A simple proof of Born's rule for statistical interpretation of quantum mechanics

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Abstract

The Born's rule to interpret the square of wave function as the probability to get a specific value in measurement has been accepted as a postulate in foundations of quantum mechanics. Although there have been so many attempts at deriving this rule theoretically using different approaches such as frequency operator approach, many-world theory, Bayesian probability and envariance, literature shows that arguments in each of these methods are circular. In view of absence of a convincing theoretical proof, recently some researchers have carried out experiments to validate the rule up-to maximum possible accuracy using multi-order interference (Sinha et al, *Science*, **329**, 418 [2010]). But, a convincing analytical proof of Born's rule will make us understand the basic process responsible for exact square dependency of probability on wave function. In this paper, by generalizing the method of calculating probability in common experience into quantum mechanics, we prove the Born's rule for statistical interpretation of wave function.

***Keywords*:** Quantum mechanics, Born's rule, Hilbert space, Eigen vectors

1. INTRODUCTION

Modern quantum theory is mainly based on two postulates. First, the generalized state $|\psi\rangle$ of a physical system can be represented by a vector in Hilbert space created by the orthogonal unit vectors (or eigen vectors). Each physically possible value of the observable represents an independent direction depicted by orthogonal eigen vectors in Hilbert space. Generalized state of the physical system can be any vector lying in this space which is a superposition of components along these eigen vectors. For example, spin of a system along (say) Z-axis can be a superposition of eigen states for up and down states ($\pm \hbar/2$). However, upon measurement, system returns either up ($+\hbar/2$) or down ($-\hbar/2$) state as these are the only physically possible values. Similarly, when position of a particle trapped in a box is represented by a wave, we mean that each position represents an eigen vector in Hilbert space and value of wave function represents the projection of sate vector on the corresponding eigen vector. Also here, although the generalized state of particle is a mixed state of position eigen vectors, measurement of position produces only one value corresponding to only one eigen vector. Then the question arises, which one should it be? Whether the probability of selection of a specific eigen vector should be proportional to the magnitude of projection of state vector on it $|\langle e_i|\psi\rangle|$ or should it be magnitude raised to some power n i.e. $|\langle e_i|\psi\rangle|^n$. To answer this question; comes the Born's rule in the form of a second postulate of quantum mechanics that assumes n=2. This is known as Born's interpretation of quantum mechanics [1]. Thus, Born's rule states that the probability to get the eigen value e_i in any experiment is given by,

$$P(e_i) = |\langle e_i|\psi\rangle|^2 \tag{1}$$

If $\psi(x)$ represents the wave function for position, probability for finding the particle at position x is $|\psi(x)|^2$ as it is same as the square of projection of state $|\psi\rangle$ on eigen vector $|x\rangle$. Since the discovery of this rule by Born in 1926, there have been numerous experiments till today to validate it. Recently, by observing null result of multi-order interference in three slit experiment, Sinha et al [2] have demonstrated that the exponent in expression for probability $|\psi|^2$ is correct up to an accuracy of 10^{-2}. These experimental observations assume significance as no convincing theoretical proof of this rule has been formulated till date. Initially, Born had proposed this rule based on intuition that light quanta and matter must behave in a similar manner and wave function might be analogous to electric field. In his Nobel lecture [3], Born stated, "Again an idea of Einstein's gave me the lead. He had tried to make the duality of particles - light quanta or photons - and waves comprehensible by interpreting the square of the optical wave amplitudes as probability density for the occurrence of photons. This concept could at once be carried over to the ψ-function: ψ^2 ought to represent the probability density for electrons (or other particles)."

Of course after this remarkable postulate of quantum theory was experimentally confirmed, there have been numerous attempts at deriving it. Among these, relative frequency or many-worlds [4-6] theory and Bayesian probability [7-9] theory are dominant in literature. Arguments in many-worlds theory which claims to derive probability from non-probabilistic axioms of quantum theory have been proved to be circular [10-16] because of hidden assumptions and preferred basis problem. The subjective Bayesian approach which declares the physical state of an object as an epistemic state (state of knowledge of observer) has been controversial [17-18] and this approach is not convincing since it cannot explain why Born's rule which is a law of nature for physical interactions should depend upon knowledge of the observer. Quantum mechanics and its rule remains equally valid where no observations are being made by us (say in distant galaxies). Zurek's mechanism of environment assisted invariance (envariance) [18-20] is also alleged to be circular by various authors [21-23] because of some fundamental assumptions that go into the derivation.

