Classical limit and quantum logic

Sebastian Fortin[1,2] – Federico Holik[1,3]

[1] Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

[2] Departament of Physics, FCEN, Universidad de Buenos Aires, Argentina.

[3] Instituto de Física de La Plata (IFLP), Argentina.

Abstract

The more common scheme to explain the classical limit of quantum mechanics includes decoherence, which removes from the state the interference terms classically inadmissible since embodying non-Booleanity. In this work we consider the classical limit from a logical viewpoint, as a quantum-to-Boolean transition. The aim is to open the door to a new study based on dynamical logics, that is, logics that change over time. In particular, we appeal to the notion of hybrid logics to describe semiclassical systems. Moreover, we consider systems with many characteristic decoherence times, whose sublattices of properties become distributive at different times.

1. Introduction

In the foundations of physics, the quest of explaining how the laws of classical mechanics arise from the laws of quantum mechanics is known as the *classical limit problem* (Cohen 1989). Generally, this limit is studied for systems that, due to its interaction with the environment, develop a process known as *quantum decoherence* (Schlosshauer 2007). The mathematical description of this phenomenon is usually based on the Schrödinger picture, in which states evolve in time, while observables are taken as constants of motion. Then, projection operators representing physical properties do not evolve in time either. As a result, the structure of the lattice of quantum properties remains the same for all time: the quantum logic associated to the system does not change (Bub 1997).

In this work, we will argue that the description of the lattice of properties in terms of the Schrödinger picture is inadequate for systems undergoing a decoherence process (and thus, it is not useful to describe the *logical* classical limit). We will show that, if the physics of the process represents a transition between quantum to classical mechanics, its logical counterpart should undergo an equivalent transition. Thus, we will propose to study the algebra of the lattice of properties from the perspective of the Heisenberg picture, in which operators representing observables, and their respective projection operators representing physical properties, evolve in time.

From this perspective, we will introduce a novel feature of the classical limit. The study of the time evolution of the projection operators associated to quantum properties in decohering systems opens the way to considering the time evolution of the whole lattice of properties. On this basis, we will study the classical limit from a logical point of view, by describing the manner in which the logical structure of properties associated to observables acquires Boolean features. In other words: we will look for a limit between quantum logic and

Boolean logic and, in this conceptual framework, we will discuss some examples and future perspectives.

2. Observables and Quantum Decoherence

The classical limit problem is usually addressed in terms of the theory of *environment induced decoherence* (EID). This program was developed by the group led by Wojciech Zurek (1982, 1991, 2003), currently at Los Alamos laboratory. According to the Schrödinger picture, a closed quantum system *U*, represented by a state $\hat{\rho}_U(t)$, evolves in time unitarily if no measurements are performed. The system *U* is partitioned into the system of interest *S*, represented by the state $\hat{\rho}_S(t) = \mathrm{Tr}_E\left(\hat{\rho}_U(t)\right)$, and the relevant rest of the world, which is interpreted as the environment *E*, represented by the state $\hat{\rho}_E(t) = \mathrm{Tr}_S\left(\hat{\rho}_U(t)\right)$. The EID approach to decoherence is based on the study of the effects due to the interaction between the quantum system *S*, considered as an open system, and its environment *E*. While *U* evolves in a unitary way, in some typical examples the subsystems may undergo a non-unitary evolution. This allows that, under certain conditions, the state $\hat{\rho}_S(t)$ becomes diagonal after a characteristic decoherence time t_D. In that case, some authors interpret this process as the essence of the classical limit of *S*.

In the framework of the EID approach, quantum decoherence is conceptualized from the point of view of the Schrödinger picture: the phenomenon of decoherence is given in terms of the state evolution. In this representation, the observables associated to the system do not evolve in time. Thus, the commutator between two observables $\hat{O}_1$ and $\hat{O}_2$ stands unchanged during the process. However, decoherence can also be approached to from the viewpoint of the observables of the system.

