Refutation of Richard Gill's Argument Against my Disproof of Bell's Theorem

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I identify a number of errors in Richard Gill's purported refutation of my disproof of Bell's theorem. In particular, I point out that his central argument is based, not only on a rather trivial misreading of my counterexample to Bell's theorem, but also on a simple oversight of a freedom of choice in the orientation of a Clifford algebra. What is innovative and original in my counterexample is thus mistaken for an error, at the expense of the professed universality and generality of Bell's theorem.

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I. INTRODUCTION

In a recent preprint [1] Richard Gill has suggested that there is an algebraic error in my disproof of Bell's theorem [2][3][4][5][6][7][8][9][10][11][12]. In what follows I show that it is in fact Gill's argument that is in error, stemming from a rather trivial misreading of my disproof, as well as from a misconstrual of a basic freedom in geometric algebra. For example, in the abstract of both versions of his preprint he refers to the quantity $-\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\wedge\mathbf{b}$ as a "bivector." Moreover, by his own admission Gill has not read any of my papers on the subject beyond the minimalist one-page paper [3]. This is unfortunate. For had he read my papers carefully he would have recognized his errors himself.

Considering these facts, let me set the stage by recalling some of the background to my counterexample on which Gill's argument relies [1]. Although based on geometric algebra [13], my counterexample is in fact a hidden variable model, with the hidden variable being the initial orientation of a parallelized 3-sphere [2][3]. In other words, the initial orientation of the physical space itself is taken as a hidden variable in the model [6]. With this in mind, consider a right-handed frame of ordered basis bivectors, $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$, and the corresponding bivector (or "even") subalgebra

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\,=\,-\,\delta_{jk}\,-\,\epsilon_{jkl}\,\boldsymbol{\beta}_l \tag{1}$$

of the Clifford algebra $Cl_{3,0}$ [4]. The latter is a linear vector space $\mathrm{I\!R}^8$ spanned by the orthonormal basis

$$\{1,\;\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z,\;\mathbf{e}_x\wedge\mathbf{e}_y,\,\mathbf{e}_y\wedge\mathbf{e}_z,\,\mathbf{e}_z\wedge\mathbf{e}_x,\;\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{2}$$

where δ_{jk} is the Kronecker delta, ϵ_{jkl} is the Levi-Civita symbol, the indices $j,\,k,\,l=x,\,y,$ or z are cyclic indices, and

$$\boldsymbol{\beta}_j\,=\,\mathbf{e}_k\wedge\mathbf{e}_l\,=\,I\cdot\mathbf{e}_j\,. \tag{3}$$

Eq. (1) is a conventional expression of bivector subalgebra, routinely used in the literature employing geometric algebra for computations [13]. From equation (1) it is easy to verify the familiar properties of the basis bivectors, such as

$$(\boldsymbol{\beta}_x)^2\,=\,(\boldsymbol{\beta}_y)^2\,=\,(\boldsymbol{\beta}_z)^2\,=\,-\,1 \tag{4}$$

$$\text{and}\quad\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,=\,-\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_x\quad\text{etc.} \tag{5}$$

Moreover, it is easy to verify that the bivectors satisfying the subalgebra (1) form a right-handed frame of basis bivectors. To this end, right-multiply both sides of Eq. (1) by $\boldsymbol{\beta}_l$, and then use the fact that $(\boldsymbol{\beta}_l)^2=-1$ to arrive at

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\,\boldsymbol{\beta}_l\,=\,+\,1\,. \tag{6}$$

The fact that this ordered product yields a positive value confirms that $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$ indeed forms a right-handed frame of basis bivectors. This is a universally accepted convention, found in any textbook on geometric algebra [13].

Suppose now $\mathbf{a}=a_j\,\mathbf{e}_j$ and $\mathbf{b}=b_k\,\mathbf{e}_k$ are two unit vectors in $\mathrm{I\!R}^3$, where the repeated indices are summed over $x,\,y,$ and z. Then the right-handed basis defined in Eq. (1) leads to

$$\{\,a_j\;\boldsymbol{\beta}_j\,\}\,\{\,b_k\;\boldsymbol{\beta}_k\,\}\,=\,-\,a_j\,b_k\,\delta_{jk}\,-\,\epsilon_{jkl}\;a_j\,b_k\;\boldsymbol{\beta}_l\,, \tag{7}$$

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which, together with (3), is equivalent to

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b})\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,-\,I\cdot(\mathbf{a}\times\mathbf{b}),\tag{8}$$

where $I=\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ is the standard trivector. Geometrically this identity describes all points of a parallelized 3-sphere.

Let us now consider a left-handed frame of ordered basis bivectors, which we also denote by $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$. It is important to recognize, however, that there is no *a priori* way of knowing that this new basis frame is in fact left-handed. To ensure that it is indeed left-handed we must first make sure that it is an ordered frame by requiring that its basis elements satisfy the bivector properties delineated in Eqs. (4) and (5). Next, to distinguish this frame from the right-handed frame defined by equation (6), we must require that its basis elements satisfy the property

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\,\boldsymbol{\beta}_l\;=\;-\,1\,.\tag{9}$$

One way to ensure this is to multiply every non-scalar element in (2) by a minus sign. Then, instead of (3), we have

$$\boldsymbol{\beta}_j\;=\;-\,\mathbf{e}_k\wedge\mathbf{e}_l\;=\;(\,-\,I\,)\cdot(\,-\,\mathbf{e}_j\,)\;=\;I\cdot\mathbf{e}_j\,,\tag{10}$$

and the condition (9) is automatically satisfied. As is well known, this was the condition imposed by Hamilton on his unit quaternions, which we now know are nothing but a left-handed set of basis bivectors [13]. Indeed, it can be easily checked that the basis bivectors satisfying the properties (4), (5), (9), and (10) compose the subalgebra

