Common cause completability of non-classical probability spaces

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Abstract

We prove that under some technical assumptions on a general, non-classical probability space, the probability space is extendible into a larger probability space that is common cause closed in the sense of containing a common cause of every correlation between elements in the space. It is argued that the philosophical significance of this common cause completability result is that it allows the defence of the Common Cause Principle against certain attempts of falsification. Some open problems concerning possible strengthening of the common cause completability result are formulated.

1 Main result

In this paper we prove a new result on the problem of common cause completability of non-classical probability spaces. A non-classical (also called: general) probability space is a pair $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is an orthocomplemented, orthomodular, non-distributive lattice and $\phi\colon \mathcal{L} \to [0, 1]$ is a countably additive probability measure. Taking $\mathcal{L}$ to be a distributive lattice (Boolean algebra), one recovers classical probability theory; taking $\mathcal{L}$ to be the projection lattice of a von Neumann algebra, one obtains quantum probability theory. A general probability space $(\mathcal{L}, \phi)$ is called common cause completable if it can be embedded into a larger general probability space which is common cause complete (closed) in the sense of containing a common cause of every correlation in it. Our main result (Proposition 4.2.3) states that under some technical conditions on the lattice $\mathcal{L}$ a general probability space $(\mathcal{L}, \phi)$ is common cause completable. This result utilizes earlier results on common cause closedness of general probability theories ([13], [6], [14]) and generalizes earlier results on common cause completability of classical probability spaces [8], [9], [24], [5], [10][Proposition 4.19], [15], [25].

The main conceptual-philosophical significance of the common cause extendability result proved in this paper is that it allows one to deflect the arrow of falsification directed against the Common Cause Principle for a very large and abstract class of probability theories. This will be discussed in section 5 in the more general context of how one can assess the status of the Common Cause Principle, viewed as a general metaphysical claim about the causal structure of the world. Further sections of the paper are organized as follows: Section 2 fixes some notation and recalls some facts from lattice theory and general probability spaces needed to formulate the main result. Section 3

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defines the notion of common cause and common cause in general probability theories and the notion of common cause completability of such theories. The main result (Proposition 4.2.3) is stated in section 4. The detailed proof of the man proposition is given in the Appendix.

2 General probability spaces

Definition 2.1. A bounded lattice $\mathcal{L}$ with units $\mathbb{0}$ and $\mathbb{1}$ is *orthocomplemented* if there is a unary operation $\perp: \mathcal{L} \to \mathcal{L}$ satisfying

1. $a \vee a^\perp = \mathbb{1}$ and $a \wedge a^\perp = \mathbb{0}$.
2. $a \leq b$ implies $b^\perp \leq a^\perp$.
3. $(a^\perp)^\perp = a$ for every $a \in \mathcal{L}$.

$\mathcal{L}$ is called *σ-complete* if $\bigvee_{i=0}^{\infty} a_i$ exists in $\mathcal{L}$ for all $a_i \in \mathcal{L}$.

Two elements $a, b \in \mathcal{L}$ are said to be *orthogonal* if $a \leq b^\perp$ and this we denote by $a \perp b$. $\mathcal{L}$ is called *orthomodular* if $a \leq b$ implies

$$b = a \vee (b \wedge a^\perp).$$

We say a and b *commutes* if

$$a = (a \wedge b) \vee (a \wedge b^\perp).$$

Definition 2.2. A map $\phi : \mathcal{L} \to [0, 1]$ on an orthomodular lattice $\mathcal{L}$ is defined to be a *probability measure* if the following two stipulations hold

1. $\phi(\mathbb{0}) = 0$ and $\phi(\mathbb{1}) = 1$.
2. Whenever $\bigvee_{i=0}^{\infty} a_i$ exists for pairwise orthogonal elements $a_i \in \mathcal{L}$ we have

$$\phi\left(\bigvee_{i=0}^{\infty} a_i\right) = \sum_{i=0}^{\infty} \phi(a_i), \qquad a_i \perp a_j \quad (i \neq j),$$

Definition 2.3.

General Probability Spaces A general probability space is a pair $(\mathcal{L}, \phi)$ where $\mathcal{L}$ is a σ-complete orthomodular lattice and ϕ is a probability measure on it.

Classical Probability Spaces If $(\mathcal{L}, \phi)$ is a general probability space and $\mathcal{L}$ is distributive then $(\mathcal{L}, \phi)$ is called semi-classical. Note that every distributive orthomodular lattice is a Boolean algebra. If $\mathcal{L}$ is isomorphic to a Boolean σ-algebra of subsets of a set X then $\mathcal{L}$ is *set-represented* and we say $(X, \mathcal{L}, \phi)$ is a classical probability space. If X is clear from the context, we omit it.

Quantum Probability Spaces Let $\mathcal{H}$ be a Hilbert space and denote the C^*-algebra of bounded linear operators in $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra then by $\mathcal{P}(\mathcal{M})$ we understand the set of projections of $\mathcal{M}$. It is known that $\mathcal{P}(\mathcal{M})$ is a complete orthomodular lattice. A *quantum probability space* is a pair $(\mathcal{P}(\mathcal{M}), \phi)$ where $\mathcal{P}(\mathcal{M})$ is the set of projections of a von Neumann algebra $\mathcal{M}$ and ϕ is a probability measure on $\mathcal{P}(\mathcal{M})$. Probability measures arise as restrictions to $\mathcal{P}(\mathcal{M})$ of *normal states* on $\mathcal{M}$.

It is clear from the definition that both classical and quantum probability spaces are special general probability spaces. Note also that classical probability measure spaces are specific cases of quantum probability spaces: If the von Neumann algebra $\mathcal{M}$ is commutative, then the quantum probability space $(\mathcal{P}(\mathcal{M}), \phi)$ is in fact a classical probability measure space (see [19] for a review of quantum probability spaces and their relation to classical probability theory). Thus all the definitions and notions involving general probability spaces have classical and quantum counterparts in a natural way.

Definition 2.4. A general probability space $(\mathcal{L}, \phi)$ is called *dense* if for any $a \in \mathcal{L}$ and $r \in \mathbb{R}$ with $0 \leq r \leq \phi(a)$ there is some $b \leq a$ such that $\phi(b) = r$.

