WHAT IS A HIGHER LEVEL SET?

DIMITRIS TSEMENTZIS

Abstract. Structuralist foundations of mathematics aim for an "invariant" conception of mathematics. But what should be their basic objects? Two leading answers emerge: higher groupoids or higher categories. In this paper I argue in favor of the former over the latter. First, I explain why to pick between them we need to ask the question of what is the correct "categorified" version of a set. Second, I argue in favor of groupoids over categories as "categorified" sets by introducing a pre-formal understanding of groupoids as abstract shapes. This conclusion lends further support to the perspective taken by the Univalent Foundations of mathematics.

1. Introduction

I am concerned with the question: what is a higher-level thing? The notion of "higher" that I will be concerned with comes from category theory: I will be talking about *categorification* (or *groupoidification* or *homotopification*). If recent developments in theoretical physics and mathematics are anything to go by, this notion will become increasingly relevant to philosophers in the coming years. But even if one is left unmoved by such developments, it is worth recalling how many philosophical problems are about determining whether a certain phenomenon is of a "higher-level" than another: are mental states "higher-level" brain states? Is colour a "higher-level" property of particle arrangements? As such, a purely philosophical investigation of this new category-theoretic notion of "higher-level" seems to be in order.

The idea of categorification emerges naturally from the development of higher category theory, i.e. the study of categories with arrows not just between objects but also between other arrows (and higher arrows, and even higher arrows etc.).[1] The first systematic study of categorification is perhaps to be found in Baez and Dolan's paper [6] who define it as "the process of finding category-theoretic analogs of ideas phrased in the language of set theory".[2] This simple vision, of taking well-established mathematical situations/structures/theorems and seeking to find their (higher) categorical analogs, has led to an explosion of activity in many diverse fields. To illustrate: in mathematics, categorification appears in fields as diverse as Lurie's pioneering program of Derived Algebraic Geometry (cf. [37]) which has led to the proof of important conjectures (cf. [38] for the proof of the Cobordism Hypothesis and cf. [24] for a recent proof of a conjecture of Weil's)

Department of Statistics and Biostatistics, Rutgers University. E-mail: dt506@rci.rutgers.edu. I would like to thank Hans Halvorson, John Burgess and Colin McLarty for comments on an earlier version of this paper. I would especially like to thank David Corfield (originally an anonymous referee) for very illuminating and thorough comments. I would also like to thank another anonymous referee for helpful suggestions. This work was partially supported by NSF CAREER-DMS-1554092 (P.I. Harry Crane).

[1] For some basic primers on "traditional" higher category theory cf. [16, 36, 58] and for a nice overview of more "modern" ∞-approaches cf. [15]. For an exposition that engages with both approaches cf. [7].

[2] Needless to say, many of these developments were inspired by Mac Lane's founding work on category theory (cf. [39] for the textbook reference) as well as Grothendieck's monumental writings (cf. [26] for a sampling of his ideas). But there are very many developments in 20^{th} century mathematics that can be traced back to these two figures. Thus, for reasons of convenience, our own story will begin with Baez and Dolan, noting that such a starting point may appear somewhat arbitrary.

and Weinberg's Program in symplectic geometry (cf. [69, 70] for an overview and [68] for some recent developments). In physics and mathematical physics, it appears the study of (topological) Quantum Field Theories (cf. [20, 21] which is credited with coining the term "categorification") and the foundations of higher gauge theories (cf. [55]) as well as the more classical work of Witten on string theory (cf. [71]). In philosophy, categorification has also begun to make inroads, as evidenced by the work of Corfield (cf. [17] especially Chapter 10 and [18] for an illuminating recent discussion of "covariance" and what categorification/groupoidification has to say about it as well as the paper [19] that grew out of it) and more recently in Halvorson's project in the philosophy of science (cf. [27, 28] for overviews).

All these developments attest to the importance of the "categorical" way of thinking and especially of the value of categorifying previously-known mathematical situations. In this paper, however, I will be concerned with more recent, and largely independent, developments in the foundations of mathematics. Namely, with the idea of coming up with a foundation for mathematics that is "natively" categorical and independent from Cantorian set theory. These developments have opened up the way to revisit one of the more fundamental questions concerning categorification: what is the categorified version of a *set*?

The plan is as follows. In Section 2, I will explain the relevance of the title question to the aforementioned recent developments in the foundations of mathematics. In Section 3 I will make the title question precise by introducing the relevant ideas from category theory and in Section 4 I will examine possible methods to pick between two different answers, namely groupoids vs. categories. In Section 5 I will develop pre-formal notions of (spatial) groupoids and (organizational) categories and argue that there is no way to pick between them by "pure metaphysics". Finally, in Section 6 I will offer an argument that (spatial) groupoids are indeed the correct notion of a next-level set in the context of the foundations of mathematics.

2. Structuralist Foundations

Structuralist foundations of mathematics aim to provide a foundational language that can express only *structural* (or *isomorphism-invariant*) properties of mathematical objects. In particular, they aim to provide a language that makes no distinction that practicing mathematicians themselves wouldn't make (e.g. between the number 2 as $\{\varnothing, \{\varnothing\}\}$ or as $\{\{\varnothing\}\}$). Such a pursuit constitutes a sort of *pragmatic* reading of Benacerraf's [9] famous observation, such as the one provided by Burgess [13, 14].[3]

As I will be using the term, there are two main contenders for a structuralist foundation: Makkai's Type-Theoretic Categorical Foundations of Mathematics (TTCFM) [40, 42] and Voevodsky's Univalent Foundations (UF) [62, 65, 66].[4] There are important similarities between the two approaches: both UF and TTCFM, for example, are to be formalized using certain variations of dependent type theory in the style of Martin Löf [49]. But the most crucial philosophical point of comparison between TTCFM and UF, and the one that is the focus of this paper, is their intended semantics: the universe of ∞-categories in the case of TTCFM and the universe of ∞-groupoids in the case of UF. The goal of this paper is to settle the title question by investigating which of these two intended semantics are more appropriate for a structuralist foundation of mathematics.

[3] Relatedly, there is a large literature on the connection between category theory and structuralism in the philosophy of mathematics. For example, see the Hellman-Awodey debate [1, 29] and the work of McLarty [50, 51].

[4] For a more detailed elaboration of the sense in which UF is a structuralist foundation see [60].

Now, in what sense are ∞-categories supposed to be the intended semantics of TTCFM and ∞-groupoids those of UF? In the case of TTCFM, its basic syntax, First-Order Logic with Dependent Sorts (FOLDS), is supposed to stand to the universe of ∞-categories (∞**Cat**) in the same relation that first-order logic stands to the universe of sets. TTCFM would then consist of an axiomatization of ∞**Cat** written out as a list of axioms in FOLDS in just the same way that ZFC is a description of the cumulative hierarchy V written out as a list of axioms in first-order logic.[5]

In the case of UF, on the other hand, it was discovered independently by Awodey and Warren [4, 67] and Voevodsky [63] that the so-called identity types of intensional Martin-Löf Type Theory (MLTT) – a system that originally had nothing to do with algebraic topology – could be interpreted as path spaces. This allows types in MLTT to be thought of as homotopy types of spaces and homotopy types of spaces are represented by ∞-groupoids.[6] This observation has given rise to the field of Homotopy Type Theory (HoTT) which studies formal systems (usually variants of MLTT) whose basic objects can be thought of as homotopy types/∞-groupoids.[7] It is in this sense that we may say that UF (formalized by some HoTT) has ∞-groupoids as its intended semantics.[8]

One may wonder, what reasons do we have for replacing V (or other well-understood conceptions of the universe of sets) with something like ∞**Cat** or ∞**Gpd**? There are both pragmatic and philosophical reasons. On the philosophical side, following Benacerraf's seminal paper [9], there has always been the question of what kind of foundation, necessarily non-set-theoretic, would be able to achieve a formalization of everyday mathematical notions that is incapable of making distinctions that are meaningless to practicing mathematicians. On the pragmatic side, and somewhat relatedly, formal systems that are unable to make such distinctions can aid in the formal verification of theorems obtained using modern algebraic techniques, which involve keeping track of a large number of isomorphisms.[9]

