On the Persistence of the Electromagnetic Field

Márton Gömöri* and László E. Szabó**

*Institute of Philosophy, Research Center for the Humanities, Hungarian Academy of Sciences, Budapest

**Department of Logic, Institute of Philosophy, Eötvös University, Budapest

Abstract

According to the standard realistic interpretation of classical electrodynamics, the electromagnetic field is conceived as a real physical entity existing in space and time. The problem we address in this paper is how to understand this spatiotemporal existence, that is, how to describe the persistence of a field-like physical entity like electromagnetic field.

First, we provide a formal description of the notion of persistence: we derive an "equation of persistence" constituting a necessary condition that the spatiotemporal distributions of the fundamental attributes of a persisting physical entity must satisfy. We then prove a theorem according to which the vast majority of the solutions of Maxwell's equations, describing possible spatiotemporal distributions of the fundamental attributes of the electromagnetic field, violate the equation of persistence. Finally, we discuss the consequences of this result for the ontology of the electromagnetic field.

1 Introduction

There is a long debate in contemporary metaphysics whether and in what sense instantaneous velocity can be regarded as an intrinsic property of an object at a given moment of time (Butterfield 2006; Hawley 2001, pp. 76–80; Sider 2001, pp. 34–35; Arntzenius 2000; Tooley 1988). What is important from this debate to our present concern is—in which there seems to be a consensus—that

> [T]he notion of velocity presupposes the persistence of the object concerned. For average velocity is a quotient, whose numerator must be the distance traversed by the given persisting object [...] So presumably, average velocity's limit, instantaneous velocity, also presupposes persistence. (Butterfield 2005, p. 257).

We will argue in this paper that the opposite is also true: persistence presupposes velocity. More precisely, in case of an extended physical object, persistence presupposes, at least, the existence of a field of local and instantaneous velocity; regardless if this local instantaneous velocity is considered as an intrinsic property of the object concerned, or not. In fact, as we will see, velocity occurs as a feature of the way in which the object persists.

In section 2 we give a formal description of persistence in terms of the spatiotemporal distributions of individuating quantities establishing the identity for the persisting object. We derive an "equation of persistence" constituting a necessary condition the distributions of individuating quantities must satisfy in every space-time point where the extended object persists. It turns out however that this condition is not necessarily satisfied by some field-like physical entities. In section 3 we discuss the case of electromagnetic field. We show that the equation of persistence is satisfied in some particular states of the electromagnetic field. This is however the exception: we will prove a theorem according to which the vast majority of the solutions of Maxwell's equations violate the equation of persistence. In section 4 we discuss the possible consequences concerning the ontology of electromagnetic field.

2 Formal description of persistence

It is common to all theories of persistence—endurantism vs. perdurantism—that a persisting entity needs to have some package of individuating properties, in terms of which one can express that two things in two different spatiotemporal regions are identical, or at least constitute different temporal parts of the same entity. Butterfield writes:

> I believe that [the criteria of identity] are largely independent of the endurantism–perdurantism debate; and in particular, that endurantism and perdurantism [...] face some common questions about criteria of identity, and can often give the same, or similar, answers to them. [...] [A]ll parties need to provide criteria of identity for objects, presumably invoking the usual notions of qualitative similarity and-or causation (Butterfield 2005, 248–289)

Without loss of generality we may assume that each of these individuating properties can be characterized as such that a certain (real valued) quantity f_i takes a certain value; more precisely, the spatiotemporal distribution of this quantity, $f_i(\mathbf{r}, t)$, takes a certain local value at a spatiotemporal locus. Accordingly, we express persistence in terms of these distributions of individuating quantities. We proceed in three heuristic steps.

I.

First we consider the persistence of a point-like physical object. The fact that a point-like object (or its temporal part), occupying a small place at point $\mathbf{r}$ in space at the moment of time t, instantiates a certain property in that moment can be expressed by the fact that the corresponding quantity $f_i(\mathbf{r}, t)$ takes a certain local value. For example: the ball in Fig. 1 can be described by the spatial distributions of two quantities, $spottedness(\mathbf{r}, t)$ and $rubberness(\mathbf{r}, t)$—taking value, say, 1 where spottedness/rubberness is instantiated and 0 otherwise.

To express the fact of persistence we will use the distributions of a given package of individuating quantities $\{f_i(\mathbf{r}, t)\}_{i=1}^{n}$. Different theories may dis-

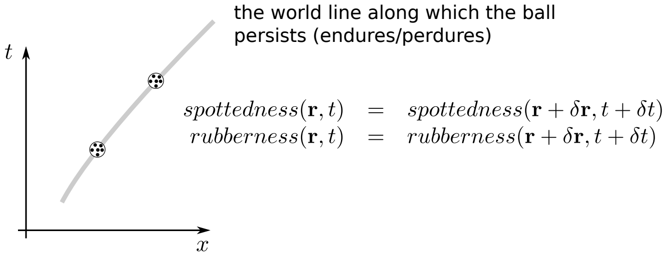

Figure 1: *A small, "point-like" ball can be individuated by its spottednes, rubberness, etc.*