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\;=\;-\,\delta_{jk}\,+\,\epsilon_{jkl}\,\boldsymbol{\beta}_l\,.\tag{11}$$

Conversely, it is easy to check that the basis bivectors defined by this subalgebra do indeed form a left-handed frame. To this end, right-multiply both sides of Eq. (11) by $\boldsymbol{\beta}_l$, and then use the property $(\boldsymbol{\beta}_l)^2=-1$ to verify Eq. (9). As is well known, this subalgebra is generated by the unit quaternions originally proposed by Hamilton [13]. It is routinely used in the textbook treatments of angular momenta, but without mentioning the fact that it defines nothing but a left-handed set of basis bivectors. It may look more familiar if we temporarily change notation and rewrite Eq. (11) as

$$\mathbf{J}_j\,\mathbf{J}_k\;=\;-\,\delta_{jk}\,+\,\epsilon_{jkl}\,\mathbf{J}_l\,.\tag{12}$$

More importantly (and especially since Richard Gill seems to have missed this point), I stress once again that there is no way to set apart the left-handed frame of basis bivectors from the right-handed frame without appealing to the intrinsically defined distinguishing conditions (6) and (9), or equivalently to the corresponding subalgebras (1) and (11). Note also that at no time within my framework the two subalgebras (1) and (11) are mixed in any way, either physically or mathematically. They merely play the role of two distinct and alternative hidden variable possibilities.

Suppose now $\mathbf{a}=a_j\,\mathbf{e}_j$ and $\mathbf{b}=b_k\,\mathbf{e}_k$ are two unit vectors in $\mathrm{I\!R}^3$, where the repeated indices are summed over x, y, and z. Then the left-handed basis equation (11) leads to

$$\{\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\}\;=\;-\,a_j\,b_k\,\delta_{jk}\,+\,\epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l\tag{13}$$

which, together with (10), is equivalent to

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b})\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,+\,I\cdot(\mathbf{a}\times\mathbf{b}),\tag{14}$$

where I is the standard trivector. Once again, geometrically this identity describes all points of a parallelized 3-sphere.

It is important to note, however, that there is a sign difference in the second term on the RHS of the identities (8) and (14). The algebraic meaning of this sign difference is of course clear from the above discussion, and it has been discussed extensively in most of my papers [4][5][6][8][12], with citations to prior literature [14]. But from the perspective of my model a more important question is: What does this sign difference mean *geometrically*? To bring out its geometric meaning, let us rewrite the identities (8) and (14) as

$$(\,+\,I\cdot\mathbf{a})(\,+\,I\cdot\mathbf{b})\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,-\,(\,+\,I\,)\cdot(\mathbf{a}\times\mathbf{b})\tag{15}$$

and

$$(\,-\,I\cdot\mathbf{a})(\,-\,I\cdot\mathbf{b})\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,-\,(\,-\,I\,)\cdot(\mathbf{a}\times\mathbf{b}),\tag{16}$$

respectively. The geometrical meaning of the two identities is now transparent if we recall that the bivector $(\,+\,I\cdot\mathbf{a})$ represents a counterclockwise rotation about the **a**-axis, whereas the bivector $(\,-\,I\cdot\mathbf{a})$ represents a clockwise rotation about the **a**-axis. Accordingly, both identities interrelate the points of a unit parallelized 3-sphere, but the identity

(15) interrelates points of a positively oriented 3-sphere whereas the identity (16) interrelates points of a negatively oriented 3-sphere. In other words, the 3-sphere represented by the identity (15) is oriented in the counterclockwise sense, whereas the 3-sphere represented by the identity (16) is oriented in the clockwise sense. These two alternative orientations of the 3-sphere is then the random hidden variable $\lambda = \pm 1$ (or the initial state $\lambda = \pm 1$) within my model.

Given this geometrical picture, it is now easy to appreciate that identity (15) corresponds to the physical space characterized by the trivector $+I$, whereas identity (16) corresponds to the physical space characterized by the trivector $-I$ [14]. This is further supported by the evident fact that, apart from the choice of a trivector, the identities (15) and (16) represent one and the same subalgebra. Moreover, there is clearly no *a priori* reason for Nature to choose $+I$ as a fundamental trivector over $-I$. Either choice provides a perfectly legitimate representation of the physical space, and neither is favored by Nature. Consequently, instead of characterizing the physical space by fixed basis (2), we can start out with two alternatively possible characterizations of the physical space by the *hidden* basis

$$\{1,\ \mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z,\ \mathbf{e}_x \wedge \mathbf{e}_y,\, \mathbf{e}_y \wedge \mathbf{e}_z,\, \mathbf{e}_z \wedge \mathbf{e}_x,\ \lambda\,(\,\mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z\,)\}, \tag{17}$$

where $\lambda = \pm 1$. Although these considerations and the physical motivations behind them have been the starting point of my program (see, for example, discussions in Refs. [4], [8], and [11]), Gill seems to have overlooked them completely.

