The Representation of Belief

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Abstract

I derive a novel sufficient condition for a belief set to be representable by a probability function: if at least one comparative confidence ordering of a certain type satisfies Scott's axiom, then the belief set used to construct that ordering is representable. This provides support for Kenny Easwaran's project of analyzing doxastic states in terms of belief sets rather than credences.

0 Introduction

Kenny Easwaran argues that Bayesian credences can be reinterpreted as mathematical tools for summarizing agential belief [1]. The idea is to ascribe belief sets to agents, and then use probability functions to represent those belief sets in a way that reaps the benefits of credences without incurring the costs.[1] In short, what Bayesians take to be a credence function is really just a representation of an agent's belief set. The mathematical idealizations of probability axioms, the infinite precision of credences, and Lockean thresholds are merely mathematical tools for analyzing belief sets and the value that agents place on truth and falsity.

So under what conditions is a belief set representable by a probability function? In partial answer, Easwaran lists several necessary conditions for representability. One is *strong coherence*: a belief set B is strongly coherent just in case there is no other belief set that is at

[1] See [1] for a detailed discussion of those costs.

least as accurate as B in every possible state of the world, but that is strictly more accurate in at least one state. Easwaran shows that if B is representable, then B is strongly coherent [1, p. 14].[2]

It would benefit Easwaran's project considerably if strong coherence were also a sufficient condition for representability. Then, by satisfying strong coherence, an agent's belief set would conform to many of the constraints that credences impose on rationality, since that set would be representable by a probability function that mimics the behavior of credences. Moreover, strong coherence itself seems like a plausible rationality constraint: if an agent is rational, then her belief set had better not be less accurate, overall, than another belief set. But unfortunately, strong coherence is not sufficient for representability: there are subsets of Boolean algebras that are strongly coherent yet unrepresentable [1, pp. 31-32].

This raises two questions. First, what conditions are sufficient for a belief set to be representable by a probability function? Second, does any such sufficient condition provide a plausible constraint on rationality, the way strong coherence does?

In this paper, I answer both questions with a new sufficient condition for representability. In Section 1, I review the basic notions which are used to articulate that condition. In Section 2, I derive representability from the sufficient condition. Finally, in Section 3, I explain why this condition seems to provide a plausible constraint on rationality.

1 Basic Notions

1.1 Belief Sets

Roughly, an agent's belief set is the set consisting of all propositions which she believes.

[2] The proof assumes that the probability function which represents B assigns a non-zero probability to every state of the world.

Definition 1 (Belief Set)**.** *Let $\mathcal{X}$ be a finite Boolean algebra. A **belief set** is a set $B \subseteq \mathcal{X}$.*

Think of the atoms[3] of the algebra as mutually exclusive states of the world. The propositions which agent A believes are disjunctive combinations of these atomic states. For example, if the atoms of the algebra are a_1, a_2, and a_3, A's belief set might contain a_1, or $a_2 \vee a_3$, or any other disjunctive combination of zero or more of the a_i.[4] The algebra is assumed to be finite (and non-empty) so as to simplify the forthcoming analysis.

The following definition states the conditions under which a belief set B is representable by a probability function. The idea, roughly, is that B is representable just in case all the propositions in B are at least as likely as not to be true, and all the propositions not in B are at least as likely as not to be false.

Definition 2 (B-Representability)**.** *Let $\mathcal{X}$ be a finite Boolean algebra, and let $B \subseteq \mathcal{X}$ be a belief set. Say that B is **b-representable** (for 'belief-representable') just in case there is a probability function Pr such that*

(i) if $Pr(p) > \frac{1}{2}$ then $p \in B$, and

(ii) if $Pr(p) < \frac{1}{2}$ then $p \notin B$.

If $Pr(p) = \frac{1}{2}$, then Pr b-represents B regardless of whether $p \in B$ or $p \notin B$.

It is a general fact that for any probability function Pr and any $p \in \mathcal{X}$, $Pr(p) > \frac{1}{2}$ if and only if $Pr(\neg p) < \frac{1}{2}$. Read through the light of this fact, definition 2 says that if a proposition is more likely than its negation then it is believed, and if a proposition is less likely than its negation then it is not believed.[5]

[3]These are the algebra's primitive propositions.

[4]The proposition which consists of zero atoms is the empty proposition, denoted $\perp$.

[5]For justification of the last clause in definition 2, which treats the case $Pr(p) = \frac{1}{2}$, see [1, p. 13].