agree in the actual content of this package, of course. We only assume that these quantities together are capable to express the fact that two things in two different space-time points are identical (endurance), or, constitute different temporal parts of the same entity (perdurance); in other words, they trace out the world-line along which the point-like object persists. Expressing identity in terms of equality of the individuating quantities in the different spatiotemporal regions, we have

$$f_i(\mathbf{r}, t) = f_i(\mathbf{r}', t') \quad (i = 1, 2, \ldots, n) \tag{1}$$

for any two points $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$ along the world-line (Fig. 1). Introducing the *average velocity* as $\mathbf{v} = \frac{\mathbf{r}' - \mathbf{r}}{t' - t}$, we can write:

$$f_i(\mathbf{r}, t) = f_i(\mathbf{r} + \mathbf{v}\delta t, t + \delta t) \quad (i = 1, 2, \ldots, n) \tag{2}$$

with $\delta t = t' - t$.

Assume that all functions in $\{f_i(\mathbf{r}, t)\}_{i=1}^n$ are smooth (if not, they can be approximated as closely as required for physics by smooth functions). Taking (2) for a small, infinitesimal interval of time, and expressing it in a differential form, we have

$$-\partial_t f_i(\mathbf{r}, t) = \nabla f_i(\mathbf{r}, t) \cdot \mathbf{v}(t) \quad (i = 1, 2, \ldots, n) \tag{3}$$

where $\mathbf{v}(t)$ is the instantaneous velocity. In components:

$$-\partial_t f_i(\mathbf{r}, t) = V_x \partial_x f_i(\mathbf{r}, t) + V_y \partial_y f_i(\mathbf{r}, t) + V_z \partial_z f_i(\mathbf{r}, t) \quad (i = 1, 2, \ldots, n) \tag{4}$$

Of course, the concrete world-line along which the object persists may be varied. Thus, equations (3) with some instantaneous velocity constitute a *necessary* condition the individuating quantities must satisfy in every space-time point where the object persists. Let us call them the *equations of point-like persistence*.

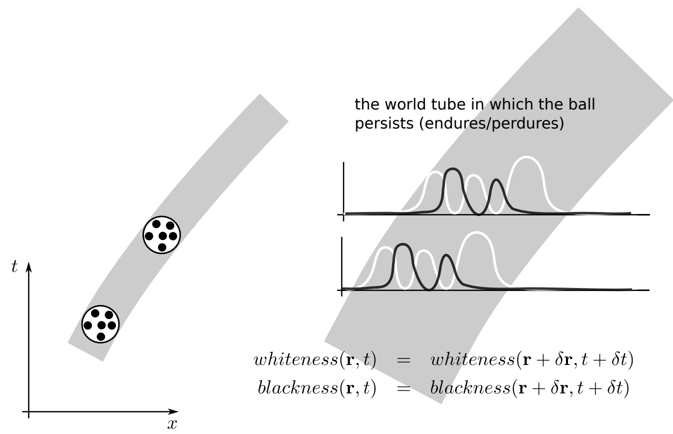

Figure 2: *A spotted ball, as an extended object, can be characterized by the distributions of whiteness and blackness*

II.

Now we make a straightforward extension of the above results to the case of an extended object. Assume that the fine-grained structure of an extended object also can be described in terms of the distributions of some, probably more fundamental, quantities (Fig. 2). And, therefore, the identity of the persisting object can be expressed in terms of an individuating package of these distributions, $\{f_i(\mathbf{r}, t)\}_{i=1}^{n}$. It is a straightforward generalization of the idea expressed in equation (3) to say that an extended object persists if there is a velocity vector $\mathbf{v}(t)$ for every moment of time, such that equation (3) is satisfied in all space-time points $(\mathbf{r}, t)$ belonging to the space-time tube swept by the extended object.

However, this describes only a particular situation when the extended object persists like a rigid body in translational motion. The instantaneous velocity $\mathbf{v}(t)$ is the same everywhere in the spatial region occupied by the object. Consequently, the spatial distributions $f_i(\mathbf{r}, t = const)$ are simply translating with a universal velocity, without deformation. Of course, generally this is not necessarily the case. For example, the ball in Fig. 3 preserves its identity even though it rotates and inflates.

III.

Concerning the general case, imagine an extended object with a more complex behavior. Let Σ_t and $\Sigma_{t+\delta t}$ denote the spatial regions occupied by the object at time t and $t + \delta t$. The object can change in various sense. Even if $\Sigma_t = \Sigma_{t+\delta t}$, the spatial distributions of its local properties may change, in the sense that for several distributions $f_i(\mathbf{r}, t) \not\equiv f_i(\mathbf{r}, t + \delta t)$. Moreover, Σ_t and $\Sigma_{t+\delta t}$ may differ not only in their location but also in size and shape. All these changes manifest themselves in the spatiotemporal distributions of local properties, that is, in the distributions $f_i(\mathbf{r}, t)$. For example, all changes, the translation, the rotation, and the inflation of the ball in Fig. 3 are expressible in terms of the distributions