As it is well known, from the point of view of the properties of the system, the fact that the commutator between two observables vanishes ($\left[\hat{O}_1, \hat{O}_2\right] = 0$) indicates that those observables are compatible: the corresponding properties can be measured simultaneously. If, on the contrary, the commutator is not zero, $\left[\hat{O}_1, \hat{O}_2\right] \neq 0$, the observables are compatible and the simultaneous measurement of the corresponding properties is not possible. The Schrödinger representation imposes that, if two observables are incompatible at the beginning of the process of decoherence ($t = 0$), then, they will remain incompatible during the entire process, up to its end ($t = t_D$). This fact should call the attention of those who wish to interpret the diagonal state $\hat{\rho}_S(t)$ as a classical state, since in a classical system there are no incompatible observables. Thus, the diagonalization of the reduced state is not sufficient to describe the quantum-to-classical transition of the system.

In the history of decoherence, alternative approaches have been proposed in order to deal with certain problems of EID, in particular, difficulties related to the study of closed systems (Diósi 1987; Milburn 1991; Casati and Chirikov 1995; Polarski and Starobinsky 1996; Adler 2004; Kiefer and Polarski 2009). Among them, we are interested in the self-induced decoherence approach, developed from the physical and philosophical point of view in several papers (Castagnino and Lombardi 2003, 2004, 2005, 2007; Castagnino 2004; Castagnino and Ordóñez 2004; Lombardi and Castagnino 2008; Castagnino and Fortin 2011a). According to the SID approach, a closed quantum system with continuous spectrum may undergo decoherence due to destructive interference, thus reaching a final state that can be interpreted as classical. The central point of this proposal consists in a shift in the perspective: instead of splitting the closed quantum system into "open system" and "environment", the division is traced between *relevant and irrelevant observables*. This mechanism allows us to analyze the time evolution of the mean value of the observables: the vanishing of the interference terms is interpreted as the result of a process of decoherence, which leads to the classical limit.

At this point, it is important to remark that, by means of the commutator between two observables $\hat{O}_1$ and $\hat{O}_2$, it is possible to build a new operator $\hat{C} = i\left[\hat{O}_1, \hat{O}_2\right]$ (Fortin and Vanni 2014). We will interpret this observable as measuring the degree of compatibility between $\hat{O}_1$ and $\hat{O}_2$: if $\hat{C} = 0$, the observables are compatible; if $\hat{C} \neq 0$, they are not. According to quantum mechanics, a closed system evolves unitarily following the Schrödinger equation; since the evolution is unitary, it is impossible that it leads to the following process:

$$\hat{C} \neq 0 \rightarrow \hat{C} = 0$$

In a recent article it has been proved that SID can produce a process of this type in the case of systems with continuous energy spectrum (Fortin and Vanni 2014). Given the incompatible observables $\hat{O}_1$ with core $O_1(\omega, \tilde{\omega})$ and $\hat{O}_2$ with core $O_2(\omega, \tilde{\omega})$, both with continuous spectrum, we can compute the commutator $\hat{C}$ as follows:

$$\hat{C}(t) = i\int_0^\infty\int_0^\infty\int_0^\infty \left(O_1(\omega, \tilde{\omega}) O_2(\tilde{\omega}, \omega') - O_2(\omega, \tilde{\omega}) O_1(\tilde{\omega}, \omega')\right) d\tilde{\omega}\; e^{i(\omega-\omega')t} \hat{E}_{\omega,\omega'}\, d\omega d\omega'$$

where $\left\{\hat{E}_{\omega,\omega'}\right\}$ is the energy basis of the space of operators. If $O_1(\omega, \tilde{\omega})$ and $O_2(\omega, \tilde{\omega})$ are regular functions, then, by appealing to the Riemann-Lebesgue theorem, it is possible to prove that (see Castagnino and Fortin 2011b)

$$\text{if } \left\langle \hat{C}(t=0) \right\rangle \neq 0 \Rightarrow \lim_{t\to\infty} \left\langle \hat{C}(t) \right\rangle = 0$$