Exploiting the natural freedom of choice in characterizing S^3 by either $+I$ or $-I$, we can now combine the identities (15) and (16) into a single hidden variable equation (at least for the computational purposes):

$$(\,\lambda I \cdot \mathbf{a})(\,\lambda I \cdot \mathbf{b}) \,=\, -\,\mathbf{a} \cdot \mathbf{b} \,-\, (\,\lambda I\,) \cdot (\mathbf{a} \times \mathbf{b}), \tag{18}$$

where $\lambda = \pm 1$ now specifies the orientation of the 3-sphere. It is important to recognize that the difference between the trivectors $+I$ and $-I$ in this equation primarily reflects the difference in the handedness of the bivector basis $\{\boldsymbol{\beta}_x,\, \boldsymbol{\beta}_y,\, \boldsymbol{\beta}_z\}$, and not in the handedness of the vector basis $\{\,\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z\}$. This should be evident from the foregone arguments, but let us bring this point home by considering the following change in the handedness of the vector basis:

$$+\,I \,=\, \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z \ \ \longrightarrow\ \ (+\,\mathbf{e}_x)(-\,\mathbf{e}_y)(+\,\mathbf{e}_z) \,=\, -\,(\,\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,) \,=\, -\,I\,. \tag{19}$$

Such a change does not induce a change in the handedness of the bivector basis, since it leaves the product $\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z$ unchanged. This can be easily verified by recalling that $\boldsymbol{\beta}_x \equiv I \cdot \mathbf{e}_x$, $\boldsymbol{\beta}_y \equiv I \cdot \mathbf{e}_y$, and $\boldsymbol{\beta}_z \equiv I \cdot \mathbf{e}_z$, and consequently

$$+\,1 \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \ \ \longrightarrow\ \ (\,-\,I\,) \cdot (+\,\mathbf{e}_x)\,(\,-\,I\,) \cdot (-\,\mathbf{e}_y)\,(\,-\,I\,) \cdot (+\,\mathbf{e}_z) \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \,=\, +1\,. \tag{20}$$

Conversely, a change in the handedness of bivector basis does not necessarily affect a change in the handedness of vector basis, but leads instead to

$$+\,1 \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \ \ \longrightarrow\ \ (\,-\,\boldsymbol{\beta}_x)\,(\,-\,\boldsymbol{\beta}_y)\,(\,-\,\boldsymbol{\beta}_z) \,=\, -\,(\,\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z\,) \,=\, -1\,, \tag{21}$$

which in turn leads us back to equation (18) via equations (8) and (14). Thus the sign difference between the trivectors $+I$ and $-I$ captured in equation (18) arises from the sign difference in the product $\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z$ and not from that in the product $\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$. It is therefore of very different geometrical significance [5]. It corresponds to the difference between two possible orientations of the 3-sphere mentioned above. It is also important to keep in mind that the combined equation (18) is simply a convenient shortcut for representing two completely independent initial states of the system, one corresponding to the counterclockwise orientation of the 3-sphere and the other corresponding to the clockwise orientation of the 3-sphere. Moreover, at no time these two alternative possibilities are mixed during the course of an experiment. They represent two independent physical scenarios, corresponding to two independent runs of the experiment. If we now use the notation $\boldsymbol{\mu} = \lambda\,I$, then the combined identity (18) takes the convenient form

$$(\,\boldsymbol{\mu} \cdot \mathbf{a})(\,\boldsymbol{\mu} \cdot \mathbf{b}) \,=\, -\,\mathbf{a} \cdot \mathbf{b} \,-\, \boldsymbol{\mu} \cdot (\mathbf{a} \times \mathbf{b}). \tag{22}$$

If one remains uncomfortable about using this unconventional identity, then there is always the option of working directly with the bivector basis themselves, and that is what I do in my one-page paper [3]. Accordingly, let us return to equations (1) and (11) and start afresh by writing the basic hidden variable equation of the model as

$$\boldsymbol{\beta}_j(\lambda)\,\boldsymbol{\beta}_k(\lambda) \,=\, -\,\delta_{jk} \,-\, \epsilon_{jkl}\,\boldsymbol{\beta}_l(\lambda)\,, \tag{23}$$

with $\boldsymbol{\beta}_j(\lambda) = \lambda\,\boldsymbol{\beta}_j$ and $\lambda = \pm 1$ as a fair coin representing the two possible orientations of the 3-sphere. Note that upon substituting $\boldsymbol{\beta}_j(\lambda) = \lambda\,\boldsymbol{\beta}_j$ and using $\lambda^2 = +1$ the above equation can also be written as

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \,=\, -\,\delta_{jk} \,-\, \lambda\,\epsilon_{jkl}\,\boldsymbol{\beta}_l\,, \tag{24}$$

which in turn, for $\lambda = +1$, specializes to

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \,=\, -\,\delta_{jk} \,-\, \epsilon_{jkl}\,\boldsymbol{\beta}_l\,. \tag{25}$$

As elementary as they are, the last three equations are a major source of confusion in Gill's preprint.

To see the equivalence of these equations with the hidden variable identity (22), let $\mathbf{a} = a_j\,\mathbf{e}_j$ and $\mathbf{b} = b_k\,\mathbf{e}_k$ be two unit vectors in $\mathrm{I\!R}^3$. Using either equation (24) or (23) and $\boldsymbol{\beta}_j(\lambda) = \lambda\,\boldsymbol{\beta}_j$ we then have

$$\{\,a_j\;\boldsymbol{\beta}_j(\lambda)\,\}\,\{\,b_k\;\boldsymbol{\beta}_k(\lambda)\,\} \,=\, \{\,\lambda\,a_j\;\boldsymbol{\beta}_j\,\}\,\{\,\lambda\,b_k\;\boldsymbol{\beta}_k\,\} \,=\, -\,a_j\,b_k\,\delta_{jk} \,-\, \lambda\,\epsilon_{jkl}\;a_j\,b_k\;\boldsymbol{\beta}_l\,, \tag{26}$$

which is equivalent to the identity (22) with $(\,\boldsymbol{\mu}\cdot\mathbf{a}) \equiv \{\,\lambda\;a_j\;\boldsymbol{\beta}_j\,\}$, $\;\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}) \equiv \{\,\lambda\;\epsilon_{jkl}\;a_j\,b_k\;\boldsymbol{\beta}_l\,\}$, etc.