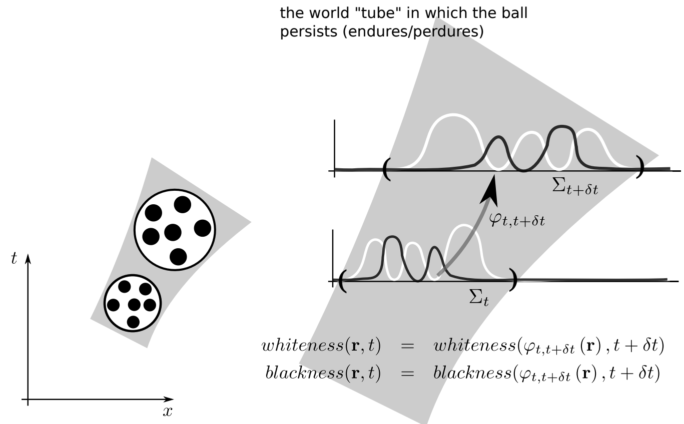

Figure 3: *The ball preserves its identity even though it may rotate or inflate*

like $whiteness(\mathbf{r},t)$ and $blackness(\mathbf{r},t)$.

Now, how can we describe the persistence of such an object? What conditions the distributions $f_i(\mathbf{r},t)$ must satisfy in order to count the two things in Σ_t and $\Sigma_{t+\delta t}$ as identical, or as two temporal parts of the same object? The conditions we are looking for have to express some similarities between the values of $f_i(\mathbf{r},t)$ in Σ_t and the values of $f_i(\mathbf{r},t+\delta t)$ in $\Sigma_{t+\delta t}$. On the basis of our previous considerations in points I and II and the examples like the inflating-rotating ball, we claim that the general form of such conditions are the following. There must exist a package of relevant, individuating distributions $\{f_i(\mathbf{r},t)\}_{i=1}^n$ and a mapping $\varphi_{t,t+\delta t} : \Sigma_t \to \Sigma_{t+\delta t}$, such that

$$f_i(\mathbf{r},t) = f_i(\varphi_{t,t+\delta t}(\mathbf{r}), t+\delta t) \qquad (i = 1,2,\ldots,n) \tag{5}$$

Notice that the only non-trivial requirement concerning $\varphi_{t,t+\delta t}$ is that it must be common for all individuating distributions $f_i(\mathbf{r},t)$. Intuitively this means that if a local part of the objects at $\mathbf{r}$ instantiates some local individuating properties then its counterpart at point $\varphi_{t,t+\delta t}(\mathbf{r})$ instantiates the same local individuating properties. But this fact by no means implies that the extended object necessarily consists of atomic entities—pointlike or non-pointlike—persisting in the sense of points **I** or **II**. Just the contrary, the general notion of persistence defined by (5) satisfies a kind of downward mereological principle: if the whole extended object persists in the sense of (5) then all (arbitrarily small) local parts of the object persist in the same sense.

Assuming that $\varphi_{t,t+\delta t}(\mathbf{r})$ is smooth and $\varphi_{t,t} = id_{\Sigma_t}$, one can express (5) in the following differential form:

$$-\partial_t f_i(\mathbf{r},t) = \nabla f_i(\mathbf{r},t) \cdot \mathbf{v}(\mathbf{r},t) \qquad (i = 1,2,\ldots,n) \tag{6}$$

where $\mathbf{v}(\mathbf{r},t) = \partial_{t'}\varphi_{t,t'}\Big|_{t'=t}(\mathbf{r})$. $\mathbf{v}(\mathbf{r},t)$ can be interpreted as the instantaneous velocity field characterizing the motion of the local part of the extended entity at the spatiotemporal locus $(\mathbf{r}, t)$.

Taking into account that the concrete mapping $\varphi_{t,t+\delta t} : \Sigma_t \to \Sigma_{t+\delta t}$ may be varied, equations (6) with some suitable instantaneous velocity field $\mathbf{v}(\mathbf{r}, t)$ constitute a *necessary* condition the distributions of individuating quantities must satisfy in every space-time point where the extended object persists. Let us call them the *equations of persistence*.

3 The Case of a General Electrodynamic System

As a concrete physical example, we will deal with an electrodynamic system, that is a coupled system of charged point particles and the electromagnetic field. The system is described by the Maxwell–Lorentz equations (for this form of the equations, see for example Gömöri and Szabó 2013):

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}, t) &= \sum_{i=1}^{n} q^i \delta\left(\mathbf{r} - \mathbf{r}^i(t)\right) && (7) \\
c^2 \nabla \times \mathbf{B}(\mathbf{r}, t) - \partial_t \mathbf{E}(\mathbf{r}, t) &= \sum_{i=1}^{n} q^i \delta\left(\mathbf{r} - \mathbf{r}^i(t)\right) \mathbf{v}^i(t) && (8) \\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0 && (9) \\
\nabla \times \mathbf{E}(\mathbf{r}, t) + \partial_t \mathbf{B}(\mathbf{r}, t) &= 0 && (10) \\
m^i \gamma\left(\mathbf{v}^i(t)\right) \mathbf{a}^i(t) &= q^i \Big\{ \mathbf{E}\left(\mathbf{r}^i(t), t\right) + \mathbf{v}^i(t) \times \mathbf{B}\left(\mathbf{r}^i(t), t\right) \\
&\quad - c^{-2} \mathbf{v}^i(t) \left(\mathbf{v}^i(t) \cdot \mathbf{E}\left(\mathbf{r}^i(t), t\right)\right) \Big\} && (11) \\
&\quad (i = 1, 2, \ldots, n)
\end{aligned}$$

where, $\gamma(\ldots) = \left(1 - \frac{(\ldots)^2}{c^2}\right)^{-\frac{1}{2}}$, q^i is the electric charge and m^i is the rest mass of the i-th particle.