That is, the observable that measures the incompatibility between two observables goes to zero from the observational point of view. This shows that, since the SID approach describes decoherence from the point of view of the mean value of any observable, it turns out to be useful to study the quantum-to-classical transition of $\hat{C}$ (see Fortin and Vanni 2014). As a concrete example, in a Mach-Zender interferometer, if $\hat{O}_1$ is the observable that measures which is the path taken by the photon, and $\hat{O}_2$ is the observable associated to the visibility of interference, then, $\hat{C}$ can be conceived as the tool to measure how compatible those observables are. In the lab, there are different observables associated with the degree of classicality; for example, the contrast of the interference fringes in the double slit experiment. When the experiment is performed and decoherence occurs, it is reasonable to expect that at the beginning $\hat{C} \neq 0$, but then, after the decoherence time, the system reaches the classical limit with $\hat{C} = 0$. And it is also expected that, in that limit, the interference fringes will accordingly vanish. Moreover, in an experiment with slow and controlled decoherence, it could be possible to measure the evolution of the observable $\hat{C}$.

EID and SID are not the only ways to account for non-unitary evolutions. A strategy to transform the unitary evolution of a closed system into a non-unitary evolution has been proposed in the cosmological context. Kiefer and Polarski (2009) adopted the Heisenberg picture for the study of the decoherence process of the universe. According to this perspective, the state $\hat{\rho}$ stands constant while the observables $\hat{O}(t)$ change in time. In this way, the observable associated to the commutator of two observables becomes a function of time, $\hat{C}(t)$. This approach allows us to study the commutator of two observables for cosmological problems. In particular, according to the *inflation model*, there was an accelerated phase of the early universe called inflation; the whole structure of the universe can be traced back to the primordial fluctuations in the *inflaton* field (Kolb and Turner 1990; Mukahnov 2005; Peacock 1990). Because of the expansion of the universe, inflaton fluctuations must be described by a time-dependent Hamiltonian:

$$\hat{H}(\eta) = \frac{1}{2}\int dk^3 \left[k\left(\hat{a}(k)\hat{a}^\dagger(k) + \hat{a}^\dagger(-k)\hat{a}(-k)\right) + i\frac{a'}{a}\left(\hat{a}^\dagger(k)\hat{a}^\dagger(k) + \hat{a}(-k)\hat{a}(-k)\right)\right]$$

where η is the conformal time, $\hat{a}(k)$, $\hat{a}^\dagger(k)$ are the annihilation operator and the creator operator respectively, and a is the scale factor of the universe. These three last elements are time dependent, and this is the reason why the Hamiltonian $\hat{H}(\eta)$ is not constant in time.

Under these conditions, it is possible to compute the commutator between the operators of position $\hat{y}(\eta)$ and momentum $\hat{p}(\eta)$ (see Kiefer and Polarski 2009):

$$\left[\hat{y}(0), \hat{p}(0)\right] \neq 0 \Rightarrow \lim_{\eta \to \infty} \left[\hat{y}(\eta), \hat{p}(\eta)\right] = 0$$

In other words, the evolution of the commutator between the operators of position and momentum shows that, under certain conditions, it vanishes for times longer than the decoherence time.

Finally, it is important to mention that the approach to decoherence based in non-Hermitian Hamiltonians was also applied to the study of the time evolution of the commutators (Fortin, Holik and Vanni 2016).

3. The Logical Perspective

As it is well known, any physical observable of a quantum system can be represented in a mathematical way as a self-adjoint operator on a Hilbert space (Ballentine 1990). The *spectral theorem* states that any self-adjoint operator $\hat{A}$ can be represented by its *projective measure* $M_A(...)$ (Reed and Simon 1972; Rèdei 1998; Lacki 2000). A projective measure assigns a projection operator to each Borel set of the real line: given the interval $I(a,b)$, $M_A(I)$ is a projection operator. This mathematical fact was interpreted by Birkhoff and von Neumann (1936) as follows. The projector $M_A(I)$ represents the empirical proposition: "the value of the observable represented by $\hat{A}$ lies in the interval I". The truth value of this proposition can be obtained experimentally by means of a yes-no test: that truth value can be tested in any particular run of the experiment, and the quantum state assigns a probability to it.