But pragmatic or philosophical motivation aside, it is certainly true that contemporary mathematics is moving away from the "extensional point of view" (to borrow a phrase of Marquis in [47]) that set theory was tailor-made to formalize and the Bourbaki approach

[5]A serious problem with this simple-sounding vision is that the notion of a (weak) ∞-category still lacks a usable and universally accepted analytic definition, viz. as a structure built out of other mathematical objects. In its most concrete instantiation ∞**Cat** has been precisified as the so-called "multitopic ω-category of multitopic ω-categories" (cf. [41, 43–45, 48]). Trying to understand that particular definition, however, is more an exercise in losing, rather than solidifying, one's intuition about ∞-categories. It is perhaps fair to say that if this kind of definition proves anything, it is some sort of inherent incompatibility between this "maximally invariant" object ∞**Cat** and a set-theoretic semantics in which we may try to express it. Of course the fact that ∞**Cat** seems to lack simple and elegant analytic foundations in set theory doesn't necessarily mean that it could not have simpler analytic foundations in some other theory.

[6]The fact that homotopy types can be represented by ∞-groupoids is usually called the Homotopy Hypothesis. Though called a "hypothesis" it is best thought of as a kind of guiding principle which can be made precise (and proven) in various ways. For a helpful overview cf. [5].

[7]For standard introductions to UF and HoTT, see [54, 57, 62, 66] and for some of the early motivation that led to UF see [63–65]. Although it makes no difference for the purposes of this paper, it should also be made clear that MLTT is not necessarily the only way to formalize UF. A more "first-order" approach can be taken using the system in [61] base on Makkai's FOLDS.

[8]As opposed to ∞**Cat**, the universe of weak ∞-groupoids has firm analytic foundations in set theory, the most well-understood of which is perhaps given by (the fibrant objects in the standard model structure on) the category **sSets** of simplicial sets, i.e. the so-called Kan complexes. Indeed, this has allowed an explicit construction of a model of HoTT (understood as intensional MLTT plus the asiom of univalence) in ZFC (plus two inaccessible cardinals) by interpreting types as Kan complexes – see [31].

[9]This indeed was Voevodsky's original motivation in developing UF, as can be seen by his (now somewhat outdated) 2006 lecture [63].

most vividly exemplified. It is thus an important challenge to figure what kind of formal systems are able to formalize this "non-extensional" point of view of the post-Bourbaki era. And in this paper we will be focusing on one specific aspect of this challenge: what should the intended semantics of such formal systems be: ∞-categories or ∞-groupoids? And this question, I submit, reduces to the following far more tractable one: "What is a higher-level set?" The main aim of this paper is to answer this question. Before we do so, it is important to get clear on the relevant notion of "higher level".

3. The notion of "higher-level"

Consider a bowl filled with raffle tickets. Someone is assigned the task of shuffling the tickets. For certain questions we might want to ask of the bowl, the extra information about who shuffled it and how will not be relevant. For example, if we want to ask: "How many tickets are in the bowl?" or "Who won the raffle?" For certain other questions, information about who did the shuffling will become relevant: "Was the shuffling thorough?" or "Why is the winning ticket smudged with barbecue sauce?" If we are interested in asking such questions then it is better to think of our object of interest not as a bowl of raffle tickets but as a bowl of raffle tickets *together with* someone assigned to shuffle it. Then the relation of the bowl-and-shuffler to the bowl is an illustration of the relation I am interested in: *the bowl-and-shuffler is a higher-level bowl.*

Consider a laptop not connected to the Internet. Then, suddenly, its owner decides to connect the laptop to the Internet. The device she now has before her is not (just) her laptop. In fact, what she has before her is the whole Internet. In being connected to a network of other laptops (and servers) the owner has access to a device over and above the laptop before her. The relation of (offline) laptop to Internet is also an illustration of the relation I am interested in: *the Internet is a higher-level laptop.*

These illustrations tack on to familiar intuitions. In moving from bowl to bowl-and-shuffler what we are doing is moving from an object to an object-with-symmetries. Namely, the "higher-level"-ness of the bowl-and-shuffler consists in keeping track of certain symmetries of the object that we had initially deemed irrelevant. This is entirely analogous to moving from a spacetime $\mathcal{S}$ (e.g. $\mathbb{R}^4$) regarded in isolation to a spacetime-with-symmetry group pair $(\mathcal{S}, G)$ (e.g. $(\mathbb{R}^4, \mathcal{L})$ where $\mathcal{L}$ is the Lorentz group). On the other hand, in moving from the laptop to the Internet what we are doing is moving from an object to a structured network of similar objects. Namely, the "higher-level"-ness of the Internet consists in being a collection of *connected* laptops (and servers). There is no shortage of controversial examples of this relation: a mind is a collection of connected neurons; colour is the way certain molecules hang together. And so on.

Since I want to view both these illustrations as examples of a similar phenomenon, a grammatical clarification is in order. When I speak of a "higher-level X" I take X to designate the class of things to which it belongs and not necessarily some specific individual in that class. A "higher-level apricot" does not ask for the higher-level version of the particular apricot on my table, but of the species "apricot" of which the apricot on my table is a particular instance. Of course, it might happen that occasionally – as with the raffle-bowl – a higher-level X might make sense even for a *specific* X. But it need not do, as the example with the laptops hopefully illustrates.[10]

Both notions of "higher-level" that the above examples are meant to illustrate originate from category theory and only properly make sense in a mathematical setting. So let us

[10]For the purposes of this paper, it is safe to assume the question "What is a higher-level X?" is an abbreviated form of the much more cacophonous: "What is type of thing that can be regarded as the higher-level version of the type of thing that X is?"

put apricots and neurons to the side for now and offer some concrete mathematical examples:

(1) An abelian category is a *higher-level* abelian group.
(2) A monoidal category is a *higher-level* monoid.

Both examples are categorifications in their own unique way. An abelian category is a category that resembles the category **Ab** of abelian groups. A monoidal category is a category together with a multiplication operation (usually denoted "$\otimes$") that behaves just like the multiplication in a monoid except its properties are defined "up to natural isomorphism." Of these examples, it is only the second (monoids to monoidal categories) that is properly speaking a *categorification*. But both examples exhibit the two essential kinds of categorification that I want to focus on below:

(1) **"Upward" Categorification:** Moving from a (structured) object (e.g. abelian group, monoid) to a (structured) *collection* of objects (e.g. abelian category, monoidal category).[11]
(2) **"Inward" Categorification**: Weakening identities (e.g. $1 \times x = x$, $g \circ f = h$) to natural isomorphisms (e.g. $\alpha_x \colon 1 \otimes x \cong x$, $g \circ f \cong h$).[12]

The details of these mathematical examples are not at all essential to what I have to say below. Although it would be interesting to try and come up once and for all with a mathematical definition of categorification, this is not what I want to do here.[13] Rather, as indicated above, in this paper I will focus on the simplest of mathematical examples, viz. that of a *set*. What is the higher-level version of a set, when "higher-level" is understood in the above-sketched manner? Immediately, this question raises difficulties that other mathematical examples do not. Firstly, if we want to upwardly categorify a set by moving to a collection of sets, we have no pre-existing "structure" to use as a guide to determine what structure to impose on this collection as we did, for instance, in the case of monoids where we had a multiplication operation. Secondly, if we want to inwardly categorify a set, then what kind of identities does a set satisfy that we could weaken to isomorphisms?

As it turns out, each of these concerns leads naturally to distinct answers to the question of what is a higher-level set: categories or groupoids. In the next section, we investigate how we could decide this question.