The space $(\mathcal{L}, \phi)$ is called *purely non-atomic* if for all $a \in \mathcal{L}$ with $0 < \phi(a)$ there exists $b < a$ such that $0 < \phi(b) < \phi(a)$.

Remark 2.5. A general probability space $(\mathcal{L}, \phi)$ is dense if and only if it is purely non-atomic.

Definition 2.6. The general probability space $(\mathcal{L}', \phi')$ is called an *extension* of $(\mathcal{L}, \phi)$ if there exists a complete ortholattice embedding h of $\mathcal{L}$ into $\mathcal{L}'$ such that

$$\phi(x) = \phi'(h(x)) \qquad \text{for all } x \in \mathcal{L} \tag{1}$$

A lattice embedding is complete if it preserves all the infinite lattice operations as well.

3 Common cause and common cause completeness in general probability spaces

Definition 3.0.1. In a general probability space $(\mathcal{L}, \phi)$, two *commuting* events a and b are said to be (positively) correlated if

$$Corr_{\phi}(a, b) \doteq \phi(a \wedge b) - \phi(a)\phi(b) > 0. \tag{2}$$

The event $c \in \mathcal{L}$ is a common cause of the correlation (2) if it commutes with both a and b and the following (independent) conditions hold:

$$\phi(a \wedge b|c) = \phi(a|c)\phi(b|c) \tag{3}$$

$$\phi(a \wedge b|c^{\perp}) = \phi(a|c^{\perp})\phi(b|c^{\perp}) \tag{4}$$

$$\phi(a|c) > \phi(a|c^{\perp}) \tag{5}$$

$$\phi(b|c) > \phi(b|c^{\perp}) \tag{6}$$

where $\phi(x|y) = \phi(x \wedge y)/\phi(y)$ denotes the conditional probability of x on condition y, and it is assumed that none of the probabilities $\phi(x)$, $x = a, b, c, c^{\perp}$ is equal to zero.

Note that taking c to be either a or b, conditions (3)–(6) are satisfied, so, formally, both a and b are common causes of the correlation (2) between a and b; intuitively however such a "common cause" is not a "proper" common cause. A common cause is *proper* if it differs from both a and b by more than measure zero. In what follows, a common cause will *always* mean a proper common cause.

Definition 3.0.2. A general probability space $(\mathcal{L}, \phi)$ is called common cause closed (complete) if $\mathcal{L}$ contains a common cause of every correlation in it, and common cause incomplete otherwise.

Both common cause complete and common cause incomplete probability spaces exist: It was shown in [3]) (also see [4] and [10][Chapter 4]) that no classical probability space with a Boolean algebra of finite cardinality can be common cause complete and that a dense classical probability space is common cause closed. The converse is not true, a classical probability space can be not purely non-atomic and still common cause closed; in fact, common cause closedness of classical probability spaces can be characterized completely: a classical probability space is common cause closed if and only if it has at most one measure theoretic atom [5].

A similar characterization of common cause closedness of general probability spaces is not known, only partial results have been obtained: It was proved by Kitayima that if $\mathcal{L}$ is a nonatomic, *complete*, orthomodular lattice and ϕ is a *completely* additive faithful probability measure then $(\mathcal{L}, \phi)$ has the denseness property; which was then shown to entail that every correlation between elements a and b that are logically independent has a common cause. This result was strengthened in [6] by showing that for $(\mathcal{L}, \phi)$ to be dense it is enough that $\mathcal{L}$ is only σ-complete and the faithful ϕ is σ-additive, and that in this case every correlation has a common cause in $\mathcal{L}$ – not just correlations between logically independent elements – hence such general probability spaces are common cause closed. Just like in the classical case, a general probability space can be common cause closed and not being purely nonatomic: it was shown in [6] that (under an additional technical condition called "Q-decomposability" of a probability measure) a general probability measure space is common cause closed if it has only one measure theoretic atom. The Q-decomposability condition could be shown to be redundant in the particular case when $(\mathcal{L}, \phi)$ is a quantum probability space: it was proved in [14] that if ϕ is a faithful normal state on the von Neumann algebra $\mathcal{N}$, then $(\mathcal{P}(\mathcal{N}), \phi)$ is common cause closed if and only if there is *at most one* ϕ-atom in the projection lattice $\mathcal{P}(\mathcal{N})$. This result entails that the standard quantum probability theory $(\mathcal{P}(\mathcal{H}), \phi)$, which has more than one measure theoretic atom, is not common cause closed. It is however still an open question whether having *at most one* measure theoretic atom is equivalent to common cause closedness of a general probability space.

If a probability space $(\mathcal{L}, \phi)$ is common cause incomplete, then the question arises whether it can be common cause completed. Common cause completion of $(\mathcal{L}, \phi)$ with respect to a set $\{(a_i, b_i) \; : \; i \in I\}$ of pairs of correlated elements is meant finding an extension $(\mathcal{L}', \phi')$ of $(\mathcal{L}, \phi)$ (Definition 2.6) such that $(\mathcal{L}', \phi')$ contains a common cause $c_i \in L'$ for every correlated pair (a_i, b_i), $i \in I$. If a probability space $(\mathcal{L}, \phi)$ is common case completable with respect to *all* the correlations in it, we simply say that $(\mathcal{L}, \phi)$ is common cause completable.

Note that the definition of extension, and in particular condition (1), implies that if $(\mathcal{L}', \phi')$ is an extension of $(\mathcal{L}, \phi)$ with respect to the embedding h, then every single correlation $Corr_p(a, b)$ in $(\mathcal{L}, \phi)$ is carried over intact by h into the correlation $Corr_{\phi'}(h(a), h(b))$ in $(\mathcal{L}', \phi')$ because

$$\begin{aligned} \phi'(h(a) \wedge h(b)) &= \phi'(h(a \wedge b)) \\ &= \phi(a \wedge b) \; > \; \phi(a)\phi(b) = \phi'(h(a))\phi'(h(b)) \end{aligned}$$

Hence, it does make sense to ask whether a correlation in $(\mathcal{L}, \phi)$ has a Reichenbachian common cause in the extension $(\mathcal{L}', \phi')$.