The following two simple facts about belief sets will be useful later. The first fact, theorem 1.1, says that if a proposition and its negation are both in B or both not in B, then they are equiprobable.

Theorem 1.1. *Let $\mathcal{X}$ be a finite Boolean algebra and let $B \subseteq \mathcal{X}$ be a belief set. If B is b-represented by a probability function Pr, then the following holds.*

(i) For every proposition $p \in B$, if $\neg p \in B$ then $Pr(p) = Pr(\neg p) = \frac{1}{2}$.

(ii) For every proposition $p \notin B$, if $\neg p \notin B$ then $Pr(p) = Pr(\neg p) = \frac{1}{2}$.

Proof. For (i): suppose that $p \in B$ and $\neg p \in B$. By definition 2, $Pr(p) \geqslant \frac{1}{2}$ and $Pr(\neg p) \geqslant \frac{1}{2}$. Since $Pr(p) = 1 - Pr(\neg p)$, it follows that $Pr(p) = Pr(\neg p) = \frac{1}{2}$.

For (ii): suppose that $p \notin B$ and $\neg p \notin B$. By definition 2, $Pr(p) \leqslant \frac{1}{2}$ and $Pr(\neg p) \leqslant \frac{1}{2}$. Again, since $Pr(p) = 1 - Pr(\neg p)$, it follows that $Pr(p) = Pr(\neg p) = \frac{1}{2}$.

$\square$

Roughly put, the second fact about belief sets—theorem 1.2—says that if two sets only differ on pairs of propositions and their negations, then they are b-represented by exactly the same probability functions.

Theorem 1.2. *Let $\mathcal{X}$ be a finite Boolean algebra and let $B \subseteq \mathcal{X}$ be a belief set. Suppose B is b-representable by the probability function Pr, and suppose $B' \subseteq \mathcal{X}$ is a belief set that satisfies the following two conditions.*

(i) If $p \in B \backslash B'$ then $\neg p \in B$.

(ii) If $p \in B' \backslash B$ then $\neg p \notin B$.

Then Pr b-represents B'.

Proof. Let $p \in \mathcal{X}$ be such that $Pr(p) > \frac{1}{2}$. Then $p \in B$ (since Pr b-represents B). Suppose $p \notin B'$. Then $\neg p \in B$ by condition (i). But then $Pr(p) = \frac{1}{2}$ by theorem 1.1, which contradicts the supposition. Therefore, $p \in B'$.

Now let $p \in \mathcal{X}$ be such that $Pr(p) < \frac{1}{2}$. Then $p \notin B$ (since Pr b-represents B). Suppose that $p \in B'$. Then $\neg p \notin B$ by condition (ii). But then $Pr(p) = \frac{1}{2}$ by theorem 1.1, which contradicts the supposition. Therefore, $p \notin B'$.

So by definition 2, Pr b-represents B'.

□

As will become clear in Section 2, this little theorem is quite important. Think of it as showing that if two belief sets are 'sufficiently similar' to each other, establishing the b-representability of one suffices to establish the b-representability of the other.

1.2 Comparative Confidence Orderings

Comparative confidence orderings encode information about whether an agent is more (or less) confident in one proposition than another.

Definition 3 (Comparative Confidence Ordering)**.** *Let $\mathcal{X}$ be a finite Boolean algebra. A* ***comparative confidence ordering*** *is a set $\succeq \subseteq \mathcal{X} \times \mathcal{X}$.*

Intuitively, if A is an agent and $p, q \in \mathcal{X}$, then $p \succeq q$ just in case A is at least as confident in the truth of p as in the truth of q. Let $p \succ q$ be shorthand for $p \succeq q$ & $q \not\succeq p$.

Comparative confidence orderings, like belief sets, can be represented by probability functions.

Definition 4 (C-Representability)**.** *Let $\mathcal{X}$ be a finite Boolean algebra, and let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$ be a comparative confidence ordering. Say that $\succeq$ is **c-representable** (for 'comparative-representable') just in case there is a probability function Pr such that for every $p, q \in \mathcal{X}$, $p \succeq q$ if and only if $Pr(p) \geqslant Pr(q)$.*

So a comparative confidence ordering is c-representable just in case there is a probability function on the underlying algebra which preserves the ordering.