Let us first give a well-known textbook example: the static and uniformly moving 'charged particle + the coupled electromagnetic field' system. First we consider the static solution when the charge q is at *rest* at point (x_0, y_0, z_0) in a given inertial frame of reference K:

$$\begin{aligned}
E_x(t,x,y,z) &= \frac{q\,(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_y(t,x,y,z) &= \frac{q\,(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_z(t,x,y,z) &= \frac{q\,(z-z_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
B_x(t,x,y,z) &= 0 \\
B_y(t,x,y,z) &= 0 \\
B_z(t,x,y,z) &= 0 \\
\varrho(x,y,z,t) &= q\delta\,(x-x_0)\,\delta\,(y-y_0)\,\delta\,(z-z_0)
\end{aligned} \tag{12}$$

where E_x, E_y, E_z and B_x, B_y, B_z are the components of the electric and magnetic field strengths respectively, and ϱ is the electric charge density.

The stationary field of a charge q moving at constant velocity $\mathbf{V} = (V,0,0)$ relative to K can be obtained (Jackson 1999, pp. 661–665) by solving the equations of electrodynamics with the time-depending source. The solution is the following:

$$\begin{aligned}
E_x(t,x,y,z) &= \frac{qX_0}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_y(t,x,y,z) &= \frac{\gamma q\,(y-y_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_z(t,x,y,z) &= \frac{\gamma q\,(z-z_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
B_x(t,x,y,z) &= 0 \\
B_y(t,x,y,z) &= -c^{-2}VE_z(t,x,y,z) \\
B_z(t,x,y,z) &= c^{-2}VE_y(t,x,y,z) \\
\varrho(x,y,z,t) &= q\delta\,(x-(x_0+Vt))\,\delta\,(y-y_0)\,\delta\,(z-z_0)
\end{aligned} \tag{13}$$

where (x_0, y_0, z_0) is the initial position of the particle at $t = 0$, $X_0 = \gamma\,(x-(x_0+Vt))$ and $\gamma = \left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}}$.

Now, it is easy to verify that both the static solution (12) and the stationary solution (13) satisfy the equations of persistence (3) with constant and homogeneous velocity field $\mathbf{V} = (0,0,0)$ and $\mathbf{V} = (V,0,0)$,[1] respectively, in the

[1]It must be pointed out that velocity $\mathbf{V}$ conceptually differs from the speed of light c. Basically, c is a constant of nature in the Maxwell–Lorentz equations, which can emerge in the solutions of the equations; and, in some cases, it can be interpreted as the velocity of propagation of changes in the electromagnetic field. For example, in our case, the stationary field of a uniformly moving point charge, in collective motion with velocity $\mathbf{V}$, can be constructed from the superposition of retarded

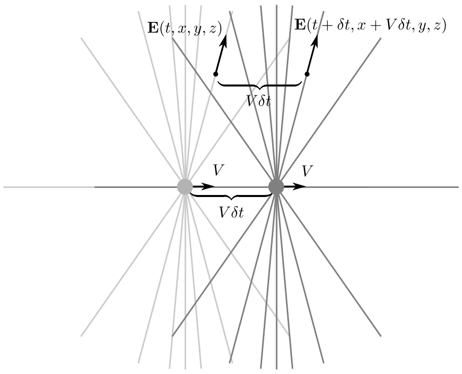

Figure 4: *The stationary field of a uniformly moving point charge is in collective motion together with the point charge*

following sense:[2]

$$
\begin{aligned}
-\partial_t \mathbf{E}(\mathbf{r},t) &= \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{V} && (14)\\
-\partial_t \mathbf{B}(\mathbf{r},t) &= \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{V} && (15)\\
-\partial_t \varrho(\mathbf{r},t) &= \nabla\varrho(\mathbf{r},t)\cdot\mathbf{V} && (16)
\end{aligned}
$$

where $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are regarded as intrinsic properties of the electromagnetic field, the components of which belong to the package of individuating properties. Or, in the more expressive form of (1),

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r}-\mathbf{V}\delta t, t-\delta t) && (17)\\
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}(\mathbf{r}-\mathbf{V}\delta t, t-\delta t) && (18)\\
\varrho(\mathbf{r},t) &= \varrho(\mathbf{r}-\mathbf{V}\delta t, t-\delta t) && (19)
\end{aligned}
$$

This picture is in complete accordance with the standard realistic interpretation of electromagnetic field:

---

potentials, in which the retardation is calculated with velocity c; nevertheless, the two velocities are different concepts. To illustrate the difference, consider the fields of a charge at rest (12), and in motion (13). The speed of light c plays the same role in both cases. Both fields can be constructed from the superposition of retarded potentials in which the retardation is calculated with velocity c. Also, in both cases, a small local perturbation in the field configuration would propagate with velocity c. But still, there is a consensus to say that the system described by (12) is at rest while the one described by (13) is moving with velocity $\mathbf{V}$ (together with K', relative to K.) A good analogy would be a Lorentz contracted moving rod: $\mathbf{V}$ is the velocity of the rod, which differs from the speed of sound in the rod.