4. What is a next-level set?

Let me recall the formal definitions of a category and a groupoid (for classic introductory texts see [2, 39] and for more philosophical perspectives on category theory see [46] as well as the forthcoming [34]). Let me clarify that nothing I say hinges on a precise understanding of these definitions. Indeed, I will also develop pre-formal definitions for both notions below and the crux of my argument will depend only on those. Nevertheless, for the sake of completeness, it is helpful to lay them out.

---

[11] I have used the term "upward" because the sense of collecting together all structures of a given kind to see what structure this collection bears does not match well with the existing terminology of *horizontal* categorification, which usually refers to producing many-object analogues of one-object categories (e.g. groupoids can be regarded as the horizontal categorification of groups because groups can be understood as one-object categories with all arrows invertible).

[12] As an alternative to "inward" we could have used the established term *vertical* categorification since what I am describing accords well with what is usually understood by the latter concept. I have avoided doing so only in order to avoid contrasting "upward" and "vertical" since these two notions do not sound distinct enough to my ear. But the reader familiar with the notion of vertical categorification may take my "inward" categorification to refer to the exact same process.

[13] For the classical such proposal cf. [6].

Definition 4.1. A **category** $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}\,\mathcal{C}$ and a collection of arrows $\mathrm{Mor}\,\mathcal{C}$ between these objects, together with a composition operation $\circ$ on arrows such that:

(1) For every $f : a \to b$ and $g : b \to c$ there is an arrow $g \circ f : a \to c$.
(2) For every object a in $\mathrm{Ob}\,\mathcal{C}$ there is an arrow $1_a : a \to a$ (called the "identity arrow") such that $1_a \circ f = f$ and $f \circ 1_b = f$ for every arrow $f : a \to b$.
(3) $\circ$ is associative.

Within a category we then say that a given arrow $f : a \to b$ is an *isomorphism* if there is another arrow $g : b \to a$ going in the other direction such that $g \circ f = 1_a$ and $f \circ g = 1_b$.

Definition 4.2. A **groupoid** is a category in which every arrow is an isomorphism.[14]

To be fully precise, I will refer to the sought-after entity as a *next*-level set, to make clear that what is sought is the notion of a set *exactly* one level up. Now, as suggested above, there are two plausible answers to the question "What is a next-level set?":

(C) A next-level set is a category

(G) A next-level set is a groupoid

How can we approach (C) and (G) as philosophical theses? A reasonable method is to state a general principle defining "higher-level"-ness and then show that this principle, when specialized to sets, yields the desired answer. Consequently, choosing between (C) and (G) reduces to choosing between these two general principles. The issue, as we shall see, cannot be settled as cleanly as that – but it is with this methodological optimism that I will begin.

In support of (C) we will consider the principle that Makkai [42] has called the *Fregean Imperative*:

(FI) A next-level X is the type of thing formed by the collection of all Xs

In support of (G) we will consider a principle that I will call *Bishop's Imperative*:[15]

(BI) A next-level X is the type of thing between each of whose constituents there are (at most) X-many identifications

The phrasing of (BI) at this level of generality becomes cumbersome, so let me try to clarify further. The essential idea is this: Suppose X is a collection of things. Then the next-level X is a collection of things such that for any two such things, the collection of ways in which they can be identified is itself a collection of type X.

Consider the following simple but crucial example. Let X be a truth-value, i.e. X is either *true* or *false*.[16] Then by (BI) a next-level truth value (call it Y) is the type of thing between each of whose constituents there are (at most) truth value-many identifications (where "truth value-many" simply means that between each pair of constituents of Y there either is ("true") or isn't ("false") an identification). Namely, Y is a collection of things each of which is uniquely separated from all the others: whether a thing in Y is the same as another thing in Y is answerable merely by a "yes" or a "no". In short,

[14] It is very important to note that a groupoid does not have to be defined in this manner (even if this is the most concise way to do so). The notion of a groupoid actually pre-dates the notion of a category and was initially seen as a generalization of the notion of a group. In other words, we are by no means forced to define a groupoid as a category with an extra property.

[15] The name is a reference to E. Bishop's view on sets as can be found e.g. in [10, 53]. But there are also obvious connections with the behaviour of identity types in (intensional) Martin-Löf type theory [49].

[16] One can thus think of X as an element of the set $\{\top, \bot\}$ or $\{0, 1\}$. But it is in no way necessary to think set-theoretically.

next-level truth values are collections of things, uniquely determined by their elements. Such entities have been known to operate under the name of "sets". Thus, according to (BI), sets are next-level truth values.

So what – if anything – can we say about (BI) and (FI)? There are two questions we must tackle:

(Q_1) Does (C) (respectively (G)) follow from (FI) (respectively (BI))?

(Q_2) Are there any distinctly philosophical reasons to prefer either (BI) or (FI)?

I will take these questions in turn.

4.1. **Does (G) follow from (BI)?.** Voevodsky [66] has put forward an argument in support of (G).[17] It goes as follows:

(1) **Categories** can be thought of as **groupoids** with extra structure. In particular, let $\mathcal{G}$ be a groupoid with G its set of objects. Then we can define a category structure $\mathcal{C}$ on $\mathcal{G}$ by taking $\mathrm{Ob}\,\mathcal{C} = G$ and regarding $\mathrm{Mor}\,\mathcal{C}$ as a function
$$\mathrm{Mor}\,\mathcal{C} \colon G \times G \to \mathbf{Set}$$
picking out a set of arrows between any two objects of G satisfying the appropriate properties.[18]
(2) **Partially ordered sets** can be thought of as **sets** with extra structure. In particular, if $(X, \leq)$ is a partially ordered set this means that for any $x, y \in X$ we have that either $x \leq y$ or not. Thus, the ordering $\leq$ on X can once again be thought of as a function
$$\leq \colon X \times X \to \{0, 1\}$$
satisfying the appropriate properties.
(3) If categories are groupoids with extra structure then lower-level categories are partially ordered sets.
(4) Therefore, next-level sets are not categories. Instead, categories are higher-level posets.

As an argument purely in support of (G), this is clearly hopeless since the analogy drawn between (1) and (2) presupposes (G). For in saying that (1) is the higher-level analogue of (2) one must already accept that groupoids are the higher-level analogues of sets. If anything, what this argument shows (successfully) is that *if* one takes next-level sets to be groupoids then one can view categories as next-level partially ordered sets. But this is not something that we care about here, nor is it something that, I think, anyone would regard as particularly controversial. Furthermore, this argument in no way relies on (BI) or on any other general principle about "higher-level"-ness. As such, Voevodsky's

[17]It might be unfair to call it an "argument" since it was not presented as one, at least in the philosophical sense of the term. Nevertheless, the kind of thinking that this "argument" represents was crucial in the development of UF. To see this, consider what Voevodsky says (also in [66]): "One of the things that made the "categories" versus "groupoid" choice so difficult for me is that I remember it being emphasized by people I learned mathematics from that the great Grothendieck in his wisdom broke with the old-schoolers and insisted on the importance of considering all morphisms and not only isomorphisms [...] and that this was one of the things that made his approach to algebraic geometry so successful."

[18]The way I've laid out Definitions 1 and 2 above should make it clear that this is not the usual way of thinking about categories. Also, note that $\mathrm{Mor}\,\mathcal{C}(a, b)$ for any given $a, b \in G$ will in general be disjoint from $\mathcal{G}(a, b)$. Namely, the morphisms of $\mathcal{C}$ have *a priori* nothing to do with the already given morphisms of $\mathcal{G}$. This kind of situation of course becomes more interesting if we have a notion of a groupoid that is defined not as a category with an extra property – and indeed such a notion (types of h-level 3, or 1-types) is available in HoTT, where it is perfectly sensible to define a notion of a category as a structure on a 1-type.

argument in itself cannot provide any justification for the thesis that from (BI) it follows that next-level sets are groupoids. (Nevertheless, the heuristic that categories are next-level partially ordered sets is a very useful one.)