Of the five axioms that are typically taken to govern comparative confidence orderings (Fitelson, p.c.), just three will be relevant here. Let $\mathcal{X}$ be a finite Boolean algebra, and let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$ be a comparative confidence ordering. Here are three conditions that seem, intuitively, like they should hold of $\succeq$ for every $p, q \in \mathcal{X}$.

(A_1) $(p \succeq q) \vee (q \succeq p)$.

(A_2) $\top \succ \bot$.

(A_3) $p \succeq \bot$.

(A_1) says that the ordering is total: for every pair of propositions, the agent is at least as confident in one as in the other. (A_2) says that the agent is strictly more confident in $\top$, the tautological proposition that contains every atom in $\mathcal{X}$, than $\bot$, the contradictory proposition that is empty. (A_3) says that for every proposition, the agent is at least as confident in that proposition as in $\bot$.

The following condition will play a central role in this paper.

(SA) (Scott's Axiom) Let $\mathcal{X}$ be a finite Boolean algebra, let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$, and let $X = \langle x_1, \ldots, x_n \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$ be arbitrary sequences of propositions in $\mathcal{X}$ of length $n \geqslant 2$. If

(i) X and Y have the same number of truths in every atom of $\mathcal{X}$,[6] and

(ii) for all $i \in [1, n)$, $x_i \succeq y_i$,

then

(iii) $y_n \succeq x_n$.

The idea of (SA) can be explained via a simple example drawn from a more familiar context: the real numbers under the usual $\geqslant$ relation. Let x_1, x_2, y_1, and y_2 be real numbers. Suppose that $x_1 + x_2 = y_1 + y_2$, which is the analogue of condition (i). Also suppose that $x_1 \geqslant y_1$, which is the analogue of condition (ii). It follows that $y_2 \geqslant x_2$,[7] which is the analogue of (iii). Thus, the $\geqslant$ relation on the reals satisfies an analogue of (SA) for the case $n = 2$.[8] One can think of (SA) as encapsulating this fact about $\geqslant$ in the context of Boolean algebras.

As discussed in Section 3, despite its complicated appearance, (SA) articulates an intuitively plausible constraint on rational belief. For if agent A is at least as confident in all the x_i as the y_i for $i \in [1, n)$ (condition (ii)), and A is strictly more confident in x_n than in y_n (the negation of condition (iii)), then A is irrational if she knows that the X propositions and the Y propositions are equally accurate (condition (i)). To put it roughly: if the X propositions and Y propositions are known by A to be equally accurate, then A better not be strictly more confident in the X propositions than in the Y propositions.

Dana Scott used (SA) to prove what is now called "Scott's Theorem" [3], one version of which is given below.

Theorem 1.3 (Scott's Theorem). *Let $\mathcal{X}$ be a finite Boolean algebra, and let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$. $\succeq$ satisfies $(A_1), (A_2), (A_3)$, and (SA) if and only if $\succeq$ is c-representable.*

[6] Formally, this is represented as $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where each x_i and y_i are understood to be characteristic functions of the corresponding elements in $\mathcal{X}$.

[7] If $x_2 > y_2$, and $x_1 \geqslant y_1$, then $x_1 + x_2 > y_1 + y_2$, which contradicts the supposition that $x_1 + x_2 = y_1 + y_2$.

[8] It is not hard to show that $\geqslant$ on the reals satisfies an analogue of (SA) for all $n \geqslant 2$.

Thus, satisfaction of (A_1), (A_2), (A_3), and (SA) is sufficient and necessary for c-representability.

Theorem 1.3 is important here because it connects (SA) to c-representability. So if c-representability connects to b-representability in the right way, then in virtue of theorem 1.3, (SA) can be used to derive a sufficient condition for b-representability. In the next section, I show how c-representability and b-representability are so connected, and then derive the sufficient condition.

2 The Sufficient Condition

Before deriving the sufficient condition for b-representability, I discuss a particular way of constructing comparative confidence orderings from belief sets. The construction proceeds in two steps. First, starting from a belief set B, I construct a partial comparative confidence ordering $\succeq_B^\star$. Second, I construct the set of all total extensions of $\succeq_B^\star$ that satisfy some relatively minor restrictions.

From now on, I assume that $\perp \notin B$ and that $\top \in B$. It is not hard to prove that if $\perp \in B$ or if $\top \notin B$, then B is not representable by a probability function.[9] Since the only belief sets of interest here are those which might be b-representable, this is a reasonable assumption to adopt.