It was shown in [8] that every common cause incomplete classical probability space is common cause completable with respect to any *finite* set of correlations. This result was strengthened by proving that classical probability spaces are common cause completable with respect to *any* set of correlations [5], i.e. every classical probability space is common cause completable. In fact, it was showed in [24] that every common cause incomplete classical probability space has an extension that is common cause closed (also see [5] and [10][Proposition 4.19]). This settled the problem of common cause completability of classical probability spaces.

For non-classical spaces, [8] proved that every quantum probability space $(\mathcal{P}(\mathcal{M}), \phi)$ is common cause completable with respect to the set of pairs of events that are correlated in the state ϕ. But common cause completability of general probability theories has remained open so far. The proposition in the next section states common cause completability of general probability theories under some technical conditions on the lattice $\mathcal{L}$.

4 Proposition on dense extension and common cause completability of certain general probability spaces

The results on common cause closedness of general probability spaces recalled in the previous section make it clear that in order to show that a general probability space $(\mathcal{L}, \phi)$ is common cause completable, it is enough to prove that $(\mathcal{L}, \phi)$ has an extension that is dense. In this section we present a method of extending certain general probability spaces to a dense one. In the case of classical and quantum probability spaces there are standard methods of extension that can be applied to obtain a dense extension of these probability theories. We discuss briefly this method in Subsection 4.1. However, this method does not seem to work in the general case; hence in Subsection 4.2 we introduce another construction of an extension which can be applied to a broader class of probability spaces and which leads to a dense extension. Finally, in Subsection 5.1 we discuss how this new construction can be applied to classical and quantum probability spaces. Here and in Subsection 4.1 we will be sketchy.

4.1 Brief overview of standard product extension of classical and quantum probability spaces into dense spaces

The main reason why classical and quantum probability theories can be extended into a dense one is that one can define the notion of tensor product of these probability spaces. In contrast, it is not that obvious how to define tensor product of general orthomodular lattices; in general such tensor products do not exits.

Classical probability spaces. Let $\mathfrak{L} = (X, \mathcal{L}, \phi)$ and $\mathfrak{X} = (I, \Lambda, \lambda)$ be two classical probability spaces and let $\mathfrak{L} \otimes \mathfrak{X}$ be their usual product. Then the function $h : \mathcal{L} \to \Lambda$ defined as

$$h(A) = I \times A$$

is an ortho-embedding which shows that $\mathfrak{L} \otimes \mathfrak{X}$ is an extension of $\mathfrak{L}$. Taking then $\mathfrak{X}$ to be dense, for example Λ is the set of Borel subsets of the unit interval I with the Lebesgue measure λ, it is not hard to see that $\mathfrak{L} \otimes \mathfrak{X}$ is dense, too. A detailed proof can be found in [24] and [10] (proof of Proposition 4.19).

Quantum probability spaces. A similar method works in the quantum case: Let $(\mathcal{P}(\mathcal{M}), \phi)$ and $(\mathcal{P}(\mathcal{N}), \psi)$ be two quantum probability spaces where $\mathcal{N}$ is a type III von Neumann algebra and ψ is a faithful normal state on $\mathcal{N}$. Then by Lemma 4 in [18] the quantum probability space $(\mathcal{P}(\mathcal{N}), \psi)$ is dense. Consider now the tensor product quantum probability space $(\mathcal{P}(\mathcal{M} \otimes \mathcal{N}), \phi \otimes \psi)$. This is an extension of $(\mathcal{P}(\mathcal{M}), \phi)$ with the embedding $h(X) = X \otimes I$. Since $\mathcal{M} \otimes \mathcal{N}$ is a type III von Neumann algebra if $\mathcal{N}$ is [12][Chapter 11], again by Lemma 4 in [18] the quantum probability space $(\mathcal{P}(\mathcal{M} \otimes \mathcal{N}), \phi \otimes \psi)$ is dense too.

4.2 Extension of σ-continuous general probability spaces

Let $(\mathcal{L}, \phi)$ be a general probability space. To state our main proposition we have to recall some further lattice theoretic notions. Let (a_i) be a sequence of elements of $\mathcal{L}$. Then

$$\liminf(a_i) \doteq \bigvee_{i=0}^{\infty} \bigwedge_{k=i}^{\infty} a_k \tag{7}$$

$$\limsup(a_i) \doteq \bigwedge_{i=0}^{\infty} \bigvee_{k=i}^{\infty} a_k. \tag{8}$$

if the right hand sides exist. We define $\mathsf{lim}(a_i)$ if and only if $\limsup(a_i) = \liminf(a_i)$, and in this case

$$\mathsf{lim}(a_i) \doteq \liminf(a_i).$$

Clearly

$$\bigvee_{i=0}^{\infty} a_i = \mathsf{lim}(a_0, a_0 \vee a_1, \ldots, \vee_{i=0}^{k} a_i, \ldots) \tag{9}$$

$$\bigwedge_{i=0}^{\infty} a_i = \mathsf{lim}(a_0, a_0 \wedge a_1, \ldots, \wedge_{i=0}^{k} a_i, \ldots), \tag{10}$$

whenever the left hand sides are defined in $\mathcal{L}$ (e.g. $\mathcal{L}$ is σ-complete).