So let us instead try to get to (G) from (BI) in a more direct manner. Let X be a set. As summarized above, X is a collection of things (elements) such that for any two elements there is only one *unique* way in which they can be identified. Put differently: for any two elements of a set we care only about *whether* they are identical and not about *in how many ways* they can be regarded as identical. Thus, according to Bishop's imperative, a next-level set would be a collection of things such that there are "set-many" ways in which any two such things can be identified.

Such a collection is now starting to sound very much like a groupoid if we think of the "set-many" identifications between any two objects as the set of isomorphisms between them (in a groupoid, recall, every arrow is an isomorphism but there is no requirement that there should be a *unique* isomorphism between any two objects: there could be "set-many" of them). The remaining difficulty is whether or not it makes sense to regard "isomorphisms" as "identifications". If it does, then we are done: (BI) demonstrates that a next-level set is a groupoid. But what licenses us to say that isomorphism is the same thing as identity? Surely we can regard isomorphic objects as identical if we so wish, but that does not mean that isomorphism *is the same thing as* identity.

Unless we have some formal principle that allows us to regard isomorphism as identical to identity, then we cannot advance any further towards (G). The axiom of univalence (UA) provides just such a formal principle – but of course we cannot invoke it here, while engaged in "pure" philosophy. For even though it is straightforward to see that (BI)+(UA) $\Rightarrow$ (G),[19] I am interested in the much stronger statement (BI) $\Rightarrow$ (G). And in arguing for this stronger statement, it is not immediately clear to me how to overcome this difficulty.

4.2. **Does (C) follow from (FI)?.** On the other hand, with his acceptance of the Fregean imperative in the background, Makkai supplies an argument in favour of (C).[20] It goes as follows:

(1) The identity relation between (abstract) sets can only be applied to their "elements" – in fact this characterizes (abstract) sets.
(2) Sets are not "elements" of other sets.
(3) You cannot ask of any two sets whether they are identical.
(4) Therefore the collection **Set** of all sets is not a set, since you cannot ask of any of its elements if they are equal.
(5) Instead you can define what it means to have an isomorphism of sets, but that requires having a category structure on **Set**.
(6) Therefore, next-level sets are categories.

The crux of the argument is in (4) where it is claimed that the kind of structure that the collection of all sets forms is the structure that allows us to formulate a criterion of identity for them, viz. isomorphism. This is because the collection **Set** of all sets is

[19]If one trusts that (BI)+(isomorphism≅identity) leads to (G) then the only thing left is to see (UA) as an axiom asserting (isomorphism≅identity). This is argued for, convincingly, by Awodey in [3]. It needs to be clarified, however, that (UA) does not *collapse* isomorphism to identity by coarsening the latter. Rather it *expands* identity to isomorphism by defining it as homotopy equivalence. In particular, it is not that there were things we could separate before asserting (UA) that we no longer can after asserting it – (UA) does not *coarsen* identity; it *defines* it.

[20]Note that "sets" here means "abstract sets" i.e. the kind of entities described e.g. by the FOLDS theory of sets in [42]. For further discussion on the distinction between "abstract" and "concrete" in this context as well as more discussion on Makkai's argument see [48].

not itself a set and since it is only possible to ask of elements of sets whether they are identical it is *impossible* (!) to ask of sets themselves whether they are identical. What we can do instead is ask whether they are isomorphic – but that requires having a category structure on **Set**! Therefore, the collection of all sets is a category and as such, by (FI), next-level sets are categories.[21,22]

Ingenious as this argument may appear, there is clearly a hidden premise operating here, namely that the structure borne by the collection of all things of type X is *the kind of structure* that allows one to establish/define a criterion of identity for things of type X. In the case of sets, this hidden premise specializes to the following: the structure borne by the collection **Set** of all abstract sets is the kind of structure that allows one to define/establish a criterion of identity for sets. This criterion of identity, Makkai assumes, is isomorphism and, Makkai claims, it is only in the setting of categories that we get an abstract definition of isomorphism between two objects (along the lines indicated after Definition 4.1).

But then we are faced with the following dilemma. If one is to understand the argument as claiming that the *definition* of what it is for something to be the next-level X is to be that kind of structure that provides the collection of all things of type X with a criterion of identity, then surely this definition is not specific enough to pick out categories as next-level sets. Indeed this description is perfectly compatible with taking groupoids to be next-level sets, since really all that is needed in order to provide individual sets with a criterion of identity is the underlying *groupoid* of **Set**. After all, if we only care about providing a criterion of *identity* then what do we need non-invertible arrows for at all?

On the other hand, if the argument is to be understood as saying that it is *because* a category structure provides a criterion of identity between elements of **Set** that **Set** is to be regarded as a category, then the argument becomes viciously circular. For in order to conclude that **Set** is a category one has to assume that it is one. More precisely, the argument would go as follows:

(1) If **Set** is a category then it provides a criterion of identity for sets
(2) If a structure on **Set** provides a criterion of identity for sets then that structure is a next-level set.
(3) Therefore **Set** is a category

In order to conclude (3) from the two conditionals (1) and (2) one must assert that **Set** is a category – but this is the desired conclusion.

[21]A very similar way of separating sets from "categories" can also be found in the work of Martin-Löf:

> A category is defined by explaining what an object of the category is and when two such objects are equal. A category need not be a set, since we can grasp what it means to be an object of a given category even without exhaustive rules for forming its objects. [...] To define a category it is not necessary to prescribe how its objects are formed, but just to grasp what an arbitrary object of the category is. ([49], pp. 21-22)

In the above quote "category" does not necessarily refer to "category" in the sense of category theory, but rather to the kind of thing signified by the symbol `Type` in type theory, i.e. the collection of all types. If type theory contains the appropriate type-formers and axioms (e.g. Π-types with the usual rules) then `Type` will also carry the structure of a category-in-the-sense-of-category-theory. But – for Martin-Löf – the structure of a category-in-the-sense-of-category-theory does not immediately "emerge" from the above-sketched way of separating "categories" from sets. (It is also worth noting that Martin-Löf uses "sets" here for what we would now call "types" in MLTT.)

[22]Very similar arguments have been put forward by several authors arguing in favour of some variant of category-theoretic structuralism in the philosophy of mathematics (e.g. [1, 32] and also [35]). The main idea there is to separate the way in which an object is "given" in category theory (schematically, abstractly, up to isomorphism) from the way in which an object is "given" in set theory (by being constructed directly out of simpler objects).

5. Next-level sets are groupoids

Neither Makkai's nor Voevodsky's arguments, I conclude, provide satisfying positive answers to (Q_1). In failing to do so, they illustrate for us the main difficulty of approaching the question through an investigation of (FI) and (BI): both principles are equally compatible with giving the opposite of the desired answer. More precisely, (FI) is compatible with (G) because **Set** also bears the structure of a groupoid[23] and (BI) is compatible with (C) since any groupoid can be regarded as category with an extra property (all arrows are invertible).

This leads us to the following crucial question: are groupoids to be regarded as categories with an extra *property* (that all arrows are invertible) or are categories to be regarded as groupoids with extra *structure* (non-invertible arrows between objects)? Clearly, this is at root a question about fundamentality. We are asking: which structure is more fundamental, groupoids or categories?

If we take this line, then the acceptance of (FI) or (BI) should not really be seen as directly justifying that next-level sets are categories or groupoids. It should instead be understood as a justification for the *fundamentality* of categories over groupoids or groupoids over categories. On this view (FI) (respectively (BI)) implies that categories (respectively groupoids) are somehow more fundamental than groupoids (respectively categories).