The ordering $\succeq_B^\star$ is constructed as follows. Let $B \subseteq \mathcal{X}$ be the belief set of agent A. To start, define the following three sets.

$$D_1 = \{\langle p, \neg p\rangle \mid p \in B\}.$$

$$D_2 = \{\langle \neg p, p\rangle \mid p \notin B\}.$$

$$D_3 = \{\langle p, \perp\rangle \mid p \in \mathcal{X}\}.$$

Then let $\succeq_B^\star = D_1 \cup D_2 \cup D_3$.

[9] Every probability function Pr assigns 0 to $\perp$ and 1 to $\top$. So if Pr b-represents B, then $\perp \notin B$ and $\top \in B$.

In virtue of containing D_1, $\succeq^{\star}_{B}$ says that for every $p \in B$, A is at least as confident in p as in $\neg p$. In virtue of containing D_2, $\succeq^{\star}_{B}$ says that for every $p \notin B$, A is at least as confident in $\neg p$ as in p. Finally, in virtue of containing D_3, $\succeq^{\star}_{B}$ says that for every $p \in \mathcal{X}$, A is at least as confident in p as in the empty proposition.

Now for the second step of the construction. Let $\mathcal{K}$ be the set of total comparative confidence orderings $\succeq$ that contain $\succeq^{\star}_{B}$ as a subset, and that also satisfy the following two conditions.

($\mathcal{C}_1$) $\langle \bot, \top \rangle \notin \succeq$.

($\mathcal{C}_2$) For all $p \in B$, if $\neg p \notin B$ then $\langle \neg p, p \rangle \notin \succeq$.

Given that $\langle \top, \bot \rangle$ is automatically in each ordering in $\mathcal{K}$ (since it is in D_3), ($\mathcal{C}_1$) simply says that in all those orderings, A is strictly more confident in $\top$ than in $\bot$. ($\mathcal{C}_2$) is a little more complex, but basically, it implies that D_1 *and* D_2 *exhaust the comparisons between propositions and their negations that orderings in* $\mathcal{K}$ *may include.* As shown in theorem 2.3, this ensures that the belief sets which can be 'read off' orderings in $\mathcal{K}$ are 'sufficiently similar' to the belief set B, for the purposes of drawing conclusions about b-representability.

The following definition provides a succinct way to refer to this construction.

Definition 5 (Constructed from B in the manner of $\mathcal{C}$)**.** *Let $\mathcal{X}$ be a finite Boolean algebra, let $B \subseteq \mathcal{X}$ be a belief set, and let $\mathcal{K}$ be the set of total comparative confidence orderings that contain $\succeq^{*}_{B}$ and that satisfy ($\mathcal{C}_1$) and ($\mathcal{C}_2$). Then $\mathcal{K}$ is the set of comparative confidence orderings* ***constructed from B in the manner of $\mathcal{C}$****.*

I now derive the sufficient condition for b-representability. To start, the following lemma shows that orderings constructed in the manner of $\mathcal{C}$ satisfy (A_1), (A_2), and (A_3).

Lemma 1. *Let $\mathcal{X}$ be a finite Boolean algebra, let $B \subseteq \mathcal{X}$ be a belief set, and let $\mathcal{K}$ be the set of comparative confidence orderings constructed from B in the manner of $\mathcal{C}$. Each $\succeq \in \mathcal{K}$ satisfies (A_1), (A_2), and (A_3).*

Proof. For (A_1): by definition 5, $\succeq$ is total. Thus, $\succeq$ satisfies (A_1).

For (A_2): $\langle \top, \bot \rangle \in \succeq^{\star}_{B} \subseteq \succeq$. By definition 5, $\succeq$ satisfies $(\mathcal{C}_1)$, and so $\langle \bot, \top \rangle \notin \succeq$. Hence, $\top \succ \bot$.

For (A_3): for each $p \in \mathcal{X}$, $\langle p, \bot \rangle \in \succeq^{\star}_{B} \subseteq \succeq$.

□

The following theorem shows that (SA) is both sufficient and necessary for an ordering in $\mathcal{K}$ to be c-representable.

Theorem 2.1. *Let $\mathcal{X}$ be a finite Boolean algebra, let $B \subseteq \mathcal{X}$ be a belief set, and let $\mathcal{K}$ be the set of comparative confidence orderings constructed from B in the manner of $\mathcal{C}$. Then for each $\succeq \in \mathcal{K}$, $\succeq$ satisfies (SA) if and only if $\succeq$ is c-representable.*

Proof. By theorem 1.3, $\succeq$ satisfies (A_1), (A_2), (A_3), and (SA) if and only if it is c-representable. By lemma 1, each $\succeq \in \mathcal{K}$ satisfies (A_1), (A_2), and (A_3). Therefore, each $\succeq \in \mathcal{K}$ satisfies (SA) if and only if it is c-representable.