Next we prove that

$$\phi(\mathsf{lim}(a_i)) = \lim_{i \to \infty} \phi(a_i),$$

whenever $\mathsf{lim}(a_i)$ is defined. In order to show this observe first, that by the orthomodular law, since $a \leq a \vee b$ we have

$$a \vee b = a \vee \big(a^{\perp} \wedge (a \vee b)\big),$$

where clearly $a \perp a^{\perp} \wedge (a \vee b)$. Hence, by an easy induction argument, it follows that if (a_i) is a sequence of elements such that $\bigvee_{i=0}^{\infty} a_i$ exists, then there is another sequence (b_i) of elements, such that for all $k \in \mathbb{N}$ we have

$$\bigvee_{i=0}^{k} a_i = \bigvee_{i=0}^{k} b_i,$$

and $b_i \perp b_j$ for $i \neq j$. It also follows that $\bigvee_{i=0}^{\infty} b_i$ exists and is equal to $\bigvee_{i=0}^{\infty} a_i$. Therefore by σ-additivity of ϕ we get

$$\phi(\bigvee_{i=0}^{\infty} a_i) = \phi(\bigvee_{i=0}^{\infty} b_i) = \sum_{i=0}^{\infty} \phi(b_i) = \lim \sum_{i=0}^{k} \phi(b_k) = \lim \sum_{i=0}^{k} \phi(a_k).$$

Hence, we proved that

$$\phi\big(\mathsf{lim}(a_0, a_0 \vee a_1, \ldots)\big) = \lim \big(\phi(a_0), \phi(a_0 \vee a_1), \ldots\big).$$

Now, if (c_i) is any sequence such that $\mathsf{lim}(c_i)$ exists, then by the definition of lim and in particular $\liminf$, there is another sequence (d_i) such that $\mathsf{lim}(c_i) = \bigvee_{i=0}^{\infty} d_i$. Using this and by the previous argument, it is not hard to see, that

$$\phi\big(\mathsf{lim}(c_i)\big) = \lim \phi(c_i).$$

Definition 4.2.1. A lattice $\mathcal{L}$ is called *σ-continuous* if the following two stipulations hold

1. $\mathsf{lim}(a_i) \wedge \mathsf{lim}(b_i) = \mathsf{lim}(a_i \wedge b_i)$.
2. $\mathsf{lim}(a_i) \vee \mathsf{lim}(b_i) = \mathsf{lim}(a_i \vee b_i)$.

whenever the limits $\mathsf{lim}(a_i)$ and $\mathsf{lim}(b_i)$ exist in $\mathcal{L}$. Note that the equation $(\mathsf{lim}(a_i))^\perp = \mathsf{lim}(a_i^\perp)$ automatically holds in orthocomplemented lattices.

An important class of σ-continuous orthomodular lattices are the finite ones.

We can now state our main result:

Proposition 4.2.2. Let $(\mathcal{L}, \phi)$ be a general probability space where $\mathcal{L}$ is σ-continuous. Then there exists a dense extension $(\mathcal{L}', \phi')$ of $(\mathcal{L}, \phi)$.

This proposition entails:

Proposition 4.2.3. Let $(\mathcal{L}, \phi)$ be a general probability space where $\mathcal{L}$ is σ-continuous. Then $(\mathcal{L}, \phi)$ is common cause completable.

The proof of Proposition 4.2.2 consists of the following 3 steps (details can be found in the Appendix):

Step 1. Using an arbitrary classical probability space $\mathfrak{X}$ we construct first an orthomodular lattice $\mathcal{S} = \mathcal{S}(\mathfrak{X},\ \mathcal{L})$ in such a way that $\mathcal{L}$ is a sub-orthomodular lattice of $\mathcal{S}$.

Step 2. We define a measure ρ on $\mathcal{S}$ in a way that $(\mathcal{S},\ \rho)$ becomes an extension of $(\mathcal{L},\ \phi)$. Unfortunately $\mathcal{S}$ is not in general σ-complete, so we need the one more step.

Step 3. We extend $\mathcal{S}$ to a σ-complete orthomodular lattice while paying attention not to ruin ρ-measurability.

5 Common cause completability and the Common Cause Principle

Establishing results on common cause completability of general probability spaces is motivated by the need to assess the status of what is known as the Common Cause Principle. The Common Cause Principle is a claim about the causal structure of the world, and it states that if there is a probabilistic correlation between two events, then either there is a direct causal link between the correlated events that explains the correlation, or there is a third event, a common cause that brings about (hence explains) the correlation. This principle goes back to Reichenbach's work that defined the notion of common cause in terms of classical probability theory [20], and the principle was sharply articulated mainly by Salmon [21], [22]. There is a huge literature on the problem of whether this principle reflects correctly the causal structure of the world (see the references in [10] and [25]). The difficulty in giving a definite answer to this question is that the Common Cause Principle has metaphysical character: it is a strictly universal claim containing two general existential statements. Standard arguments well known from the history of philosophy show that such general claims can be neither verified nor falsified conclusively; thus, as it was argued in [10] (Chapters 1 and 10) and [17], the only option one has when it comes to the problem of assessing the epistemic status of the principle is to have a look at the best available evidence relevant for the principle and see whether they are in harmony with the principle or not. Such evidence is provided by the empirical sciences.

Some of the (experimentally testable and confirmed) correlations are predicted by physical theories that apply non-classical probability spaces to describe the phenomena in their domain: Standard non-relativistic quantum mechanics of finite degrees of freedom is based on non-classical probability theory of the form $(\mathcal{P}(\mathcal{H}), \phi)$, where $\mathcal{P}(\mathcal{H})$ is the lattice of all projections of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ consisting of all bounded operators on Hilbert space $\mathcal{H}$, and ϕ is a quantum state.

This type of quantum theory predicts the notorious EPR correlations [10][Chapter 9]. Relativistic quantum field theory (in the so called algebraic approach [11], [7], [1]) is based on probability theory of the form $(\mathcal{P}(\mathcal{N}), \phi)$, where $\mathcal{P}(\mathcal{N})$ is the projection lattice of a type III von Neumann algebra and ϕ is a normal state on $\mathcal{N}$. Quantum field theory predicts an abundance of correlations between observables that are localized in spacelike separated spacetime regions. This is a consequence of violation of Bell's inequality in quantum field theory (for a review of the relevant theorems and references see [10][Chapter 8]). Given these correlations predicted by physical theories using non-classical probability spaces, the problem arises whether these correlations are in harmony with the Common Cause Principle. This is a very subtle and complicated matter. One has to specify very carefully what precisely the harmony would be and, given a specification, one can try to show that harmony obtains (or not).