However in this paper, we prove the Born's rule for statistical interpretation of quantum mechanics by generalizing the method of calculating probability in common experience into quantum mechanics.

2. PROOF OF BORN'S RULE

Let us consider a basket containing total 12 balls out of which 4 are red balls, 4 are yellow balls and 4 are green. If we randomly pick a single ball from the basket, then the probability that it is red is (4/12). In this case, the probability of a particular outcome is calculated by the ratio of contribution of constituent part (i.e. number of red balls) to the total contributions in the system (i.e. total number of balls of any color). This method of calculating the probability becomes successful because in this case, *total characteristic of the system is* ***scalar*** *addition of the contributions of its constituent parts*. For example if we increase the number of red balls, total number of colored balls also increases in a linear manner with the scalar value of the changing contribution. However, suppose the color painted on the balls is such a chemical that when they are placed side by side, they react with each other and all the balls become colorless or some different color. Then, of course the probability of getting a red ball will not be (4/12). Thus the common method of calculating the probability fails because the scalar like dependence of the total on its parts is destroyed. So, we can state the generalized law of probability as,

"In any selection, the ratio of constituent part to the total value can be taken as a probability only when the total system is expressed as scalar sum of the contributions of its constituent parts".

Now, we apply the above generalized law of probability to quantum mechanics. Suppose the wave function of a quantum mechanical system is ψ. It can be represented as a generalized

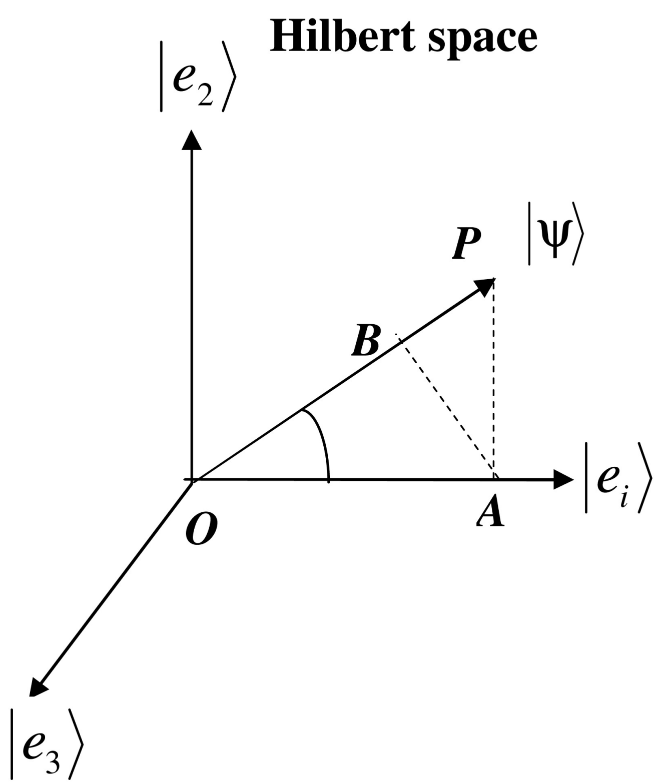

Fig. 1 Hilbert space analysis in which perpendiculars (like AB) are drawn from tips of components along eigen vectors to the normalized general state of the system $|\psi\rangle$ to know the contributions (like OB) along the direction of $|\psi\rangle$

state vector $|\psi\rangle$ in Hilbert space generated by the components along orthogonal eigen vectors. Thus $|\psi\rangle$ is a vector sum of the components along each eigen vector. As shown in Fig.1, if i^{th} eigen vector is $|e_i\rangle$, then projection $\overrightarrow{OA}$ of state $|\psi\rangle$ on $|e_i\rangle$ is calculated by applying projection operator on $|\psi\rangle$. Thus,

$$\overrightarrow{OA} = |e_i\rangle\langle e_i|\psi\rangle \tag{2}$$

And $$|\psi\rangle = \sum_i |e_i\rangle\langle e_i|\psi\rangle = |e_1\rangle\langle e_1|\psi\rangle + |e_2\rangle\langle e_2|\psi\rangle + \ldots\ldots.. \tag{3}$$