These formal aspects of quantum theory constitute the elemental bricks out of which the entire building of its rigorous formulation is erected; this task was achieved by von Neumann (1932) in his famous *Mathematical Foundations of Quantum Mechanics*. Importantly enough, the same kind of analysis can be performed for classical probabilistic theories, and further research showed that this approach can be extended to quantum field theory and quantum statistical mechanics. The algebraic structure of the quantum mechanical propositions was called *quantum logic* after the famous paper by Birkhoff and von Neumann (1936). As it is well known, those propositions can be endowed with an *orthomodular lattice* structure (Kalmbach 1983). Additionally, a solid axiomatic foundation for quantum mechanics can be used to explain in an operational way many important features of the Hilbert space formalism (Varadarajan 1968; Stubbe and Van Steirteghem 2007; see also Holik et al. 2013, 2014, 2015 for more recent developments, and for the relationship between the quantum-logical approach and quantum probability theory). But the feature relevant to our discussion is that the logic associated to all varieties of quantum theories is not Boolean, due to the fact that it is not distributive. This implies a very deep structural difference between classical and quantum theories.

Quantum states are, in its formal essence, measures that assign probabilities to all the different empirical propositions. For example, if we want to know the probability of observing the value of the observable $\hat{A}$ in the interval I, given that the system is prepared in the state $\hat{\rho}$, the Born rule states that this quantity is given by $Tr\left[\rho M_A(I)\right]$. According to the traditional Schrödinger picture, unitary evolutions induce time transformations between states. But, according to the Heisenberg picture, observables are transformed, and this transformation induces an action on their respective spectral measures. This in turn implies that the actual properties (i.e., those involved in propositions whose truth is endowed with probability equal to one) also evolve in time. In other words, unitary time evolutions are represented by *automorphisms* on the quantum logic (they are just “rotations” in the projective geometry of the Hilbert space). More general evolutions (such as the non-unitary evolutions associated to measurements or to decoherence processes) are represented by Kraus operators, and also induce concomitant maps on the quantum logic.

But although all possible kinds of time evolutions can be described in the rigorous approach to quantum theory, decoherence poses a conceptual problem in the following sense. Let us suppose that we start with a system that is completely quantum, with its associated orthomodular lattice of projection operators. If the system undergoes a classical limit process, the lattice associated to the final stage should be classical (i.e., Boolean). Therefore, if we want to describe faithfully the classical limit, we should have at hand a time ordered family of logics, starting from a quantum one, and ending up with a classical one. This is the problem that we are going to address in the next section. Transitions between logics were studied (see, for example, Aerts et al. 1993), but not in relation to decoherence and the classical limit. In the present work, we are interested in the philosophical implications of assuming a non-unitary time evolution to induce a continuous family of logics to describe the process of the classical limit. As we will see, this perspective leads to a better understanding of this physical process, and is also useful to cope with hybrid systems.

4. The Classical Limit from the Logical Point of View

In order to be able to describe the classical limit from a logical point of view, let us consider a quantum system that evolves in a non-unitary way, and a set of relevant observables represented by self-adjoint operators, $O=\left\{\hat{O}_1,\hat{O}_2,\hat{O}_3,...,\hat{O}_N\right\}$. Let us also consider the algebra $V(0)$ generated by O at time $t=0$. We also assume that some of the observables of O are incompatible: for some *i* and *j*, we initially have $\left[\hat{O}_i,\hat{O}_j\right]\neq 0$. In a system with these features, the condition for the classical limit –according to the Heisenberg picture– is given by the following evolution:

$$\forall i,\, j,\quad \left[\hat{O}_i(0),\hat{O}_j(0)\right]\neq 0 \rightarrow \left[\hat{O}_i(t_D),\hat{O}_j(t_D)\right]=0$$

As time passes, the evolving operators generate a family of algebras $V(t)$. The final algebra, $V(t_D)$ is a Boolean algebra since, if the classical limit is reached successfully, the final set of generating operators will be a set of pairwise commutative operators. That is: initially incompatible observables become compatible after the decoherence time. The algebras $V(t)$ have associated orthomodular lattices $L_{V(t)}$: the classical limit is expressed by the fact that, while $L_{V(0)}$ is a non-distributive lattice of projectors, $L_{V(t_D)}$ is a Boolean one. In this way, we obtain an adequate description of the logical evolution of a quantum system.