Next, let us define the measurement results observed by Alice and Bob within S^3 as

$$S^3 \ni \mathscr{A}(\mathbf{a},\,\lambda) \,=\, \{-\,a_j\;\boldsymbol{\beta}_j\,\}\,\{\,a_k\;\boldsymbol{\beta}_k(\lambda)\,\} \,=\, (\,-I\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{a}) \,=\, \begin{cases} +1 & \text{if} \quad \lambda \,=\, +1 \\ -\,1 & \text{if} \quad \lambda \,=\, -\,1 \end{cases} \tag{27}$$

and

$$S^3 \ni \mathscr{B}(\mathbf{b},\,\lambda) \,=\, \{\,b_j\;\boldsymbol{\beta}_j(\lambda)\,\}\,\{+\,b_k\;\boldsymbol{\beta}_k\,\} \,=\, (\,\boldsymbol{\mu}\cdot\mathbf{b})(\,+I\cdot\mathbf{b}) \,=\, \begin{cases} -\,1 & \text{if} \quad \lambda \,=\, +\,1 \\ +1 & \text{if} \quad \lambda \,=\, -\,1\,. \end{cases} \tag{28}$$

It is important to note that these measurement results are generated as products of two numbers, the fixed bivectors $(\,I\cdot\mathbf{n})$ times the random bivectors $(\,\boldsymbol{\mu}\cdot\mathbf{n})$. In other words, as discussed in greater detail in Refs. [2] and [5], they are generated with *different* bivectorial scales of dispersion for each measurement directions $\mathbf{a}$ and $\mathbf{b}$. Consequently, as discovered by Galton and Pearson [15] over a century ago, the correct correlation between the raw numbers $\mathscr{A}$ and $\mathscr{B}$ (as observed and manipulated by experimentalists) can only be inferred – *theoretically* – by calculating the covariance of the corresponding standardized variables $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}^i)\;\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}^i)\right] \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,(\,\boldsymbol{\mu}^i\cdot\mathbf{a})(\,\boldsymbol{\mu}^i\cdot\mathbf{b})\right] \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,. \tag{29}$$

Here the last equality immediately follows from the identity (22), and the standardized variables are calculated as

$$\begin{aligned} A(\mathbf{a},\,\boldsymbol{\mu}) &= \frac{\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}) \,-\, \overline{\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})}}{\sigma(\mathscr{A})} \\ &= \frac{\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}) \,-\, 0}{(-\,I\cdot\mathbf{a}\,)} \,=\, (\,+\,\boldsymbol{\mu}\cdot\mathbf{a}\,) \end{aligned} \tag{30}$$

and

$$\begin{aligned} B(\mathbf{b},\,\boldsymbol{\mu}) &= \frac{\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}) \,-\, \overline{\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})}}{\sigma(\mathscr{B})} \\ &= \frac{\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}) \,-\, 0}{(+\,I\cdot\mathbf{b}\,)} \,=\, (\,+\,\boldsymbol{\mu}\cdot\mathbf{b}\,), \end{aligned} \tag{31}$$

with $\sigma(\mathscr{A}) = (-\,I\cdot\mathbf{a}\,)$ and $\sigma(\mathscr{B}) = (+\,I\cdot\mathbf{b}\,)$, respectively, being the standard deviations in the results $\mathscr{A}$ and $\mathscr{B}$. The above result may be seen more transparently by recalling that $(\,\boldsymbol{\mu}\cdot\mathbf{a}) \equiv \{\,\lambda\;a_j\;\boldsymbol{\beta}_j\,\}$ and $(\,\boldsymbol{\mu}\cdot\mathbf{b}) \equiv \{\,\lambda\;b_k\;\boldsymbol{\beta}_k\,\}$, so that

$$\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\;a_j\;\boldsymbol{\beta}_j\,\}\;\{\,\lambda^i\;b_k\;\boldsymbol{\beta}_k\,\}\right] \,=\, -\,a_j\,b_j \,-\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\;\epsilon_{jkl}\;a_j\,b_k\;\boldsymbol{\beta}_l\,\}\right] \,=\, -\,a_j\,b_j \,+\, 0 \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \tag{32}$$

$$\text{and} \qquad \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\;a_j\;\boldsymbol{\beta}_j\,\}\right] \,=\, 0 \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\;b_k\;\boldsymbol{\beta}_k\,\}\right]. \tag{33}$$

It is important to remember that what is being summed over here are points of a parallelized 3-sphere representing the outcomes of completely independent experimental runs in an EPR-Bohm experiment. In statistical terms what these results are then showing is that correlation between the raw numbers $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}) \,=\, (-\,I\cdot\mathbf{a}\,)\,(\,+\,\boldsymbol{\mu}\cdot\mathbf{a}\,) \,=\, \pm\,1 \in S^3$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}) \,=\, (+\,I\cdot\mathbf{b}\,)\,(\,+\,\boldsymbol{\mu}\cdot\mathbf{b}\,) \,=\, \pm\,1 \in S^3$ is $-\,\mathbf{a}\cdot\mathbf{b}$. According to Bell's theorem this is mathematically impossible. Further physical, mathematical, and statistical details of this "impossible" result can be found in Refs. [3] and [5].

II. A FALLACY OF MISPLACED CONCRETENESS[1]

With this background, we are now in a position to appreciate where the confusion in Gill's argument stems from. To begin with, he has failed to understand what the hidden variable λ is in my model. He does not seem to realize that rejecting the λ described above as a hidden variable (for that is what his argument boils down to) is equivalent to rejecting the professed universality and generality of Bell's theorem. This is, however, not so easy to see if one insists on neglecting my substantive papers on the subject and concentrates solely on the minimalist one-page paper.