But what exactly is "fundamentality" even supposed to mean here? It seems hopeless to give it a precise *technical* meaning (e.g. by saying that X is more fundamental than Y if Y is the same type of thing as X but with an extra property) without deciding the question in advance. Furthermore, any such technical criterion will depend on some kind of fixed background theory (e.g. set theory) in which both groupoids and categories will already have been defined (e.g. as structured sets). But choosing such a background theory seems to make the question meaningless, since the notions we want to compare are now defined in terms of something even more fundamental. For example, if we define both groups and topological spaces as structured sets does it make sense to ask which is more fundamental? Or even to try and come up with a technical set-theoretic criterion to make the question precise?[24] It seems equally hopeless to try to understand "fundamentality" in terms of historical priority, viz. by figuring out whether groupoids or categories arose first in mathematical practice.[25]

On the other hand, if we are going to rely on a *non-technical* and *non-historical* notion of fundamentality this would seem to depend on having some kind of pre-formal grasp of categories or groupoids, analogous to how in the case of sets we have some pre-formal grasp of what it means for something to be a collection of things. Moreover, since both (BI) and (FI) can plausibly be read as pre-formal principles it makes more sense to see them as being applicable to pre-formal notions.[26]

---

[23] Simply take the underlying groupoid (or "core") of the category of sets and functions.

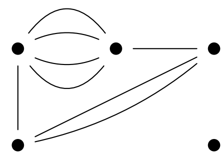
[24] There has been some recent literature in the philosophy of science on formal ways to compare the "amount of structure" in a theory – see [8, 59] and references therein. However, I do not think that any of the methods used there would be applicable to the present situation.

[25] For the interested reader, groupoids first appeared in the work of Brandt [11] long before categories were defined. According to Ronnie Brown, there is apparently a rumour that Mac Lane and Eilenberg (the inventors of category theory) were inspired by groupoids in their definition of a category although this has been denied by Eilenberg (cf. [12], p. 118). An anonymous referee informed me that Mac Lane also denied this rumour in private communication.

[26] At least in the case of categories, several arguments have been proposed over the years against the possibility of such a pre-formal account that does not *on some level* depend on some (naive/pre-formal) set theory. Feferman [22] argued that the notion of "collections and operations" is required in order to talk about categories and so any pre-formal talk of categories assumes some pre-formal set theory. The debate

So I will now go on to develop a pre-formal account of both categories and groupoids. I will then argue that with this pre-formal account in hand, it is indeed true that (BI) implies that (pre-formal) groupoids should be regarded as next-level (pre-formal) sets and similarly for (FI). As is clear, to do so I will rely on a notion of a pre-formal set. By this I will mean simply the naïve idea of a set as a *collection* of *things*. These "things" ought to be thought of as "structureless" in a naive sense: there is nothing we can say about them other than the fact that they are parts of the collection at hand. The pre-formal notion of set I have in mind, then, is that of a "purely extensional" collection: an *abstract* set.[27]

5.1. **Pre-formal Groupoids.** The basic idea is this: a groupoid is a *shape*. As with any shape, it has edges and it has vertices although we allow there to be multiple edges between any two vertices. Importantly, the edges are not *directed*. Intuitively this means that if you can walk from one vertex to another along an edge, then you can just as easily walk back along the same edge. In short, a groupoid is a collection of things (which we can regard as points) some of which are connected by a collection of edges (which we regard as lines or paths). A pre-formal groupoid thus looks something like this:



I think it is fairly intuitive that pictures like the above represent objects intuitively comprehensible as shapes and I will not argue for it any further. This is all my notion of a pre-formal groupoid consists in: a graph with undirected edges such that there can be multiple edges between any two nodes. This captures enough of the intuition that groupoids are mathematical structures encoding the homotopy types of (not necessarily path-connected) topological spaces. Although groupoids are not used exclusively for these purposes, it is still fair to say that most of their uses in one way or another reflect this intuition. And it is only this intuition on which we will rely.[28]

It is also important to note that this pre-formal notion leaves out a crucial part of the formal definition of groupoids, namely the existence of identities and composites together with coherence conditions dictating how they interact. I justify this omission for three reasons: firstly, because it is to some extent implicit (pre-formally) in the notion of a shape; secondly, because these algebraic conditions are not essential to the points I make below; thirdly, because in the case of 1-groupoids these "coherence conditions" are very few and can easily be written down.

continues all the way to the present day: see [23, 33, 48, 52] for some of the most recent installments. My own attitude to such objections is that the notions of collection and operation pre-dated by some margin even the earliest formal use of "abstract" sets in mathematics. Therefore, to rely on concepts such as "collection", "thing" and "operation" when laying out a pre-formal account of anything (including sets) should be granted as the bare minimum if any kind of philosophy of mathematics is to take place at all.

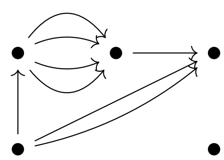
[27] I take it that this concept is intuitive enough to require no further analysis. It is a different issue whether this notion of an abstract set is compatible with the standard Zermelian conception of sets, but I will say no more about this here. I am here interested in a *comparison* of categories and groupoids modulo some fixed common conception of set, and changing this conception of sets does not affect the comparison being made, nor any of the conclusions I reach. My account could easily be modified to accommodate a more (or even less) sophisticated understanding of a pre-formal set, since what I care about here is to compare (C) and (G). Insofar as they depend on the *same* account of pre-formal sets (whatever that may be), the arguments below can be run just the same.

[28] For more on the applicability of groupoids see [12] and for a more detailed elaboration of a similar point of view of pre-formal groupoids as "forms" see [47].

5.2. **Pre-formal Categories.** The immediate temptation is to take categories to be shapes just like above but also with a notion of direction in their edges. But the situation is more complicated. (Moreover, to proceed like this would arguably decide the question of fundamentality: for certainly then groupoids would come out more fundamental than categories in the same way that manifolds can be seen as more fundamental than oriented manifolds.)

In the case of groupoids there is a clear *spatial* aspect that motivates their definition, which is also why it is reasonable to call them "shapes". In particular, it is faithful to the way groupoids are being used to regard the dots as points and the lines as (undirected) paths. On the other hand, the motivation for the definition of categories has to do with the *organization* of algebraic and topological structures into structured collections.[29]

Therefore, categories are best thought of, pre-formally, as *collections* of *structures* and not as directed graphs. A particular feature of a "structure" is that it is exemplified by many (concrete) instantiations. To recognize that the same structure is being instantiated by two distinct objects is to recognize that these two objects are related in at least one way; namely, that they can be seen to bear a common structure. Every such way of relating these two objects we can represent with an arrow. When structures are represented by points and ways of relating them by arrows then we get the usual picture of a category as something that looks like this:



But the above picture should not, I urge, be viewed as a *graph* in the usual mathematical sense of a set of vertices and a set of edges etc. *Formally*, of course, one can do so. But the pre-formal understanding of categories I have just sketched goes beyond the pre-formal understanding of a "directed graph". Rather, a pre-formal category in my sense is a "directed graph" in which nodes are to be understood as structures and arrows are to be understood as (witnesses of) structure-similarity between these structures. A pre-formal category is therefore a collection of structures together with a network of structural similarities between them.

This discussion reveals the oft-neglected fact that categories lead a double life. They can be understood, firstly, *as* mathematical structures or, secondly, as collections *of* mathematical structures. More precisely, categories can be regarded as algebraic/combinatorial objects in their own right (transitive reflexive directed graphs with extra properties) or as formalizations of collections of such objects (e.g. **Set**, **Grp**). The first aspect has proved extremely useful in algebraic topology, e.g. to encode the essentially combinatorial nature of certain topological constructions like CW-complexes. The second aspect is historically also related to algebraic topology but has found applicability in many other fields and is used primarily for *organizational* reasons. We may call the former the *spatial* aspect of categories and the latter the *organizational* aspect.[30] In those terms, the notion of pre-formal categories outlined above corresponds to the *organizational* aspect.

[29]This was done in order to formalize certain constructions that were becoming ubiquitous in algebraic topology at the time. In particular, to formalize the idea that algebraic information could be attached to topological spaces in a *natural* or *compatible* manner. As Mac Lane is said to have remarked, category theory was invented not in order to study functors, but in order to study natural transformations.