□

The remaining theorems connect the c-representability of orderings in $\mathcal{K}$ to the b-representability of the belief set from which $\mathcal{K}$ was constructed. The connection relies on a new notion: that of a belief set *induced* by a comparative confidence ordering.

Definition 6 (Induced Belief Set)**.** *Let $\mathcal{X}$ be a finite Boolean algebra, and let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$ be a comparative confidence ordering. Let $B_{\succeq}$ be the belief set which consists of all and only the propositions p such that $p \succ \neg p$. Call $B_{\succeq}$ the **belief set induced by** $\succeq$.*

The following theorem connects comparative confidence orderings to the belief sets they induce. More specifically, if an ordering induces a belief set, then any probability functions that c-represent the former must b-represent the latter.

Theorem 2.2. *Let $\succeq \subseteq \mathcal{X} \times \mathcal{X}$ be a comparative confidence ordering that induces the belief set $B_{\succeq}$. Let Pr be a probability function that c-represents $\succeq$. Then Pr b-represents $B_{\succeq}$.*

Proof. If $Pr(p) > \frac{1}{2}$ then $2 \cdot Pr(p) > 1 = Pr(p) + Pr(\neg p)$, and so $Pr(p) > Pr(\neg p)$. By definition 4, $p \succeq \neg p$. Also by definition 4, $\neg p \not\succeq p$: for if $\neg p \succeq p$, then $Pr(\neg p) \geqslant Pr(p)$. Therefore, $p \succ \neg p$. So by definition 6, $p \in B_{\succeq}$.

If $Pr(p) < \frac{1}{2}$ then $2 \cdot Pr(p) < 1 = Pr(p) + Pr(\neg p)$, and so $Pr(p) < Pr(\neg p)$. By definition 4, $\neg p \succeq p$, so $p \not\succ \neg p$. Definition 6 implies that $p \notin B_{\succeq}$.

Therefore, by definition 2, Pr b-represents $B_{\succeq}$.

$\square$

The next theorem shows that if B is used to construct a set $\mathcal{K}$ of comparative confidence orderings in the manner of $\mathcal{C}$, then each comparative confidence ordering induces a belief set that is 'sufficiently similar' to B.[10]

Theorem 2.3. *Let $\mathcal{X}$ be a finite Boolean algebra, let $B \subseteq \mathcal{X}$ be a belief set, and let $\mathcal{K}$ be the set of comparative confidence orderings constructed from B in the manner of $\mathcal{C}$. Suppose that some $\succeq \in \mathcal{K}$ induces the belief set $B_{\succeq}$. Then the following two conditions hold:*

(i) $B_{\succeq} \subseteq B$, and

(ii) if $B \nsubseteq B_{\succeq}$, then for every $p \in B \backslash B_{\succeq}$, $\neg p \in B \backslash B_{\succeq}$.

Proof. For (i): let $p \in B_{\succeq}$. By definition 6, $p \succ \neg p$, and so $\langle \neg p, p \rangle \notin \succeq$. It follows that $p \in B$, for if $p \notin B$, then $\langle \neg p, p \rangle \in D_2$. But then $\langle \neg p, p \rangle \in \succeq$, which is a contradiction.

[10]In fact, each ordering in $\mathcal{K}$ induces the very same belief set.

For (ii): suppose that $B \nsubseteq B_{\geq}$, and let $p \in B \backslash B_{\geq}$. By definition 6, $p \in B_{\geq}$ if and only if $p \succ \neg p$. Since $p \notin B_{\geq}$, either $\langle p, \neg p \rangle \notin \, \succeq$ or $\langle \neg p, p \rangle \in \, \succeq$. The former is impossible: since $p \in B$, it follows that $\langle p, \neg p \rangle \in D_1$, and thus, $\langle p, \neg p \rangle \in \, \succeq$. So $\langle \neg p, p \rangle \in \, \succeq$. Now, if $\neg p \notin B$, then $\succeq$ does not satisfy $(\mathcal{C}_2)$. Therefore, $\neg p \in B$.