Secondly, Gill has missed the reciprocal relation between the two sets of basis defined in equations (23) and (25):

$$\boldsymbol{\beta}_j(\lambda) \,=\, \lambda\,\boldsymbol{\beta}_j \iff \boldsymbol{\beta}_j \,=\, \lambda\,\boldsymbol{\beta}_j(\lambda). \tag{34}$$

This relation holds simply because $\lambda = \pm 1$ implies $\lambda^2 = +1$. Just as the relation $\boldsymbol{\beta}_j(\lambda) = \lambda\boldsymbol{\beta}_j$ encapsulates the randomness of the basis $\{\boldsymbol{\beta}_j(\lambda)\}$ with respect to the basis $\{\boldsymbol{\beta}_j\}$, the reciprocal relation $\boldsymbol{\beta}_j = \lambda\boldsymbol{\beta}_j(\lambda)$ encapsulates the randomness of the basis $\{\boldsymbol{\beta}_j\}$ with respect to the basis $\{\boldsymbol{\beta}_j(\lambda)\}$. As we shall soon see, oversight of this very simple reciprocal relationship between the bases $\{\boldsymbol{\beta}_j\}$ and $\{\boldsymbol{\beta}_j(\lambda)\}$ invalidates the central contention of Gill rather trivially.

The third source of confusion in Gill's preprint is the equation (23) defined above, namely

$$\boldsymbol{\beta}_j(\lambda)\,\boldsymbol{\beta}_k(\lambda) \,=\, -\,\delta_{jk} \,-\, \epsilon_{jkl}\,\boldsymbol{\beta}_l(\lambda)\,. \tag{35}$$

This equation, which purportedly appears as equation (2) in his preprint, is *incorrectly* presented therein as

$$\boldsymbol{\beta}_j(\lambda)\,\boldsymbol{\beta}_k(\lambda) \,=\, -\,\delta_{jk} \,-\, \lambda\,\epsilon_{jkl}\,\boldsymbol{\beta}_l(\lambda)\,. \tag{36}$$

He attributes this equation to me, but it has nothing to do with my model. Gill's mistake here is not as innocuous as it may seem at first sight. What he has failed to recognize is that in the former equation λ represents a hidden variable (*i.e.*, the initial orientation of a parallelized 3-sphere), whereas in the latter equation it represents a mere convention. This reveals that Gill has *fundamentally* misunderstood my model [4]. Thus, mistaken reading of (35), neglect of the significance of (34), neglect of all my substantive papers but the one-page summary paper, and unfamiliarity with elementary Clifford algebra seems to have led Gill to erroneously conclude that there is an error in my paper.

There is yet another reason why Gill has been led to this fallacious conclusion. Deviating from the explicit physical model described above, he insists on unnaturally identifying the actually observed numbers $\mathscr{A}$ and $\mathscr{B}$ with the hidden variable λ by *illegally* treating $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ as purely algebraic variables rather than statistical variables. But such an identification is anathema, not only from the physical point of view, but also from the statistical point of view. $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ are two *different* functions of the random variable λ. What is more, they necessarily describe two *statistically independent events* occurring within a parallelized 3-sphere. Therefore the joint probability of their occurrence is given by $P(\mathscr{A}\text{ and }\mathscr{B}) = P(\mathscr{A}) \times P(\mathscr{B}) \leq \frac{1}{2}$. And their product, $\mathscr{A}\mathscr{B}(\mathbf{a},\,\mathbf{b},\,\lambda)$, which itself is necessarily a *different* random variable, is guaranteed to be equal to -1 only for the case $\mathbf{a} = \mathbf{b}$. For all other $\mathbf{a}$ and $\mathbf{b}$, $\mathscr{A}\mathscr{B}$ will inevitably alternate between the values -1 and $+1$, since the numbers $\mathscr{A}$ and $\mathscr{B}$ are being generated with *different* bivectorial scales of dispersion. This is evidently confirmed by the correlation $-\mathbf{a}\cdot\mathbf{b}$ derived in Eq. (32).

Notwithstanding, let us play along Gill's illegal game to see where it leads. Let us unnaturally and unphysically set

$$\mathscr{A}(\mathbf{a},\,\lambda) \,=\, +\,\lambda \quad\text{and}\quad \mathscr{B}(\mathbf{b},\,\lambda) \,=\, -\,\lambda \tag{37}$$

so that $\mathscr{A}\mathscr{B} = -1$ for all $\mathbf{a}$ and $\mathbf{b}$. Now, to begin with, this immediately leads to $\sigma(\mathscr{A}) = \sigma(\mathscr{B}) = +1$. This is yet another indication that the above identification is not only illegal, but has also nothing to do with my model. What is more, since the unnatural identification inevitably leads to the conclusion that $\mathscr{A}\,\mathscr{B} = -1$ for all $\mathbf{a}$ and $\mathbf{b}$, it is at variance, not only with the basic rules of statistical inference, but also with the basic topological properties of the 3-sphere [2][5][6]. More specifically, what Gill has failed to recognize is that, as noted above, $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ are generated within my model with *different* bivectorial scales of dispersion, and hence the correct correlation between them can be inferred only by calculating the covariation of the corresponding standardized variables $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}^i)\,\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}^i)\right] \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,(\,\boldsymbol{\mu}^i\cdot\mathbf{a})(\,\boldsymbol{\mu}^i\cdot\mathbf{b})\right] \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,. \tag{38}$$

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[1] A fallacy of neglecting the degree of abstraction involved in a thought that leads to an unwarranted conclusion about a concrete entity.