[2] In $\mathsf{D}\mathbf{E}(\mathbf{r},t)$ and $\mathsf{D}\mathbf{B}(\mathbf{r},t)$, D denotes the spatial derivative operator (Jacobian for variables x, y and z). That is, in components we have:

$$
\begin{aligned}
-\partial_t E_x(\mathbf{r},t) &= V_x\partial_x E_x(\mathbf{r},t) + V_y\partial_y E_x(\mathbf{r},t) + V_z\partial_z E_x(\mathbf{r},t)\\
-\partial_t E_y(\mathbf{r},t) &= V_x\partial_x E_y(\mathbf{r},t) + V_y\partial_y E_y(\mathbf{r},t) + V_z\partial_z E_y(\mathbf{r},t)\\
&\vdots\\
-\partial_t B_z(\mathbf{r},t) &= V_x\partial_x B_z(\mathbf{r},t) + V_y\partial_y B_z(\mathbf{r},t) + V_z\partial_z B_z(\mathbf{r},t)\\
-\partial_t \varrho(\mathbf{r},t) &= V_x\partial_x \varrho(\mathbf{r},t) + V_y\partial_y \varrho(\mathbf{r},t) + V_z\partial_z \varrho(\mathbf{r},t)
\end{aligned}
$$

> In the standard interpretation of the formalism, the field strengths **B** and **E** are interpreted realistically: The interaction between charged particles are mediated by the electromagnetic field, which is ontologically on a par with charged particles and the state of which is given by the values of the field strengths. (Frisch 2005, p. 28)

Thus, in this particular example the necessary conditions of the persistence of the 'particle + electromagnetic field' system are clearly satisfied.

But, this example obviously represents a special electrodynamic configuration. Indeed, equations (14)–(15) imply that

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r}-\mathbf{V}t) \tag{20}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}-\mathbf{V}t) \tag{21}$$

with some time-independent $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$. In other words, the field must be a stationary one, that is, a translation of a static field with velocity $\mathbf{V}$. In fact, this corresponds to the very special "rigid" way of persistence we described in point **II** of the previous section. But, (20)–(21) is certainly not the case for a general solution of the equations of classical electrodynamics. The behavior of the field can be much more complex. Whatever this complex behavior is, one might hope that it satisfies the general form of persistence described in point **III**; that is, the equations of persistence are satisfied with a more general local and instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$:

$$-\partial_t\mathbf{E}(\mathbf{r},t) = \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) \tag{22}$$

$$-\partial_t\mathbf{B}(\mathbf{r},t) = \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) \tag{23}$$

$$-\partial_t\varrho(\mathbf{r},t) = \nabla\varrho(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r},t) \tag{24}$$

In other words, if, as it is usually believed, the electromagnetic field is a real persisting physical entity, existing in space and time, then for all possible solutions of the Maxwell–Lorentz equations (7)–(11) there must exist, at least, a local instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying (22)–(23). That is, substituting an arbitrary solution[3] of (7)–(11) into (22)–(23), the overdetermined system of equations must have a solution for $\mathbf{v}(\mathbf{r},t)$.

One encounters however the following difficulty:

Theorem 1. *There exists a solution of the coupled Maxwell–Lorentz equations (7)–(11) for which there cannot exist a local instantaneous velocity field* $\mathbf{v}(\mathbf{r},t)$ *satisfying the persistence equations (22)–(23).*

Proof. As a proof, we give a surprisingly simple example. Consider the electric field in a parallel-plate capacitor being charged up by a constant current. The

[3] Without entering into the details, it must be noted that the Maxwell–Lorentz equations (7)–(11), exactly in this form, have *no* solution. The reason is that the field is singular at precisely the points where the coupling happens: on the trajectories of the particles. The generally accepted answer to this problem is that the real source densities are some "smoothed out" Dirac deltas, determined by the physical laws of the internal worlds of the particles—which are, supposedly, outside of the scope of classical electrodynamics. With this explanation, for the sake of simplicity we leave the Dirac deltas in the equations. Since our considerations here focuses on the electromagnetic field, satisfying the four Maxwell equations, we must only assume that there is a coupled dynamics—approximately described by equations (7)–(11)—and that it constitutes an initial value problem. In fact, Theorem 2 could be stated in a weaker form, by leaving the concrete form and dynamics of the source densities unspecified.