By definition 6, $\neg p \in B_{\geq}$ if and only if $\neg p \succ p$. As was already shown, $\langle p, \neg p \rangle \in \, \succeq$. Therefore $\neg p \nsucc p$, and so $\neg p \notin B_{\geq}$.

Thus, $\neg p \in B \backslash B_{\geq}$.

$\square$

The set $B_{\geq}$ is 'sufficiently similar' to B in the sense that if they differ on p, they differ on $\neg p$ too. That is, the only propositions on which $B_{\geq}$ and B differ are pairs of propositions and those propositions' negations.

At long last, here is the sufficient condition for b-representability.

Theorem 2.4 (Sufficient Condition for B-Representability)**.** *Let $\mathcal{X}$ be a finite Boolean algebra, let $B \subseteq \mathcal{X}$ be a belief set, and let $\mathcal{K}$ be the set of comparative confidence orderings constructed from B in the manner of $\mathcal{C}$. Suppose that some $\succeq \, \in \mathcal{K}$ satisfies (SA). Then B is b-representable.*

Proof. By theorem 2.1, $\succeq$ is c-representable by some probability function Pr. Let $B_{\geq}$ be the belief set induced by $\succeq$. By theorem 2.2, Pr b-represents $B_{\geq}$.

By theorem 2.3, $B_{\geq} \subseteq B$. Since $B_{\geq} \backslash B = \varnothing$, condition (i) of theorem 1.2 holds trivially. In addition, if $p \in B \backslash B_{\geq}$, then $B \nsubseteq B_{\geq}$. So by theorem 2.3, for every $p \in B \backslash B_{\geq}$, $\neg p \in B \backslash B_{\geq}$. Thus, for every $p \in B \backslash B_{\geq}$, $\neg p \notin B_{\geq}$, and so condition (ii) of theorem 1.2 also holds. It therefore follows from theorem 1.2 that since Pr b-represents $B_{\geq}$, Pr b-represents B as well.

$\square$

The sufficient condition says that if the set of comparative confidence orderings constructed from B in the manner of $\mathcal{C}$ includes at least one ordering that satisfies Scott's axiom, then B is b-representable.[11]

3 The Sufficient Condition as a Constraint on Rationality

Recall that according to Easwaran, a good sufficient condition for b-representability should provide a plausible constraint on rationality: it should be the sort of constraint that a rational agent satisfies. In fact, the sufficient condition in theorem 2.4 may fit the bill. That condition can be divided into two parts—(SA) and the $\mathcal{C}$ construction—which together amount to a plausible rationality constraint.

First, consider the relationship between (SA) and comparative confidence orderings which rational agents can have, a relationship which was discussed briefly in Section 1.2. Suppose that agent A is at least as confident in x_1 as in y_1, at least as confident in x_2 as in y_2, and . . . and at least as confident in x_{n-1} as y_{n-1}. In other words, A satisfies condition (ii) of (SA). Suppose, furthermore, that A is strictly more confident in x_n than in y_n. That is, A violates condition (iii) of (SA). Then for A to be rational, it had better *not* be true that A knows that collectively, the x_i are just as accurate as the y_i; it better not be true that (i) of (SA) holds. In other words, the collection of x_is had better be strictly more accurate than the collection of y_is in at least *some* situations. For if not, then intuitively, A is strictly more confident in the collection of x_is than the collection of y_is,[12] despite the fact that A knows the two collections contain the same number of truths. So it seems like an agent's comparative confidence ordering should always satisfy (SA), if she is to be rational.[13]

[11] There is also a necessary condition for b-representability that is closely related to—though distinct from—the sufficient condition of theorem 2.4.

[12] It is not hard to precisely define the intuitive notion of being 'more confident' in one set of propositions than in another. Say that A is more confident in the collection of x_is of X than the collection of y_is of Y just in case $x_i \succeq y_i$ for each $i \in [1, n]$, and for some $j \in [1, n]$, $x_j \succ y_j$.

[13] James Hawthorne gives some reasons for thinking that a rational agent should have a comparative confidence ordering that either satisfies, or is extendible to, a condition he calls (X) [2, pp. 60-61]. Roughly put, an agent's total comparative confidence ordering satisfies (X) just in case there is some way to partition the space of possibilities into equally plausible states such that the agent has very little confidence in any one of them. It can be shown that given a couple plausible conditions, (X) implies (SA). Thus, to the extent that (X) amounts to a necessary condition for rationality, (SA) does too.