A possible (and very natural [10] (Chapters 1 and 10), [17]) line of reasoning aimed at assessing the Common Cause Principle in this spirit is the following: Suppose a theory T applies a non-classical probability theory $(\mathcal{L}, \phi)$ to describe phenomena and predicts correlation between elements a, b in $\mathcal{L}$ such that, according to T, there is no (cannot be) a direct causal connection between a and b. The first question one would want to ask then: Is there a common cause c in $\mathcal{L}$ of the correlation between a and b? If there is, then theory T is a confirming evidence in favor of the Common Cause Principle: T is then a causally complete theory that can give an explanation of the correlations it predicts. We know from results about common cause incompleteness referred to earlier in this paper that it can happen that there is no common cause in $\mathcal{L}$ of the correlation – i.e. that $(\mathcal{L}, \phi)$ is common cause incomplete. This makes T a potentially disconfirming evidence for the Common cause Principle. But the mere fact that T is common cause incomplete does not falsify the Common Cause Principle because one can mount the following defence: The general probability theory $(\mathcal{L}, \phi)$ applied by T is just too meager, and there might exist "hidden" common causes of this correlation – hidden in the sense of being part of a larger probability theory $(\mathcal{L}', \phi')$ that extends $(\mathcal{L}, \phi)$. The conceptual significance of common cause extendability of general probability spaces should now be clear: common cause extendability entails that this kind of defence of the Common Cause Principle against potential falsifiers is *always* possible – and it is possible for an extremely wide class of general probability theories. This broad class includes both classical and quantum probability theories in particular.

The general common cause completability result and the above reasoning lead to the question of whether common cause completability also holds under more stringent conditions. A particularly relevant type of additional conditions are the ones that express "locality" understood as constraints on a physical theory T that make T to be compatible with the principles of the theory of relativity. A special such case is the quantum probability theory $(\mathcal{P}(\mathcal{N}), \phi)$ with a type III von Neumann algebra $\mathcal{N}$: This quantum probability measure space theory describes relativistic quantum fields and it has a further internal (so called "quasi-local") structure. This local structure makes it possible to impose further, physically motivated locality conditions on the common causes. Under this further condition it becomes a highly non-trivial problem to decide whether the theory is common cause complete. One difficulty of this problem is that the locality conditions can be defined in different ways and causal completeness seems to depend sensitively on how the locality conditions are defined. The most natural locality condition leads to the problem of causal closedness of quantum field theory [16] that is still open. It could be proved however that quantum field theory is causally complete with respect to a notion of weakly localized common causes [18] (see also the extensive discussion in Chapter 8 of [10]). Since we do not have new results on local common causes in this paper, we do not give here the precise definitions of locality.

A further direction of possible research concerns strengthening the common cause completability results in this paper by taking into account the "type" of the common cause. A common cause c is said to have the type characterized by five positive real numbers $(r_c, r_{a|c}, r_{a|c^\perp}, r_{b|c}, r_{b|c^\perp})$ if these numbers are equal with the probabilities of events indicated by the subscript of the numbers, i.e. if $r_c = \phi(c)$, $r_{a|c} = \phi(a|c)$, $r_{a|c^\perp} = \phi(a|c^\perp)$, $r_{b|c} = \phi(b|c)$, $r_{b|c^\perp} = \phi(b|c^\perp)$ (Definition 3.6 in [10]). One then can define *strong* common cause closedness of a general probability space $(\mathcal{L}, \phi)$ by requiring that for any correlation in this space and for any given possible type there exists in $\mathcal{L}$ a common cause of the given type. A general probability space can then be defined to be *strongly* common cause completable if it has an extension that is *strongly* common cause closed. It is not known whether general probability spaces are strongly common cause completable (cf. Problem 6.2 in [10]).

Appendix

Step 1.

Construction of $\mathcal{S}(\mathfrak{X},\ \mathcal{L})$. Let $\mathfrak{X} = (X,\ \Sigma,\ \mu)$ be a classical probability measure space, that is, Σ is a σ-algebra of subsets of X and μ is a probability measure on Σ, and let $\mathcal{L}$ be an orthomodular lattice.

For an element $a \in \mathcal{L}$ and a subset $B \subseteq X$ we define $\chi^a_B : B \to \mathcal{L}$ as

$$\chi^a_B(x) \doteq a.$$

Next, we define *step functions*. A function $p : X \to \mathcal{L}$ is called a *step function* if it is of the following form

$$p = \bigsqcup_{i=0}^{\infty} \chi^{a_i}_{B_i},$$

where $a_i \in \mathcal{L}$, $B_i \subseteq X$ is measurable, i.e. $B_i \in \Sigma$ and $X = \bigsqcup_{i=0}^{\infty} B_i$ is a disjoint partition of X. So a step function is a function whose domain can be partitioned into countably many measurable sets and the function is constant on each partition.

Definition 5.0.1. For a classical probability space $\mathfrak{X} = (X,\ \Sigma,\ \mu)$ and an orthomodular lattice $\mathcal{L}$ we define

$$\begin{aligned} \mathcal{F}(X, \mathcal{L}) &\doteq \big\{p :\ p\colon X \to \mathcal{L} \text{ is a function}\big\} \\ \mathcal{S}(\mathfrak{X}, \mathcal{L}) &\doteq \big\{p :\ p\colon X \to \mathcal{L} \text{ is a step function}\big\}. \end{aligned}$$

$\mathcal{F}(X,\ \mathcal{L})$ is an orthomodular lattice with the pointwise operations as follows: Suppose $f \in \{\wedge, \vee, \perp, 0, 1\}$ is an ℓ-ary operation and let $p_i \in \mathcal{F}(X, \mathcal{L})$ for $i < \ell$. Let the element $f^{\mathcal{F}}(p_0, \dots, p_{\ell-1})$ be defined as

$$f^{\mathcal{F}}(p_0, \dots, p_{\ell-1})(x) \doteq f^{\mathcal{L}}(p_0(x), \dots, p_{\ell-1}(x)),$$

for all $x \in X$. Note that $\mathcal{F}(X,\ \mathcal{L})$ is nothing else but the power $\mathcal{L}^X$.

Lemma 5.0.2.

(1) $\mathcal{S}(\mathfrak{X},\ \mathcal{L})$ is a subalgebra of $\mathcal{F}(X,\ \mathcal{L})$ and hence it is an orthomodular lattice.