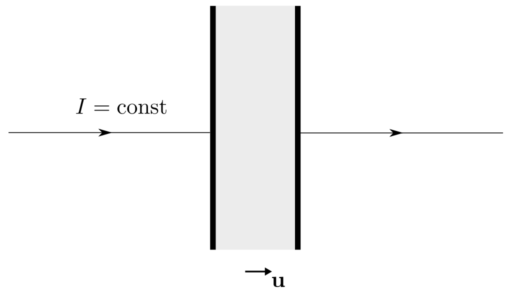

Figure 5: *Parallel-plate capacitor charged up by a constant current*

electric field strength is:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{u}t \tag{25}$$

where $\mathbf{u}$ is a constant vector determined by the current and the properties of the capacitor (Fig. 5). It is easy to check that there is no space-time point $(\mathbf{r}, t)$ where $\mathbf{E}(\mathbf{r}, t)$ would satisfy the equation of persistence (22) with some velocity $\mathbf{v}(\mathbf{r}, t)$. □

One might think that this is an exceptional case, due to the idealization of the real physical situation. But, as the next theorem shows, this is not so exceptional.

Theorem 2. *There is a dense subset of solutions of the coupled Maxwell–Lorentz equations (7)–(11) for which there cannot exist a local instantaneous velocity field* $\mathbf{v}(\mathbf{r}, t)$ *satisfying the persistence equations (22)–(23).*

Proof. The proof is almost trivial for a locus $(\mathbf{r}, t)$ where there is a charged point particle. However, in order to avoid the eventual difficulties concerning the physical interpretation, we are providing a proof for a point $(\mathbf{r}_*, t_*)$ where there is assumed no source at all.

Consider a solution $\left(\mathbf{r}^1(t), \mathbf{r}^2(t), \ldots, \mathbf{r}^n(t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)\right)$ of the coupled Maxwell–Lorentz equations (7)–(11), which satisfies (22)–(23). At point $(\mathbf{r}_*, t_*)$, the following equations hold:

$$\begin{aligned}
-\partial_t \mathbf{E}(\mathbf{r}_*, t_*) &= \mathrm{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) && (26)\\
-\partial_t \mathbf{B}(\mathbf{r}_*, t_*) &= \mathrm{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) && (27)\\
\partial_t \mathbf{E}(\mathbf{r}_*, t_*) &= c^2 \nabla \times \mathbf{B}(\mathbf{r}_*, t_*) && (28)\\
-\partial_t \mathbf{B}(\mathbf{r}_*, t_*) &= \nabla \times \mathbf{E}(\mathbf{r}_*, t_*) && (29)\\
\nabla \cdot \mathbf{E}(\mathbf{r}_*, t_*) &= 0 && (30)\\
\nabla \cdot \mathbf{B}(\mathbf{r}_*, t_*) &= 0 && (31)
\end{aligned}$$

Without loss of generality we can assume—at point $\mathbf{r}_*$ and time t_*—that operators $\mathrm{D}\mathbf{E}(\mathbf{r}_*, t_*)$ and $\mathrm{D}\mathbf{B}(\mathbf{r}_*, t_*)$ are invertible and $v_z(\mathbf{r}_*, t_*) \neq 0$.

Now, consider a 3×3 matrix J such that

$$J = \begin{pmatrix} \partial_x E_x(\mathbf{r}_*, t_*) & J_{xy} & J_{xz} \\ \partial_x E_y(\mathbf{r}_*, t_*) & \partial_y E_y(\mathbf{r}_*, t_*) & \partial_z E_y(\mathbf{r}_*, t_*) \\ \partial_x E_z(\mathbf{r}_*, t_*) & \partial_y E_z(\mathbf{r}_*, t_*) & \partial_z E_z(\mathbf{r}_*, t_*) \end{pmatrix} \tag{32}$$

with

$$J_{xy} = \partial_y E_x(\mathbf{r}_*, t_*) + \lambda \tag{33}$$

$$J_{xz} = \partial_z E_x(\mathbf{r}_*, t_*) - \lambda \frac{v_y(\mathbf{r}_*, t_*)}{v_z(\mathbf{r}_*, t_*)} \tag{34}$$

by virtue of which

$$\begin{aligned} J_{xy} v_y(\mathbf{r}_*, t_*) + J_{xz} v_z(\mathbf{r}_*, t_*) &= v_y(\mathbf{r}_*, t_*) \partial_y E_x(\mathbf{r}_*, t_*) \\ &\quad + v_z(\mathbf{r}_*, t_*) \partial_z E_x(\mathbf{r}_*, t_*) \end{aligned} \tag{35}$$

Therefore, $J\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*)$. There always exists a vector field $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ such that its Jacobian matrix at point $\mathbf{r}_*$ is equal to J. Obviously, from (30) and (32), $\nabla \cdot \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) = 0$. Therefore, there exists a solution of the Maxwell–Lorentz equations, such that the electric and magnetic fields $\mathbf{E}_{\lambda}(\mathbf{r}, t)$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t)$ satisfy the following conditions:[4]

$$\mathbf{E}_{\lambda}(\mathbf{r}, t_*) = \mathbf{E}^{\#}_{\lambda}(\mathbf{r}) \tag{36}$$

$$\mathbf{B}_{\lambda}(\mathbf{r}, t_*) = \mathbf{B}(\mathbf{r}, t_*) \tag{37}$$

At $(\mathbf{r}_*, t_*)$, such a solution obviously satisfies the following equations:

$$\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = c^2 \nabla \times \mathbf{B}(\mathbf{r}_*, t_*) \tag{38}$$

$$-\partial_t \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*) = \nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \tag{39}$$

therefore

$$\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = \partial_t \mathbf{E}(\mathbf{r}_*, t_*) \tag{40}$$