4.1 Semiclassical systems from the logical point of view.

The condition that imposes that all observables of the system must be commutative is equivalent to that of the diagonalization of the state operator, and it is necessary in the case of quantum systems that become completely classical. Notwithstanding, if this condition is strictly applied to any case of classical limit, it leaves no room for the description of the majority of everyday systems, some of which of great importance, such as transistors or squids (Clarke and Braginski 2004). As an example, let us suppose that we go to an electronics store to buy a transistor. The salesman will first find its location in the shelves, and then will take it with his hand in order to put it in a bag and, finally, to give it to us. From this point of view and for all practical purposes, the transistor *behaves classically*: it is an object that can be located in space an time, and can be manipulated by classical means. However, when connected to a circuit, well-known quantum effects of our interest take place on it; for example, consider the tunnel effect of the electrons inside it. This means that a transistor is an object such that some of its observables behave classically, while some others behave in a quantum way: physicists refer to objects of this kind as *semiclassical*.

Our approach of the classical limit allows us to account for these cases. In the semiclassical situation, instead of the above strong condition, the condition turns out to be:

$$\exists i, j, \quad \left[\hat{O}_i(0), \hat{O}_j(0)\right] \neq 0 \rightarrow \left[\hat{O}_i(t_D), \hat{O}_j(t_D)\right] = 0$$

In other words, there are some observables that begin as incompatible and become compatible through the evolution. But there may be also observables that are incompatible at the beginning, and remain incompatible after the decoherence time. From a logical viewpoint, this implies that the lattices of properties associated to this kind of systems are *hybrid* lattices.

The focus on hybrid lattices is of particular importance, because it is reasonable to suppose that, if successfully developed, quantum computers will be semiclassical systems in their very nature, represented by hybrid lattices. This is manifested by the fact that some relevant quantum algorithms possess classical and quantum elements in the process of computation (see, for example, Shor 1997). Thus, a hybrid logic might be useful not only to describe the logical architecture of a quantum computer in a conceptual way, but also to cope with the problems related to decoherence.

4.2 Transitions using many steps

Up to this point we have considered quantum systems that become classical after a decoherence time t_D; in this way, we explained the transition from a quantum logic to a Boolean logic. But we have not explored in detail the intermediate steps of this transition. One way to do this is to consider systems with several characteristic times.

There are a number of examples of physical systems that reach the classical limit in several stages. From the point of view of the state operator, this means that its different non-diagonal components vanish at different characteristic times (Fortin, Holik and Vanni 2016). A concrete example of such a system is that of a harmonic oscillator embedded in a bath of oscillators (Castagnino and Fortin 2012). In this case, the compatibility condition between different observables is fulfilled at different times as follows:

$$\begin{gathered}
\left[\hat{O}_1(0),\hat{O}_2(0)\right]\neq 0\rightarrow\left[\hat{O}_1(t_\alpha),\hat{O}_2(t_\alpha)\right]=0\\
\left[\hat{O}_1(0),\hat{O}_3(0)\right]\neq 0\rightarrow\left[\hat{O}_1(t_\beta),\hat{O}_3(t_\beta)\right]=0\\
\cdots\\
\forall i,j,\quad\left[\hat{O}_i(0),\hat{O}_j(0)\right]\neq 0\rightarrow\left[\hat{O}_i(t_D),\hat{O}_j(t_D)\right]=0
\end{gathered}$$

To put it into words: among all the observables that are incompatible at the beginning of the process, some become compatible at time t_α, others become compatible at time t_β, and so on. If the classical limit is reached, at the end of the process all the observables will commute with each other. In the logical language of lattices introduced above, this many-step process can be described by stating that the different parts of the evolving lattice will become distributive at different times.

