (1)

A2. $\big[A \to (B \to C)\big] \to \big[(A \to B) \to (B \to C)\big]$ (2)

A3. $\big[A \to (B \to C)\big] \to \big[B \to (A \to C)\big]$ (8)

A4. $(A \to B) \to (\neg B \to \neg A)$ (28)

A5. $\neg\neg A \to A$ (31)

A6. $A \to \neg\neg A$ (41)

Predicate calculus

A7. $c \equiv d \to [f(c) \to f(d)]$ (52)

A8. $c \equiv c$ (54)

A9. $\forall x \varphi(x) \to \varphi(t)$ if t is free for x in $\varphi(x)$ (58)

Inference Rules

i. Modus ponens (MP) (§6)

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}$$

The resut presented here grew out of professor Haim Gaifman's Philosophy of Logic class in Fall 2009 at Columbia Unviersity, who independently proved the same claim using a different method.

[1]Frege, Gottlob, 1879, *Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle a. S.: Louis Nebert. Translated as Concept Script, a formal language of pure thought modeled upon that of arithmetic, by S. Bauer-Mengelberg in J. van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Cambridge, Harvard University Press, 1967.

[2]The equation/section numbers at the end of each line refer to the original propositions or sections numbers appeared in the *Begriffsschrift*.

ii. Frege's universal generalization rule (FUG) (§11)

$$\frac{\vdash A \to \varphi(x)}{\vdash A \to \forall x \varphi(x),} \quad \text{provided that } x \text{ is not free in } A.$$

iii. Universal Generalization (UG) (§11)

$$\frac{\vdash \varphi(x)}{\vdash \forall x \varphi(x).}$$

3. The following axiom in standard first-order logic is absent from Frege's system:

$$\forall x\big(\varphi(x) \to \psi(x)\big) \to \big(\forall x \varphi(x) \to \forall x \psi(x)\big) \qquad (\star)$$

We show that ($\star$) is derivable from the stated axioms and rules. In order to do so, we first establish the following lemma.

Lemma. Suppose that $\Gamma, A \vdash B$, no application of FUG or UG to a wff has as its quantified variable a free variable of A.[3] Then $\Gamma \vdash A \to B$.

Proof. Let $\alpha_0, \dots, \alpha_{n-1}$ be a proof of B from Γ and A. We show that, for every $i < n$,

$$\Gamma \vdash A \to \alpha_i.$$

i. It is easy to show, by choosing proper tautologies, that the lemma holds for the cases in which α_i is either an axiom or a member of $\Gamma \cup \{A\}$, or can be derived from previous steps by MP.

ii. If α_i is obtained from α_j by FUG, where $\alpha_j = \alpha_r \to \alpha_k(x)$ for some $r, k < j < i$, and x is not free in α_r, that is, $\alpha_i = \alpha_r \to \forall x \alpha_k(x)$. And by the assumption of the lemma we know that x is not free in A. We want to show $\Gamma \vdash A \to \big(\alpha_r \to \forall x \alpha_k(x)\big)$, namely, $\Gamma \vdash A \to \alpha_i$. Note that, by the inductive hypothesis, $\Gamma \vdash A \to \alpha_r$ and $\Gamma \vdash A \to \alpha_k$. For the latter, we have that x is not free in A, then by FUG we get $\Gamma \vdash A \to \forall x \alpha_k(x)$. These together with the tautology $(A \to B) \to \big[(A \to C) \to \big(A \to (B \to C)\big)\big]$ give us $\Gamma \vdash A \to \big(\alpha_r \to \forall x \alpha_k(x)\big)$, which is what we want.

iii. If α_i is obtained from α_j by UG, that is, $\alpha_i = \forall x \alpha_j$ for some $j < i$. By the inductive hypothesis $\Gamma \vdash A \to \alpha_j$, and by the assumption of the lemma that x is not free in A, we therefore have, by FUG, $\Gamma \vdash A \to \forall x \alpha_j$, and hence $\Gamma \vdash A \to \alpha_i$. □

Theorem. For any wffs φ and ψ, show that ($\star$) holds.

Proof. (1) $\forall x\big(\varphi(x) \to \psi(x)\big)$ Hyp
(2) $\forall x \varphi(x)$ Hyp
(3) $\varphi(x) \to \psi(x)$ (1) and A9
(4) $\varphi(x)$ (2) and A9
(5) $\psi(x)$ (3), (4), and MP
(6) $\forall x \psi(x)$ (5) and UG
(7) $\forall x\big(\varphi(x) \to \psi(x)\big), \forall x \varphi(x) \vdash \forall x \psi(x)$ (1)-(6)
(8) $\vdash \forall x\big(\varphi(x) \to \psi(x)\big) \to \big(\forall x \varphi(x) \to \forall x \psi(x)\big)$ (7) and Lemma (twice)
□

[3] That is to say, no variable in a wff which becomes quantified by applying FUG or UG is a free variable in A. Actually, we can broaden the application of this lemma by considering the possibility that A does not play any role in proving B, then in this case the quantified variable can be a free variable in A. The proof for this case is easy.