As a little reflection shows, if $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$, that is J, happened to be not invertible, then one can choose a *smaller* λ such that $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$ becomes invertible (due to the fact that $\mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)$ is invertible), and, at the same time,

$$\nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \neq \nabla \times \mathbf{E}(\mathbf{r}_*, t_*) \tag{41}$$

Consequently, from (40) , (34) and (26) we have

$$-\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{42}$$

and $\mathbf{v}(\mathbf{r}_*, t_*)$ is uniquely determined by this equation. On the other hand, from (39) and (41) we have

$$-\partial_t \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*) \neq \mathsf{D}\mathbf{B}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{43}$$

[4] $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t_*)$ can be regarded as the initial configurations at time t_*; we do not need to specify a particular choice of initial values for the sources.

because $\mathrm{D}\mathbf{B}(\mathbf{r}_*, t_*)$ is invertible, too. That is, for $\mathbf{E}_\lambda(\mathbf{r}, t)$ and $\mathbf{B}_\lambda(\mathbf{r}, t)$ there is no local and instantaneous velocity at point $\mathbf{r}_*$ and time t_*.

At the same time, λ can be arbitrary small, and

$$\lim_{\lambda \to 0} \mathbf{E}_\lambda(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \tag{44}$$

$$\lim_{\lambda \to 0} \mathbf{B}_\lambda(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) \tag{45}$$

Therefore solution $\left(\mathbf{r}^1_\lambda(t), \mathbf{r}^2_\lambda(t), \ldots, \mathbf{r}^n_\lambda(t), \mathbf{E}_\lambda(\mathbf{r}, t), \mathbf{B}_\lambda(\mathbf{r}, t)\right)$ can fall into an arbitrary small neighborhood of $\left(\mathbf{r}^1(t), \mathbf{r}^2(t), \ldots, \mathbf{r}^n(t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)\right)$.[5] □

4 Ontology of Classical Electrodynamics

The consequence of this result is embarrassing: the two *fundamental* electrodynamic quantities, the field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, do not satisfy the equations of persistence (6). Therefore, the electromagnetic field individuated by the field strengths cannot be regarded as a persisting physical object; in other words, electromagnetic field—for example, the field within the capacitor in Fig. 5—cannot be regarded as being a real physical entity existing in space and time. This seems to contradict to the usual realistic interpretation of classical electrodynamics. So, there are three options.

(i) One can abandon the realist understanding of electrodynamics: There is no such a persisting physical entity as "electromagnetic field".

(ii) Although, we think, in point **III** we formulated the most general form of how an extended physical object can persist, one may try to imagine a more sophisticated way of persistence.

(iii) Electromagnetic field is a real physical entity, persisting in the sense we formulated persistence in point **III**, but it cannot be individuated by the field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. That is, there must exist some quantities other than the field strengths, perhaps outside of the scope of classical electrodynamics, individuating the electromagnetic field. This suggests that classical electrodynamics is an ontologically incomplete theory.

How to conceive properties, different from the field strengths, which are capable of individuating the electromagnetic field? One might think of them as some "finer", more fundamental, properties of the field, not only individuating it as a persisting extended object, but also determining the values of the field strengths. However, the following easily verifiable theorem shows that this determination cannot be so simple:

[5]Notice that our investigation has been concerned with the general laws of Maxwell–Lorentz electrodynamics of a coupled particles + electromagnetic field system. The proof was essentially based on the presumption that all solutions of the Maxwell–Lorentz equations, determined by *any* initial state of the particles + electromagnetic field system, corresponded to physically possible configurations of the electromagnetic field. It is sometimes claimed, however, that the solutions must be restricted by the so called retardation condition, according to which all physically admissible field configurations must be generated from the retarded potentials belonging to some pre-histories of the charged particles (Jánossy 1971, p. 171; Frisch 2005, p. 145). There is no obvious answer to the question of how Theorem 2 is altered under such additional condition.

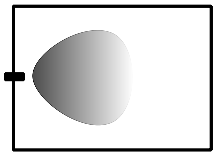

Figure 6: *A puff of gas is sprayed into an empty room through a little pipe*

Theorem 3. *Let $\{f_i(\mathbf{r},t)\}_{i=1}^n$ be a package of quantities for which there exists a local instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying the equations of persistence* (6) *in a given space-time region. If a quantity Φ is a functional of the quantities $f_1, f_2, \ldots, f_n$ in the following form:*

$$\Phi(\mathbf{r},t) = \Phi\left(f_1(\mathbf{r},t), f_2(\mathbf{r},t), ..., f_n(\mathbf{r},t)\right)$$

then Φ also obeys the equation of persistence

$$-\partial_t \Phi(\mathbf{r},t) \quad = \quad \nabla\Phi(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$$

with the same local instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$, within the same space-time region.