(2) $\mathcal{L}$ can be completely embedded into $\mathcal{S}(\mathfrak{X},\ \mathcal{L})$.

Proof. **(1)** For simplicity denote $\mathcal{S}(\mathfrak{X},\ \mathcal{L})$ and $\mathcal{F}(X,\ \mathcal{L})$ respectively by $\mathcal{S}$ and $\mathcal{F}$. It is clear that $\mathcal{S} \subseteq \mathcal{F}$, thus we have to prove that $\mathcal{S}$ is closed under the operations of $\mathcal{F}$.

Suppose $f \in \{\wedge, \vee, \perp, \mathtt{0}, \mathtt{1}\}$ is an ℓ-ary operation and let $p_i \in \mathcal{S}$ for $i < \ell$. By definition of a step function, each p_i can be written in the following form

$$p_i = \bigsqcup_{j=0}^{\infty} \chi_{B_{i,j}}^{a_{i,j}},$$

where $a_{i,j} \in \mathcal{L}$, $B_{i,j} \in \Sigma$, such that for all i we have

$$X = \bigsqcup_{j=0}^{\infty} B_{i,j}.$$

Using that Σ is closed under countable intersections and unions, one can find a partition

$$X = \bigsqcup_{j=0}^{\infty} C_j$$

where $C_i \in \Sigma$ for all i and this partition is a common refinement of the partitions $\{B_{i,j}\}$. Thus we can conclude that each p_i is constant on each C_i:

$$p_i = \bigsqcup_{j=0}^{\infty} \chi_{C_j}^{a_{i,j}},$$

Then it is easy to see, that $f^{\mathcal{F}}(p_0, \ldots, p_{\ell-1})$ is also constant on each C_j:

$$f^{\mathcal{F}}(p_0, \ldots, p_{\ell-1}) = \bigsqcup_{j=0}^{\infty} \chi_{C_j}^{f^{\mathcal{L}}(a_{0,j}, \ldots, a_{\ell-1,j})}.$$

But this is a step function, hence belongs to $\mathcal{S}$. Thus we proved that $\mathcal{S}$ is closed under the operations of $\mathcal{F}$, and therefore it is a subalgebra of $\mathcal{F}$. Since subalgebras of orthomodular lattices are orthomodular lattices the proof is complete.

(2) Let $h : \mathcal{L} \to \mathcal{S}$ be defined as

$$h(a) \quad \doteq \quad \chi_X^a. \tag{11}$$

Then h is an embedding which preserves every operation (infinite ones as well). Checking this is a routine and omitted. ∎

Step 2.

Construction of $(\mathcal{S}, \rho)$. Let $(\mathcal{L},\ \phi)$ be a general and let $\mathfrak{X} = (X,\ \Sigma,\ \mu)$ be a classical probability space. Let $\mathcal{F} = \mathcal{F}(X, \mathcal{L})$ and $\mathcal{S} = \mathcal{S}(\mathfrak{X}, \mathcal{L})$ be as above. A function $f : X \to \mathcal{L}$ is called *μ-integrable* if

$$\phi \circ f : X \to [0, 1]$$

is (Σ, Λ)-measurable, where Λ is the Lebesgue σ-algebra of subsets of the unit interval. Then, because $\phi \circ f$ is non-negative, the integral $\int_X \phi \circ f \, d\mu$ exists. For μ-integrable functions $f \in \mathcal{F}$ we define

$$\rho(f) \doteq \int_X \phi \circ f \, d\mu.$$

Every step function is μ-integrable, hence $\rho(f)$ is defined for every element $f \in \mathcal{S}$. We prove that ρ is a probability measure on $\mathcal{S}$.

Lemma 5.0.3. Suppose that $f_0, f_1, \ldots$ are μ-integrable functions and suppose $\mathsf{lim}(f_i)$ exists in $\mathcal{F}$. Then

$$\rho\big(\mathsf{lim}(f_i)\big) \quad = \quad \lim_{i\to\infty} \rho(f_i). \tag{12}$$

Specifically, $\mathsf{lim}(f_i)$ is μ-integrable and ρ is a probability measure on $\mathcal{S}$.

Proof. Fix an arbitrary $x \in X$. Then

$$\phi\big(\mathsf{lim} f_i(x)\big) \quad = \quad \lim \phi(f_i(x)), \tag{13}$$

because ϕ is a measure on $\mathcal{L}$ and hence

$$\phi \circ \left(\mathsf{lim} f_i\right) \quad = \quad \lim \phi \circ f_i. \tag{14}$$

Then by definition

$$\rho\big(\mathsf{lim} f_i\big) = \int \phi \circ \left(\mathsf{lim} f_i\right) = \int \lim \phi \circ f_i \overset{!}{=} \lim \int \phi \circ f_i = \lim \rho(f_i),$$

where the equality marked with ! is a consequence of the monotone convergence theorem. $\rho(\mathsf{1}) = 1$ and $\rho(\mathsf{0}) = 0$, so ρ is normalized. ∎

Next, we prove that $(\mathcal{S},\ \rho)$ is an extension of $(\mathcal{L},\ \phi)$.

Lemma 5.0.4. $(\mathcal{S},\ \rho)$ is an extension of $(\mathcal{L},\ \phi)$.

Proof. We need to show that ρ extends $\phi \circ h$, where h is defined by (11):

$$\rho(h(a)) = \int_X \phi \circ \chi_X^a \, d\mu = \int_X \phi(a) \, d\mu = \phi(a) \cdot \int_X 1 \, d\mu = \phi(a)$$

∎

Step 3.

σ-complete extension of $(\mathcal{S}, \rho)$. Observe that $(\mathcal{S}, \rho)$ is not necessarily a general probability space because $\mathcal{S}$ is not, in general, σ-complete (albeit ρ is σ-additive). In what follows we prove that $\mathcal{S}$ can be extended to a σ-complete orthomodular lattice $\mathcal{Q}$ in such a way that $(\mathcal{Q}, \rho)$ is a general probability space (and thus an extension of $(\mathcal{L}, \phi)$).