I have explained the relationship between raw scores and standard scores in greater detail in Ref. [5], with explicit calculations for the optical EPR correlations observed in both Orsay and Innsbruck experiments [16].

But I am digressing. Let us continue to play along Gill's unnatural game and recalculate the correlation between $\mathscr{A}$ and $\mathscr{B}$. For completeness, let us first rewrite how this correlation has been calculated in my one-page paper [3]:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \frac{\lim_{\,n\,\gg 1}\left\{\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^i)\,\mathscr{B}(\,\mathbf{b},\,\lambda^i)\right\}}{\{-\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\}} \,=\, \lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\frac{\mathscr{A}(\mathbf{a},\,\lambda^i)\,\mathscr{B}(\mathbf{b},\,\lambda^i)}{\{-\,\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\}}\right] \tag{39}$$

$$=\,\lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,\mathscr{A}(\mathbf{a},\,\lambda^i)\,\mathscr{B}(\mathbf{b},\,\lambda^i)\}\,\{-\,b_k\,\boldsymbol{\beta}_k\,\}\right] \,=\, \lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\left\{\,a_j\,\boldsymbol{\beta}_j(\lambda^i)\,\right\}\,\left\{\,b_k\,\boldsymbol{\beta}_k(\lambda^i)\,\right\}\right] \tag{40}$$

$$=\,-\,a_j\,b_j\,-\,\lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\left\{\,\lambda^i\,\epsilon_{jkl}\,\,a_j\,b_k\,\boldsymbol{\beta}_l\,\right\}\right] \,=\, -\,a_j\,b_j\,+\,0\,=\,-\,\mathbf{a}\cdot\mathbf{b}\,, \tag{41}$$

where I have used Eq. (23) and $\boldsymbol{\beta}_j(\lambda)\,=\,\lambda\,\boldsymbol{\beta}_j$ from Eq. (34) in the last two lines of the derivation.

Now, complying with Gill's specious demand, if we set $\mathscr{A}\mathscr{B}\,=\,-\,1$ for all $\mathbf{a}$ and $\mathbf{b}$, then instead of the above derivation we have the following:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \frac{\lim_{\,n\,\gg 1}\left\{\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^i)\,\mathscr{B}(\,\mathbf{b},\,\lambda^i)\right\}}{\{-\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\}} \,=\, \lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\frac{-1}{\{-\,\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\}}\right] \tag{42}$$

$$=\,\lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,a_j\,\boldsymbol{\beta}_j\,\}\,\{-1\}\,\{-\,b_k\,\boldsymbol{\beta}_k\,\}\right] \,=\, \lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\left\{\,a_j\,\boldsymbol{\beta}_j\,\right\}\,\left\{\,b_k\,\boldsymbol{\beta}_k\,\right\}\right] \tag{43}$$

$$=\,-\,a_j\,b_j\,-\,\lim_{n\,\gg 1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\left\{\,\lambda^i\,\epsilon_{jkl}\,\,a_j\,b_k\,\boldsymbol{\beta}_l(\lambda)\,\right\}\right] \,=\, -\,a_j\,b_j\,+\,0\,=\,-\,\mathbf{a}\cdot\mathbf{b}\,, \tag{44}$$

where I have used Eq. (25) and $\boldsymbol{\beta}_j = \lambda\,\boldsymbol{\beta}_j(\lambda)$ from Eq. (34) in the last two lines of derivation, which is equivalent to using Eq. (24). Evidently, the correlation between $\mathscr{A}$ and $\mathscr{B}$ does not change, even from this unnatural perspective.

One may wonder, however, about my use of the relation $\boldsymbol{\beta}_j\,=\,\lambda\,\boldsymbol{\beta}_j(\lambda)$ in this derivation in addition to using Eq. (25). After all, are not the basis $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$ supposed to be the "fixed" bivector basis and the basis $\{\boldsymbol{\beta}_x(\lambda),\,\boldsymbol{\beta}_y(\lambda),\,\boldsymbol{\beta}_z(\lambda)\}$ dependent on the hidden variable λ? We must not forget what we did, however, in the first line of the derivation, in order to arrive at the last two lines. Complying with Gill's specious demand, we artificially treated the statistical variables $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ as if they were purely algebraic variables, and forced the value of their product $\mathscr{A}\mathscr{B}(\mathbf{a},\,\mathbf{b},\,\lambda)$ to be equal to -1 for all $\mathbf{a}$ and $\mathbf{b}$. In other words, we surreptitiously assumed that the numbers $\mathscr{A}$ and $\mathscr{B}$ are completely dispersion-free. But how can that be? Given their definitions (27) and (28), it is clear that $\mathscr{A}$ and $\mathscr{B}$ are *not* generated as dispersion-free numbers, at least from the perspectives of the detectors $(\,-I\cdot\mathbf{a})$ and $(\,+I\cdot\mathbf{b})$:

$$\mathscr{A}\,=\,(\,-I\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{a}) \tag{45}$$

$$\text{and}\quad \mathscr{B}\,=\,(\,\boldsymbol{\mu}\cdot\mathbf{b})(\,+I\cdot\mathbf{b}). \tag{46}$$