A concrete example should help make the point clear. Suppose Emily the policewoman is chasing Dick the thief. Dick, who is a block ahead of her, turns a corner. When Emily reaches the corner, she turns, runs to the end of the block, and sees three directions in which Dick could have run: LEFT, RIGHT, or STRAIGHT. She is more confident that Dick went LEFT than that he went STRAIGHT.[14] So intuitively, she should be more confident in $\text{LEFT} \vee \text{RIGHT}$ than in $\text{STRAIGHT} \vee \text{RIGHT}$.

(SA) gets this intuition correct. Let $\mathcal{X}$ be the Boolean algebra consisting of three atoms: LEFT, RIGHT, and STRAIGHT. Let $X = \langle \text{LEFT}, \text{STRAIGHT} \vee \text{RIGHT} \rangle$ and let $Y = \langle \text{STRAIGHT}, \text{LEFT} \vee \text{RIGHT} \rangle$. Since X and Y have the same number of truths in each atom of $\mathcal{X}$,[15] condition (i) of (SA) is satisfied. Since Emily's doxastic state is such that $\text{LEFT} \succeq \text{STRAIGHT}$, condition (ii) is satisfied too. Therefore, given (SA), $\text{LEFT} \vee \text{RIGHT} \succeq \text{STRAIGHT} \vee \text{RIGHT}$; this is just condition (iii).

Moreover, it is possible to show that for each way Emily could violate (SA), her doxastic state is intuitively irrational. This is done by listing all the different ways of satisfying the antecedent of (SA) while violating the consequent. For each, the violation of (SA) looks as irrational as if Emily's comparative confidence ordering implied both $\text{LEFT} \succeq \text{STRAIGHT}$ and $\text{STRAIGHT} \vee \text{RIGHT} \succ \text{LEFT} \vee \text{RIGHT}$.[16]

Second, consider the types of comparative confidence orderings which the $\mathcal{C}$ construction generates. That construction allows A to have almost any comparative confidence ordering, given her belief set, that satisfies a fairly permissive cluster of restrictions. In virtue of D_1, it requires that A be at least as confident in her beliefs as in those beliefs' negations. In virtue of D_2, it requires that A be no more confident in the beliefs she does not have than in the negations of those non-beliefs. In virtue of D_3, it requires that A be at least as confident

[14] Dick is out of sight, and he is not fast enough to run two blocks in the time it takes Emily to run one.

[15] Regardless of which atom is taken to be actual, the number of truths in U and the number of truths in V are both one. For example, if LEFT is taken to be actual, then exactly one proposition in U is true (the proposition LEFT) and exactly one proposition in V is true (the proposition $\text{LEFT} \vee \text{RIGHT}$).

[16] This is not to suggest that (SA) is the only rationality constraint that gets these intuitions right. Rather, this is just to illustrate why (SA) seems like a plausible rationality constraint. There could certainly be others.

in every proposition as in $\perp$. In virtue of ($\mathcal{C}_1$) and ($\mathcal{C}_2$), it requires that A be strictly more confident in her beliefs than in their negations, whenever the negations are not in A's belief set. All of these restrictions seem reasonable. Rational agents should satisfy them.[17]

With all that as background, the sufficient condition in theorem 2.4 seems like a rationality constraint. For if A's belief set is rational, then it had better *not* be the case that for every ordering to which A's belief set could give rise (that is consistent with the $\mathcal{C}$ construction), A is strictly more confident in one of two equally accurate collections of propositions. That is just another way of saying that A's belief set had better give rise to a comparative confidence ordering that obeys the rather minimal restrictions of the $\mathcal{C}$ construction, while also satisfying (SA).

4 Conclusion

Theorem 2.4 states a sufficient condition for b-representability, and there are reasons for thinking that this condition provides a plausible constraint on rationality. Moreover, by satisfying that rationality constraint, an agent's belief set conforms to many of the constraints on rationality that credences impose (since by theorem 2.4, there exists a probability function that b-represents the belief set). This does not prove, of course, that credences are just mathematical representations of agential belief. But it is suggestive.

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[17]This is not to say that A's comparative confidence ordering must be total, to be rational. It is only to say that A's comparative confidence ordering must be *extendible* to a total ordering without violating ($\mathcal{C}_1$) and ($\mathcal{C}_2$). See [2, p. 59] for an explanation of why partial comparative confidence orderings should be extendible to total orderings in this way, if the agent in question is to be rational.

References

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