[30]To clarify, my "spatial categories" are not to be confused with the study of "categories of (generalized) spaces" in the vein of e.g. [30].

This dual aspect also clearly makes sense in the case of groupoids: there are *spatial* and *organizational* notions of a groupoid, understood in an exactly analogous manner.[31] The distinction I have drawn between pre-formal groupoids and pre-formal categories can thus be summarized as follows: my pre-formal groupoids are spatial whereas my pre-formal categories are organizational. This difference will be essential in the argument presented below.

5.3. **Bishop or Frege?** We can now revisit (Q_1): does (FI)/(BI) imply that next-level sets are categories/groupoids? Let me start with (FI). A next-level set, according to (FI), is the *kind of thing* exemplified by the collection of all sets. This collection of all sets consists of objects with a certain "purely extensional" structure, but they are certainly structures of some sort. And the way we have of relating these purely extensional structures is by pairing up their elements, viz. by defining functions between them. And this, indeed, gives rise exactly to a pre-formal category as outlined above.

On the other hand, according to (BI), a next-level set is the *kind of thing* between each of whose constituents there is (at most) a *set* of identifications. This means that a next-level set is a collection of nodes together with, for any pair of nodes, a (possibly empty) set of "identifications". If we take these "identifications" to be structureless "edges" we get exactly the picture of the pre-formal groupoid I outlined above. Namely, between any two nodes there will be a certain number of lines connecting them. As such, both (FI) and (BI) render support to (C) and (G) when the terms "category" and "groupoid" are understood in the pre-formal way I have indicated: according to (FI) next-level sets are (organizational) categories whereas according to (BI) next-level sets are (spatial) groupoids.

We are now ready to shift our attention to (Q_2): what can we say about (FI) and (BI) as self-standing principles? To my mind, (FI) is susceptible to one main objection: the collection of things of type X usually bears many natural (non-equivalent) structures. For example, as argued above, the collection of all sets does not *just* bear the structure of a category with arrows given by the functions between sets. Why should the structure of **Set** somehow uniquely *emerge* when we collect all sets together? And even if it did, why should we choose functions as the arrows? There are alternatives after all: for instance we have the category **Rel** with objects the same as **Set** but with arrows all the (not necessarily functional) relations between sets. And **Rel** is perhaps then not best understood as bearing the structure of a category, but rather of a $(2,1)$-category (a 2-category in which all 2-cells are invertible). Indeed **Rel** can also be seen as a "(2)-category with duals" which in [6] was a notion that was considered as a "third alternative" between higher groupoids and higher categories.[32] As such, it seems unlikely that (FI) alone will be able to pin down the category structure of **Set** as uniquely emerging from the process of gathering all sets together. On the whole, (FI) seems to be too "loose" a guiding principle.

As for (BI), the main difficulty is how to understand the passage from the "empty" collection to the "singleton" collection. Recall that, according to (BI), a lower-level set is a

[31]As an anonymous referree has pointed out, the organizational aspect of groupoids was already present in Brandt's original definition, since Brandt's groupoids were meant to classify quadratic forms. I am certainly not denying that organizational notions of groupoids make sense. But I will here be focusing only the contrast between organizational categories and spatial groupoids. And in any case my argument against organizational categories in Section 6 applies equally well to organizational groupoids.

[32]I thank an anonymous referee for pointing this out to me.

truth value. So what then is a lower-level truth value? Let us once again reverse-engineer:

($*$) A lower-level truth value is the type of thing Y such that there can be at most Y-many identifications between elements of a truth value.

The phrase "elements of a truth value" raises the obvious question: how can the empty (or "false") truth value *have* an element? The answer is that it cannot. The statement of ($*$) only makes sense for the inhabited (or "true") truth value. And there the answer is clear: there can only be a single unique identification between the single element of the inhabited truth-value. As such, a lower-level truth value is a singleton. Thus, according to (BI), the notion of a singleton is more fundamental (in the sense of being of a lower level) than the empty collection. Put more provocatively, but less accurately: non-existence is a higher-level form of existence.[33]

At this point, in saying such things, we may begin to wonder whether we have bumped our head against the limits of our language (to borrow a phrase of Wittgenstein's). After all, we are translating into natural language constructions that only properly make sense in formal systems and we cannot expect the sense of these constructions to survive that transition. But neither should we surrender to uninterpreted formalism. So what I'll say is that such considerations as the above might prove decisive for someone holding a particular metaphysical view (such a person will also, of course, need to have a pre-formal account of all the terms being used, i.e. "singleton", "collection", "truth value" etc.). For example, a so-called compositional nihilist might not want to countenance (FI) purely because it considers **Set** as a thing over and above its individual constituents (i.e. sets). Someone else might think that it is absurd to give ontological priority to the singleton set over the empty set as (BI) seems to force us to do. But absent some such metaphysical thesis in the background, I don't see any philosophically decisive reason to pick either (BI) or (FI).

We have thus reached an impasse. Part of the reason for this must surely come down to the fact that it is very likely that there is no philosophically decisive reason to pick one notion of next-level set over another when the question is posed at this level of abstraction, without any further constraints. Indeed, I certainly am of the opinion that there is no unique objectively correct such notion waiting to be discovered in some Platonic realm. However, as I alluded to in section 2, there may be a decisive reason to pick one notion over the other if we ask the question in a specific enough context, as I will now go on to do.

6. Groupoids as next-level sets in Foundations

In order to answer (Q_2) in any interesting way, we must embed it in a concrete mathematical context. This will provide a way out of the impasse reached in the previous subsection. The most relevant context here is the foundations of mathematics. And since (FI) and (BI) are really best understood as mathematical justifications for (C) and (G) respectively the better question to ask (given what has been said so far) is this:

(Q_3) Should we pick spatial groupoids or organizational categories (and the hierarchies they generate) as the basic objects of our (structuralist) foundations?

[33] If this discussion is straining the reader's credulity allow me to clarify that what I am doing here is informally describing the hierarchy of n-types in as formalized in (a given) homotopy type theory. What is here expressed in prose form has been fully formalized, cf. [62], Chapter 7.

And here, finally, I believe a clear answer emerges: we should prefer spatial groupoids over organizational categories.

The decisive objection against organizational categories as the building block in a hierarchy of basic objects for a foundation of mathematics is that, as the name suggests, the very notion of an organizational category pre-supposes that it *organizes* something. So if an organizational category is to be regarded as a basic structure of a foundation, then the basic structures of that foundation will themselves be structures organizing other structures. This is like saying that the foundations of a house are not the bricks that go into its construction but the cardboard boxes in which these bricks are stacked and transported. If categories are to be fruitfully regarded as the basic structures of a foundational system then it cannot be merely by virtue of the fact that they provide a way to organize other, previously constructed, structures. The basic objects of any foundations should simply *be* the basic structures that one can then organize into whatever shape or form one wishes.

Thus, in a hierarchy of basic objects, the lower-level objects should be understandable as degenerate versions of the higher-level ones and not as entirely distinct kinds of things (as distinct as, say, bricks are from the cardboard boxes that store them). Does it really make sense to say that sets are degenerate (discrete) categories *because* the collection of all sets forms a category? As far as I am concerned it does not: bricks are not degenerate versions of cardboard boxes. On the other hand, it seems to me far more intuitive to say that a set is a degenerate (spatial) groupoid. For in this case we are talking about a degenerate shape and there has hardly ever been a problem understanding parts of shapes as degenerate versions of more complicated ones, e.g. lines as degenerate triangles. And, of course, exactly the same argument can be made against organizational groupoids.

Thus, if categories are to be regarded as the basic building blocks of a (hierarchy of) basic objects, it must be in their spatial aspect, i.e. by being regarded as (transitive, reflexive) directed graphs with an associative composition operation. But there are now two serious problems. Firstly, (FI) no longer applies: if we were to apply (FI) to sets we would not end up with spatial categories as next-level sets. For why would a transitive, reflexive, directed graph be the type of thing we get when we gather all sets together? As such, we are no longer in the domain of TTCFM, but in the domain of some unknown foundation based on a hierarchy of spatial (higher) categories.