The scalar $\mathscr{A}$ is generated with a bivectorial scale of dispersion $(\,-I\cdot\mathbf{a})$ due to randomness within $(\,\boldsymbol{\mu}\cdot\mathbf{a})$, and the scalar $\mathscr{B}$ is generated with a bivectorial scale of dispersion $(\,+I\cdot\mathbf{b})$ due to randomness within $(\,\boldsymbol{\mu}\cdot\mathbf{b})$. Thus, clearly, $\mathscr{A}$ and $\mathscr{B}$ can be treated as dispersion-free *only* from the perspectives of the spins $(\,\boldsymbol{\mu}\cdot\mathbf{a})$ and $(\,\boldsymbol{\mu}\cdot\mathbf{b})$ themselves rather than from those of the detectors $(\,-I\cdot\mathbf{a})$ and $(\,+I\cdot\mathbf{b})$. In other words, fixing the value of the product $\mathscr{A}\mathscr{B}$ amounts to viewing the correlation between $\mathscr{A}$ and $\mathscr{B}$ from the perspectives of the spins rather than those of the detectors [2]. But from the perspectives of the spins the detectors are *not* fixed but alternate their handedness between left and right, precisely as dictated by the relation $\boldsymbol{\beta}_j\,=\,\lambda\,\boldsymbol{\beta}_j(\lambda)$ we have used in deriving the correlation (44). And as we discussed above, this relation encapsulates the randomness within $(\,-I\cdot\mathbf{a})$ *relative* to the spin $(\,\boldsymbol{\mu}\cdot\mathbf{a})$. Moreover, within my model the scalars $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{A}\mathscr{B}$ and the bivectors $(\,-I\cdot\mathbf{a})$, $(\,+I\cdot\mathbf{b})$, $(\,\boldsymbol{\mu}\cdot\mathbf{a})$, and $(\,\boldsymbol{\mu}\cdot\mathbf{b})$ are all supposed to be different points of a parallelized 3-sphere. Thus the above derivation once again reinforces the central view of my program that EPR correlations are nothing but correlations among the points of a parallelized 3-sphere, regardless of its algebraic representation. Consequently, it is not at all surprising that algebraic consistency continues to hold between the natural perspective advocated in Refs. [2] and [5] and the unnatural perspective insisted upon by Gill.

III. CONCLUSION

The argument of Gill against my disproof of Bell's theorem is based on what Whitehead would have called a fallacy of misplaced concreteness. Instead of trying to understand my model from a natural, physical perspective, Gill insists on understanding it from an abstract, unphysical perspective, and claims that that exposes an algebraic error in my paper. However, it turns out that even this misplaced strategy fails, because my model passes his peculiar algebraic test with flying colors. Put differently, Gill's argument is misguided on more than one counts. To begin with, it is based on a trivial misreading of my local-realistic model, as well as on an oversight of a freedom of choice in the orientation of a parallelized 3-sphere. In addition to this, there are a number of elementary mathematical errors in his argument which by themselves are sufficient to undermine his conclusions. Upon his request I have tried to ignore these errors and concentrate on his central argument only. It is however worth noting that, trivial as they may be, the errors start building up right from the abstract of his preprint. For instance, in his abstract he states that "Correctly computed, [my] standardized correlation are the bivectors $-\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\wedge\mathbf{b}\dots$" This statement is nonsensical even in the corrected second version of his preprint. More importantly, I have shown that Gill's argument stems from an erroneous reading of my central equation (23), not recognizing the significance of the reciprocity relation (34) implicit in my papers, not reading my substantive papers but only the one-page summary paper, and unfamiliarity with basic Clifford algebra. This leads him to erroneously conclude that there is an error in my paper. I hope I have succeeded in demonstrating that this conclusion is false. More specifically, the EPR correlation predicted by my local-realistic model are precisely

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\;=\;\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}^i)\;\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}^i)\right]\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,, \tag{47}$$

where $S^3\ni\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})=\pm\,1$ and $S^3\ni\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})=\pm\,1$ are the unadorned raw scores observed by Alice and Bob. Given this clear and straightforward result one may wonder why Gill ends up getting a different result. The answer is quite simple. He is working with a counterfeit of my model, with little or no incentive to understand the real model. It is therefore not all that surprising that he ends up getting the result $\mathcal{E}(\mathbf{a},\,\mathbf{b})=\text{nonsense}$, instead of $\mathcal{E}(\mathbf{a},\,\mathbf{b})=-\,\mathbf{a}\cdot\mathbf{b}$.

Finally, it is worth noting that, contrary to Gill's failed strategy, my model [3] is based on a substantive physical hypothesis. I hypothesize that the space we live in respects the symmetries and topologies of a parallelized 3-sphere, which is one of the infinitely many fibers of a parallelized 7-sphere. The EPR correlations are thus correlations among the points of a parallelized 3-sphere, whereas quantum correlations in general are correlations among the point of a parallelized 7-sphere. My program thus goes far deeper and well beyond the narrow confines of Bell's theorem [2].

Acknowledgments

I wish to thank Richard Gill for nearly a month-long correspondence about my one-page paper, building up to his critique. I regret not having persuaded him so far, but hope to do better with this formal response to his critique. I also wish to thank the Foundational Questions Institute (FQXi) for supporting this work through a Mini-Grant.

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Appendix A: Refutation of Richard Gill's New Argument Against my Disproof of Bell's Theorem

Richard Gill has suggested online that the derivation of the identities (8) and (14) above can be questioned because of my use of the same notation $\boldsymbol{\beta}_i$ for both the right- and left-handed bivectors. To dispel any such doubt, let me rederive those identities here more carefully using different notations for the right- and the left-handed bivectors. To this end, consider a right-handed frame of ordered basis bivectors, $\{\boldsymbol{\alpha}_x,\, \boldsymbol{\alpha}_y,\, \boldsymbol{\alpha}_z\}$, and the corresponding bivector sub-algebra

$$\boldsymbol{\alpha}_i\,\boldsymbol{\alpha}_j \,=\, -\,\delta_{ij} \,-\, \epsilon_{ijk}\,\boldsymbol{\alpha}_k \tag{A1}$$

of the Clifford algebra $Cl_{3,0}$. The latter is a vector space, $\mathrm{I\!R}^8$, spanned by the ordered set of graded orthonormal basis