Lemma 5.0.5. Let $\mathcal{B} \leq \mathcal{F}$ be a subalgebra of $\mathcal{F}$ such that every element of $\mathcal{B}$ is μ-integrable. Let $\{a_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}$ be a sequence such that $\mathsf{lim}(a_i)$ exists (in $\mathcal{F}$). Let

$$\mathcal{Y} = \langle \mathcal{B}, \mathsf{lim}(a_i) \rangle$$

be the generated subalgebra of $\mathcal{F}$. Then every element of $\mathcal{Y}$ is μ-integrable.

Proof. For simplicity, let $a = \mathsf{lim}(a_i)$. If $a \in \mathcal{B}$ then there is nothing to prove. Since $\mathcal{Y}$ is a generated algebra, every element $y \in \mathcal{Y}$ can be written in the form

$$y = t(b_0, \ldots, b_n),$$

where $b_i \in \mathcal{B} \cup \{a\}$ and t is a term in the algebraic language of $\mathcal{B}$. Clearly, if a doesn't occur in t, then $y \in \mathcal{B}$. So we may assume that

$$y = t(\bar{b}, a).$$

We need to prove that $t(\bar{b}, a)$ is μ-integrable. This we do by induction on the complexity of the term t. If t is a constant $b \in \mathcal{B}$ then it is μ-integrable since it belongs to $\mathcal{B}$. If t is the constant a, then it is μ-integrable by Lemma 5.0.3.

Now, suppose that $t_1(\bar{b}_1, a)$, …, $t_n(\bar{b}_n, a)$ are μ-integrable and let $f \in \{\wedge, \vee, \perp\}$ be any n-ary operation. Then, by continuity of $\mathcal{L}$, we have

$$f\Big(t_1(\bar{b}_1, a), \ldots, t_n(\bar{b}_n, a)\Big) = \mathsf{lim}_{i\to\infty} f\Big(t_1(\bar{b}_1, a_i), \ldots, t_n(\bar{b}_n, a_i)\Big),$$

which is a limit of elements of $\mathcal{B}$, hence by Lemma 5.0.3, it is μ-integrable. ∎

Define now a set K of subalgebras of $\mathcal{F}$ as follows:

$$\mathsf{K} = \big\{\mathcal{B} : \mathcal{S} \leq \mathcal{B} \leq \mathcal{F} \text{ and every elements of } \mathcal{B} \text{ are } \mu-\text{integrable}\big\}.$$

If $\mathcal{B}_\alpha$, $(\alpha < \kappa)$ is a chain from K, then $\bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$ also belongs to K, whence the assumption of Zorn's lemma is satisfied. Consequently, there exists a maximal subalgebra $\mathcal{Q} \in \mathsf{K}$. Because every elements of $\mathcal{Q}$ are μ-integrable the measure ρ is defined on $\mathcal{Q}$. We claim that $(\mathcal{Q}, \rho)$ is a general probability space.

Lemma 5.0.6. $(\mathcal{Q}, \rho)$ is a general probability space.

Proof. We only have to show that $\mathcal{Q}$ is σ-complete. Let $a_i \in \mathcal{Q}$ for $i \in \mathbb{N}$ such that $\bigvee_{i=0}^{N} a_i \in \mathcal{Q}$ for all natural number N. We should prove

$$\bigvee_{i=0}^{\infty} a_i \in \mathcal{Q}.$$

For if not, by Lemma 5.0.5, it follows that

$$\mathcal{Q} \leq \langle \mathcal{Q}, \bigvee_{i=0}^{\infty} a_i \rangle$$

is a proper superalgebra of $\mathcal{Q}$ such that any of its elements are μ-integrable. This contradicts to the maximality of $\mathcal{Q}$. ∎

Lemma 5.0.7. If $\mathfrak{X} = (X, \Sigma, \mu)$ is dense then so is $(\mathcal{Q}, \rho)$.

Proof. It is enough to prove that for all $f \in \mathcal{Q}$ with $0 < \rho(f)$ there exists $g \in \mathcal{Q}$ such that $g \leq f$ and $0 < \rho(g) < \rho(f)$ since using σ-completeness of $\mathcal{Q}$ and σ-additivity of ρ this ensures denseness.

Suppose f is a step function. Then there exists a measurable subset A of X (i.e. $A \in \Sigma$) with $\mu(A) > 0$ such that if

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ \mathsf{o} & \text{otherwise} \end{cases}$$

then g is a step function with $\rho(g) > 0$ and $g \leq f$. By denseness of μ, there is a smaller set $B \subset A$ with $0 < \mu(B) < \mu(A)$. Let

$$f'(x) = \begin{cases} \mathsf{o} & \text{if } x \in A \smallsetminus B \\ g(x) & \text{otherwise} \end{cases}$$

Then $0 < \rho(f') < \rho(f)$ and $f' \prec f$ is in S.

In the general case if $f \in \mathcal{Q}$ then by the construction of $\mathcal{Q}$ there is some step function $f' \in \mathcal{Q}$ with $f' \leq f$ and $0 < \rho(f') \leq \rho(f)$. Applying the previous argument to f' completes the proof. ∎

Proof of Proposition 4.2.2. Let $(\mathcal{L}, \phi)$ be a σ-continuous general probability space. Then $(\mathcal{Q}, \rho)$ constructed above with the choice $\mathfrak{X} = ([0, 1], \Lambda, \lambda)$ is a dense extension of $(\mathcal{L}, \phi)$. ∎

5.1 The extension in the classical and quantum case

We illustrate the extension method presented in Subsection 4.2 by showing how it works in classical and quantum probability spaces. Since for these cases a simpler method exist (see Subsection 4.1) we will be brief and sketchy.

Classical Case. Since every distributive lattice is σ-continuous, Proposition 4.2.2 can directly applied to semi-classical spaces. For classical spaces it is enough to observe that the space $(\mathcal{Q}, \rho)$ is set-represented provided $(\mathcal{L}, \rho)$ is set-represented. A somewhat detailed form of the construction in this case can be found in [5].