Since eigen vectors are not collinear (they are orthogonal), total is NOT equal to the scalar sum of magnitudes of constituent parts i.e.

$$\||\psi\rangle\| \neq |\langle e_1|\psi\rangle| + |\langle e_2|\psi\rangle| + \ldots\ldots..$$

Because of this failure, as per the generalized law of probability formulated earlier, in case of selection, $\dfrac{|\langle e_1|\psi\rangle|}{\||\psi\rangle\|}$ cannot represent the probability.

However, to express Eq. (3) as a scalar sum of individual parts, let us calculate the contribution of each component vector along the direction of $|\psi\rangle$. For example, contribution of vector $\overrightarrow{OA}$ along OP is $\overrightarrow{OB}$ i.e. projection of $\overrightarrow{OA}$ on $|\psi\rangle$. Thus,

$$\overrightarrow{OB} = |\psi\rangle\langle\psi|\,\overrightarrow{OA}$$

Using Eq. (2) in above expression,

$$\overrightarrow{OB} = |\psi\rangle\langle\psi|e_i\rangle\langle e_i|\psi\rangle$$

Or $$\overrightarrow{OB} = |\psi\rangle|\langle e_i|\psi\rangle|^2$$

Now, total system vector is given by sum of contributions of each component along same direction $|\psi\rangle$. Thus,

$$|\psi\rangle = \sum_i |\psi\rangle|\langle e_i|\psi\rangle|^2 = |\psi\rangle|\langle e_1|\psi\rangle|^2 + |\psi\rangle|\langle e_2|\psi\rangle|^2 + \ldots\ldots\ldots. \tag{4}$$

Since each term on left and right hand side of above equation are pointing along same direction (i.e. **collinear**), we can ignore the directions and write the magnitudes only. Taking magnitude of $|\psi\rangle$ as 1 (for normalized wave function), we get,

$$Total\ \ system = 1 = \sum_i \left|\langle e_i | \psi \rangle\right|^2 = \left|\langle e_1 | \psi \rangle\right|^2 + \left|\langle e_2 | \psi \rangle\right|^2 + ... \tag{5}$$

Thus, in Eq. (5) we have expressed the total system as a ***scalar sum*** of the contributions of constituent parts. So, as per the generalized law of probability formulated earlier, we can say that, the magnitude of probability of getting eigen value e_i in a selection is given by,

$$P(e_i) = \left|\langle e_i | \psi \rangle\right|^2 \tag{6}$$

If $\psi(x)$ is wave function in position space, $|\psi(x)|$ is same as the magnitude of projection of state $|\psi\rangle$ on eigen vector $|x\rangle$ i.e. $|\psi(x)| = |\langle x|\psi\rangle|$. So,

$$P(x) = |\psi(x)|^2 \tag{7}$$

Thus, the Born's law which had been taken as a postulate of quantum theory for statistical interpretation of wave function can be proved by application of classical law of probability and the exponent in Eq. (6) is found to be exactly two. So, there is no more need to carry out experiments like the recently reported one [2] to confirm the absolute correctness of this exponent.

3. CONCLUSION

Since the days of formulations of quantum mechanics, the Born's rule to interpret the square of wave function as the probability to get a specific value in any measurement has been accepted as a fundamental postulate. Although there have been many attempts till date to derive this rule theoretically, all of these methods have been proved circular in literature [10-18, 21-23]. In absence of a convincing theoretical proof for Born's rule, importance of experiments to validate the rule up-to maximum possible accuracy has become significant and one such experimental result was reported recently [2]. However, in this paper, by generalizing the method of calculating probability in common experience into quantum mechanics, we have proved the Born's rule for statistical interpretation of wave function. We find that the probability becomes exactly the square of wave function because the general state of the system is made up of *orthogonal* components along eigen vectors in Hilbert space while the probability is applicable only when the total system is expressed as a *scalar sum* of individual contributions.

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