Secondly, if we are indeed to view categories as directed graphs then these clearly seem to be less fundamental than spatial groupoids since the latter can roughly be understood as undirected graphs. So if we are to compare spatial categories and spatial groupoids with respect to their suitability to be the next-level sets in a foundational hierarchy then it would seem much more natural to regard categories as groupoids with extra structure (namely a *non-invertible* direction). But then building a foundation whose basic building blocks are spatial categories and then defining spatial groupoids by adding more properties, seems analogous to axiomatizing the universe of partially ordered sets (i.e. defining something like a "ZF *poset* theory") in order to then define sets within it as those posets with a trivial ordering.

This is where Voevodsky's aforementioned "argument" helps clarify the situation: in traditional set-theoretic foundations it would seem absurd to try and axiomatize the "cumulative hierarchy of partially-ordered sets". In the realm of structuralist foundations, it should strike us as equally absurd to try and axiomatize the "hierarchy of higher categories". We should, instead, try to axiomatize the hierarchy of higher groupoids, i.e. the hierarchy of *homotopy types*. And then higher categories should be studied as structures defined *on* those higher groupoids, just like partially ordered sets are studied as structures on sets.

However, there is an important objection to consider here.[34] It seems as though I have stacked the cards in my favour by taking my pre-formal notion of a spatial category to be a something like a directed graph, which is to be contrasted with the (less structured) undirected graphs that form groupoids. But why should directionality be seen as less fundamental? What is it that binds us to this way of looking at things? After all, we can very well imagine cashing out pre-formal groupoids in terms of an invertible direction, i.e. along the lines of the usual definition of groupoids as categories where all arrows are isomorphisms. In other words, why couldn't there be some *independent* spatial motivation for categories? Indeed, such a perspective is outlined by Grandis in [25] and is also fairly intuitive pre-mathematically, e.g. by thinking in terms of sloping paths where it takes more effort to go up than come down.

From a purely philosophical point of view this seems to me a bullet that I have to bite, since I do concede that there could be alternative pre-formal accounts of groupoids and categories in which it is not so clear that spatial categories are less fundamental than spatial groupoids. Thus, if someone fundamentally disagrees with the way in which I have cashed out my pre-formal notions of groupoid and category, there is little I can do other than to say that my argument above is to be taken as conditional upon the *particular* way I have presented of cashing out groupoids and categories pre-formally. For I do not myself see a way of philosophically settling whether "directionality" or "reversibility" is a more fundamental notion.

But from a *pragmatic* point of view, there is more to be said concerning the choice between spatial groupoids and spatial categories. There are strong pragmatic considerations that corroborate with the choice of the former. In particular, intensional Martin-Löf Type Theory (MLTT) provides a remarkably simple description of the structure of homotopy types via the four rules for identity types. No such description is currently available for *directed* homotopy types, i.e. types that natively bear the structure of ∞-categories rather than ∞-groupoids. So insofar as the debate between spatial (higher) groupoids vs. spatial (higher) categories is made precise as the debate between homotopy types vs. directed homotopy types, then the availability of well-studied formal systems which formalize the former (e.g. intensional MLTT) but not the latter should certainly be seen as an important pragmatic reason to prefer the former in the context of the foundations of mathematics, even without the kind of philosophical justification that I have endeavored to provide. Of course, the deeper reason for this asymmetry has yet to be discovered. As such, the existence of a formal system that axiomatizes directed homotopy types just like HoTT formalizes homotopy types is certainly within the realm of possibility – and such a formal system would render my point moot. However, no such formal system currently exists; and until such time as one appears, this pragmatic point will remain a strong consideration in support of the conclusion that groupoids are correct notion of a higher-level set in the context of structuralist foundations of mathematics.[35]

7. Conclusion

I have argued on philosophical grounds that the correct notion of a higher level set in the context of the foundations of mathematics is a groupoid, not a category. This would be an entirely uninteresting conclusion if there weren't any foundations of mathematics that took the question of next-level sets as a serious (indeed axiomatic) design constraint. As outlined in Section 2, there have recently been two such foundational proposals: UF

[34] I owe it to an anonymous referee.

[35] For a relevant discussion on directed homotopy theory cf. [56].

and TTCFM. Everything I've said in this paper now provides strong reasons to pick the former over the latter.

But I also believe that the significance of the idea of a higher-level set being a groupoid is not limited to the domain of structuralist foundations of mathematics. It is relevant, I submit, to far broader philosophical concerns because it provides philosophy with an entirely new notion of a *higher-level* collection of things to consider. Among other things, this allows us to re-visit one of the core ideas of logic since antiquity: that to define a *property* (or *concept*) is to define a *feature* common to a multitude of objects.[36] A higher-level property is then a feature common to a multitude of properties and so on. But what if a higher-level property were simply one in which we *tracked* the ways in which collections of objects share a feature, rather than merely assert that they do? Then we get, essentially, a notion of a higher-level property being a *shape* of some sort. Groupoids (possibly higher ones) in UF provide a formalization of this idea. It is imperative that the philosophical implications of this formal development be explored.

References

[1] S. Awodey, *An answer to Hellman's question: 'Does category theory provide a framework for mathematical structuralism?'* Philosophia Mathematica **12 (1)** (2004), 54–64.

[2] ______, *Category theory*, OUP Oxford, 2010.

[3] ______, *Structuralism, invariance and univalence*, Philosophia Mathematica **22 (1)** (2014), 1–11.

[4] S. Awodey and M. A. Warren, *Homotopy theoretic models of identity types*, Math. Proc. Camb. Phil. Soc. **146** (2009), no. 45, 45–55.

[5] J. Baez, *The homotopy hypothesis*: *Slides of a talk at the Fields Institute*, 2007. http://math.ucr.edu/home/baez/homotopy/homotopy.pdf (accessed November 2016).

[6] J. Baez and J. Dolan, *Categorification*, arXiv.math:9802029 [math.QA] (1998).

[7] J. Baez and M. Shulman, *Lectures on n-categories and cohomology*, Towards Higher categories, The IMA Volumes in mathematics 152, Baez, J. and May, P. eds., 2010, pp. 1–68.

[8] T. W. Barrett, *On the structure of classical mechanics*, The British Journal for the Philosophy of Science **66** (2015), no. 4, 801–828.

[9] P. Benacerraf, *What numbers could not be*, The Philosophical Review **74 (1)** (Jan. 1965), 47–73.

[10] E. Bishop, *Foundations of constructive analysis*, McGraw-Hill, Oxford, 1967.

[11] H. Brandt, *Über eine verallgemeinerung des gruppenbegriffes*, Mathematische Annalen **96** (1927), no. 1, 360–366.

[12] R. Brown, *From groups to groupoids: A brief survey*, Bull. London Math. Soc. **19** (1987), 113–134.

[13] J. Burgess, *Putting structuralism in its place*, 2013. Manuscript.

[14] ______, *Rigor and structure*, Oxford University Press, Oxford, 2014.

[15] O. A. Camarena, *A whirlwind tour of the world of (∞,1)-categories*, arXiv:1303.4669 [math.CT] (2013).

[16] E. Cheng and A. Lauda, *Higher-dimensional categories: an illustrated guide book*, Preprint (2004). http://www.cheng.staff.shef.ac.uk/guidebook/guidebook-new.pdf (accessed Novermber 2016).

[17] D. Corfield, *Towards a philosophy of real mathematics*, Cambridge University Press, Cambridge, 2013.

[18] ______, *The covariance of coloured balls*: *n-category cafe blog post*, 2015. https://golem.ph.utexas.edu/category/2013/11/the_covariance_of_coloured_bal.html (accessed November 2016).

[19] ______, *Expressing 'the structure of' in homotopy type theory*, 2016. https://dl.dropboxusercontent.com/u/16936016/The_Structure.pdf (accessed November 2016).