Quantum Case. Fix a quantum probability space $(\mathcal{P}(\mathcal{M}),\ \phi)$, where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra acting on the separable Hilbert space $\mathcal{H}$, $\mathcal{P}(\mathcal{M})$ is the lattice of projections of $\mathcal{M}$ and ϕ a normal state on $\mathcal{M}$. We may assume, in fact, that ϕ is a normal state on $\mathcal{B}(\mathcal{H})$. Further fix a classical probability space $\mathfrak{X} = (X,\ \Sigma,\ \mu)$, where X is a set, Σ is a σ-algebra of subsets of X and μ is a probability measure on Σ. Throughout we will have $\mathfrak{X} = ([0,1],\ \Lambda,\ \lambda)$ in mind, where Λ is the σ-algebra of Lebesgue-measurable subsets of the unit interval, and λ is the Lebesgue measure.

We start by recalling some definitions and facts from Hilbert space theory, e.g. from [2], [12] and [23].

Definition 5.1.1. A function $s : X \to \mathcal{H}$ is called *simple* if it is of the following form:

$$s = \sum_{i=1}^{\ell} h_i \chi_{E_i},$$

where $\ell \in \mathbb{N}$, $h_i \in \mathcal{H}$ and the sets E_i form a partition of X such that each E_i is measurable ($E_i \in \Sigma$). χ_E is the characteristic function of the set E.

A function $u : X \to \mathcal{H}$ is called *strongly measurable*, if there is a sequence $\{s_n\}_{i\in\mathbb{N}}$ of simple functions such that

$$\lim \|s_n(x) - u(x)\|_{\mathcal{H}} = 0 \quad \text{for } \mu\text{-almost all } x \in X.$$

If $u : X \to \mathcal{H}$ is strongly measurable, then the map $x \mapsto \|u(x)\|_{\mathcal{H}}$ is measurable in the classical sense, i.e. (Σ, Λ)-measurable.

Definition 5.1.2. We define $\mathfrak{H} = L^2(\mathfrak{X}, \mathcal{H})$ as follows:

$$L^2(\mathfrak{X}, \mathcal{H}) = \Big\{ u : X \to \mathcal{H} :\ u \text{ is strongly measurable and } \|u\|_{\mathfrak{H}} < \infty \Big\},$$

where $\|u\|_{\mathfrak{H}}$ for a strongly measurable $u : X \to \mathcal{H}$ is defined as

$$\|u\|_{\mathfrak{H}} = \left(\int_X \|u(x)\|_{\mathcal{H}}^2 \, d\mu(x) \right)^{1/2}.$$

As usual, we identify functions which are almost everywhere the same. If, for the elements $x, y \in \mathfrak{H}$ we let

$$(x, y) = \int_X (x(t), y(t))_{\mathcal{H}} \, d\mu(t),$$

then this defines a scalar product which generates the norm $\|\cdot\|_{\mathfrak{H}}$.

By Theorem 2.110 of [2], $(\mathfrak{H}, \|\cdot\|_{\mathfrak{H}})$ is a Banach space in which the set of simple functions is dense. Consequently, $\mathfrak{H}$ is a Hilbert space. Further, if X is a separable metric space, μ is a Radon measure and $\mathcal{H}$ is separable, then $\mathfrak{H}$ is separable, too. This will be the case when $\mathfrak{X}$ is the Lebesgue-space.

If $\mathfrak{H}$ is separable, then it is the direct integral of $\{\mathcal{H}\}_{x\in X}$ over $\mathfrak{X}$ in the sense of Definition 14.1.1 of [12]:

$$\mathfrak{H} = \int_X^{\oplus} \mathcal{H}\, d\mu.$$

For the rest part of this section, we assume that $\mathfrak{H}$ is separable.

Definition 5.1.3. An operator $T \in \mathcal{B}(\mathfrak{H})$ is said to be *decomposable* if there are operators $T_x \in \mathcal{B}(\mathcal{H})$ for $x \in X$ such that for each $a \in \mathfrak{H}$ we have

$$(Ta)(x) = T_x a(x) \quad \text{for almost all } x \in X.$$

In this case, the system $\langle T_x : x \in X\rangle$ is called a *decomposition* of T and we write

$$T = \langle T_x : x \in X\rangle.$$

Components of the decomposition are almost everywhere unique.

In particular, the identity operator $I_{\mathfrak{H}} \in \mathcal{P}(\mathfrak{H})$ is decomposable with the decomposition

$$I_{\mathfrak{H}} = \langle I_{\mathcal{H}} :\ x \in X\rangle.$$

In a similar manner we can define for each $T \in \mathcal{B}(\mathcal{H})$ an operator $h(T)$ as follows:

$$h(T) = \langle T :\ x \in X\rangle,$$

that is, for $a \in \mathfrak{H}$ the action of $h(T)$ is defined as

$$(h(T)a)(x) \doteq Ta(x) \quad \text{for all } x \in X.$$

It is not hard to see that $h(T) \in \mathcal{B}(\mathfrak{H})$ for all $T \in \mathcal{B}(\mathcal{H})$.

By Theorem 14.1.10 of [12], the set $\mathcal{R} \subseteq \mathcal{B}(\mathfrak{H})$ of decomposable operators is a von Neumann algebra acting on $\mathfrak{H}$. Now the map $h : \mathcal{B}(\mathcal{H}) \hookrightarrow \mathcal{R}$ is a $*$-algebra embedding with $h[\mathcal{P}(\mathcal{H})] \subseteq \mathcal{P}(\mathcal{R})$ and, in particular, as a function $h : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{R})$ it is a lattice embedding preserving $\perp$ as well.

Finally, we define a normal state ψ of $\mathcal{R}$ as follows:

$$\psi(T) \doteq \int_X \phi(T_x)\, d\mu(x) \quad \text{for all } T = \langle T_x :\ x \in X\rangle \in \mathcal{R}.$$

Then $(\mathcal{P}(\mathcal{R}),\ \psi)$ is an extension of $(\mathcal{P}(\mathcal{M}),\ \phi)$ in the sense of Definition 2.6. If $\mathfrak{X}$ is dense then so is $(\mathcal{P}(\mathcal{R}),\ \psi)$.

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