5. Conclusions

Since the very beginnings of quantum mechanics, many attempts have been made to recover the laws of classical physics from quantum mechanics through a classical limiting process. This classical limit must do the job of turning a quantum system –described by a quantum logic at $t=0$– into a classical system –described by a Boolean logic at the end of the limiting process, at t_D in the case of decoherence. The dynamical characteristics of the quantum-to-classical transition were extensively studied in the physical literature. However, from a logical perspective, the quantum-to-Boolean transition was usually merely understood as a jump from a quantum logic at $t=0$ to a Boolean one at t_D. Accordingly, researchers did not pay attention to the logical structures associated to the system in times belonging to the interval $(0,t_D)$. As an example of this non-trivial logical structure, we presented physical systems with different characteristic times, which, as a consequence, reach the classical limit in many steps. This shows that the study of the logical features of intermediate times in a quantum-to-classical limiting process may exhibit a rich and non-trivial dynamical structure.

In this work, we described the decoherence process by appealing to the Heisenberg's picture. We argued that it is the proper framework for studying the quantum-to-Boolean transition. With this useful tool, we analyzed the transition in three different cases: (i) logical classical limit in systems with one characteristic time; (ii) systems that change from a quantum logic to a hybrid semiclassical logic; and (iii) systems with many characteristic decoherence times, whose sublattices become distributive at different times. The description of the classical limit presented in this short work does not claim to be exhaustive or complete. But it intends to be the kickoff for the study of a largely unexplored area of the logical structure of quantum systems. Studies of this kind might be of great help in the understanding of the new technologies associated to quantum computers (which involve hybrid logics) and to general quantum information processing tasks.

6. References

Adler, Stephen 2004. *Quantum Theory as an Emergent Phenomenon*. Cambridge: Cambridge University Press.

Aerts, Diederik, Thomas Durt, and Bruno Van Bogaert 1993. "Quantum Probability, the Classical Limit and Nonlocality." In *Symposium on the Foundations of Modern Physics 1992: The Copenhagen Interpretation and Wolfgang Pauli*, ed. K. V. Laurikainen and C. Montonen, 35-56. Singapore: World Scientific.

Ballentine, Leslie 1990. *Quantum Mechanics*. New York: Prentice Hall.

Birkhoff, George, and John von Neumann 1936. "The Logic of Quantum Mechanics." *Annals of mathematics* 37: 823-843.

Bub, Jeffrey 1997. *Interpreting the Quantum World*. Cambridge: Cambridge University Press.

Casati, Giulio, and Boris Chirikov 1995. "Quantum Chaos: Unexpected Complexity." *Physica D* 86: 220-237.

Castagnino, Mario 2004. "The Classical-Statistical Limit of Quantum Mechanics." *Physica A* 335: 511-517.

Castagnino, Mario, and Sebastian Fortin 2011a. "New Bases for a General Definition of the Moving Preferred Basis." *Modern Physics Letters A* 26: 2365-2373.

—— 2011b. "Formal Features of a General Theoretical Framework for Decoherence in Open and Closed Systems." *International Journal of Theoretical Physics* 52, 1379-1398.

—— 2012. "Non-Hermitian Hamiltonians in Decoherence and Equilibrium Theory." *Journal of Physics A* 45: 444009.

Castagnino, Mario, and Olimpia Lombardi 2003. "The Self-Induced Approach to Decoherence in Cosmology." *International Journal of Theoretical Physics* 42: 1281-1299.

—— 2004. "Self-Induced Decoherence: A New Approach." *Studies in History and Philosophy of Modern Physics* 35: 73-107.

—— 2005. "Self-Induced Decoherence and the Classical Limit of Quantum Mechanics." *Philosophy of Science* 72: 764-776.

—— 2007). "Non-Integrability and Mixing in Quantum Systems: On the Way to Quantum Chaos." *Studies in History and Philosophy of Modern Physics* 38: 482-513.

Castagnino, Mario, Adolfo Ordóñez 2004. "Algebraic Formulation of Quantum Decoherence." *International Journal of Theoretical Physics* 43: 695-719.

Clarke, John, and Alex Braginski 2004. *The SQUID Handbook: Fundamentals and Technology of SQUIDs and SQUID Systems, Volume I*. Weinheim: Wiley-VCH.