Therefore, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ cannot supervene pointwise upon some more fundamental individuating quantities satisfying the persistence equations. However, they might supervene in some non-local sense. For example, imagine that $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ provide only a course-grained characterization of the field, but there exist some more fundamental fields $\tilde{\mathbf{E}}(\mathbf{r},t)$ and $\tilde{\mathbf{B}}(\mathbf{r},t)$, such that

$$\mathbf{E}(\mathbf{r},t) = \int\limits_{\Omega_\mathbf{r}} \tilde{\mathbf{E}}\left(\mathbf{r}',t\right) d^3(\mathbf{r}') \tag{46}$$

$$\mathbf{B}(\mathbf{r},t) = \int\limits_{\Omega_\mathbf{r}} \tilde{\mathbf{B}}\left(\mathbf{r}',t\right) d^3(\mathbf{r}') \tag{47}$$

where Ω_r is a neighborhood of $\mathbf{r}$. In this case, the more fundamental quantities $\tilde{\mathbf{E}}(\mathbf{r},t)$ and $\tilde{\mathbf{B}}(\mathbf{r},t)$ may satisfy the equations of persistence, while $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, supervening on $\tilde{\mathbf{E}}(\mathbf{r},t)$ and $\tilde{\mathbf{B}}(\mathbf{r},t)$, may not.

It is worthwhile to mention that one has very similar situation in the case of continuum mechanics. Consider the following simple example. A puff of gas is sprayed into an empty room through a little pipe (Fig. 6). As the gas is spreading, the density of the gas $\varrho(x,t)$ is continuously decreasing in every point of the region occupied by the gas (Fig. 7). Consequently, $\varrho(x,t)$ does not satisfy the equation of persistence. This means that density distribution, which is one of the basic quantities of the continuum mechanical description of the gas, cannot be in the package of intrinsic properties individuating the gas.

In contrast, assuming that the gas consists of a huge number of small rigid particles, the *fine-grained* density distribution $\tilde{\varrho}(x,t)$ looks like as depicted in (Fig. 7) and satisfies the equation of persistence (6) with a suitable local and instantaneous velocity field, the value of which at every point in a region occupied by a particle is equal to the instantaneous velocity of the particle concerned. The course-grained density supervenes on the fine-grained density;

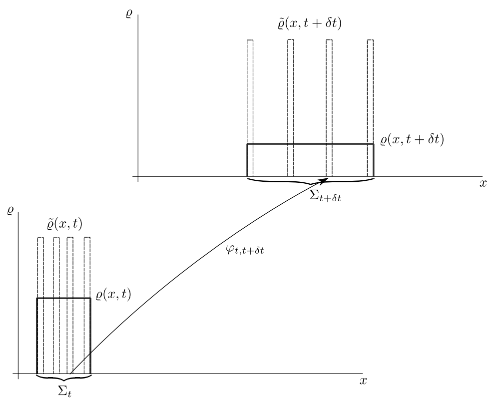

Figure 7: *The density of the gas $\varrho(x,t)$ is continuously decreasing in every point of the region Σ_t occupied by the gas. Consequently, $\varrho(x,t)$ does not satisfy the equation of persistence. In contrast, the fine-grained density distribution $\tilde{\varrho}(x,t)$, reflecting the molecular structure of the gas, satisfies the equation of persistence*

not pointwise, but in the style of (46)–(47):

$$\varrho(\mathbf{r},t) = \frac{1}{\Omega}\int\limits_{\Omega_{\mathbf{r}}} \tilde{\varrho}(\mathbf{r}',t)d^3(\mathbf{r})$$

where $\Omega_{\mathbf{r}}$ denotes a sphere of volume Ω with center $\mathbf{r}$, large enough relative to the fine-grained structure, but small enough to have a meaningful smooth approximation.

Thus, the continuum mechanical description of the gas in terms of the course-grained quantities is ontologically incomplete. This incomplete description can be completed by appealing to the fine-grained structure of the gas (cf. Murdoch 2012, Chapter 3; Batterman 2006). The perplexing question is: what could be a similar fine-grained structure of a classical electromagnetic field?

References

Arntzenius, Frank (2000): Are There Really Instantaneous Velocities? *The Monist* **83**, pp. 187–208.

Batterman, Robert W. (2006): Hydrodynamics versus Molecular Dynamics: Intertheory Relations in Condensed Matter Physics, *Philosophy of Science* **73**, pp. 888-904.

Butterfield, Jeremy (2005): On the Persistence of Particles, *Foundations of Physics* **35**, pp. 233–269.

Butterfield, Jeremy (2006): The Rotating Discs Argument Defeated, *British Journal for the Philosophy of Science* **57**, pp. 1–45.

Frisch, Mathias 2005: *Inconsistency, Asymmetry, and Non-Locality*, Oxford: Oxford University Press.

Gömöri, Márton and Szabó, László E. (2013): Operational understanding of the covariance of classical electrodynamics, *Physics Essays* **26**, pp. 361–370.

Hawley, Katherine (2001): *How Things Persist*. Oxford: Oxford University Press.

Jackson, John D. (1999): *Classical Electrodynamics (Third edition).* Hoboken, NJ: John Wiley & Sons.

Murdoch, A. Ian 2012: *Physical foundations of continuum mechanics*. Cambridge: Cambridge University Press.

Sider, Theodore 2001: *Four-Dimensionalism*. Oxford: Oxford University Press.

Tooley, Michael 1988: In Defense of the Existence of States of Motion, *Philosophical Topics* **16**, pp. 225–254.