[20] L. Crane, *Clock and category: Is quantum gravity algebraic?* Journal of Mathematical Physics **36** (1995), no. 11, 6180–6193.

[21] L. Crane and I. B. Frenkel, *Four-dimensional topological quantum field theory, hopf categories, and the canonical bases*, Journal of Mathematical Physics **35** (1994), no. 10, 5136–5154.

[36] Whether one wants to be a realist, anti-realist or whatever else about the nature of these objects does not change the particular definition of "concept", only what we mean by "object".

[22] S. Feferman, *Categorical foundations of category theory*, Logic, Foundations of Mathematics and Computability Theory (Proc. Fifth International Congress in Logic, Methodology and Philosophy of Science, Univ. Western Ontario, London, Ont., 1975, Part I. Univ. Western Ontario Ser. Philos. Sci. **9** (1977), 149–169–348.

[23] ______, *Foundations of unlimited category theory: What remains to be done*, The Review of Symbolic Logic **6** (2013), no. 1, 6–16.

[24] D. Gaitsgory and J. Lurie, *Weil's conjecture for function fields*, Preprint (2014). http://www.math.harvard.edu/~lurie/papers/tamagawa.pdf (accessed November 2016).

[25] M. Grandis, *Directed Algebraic Topology: Models of non-reversible worlds*, Vol. 13, Cambridge University Press, 2009.

[26] A. Grothendieck, *Esquisse d'un programme*, London Mathematical Society Lecture Note Series (1997), 5–48. http://www.landsburg.com/grothendieck/EsquisseEng.pdf (accessed November 2016).

[27] H. Halvorson, *What scientific theories could not be*, Philosophy of Science **79** (2012), no. 2, 183–206.

[28] H. Halvorson and D. Tsementzis, *Categories of scientific theories*, available at http://philsci-archive.pitt.edu/11923/2/Cats.Sci.Theo.pdf. Forthcoming in "Categories for the Working Philosopher" (Elaine Landry ed.)

[29] G. Hellman, *Does category theory provide a framework for mathematical structuralism?* Philosophia Mathematica **11** (2003), no. 2, 129–157.

[30] P. T. Johnstone, *On a topological topos*, Proc. London Math. Soc. (3) **38** (1979), no. 2, 237–271. MR531162 (80i:18010)

[31] K. Kapulkin, P. Lumsdaine, and V. Voevodsky, *The simplicial model of univalent foundations*, arXiv:1211.2851 [math.LO] (2014).

[32] E. Landry, *How to be a structuralist all the way down*, Synthese **179** (2011), no. 3, 435–454.

[33] ______, *The genetic versus the axiomatic method: Responding to feferman 1977*, The Review of Symbolic Logic **6** (2013), no. 1, 24–51.

[34] E. Landry (ed.) *Categories for the working philosopher*, OUP, 2016.

[35] F. W. Lawvere, *An elementary theory of the category of sets*, Reprints in Theory and Applications of Categories **12** (2005), 1–35.

[36] T. Leinster, *Higher operads, higher categories*, Vol. 298, Cambridge University Press, 2004.

[37] J. Lurie, *Derived algebraic geometry*, Ph.D. Thesis, 2004.

[38] ______, *On the classification of topological field theories*, Current developments in mathematics **2008** (2009), 129–280.

[39] S. Mac Lane, *Categories for the working mathematician*, Springer, 1998.

[40] M. Makkai, *First order logic with dependent sorts with applications to category theory*, 1995. http://www.math.mcgill.ca/makkai/folds/foldsinpdf/FOLDS.pdf (accessed November 2016).

[41] ______, *The multitopic ω-category of all multitopic ω-categories* (2004). http://www.math.mcgill.ca/makkai/mltomcat04/mltomcat04.pdf (accessed November 2016).

[42] ______, *The theory of abstract sets based on first-order logic with dependent types* (2013). http://www.math.mcgill.ca/makkai/Various/MateFest2013.pdf (accessed November 2016).

[43] M. Makkai, C. Hermida, and J. Power, *On weak higher dimensional categories i-1*, Journal of Pure and Applied Algebra **154** (2000), 221–246.

[44] ______, *On weak higher dimensional categories i-2*, Journal of Pure and Applied Algebra **157** (2001), 247–277.

[45] ______, *On weak higher dimensional categories i-3*, Journal of Pure and Applied Algebra **166** (2002), 83–104.

[46] J-P. Marquis, *From a geometrical point of view: A study of the history and philosophy of category theory*, Vol. 14, Springer Science & Business Media, 2008.

[47] ______, *Mathematical forms and forms of mathematics: leaving the shores of extensional mathematics*, Synthese **190** (2013), no. 12, 2141–2164.

[48] ______, *Categorical foundations of mathematics: Or how to provide foundations for* abstract *mathematics*, The Review of Symbolic Logic **6** (March 2013), no. 1.

[49] P. Martin-Löf, *Intuitionistic type theory*, Bibliopolis, 1984.

[50] C. McLarty, *Numbers can be just what they have to*, Nous **27** (1993), no. 4, 487–498.

[51] ______, *Exploring categorical structuralism*, Philos. Math. (3) **12** (2004), no. 1, 37–53. MR2033719 (2005b:03018a)

[52] ______, *Foundations as truths which organize mathematics*, The Review of Symbolic Logic **6** (2013), no. 1, 76–87.

[53] E. Palmgren, *Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets*, Annals of Pure and Applied Logic **163** (2012), no. 10, 1384–1399.

[54] Á. Pelayo and M. Warren, *Homotopy type theory and Voevodsky's univalent foundations*, Bulletin of the American Mathematical Society **51** (2014), no. 4, 597–648.

[55] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, arXiv:1310.7930 [math-ph] (2013).

[56] M. Shulman, *Directed homotopy type theory*: *n-category cafe blog post*, 2012.

[57] ———, *Homotopy type theory: A synthetic approach to higher equalities*, Categories for the working philosopher, forthcoming. http://home.sandiego.edu/~shulman/papers/synhott.pdf (accessed November 2016).

[58] R. Street, *Higher categories, strings, cubes and simplex equations*, Applied Categorical Structures **3** (1995), no. 1, 29–77.

[59] N. J. Teh and D. Tsementzis, *Theoretical equivalence in classical mechanics and its relationship to duality*, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics (2016).

[60] D. Tsementzis, *Univalent foundations as structuralist foundations*, available at http://philsci-archive.pitt.edu/12070/. Forthcoming in Synthese.

[61] ———, *Homotopy model theory i: Syntax and semantics*, arXiv:1603.03092 [math.LO] (2016).

[62] Univalent Foundations Program, *Homotopy type theory: Univalent foundations of mathematics* (2013). http://homotopytypetheory.org/book (accessed November 2016).

[63] V. Voevodsky, *Foundations of mathematics and homotopy theory* (2006). https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/VV%20Slides.pdf (accessed November 2016).

[64] ———, *The equivalence axiom and univalent models of type theory*, Talk at CMU (2010).

[65] ———, *Univalent foundations project* (2010). http://www.math.ias.edu/vladimir/files/univalent_foundations_project.pdf (accessed November 2016).

[66] ———, *Univalent Foundations (lecture at the IAS)* (2014). https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2014_IAS.pdf (accessed November 2016).

[67] M. A. Warren, *Homotopy theoretic aspects of constructive type theory*, Ph.D. Thesis, 2008.

[68] K. Wehrheim and C. Woodward, *Functoriality for lagrangian correspondences in floer theory*, arXiv preprint arXiv:0708.2851 (2007).

[69] A. Weinstein, *Symplectic geometry*, Bulletin of the American mathematical society **5** (1981), no. 1, 1–13.

[70] ———, *The symplectic "category"*, Differential geometric methods in mathematical physics, 1982, pp. 45–51.

[71] E. Witten, *Khovanov homology and gauge theory*, Geometry & Topology Monographs **18** (2012), 291–308.