Cohen, David 1989. *An Introduction to Hilbert Space and Quantum Logic*. Berlin: Springer-Verlag.

Diósi, Lajos 1987. "A Universal Master Equation for the Gravitational Violation of Quantum Mechanics." *Physics Letters A* 120: 377-381.

Fortin, Sebastian, and Leonardo Vanni 2014. "Quantum Decoherence: a Logical Perspective." *Foundations of Physics* 44: 1258-1268.

Fortin, Sebastian, Federico Holik, and Leonardo Vanni 2016. "Non-Unitary Evolution of Quantum Logics." *Springer Proceedings in Physics*, forthcoming.

Holik, Federico, and Angelo Plastino 2015. "Quantum Mechanics: A New Turn in Probability Theory." In *Contemporary Research in Quantum Systems*, ed. Z. Ezziane, 399-414, New York: Nova Publishers.

Holik, Federico, César Massri, Angelo Plastino, and Leandro Zuberman 2013. "On the Lattice Structure of Probability Spaces in Quantum Mechanics." *International Journal of Theoretical Physics* 52: 1836-1876.

Holik, Federico, Angelo Plastino, and Manuel Sáenz 2014. "A Discussion on the Origin of Quantum Probabilities." *Annals of Physics* 340: 293-310.

Kalmbach, Gudrun 1983. *Orthomodular Lattices*. San Diego: Academic Press.

Kiefer, Claus, and David Polarski 2009. “Why Do Cosmological Perturbations Look Classical to Us?” *Advanced Science Letters* 2: 164-173.

Kolb, Edward, and Michael Turner 1990. *The Early Universe*. Reading MA: Addison-Wesley.

Lacki, Jan 2000. “The Early Axiomatizations of Quantum Mechanics: Jordan, von Neumann and the Continuation of Hilbert's Program.” *Archive for History of Exact Sciences* 54: 279-318.

Lombardi, Olimpia, and Mario Castagnino 2008. “A Modal-Hamiltonian Interpretation of Quantum Mechanics.” *Studies in History and Philosophy of Science* 39: 380-443.

Milburn, Gerard 1991. “Intrinsic Decoherence in Quantum Mechanics.” *Physical Review A* 44: 5401-5406.

Mukhanov, Viatcheslav 2005. *Physical Foundations of Cosmology*. Cambridge: Cambridge University Press.

Peacock, John 1999. *Cosmological Physics*. Cambridge: Cambridge University Press.

Polarski, David, and Aleksei Starobinsky 1996. “Semiclassicality and Decoherence of Cosmological Perturbations.” *Classical and Quantum Gravity* 13: 377-392.

Règdei, Mikos 1998. *Quantum Logic in Algebraic Approach*. Dordrecht: Kluwer Academic Publishers.

Reed, Michel, and Barry Simon 1972. *Methods of Modern Mathematical Physics I: Functional Analysis*. New York: Academic Press.

Schlosshauer, Maximilian 2007. *Decoherence and the Quantum-to-Classical Transition*. Berlin: Springer.

Shor, Peter 1997. “Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithm on a Quantum Computer.” *SIAM Journal on Computing* 26: 1484-1509.

Stubbe, Isar, and Bart Van Steirteghem 2007. “Propositional Systems, Hilbert Lattices and Generalized Hilbert Spaces.” In *Handbook of Quantum Logic Quantum Structures:*

Quantum Structures, ed. K. Engesser, D.M. Gabbay and D. Lehmann, 477-523. Amsterdam: Elsevier.

Varadarajan, Veeravalli 1968. *Geometry of Quantum Theory I*. Princeton: van Nostrand.

von Neumann, John 1932. *Mathematische Grundlagen der Quantenmechanik*. Heidelberg: University Press.

Zurek, Wojciech 1982. "Environment-Induced Superselection Rules." *Physical Review D* 26: 1862-1880.

—— 1991. "Decoherence and the Transition from Quantum to Classical." *Physics Today* 44: 36-44.

—— 2003. "Decoherence, Einselection, and the Quantum Origins of the Classical." *Reviews of Modern Physics* 75: 715-775.

Word count: 4298