$$\{1,\;\; \mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z,\;\; \mathbf{e}_y\wedge\mathbf{e}_z,\, \mathbf{e}_z\wedge\mathbf{e}_x,\, \mathbf{e}_x\wedge\mathbf{e}_y,\;\; \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{A2}$$

where δ_{ij} is the Kronecker delta, ϵ_{ijk} is the Levi-Civita symbol, the indices $i,\, j,\, k = x,\, y,$ or z are cyclic indices, and

$$\boldsymbol{\alpha}_i \,=\, \mathbf{e}_j\wedge\mathbf{e}_k \,=\, I\cdot\mathbf{e}_i\,, \tag{A3}$$

with $I := \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z$ being a volume form of physical space. Eq. (A1) is a standard definition of bivector subalgebra, routinely used in geometric algebra [**A**1]. From it, it is easy to verify the basic properties of the basis bivectors, such as

$$(\boldsymbol{\alpha}_x)^2 \,=\, (\boldsymbol{\alpha}_y)^2 \,=\, (\boldsymbol{\alpha}_z)^2 \,=\, -\,1 \tag{A4}$$

$$\text{and}\quad \boldsymbol{\alpha}_x\,\boldsymbol{\alpha}_y \,=\, -\;\boldsymbol{\alpha}_y\,\boldsymbol{\alpha}_x \quad \text{etc.} \tag{A5}$$

Moreover, it is easy to verify that the bivectors satisfying the subalgebra (A1) form a right-handed frame of basis bivectors. To check this, right-multiply both sides of Eq. (A1) by $\boldsymbol{\alpha}_k$, and then use the fact that $(\boldsymbol{\alpha}_k)^2 = -1$ to arrive at

$$\boldsymbol{\alpha}_i\,\boldsymbol{\alpha}_j\,\boldsymbol{\alpha}_k \,=\, +\,1\,. \tag{A6}$$

The fact that this ordered product yields a positive value confirms that $\{\boldsymbol{\alpha}_x,\, \boldsymbol{\alpha}_y,\, \boldsymbol{\alpha}_z\}$ indeed forms a right-handed frame of basis bivectors. This is a universally accepted convention, easily found in any textbook on geometric algebra.

Suppose now $\mathbf{a} = a_i\,\mathbf{e}_i$ and $\mathbf{b} = b_j\,\mathbf{e}_j$ are two unit vectors in $\mathrm{I\!R}^3$, expanded in right-handed basis $\{\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z\}$, where the repeated indices are summed over $x,\, y,$ and z. Then the right-handed set of graded basis defined in (A1) leads to

$$\{\,a_i\;\boldsymbol{\alpha}_i\,\}\,\{\,b_j\;\boldsymbol{\alpha}_j\,\} \,=\, -\,a_i\,b_j\,\delta_{ij} \,-\, \epsilon_{ijk}\;a_i\,b_j\;\boldsymbol{\alpha}_k\,, \tag{A7}$$

which, together with (A3) (which says that both $+\mathbf{e}_i$ and $\boldsymbol{\alpha}_i$ form right-handed frames), is equivalent to the identity

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}), \tag{A8}$$

where $I = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ is the standard trivector. Geometrically this identity describes all points of a parallelized 3-sphere.

Let us now consider a left-handed frame of ordered basis bivectors, which we denote by $\{\boldsymbol{\beta}_x,\, \boldsymbol{\beta}_y,\, \boldsymbol{\beta}_z\}$. It is important to recognize, however, that there is no *prior* way of knowing that this new basis frame is in fact left-handed. To ensure that it is indeed left-handed we must first make sure that it is an ordered frame by requiring that its basis elements satisfy the bivector properties analogous to those delineated in Eqs. (A4) and (A5). Next, to distinguish this frame from the right-handed frame defined by equation (A6), we must require that its basis elements respect the property

$$\boldsymbol{\beta}_i\,\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \,=\, -\,1\,. \tag{A9}$$

One way to ensure this is by multiplying all vector and bivector elements in the basis set (A2) by a minus sign, giving

$$\{1,\;\; -\,\mathbf{e}_x,\, -\,\mathbf{e}_y,\, -\,\mathbf{e}_z,\;\; -\,\mathbf{e}_y\wedge\mathbf{e}_z,\, -\,\mathbf{e}_z\wedge\mathbf{e}_x,\, -\,\mathbf{e}_x\wedge\mathbf{e}_y,\;\; \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}. \tag{A10}$$

As we shall soon see, this choice of the basis leads us to the incorrect result (A14) for our purposes, because we have not changed the sign of the pseudoscalar $I := \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z$ in addition to that of the scalar 1. Consequently, although the four-dimensional even and odd subalgebras are now left-handed, the full eight-dimensional algebra remains right-handed, because we have changed the signs of only an *even* number of its elements, namely three vectors plus three bivectors, comprising six elements in total. Another way to see this is by noting that the determinant of the matrix that transforms the basis (A2) into (A10) is $(-1)^6 = +1$. On the other hand, instead of the relation (A3) we now have

$$\boldsymbol{\beta}_i \,=\, -\,\mathbf{e}_j\wedge\mathbf{e}_k \,=\, I\cdot(\,-\,\mathbf{e}_i\,)\,, \tag{A11}$$

and therefore the condition (A9) above is automatically satisfied. As is well known, this was the condition imposed by Hamilton on his unit quaternions, which we now know are nothing but a left-handed set of basis bivectors. It can be easily checked that the basis bivectors satisfying the properties (A4), (A5), (A9), and (A11) compose the subalgebra

$$\boldsymbol{\beta}_i\,\boldsymbol{\beta}_j \,=\, -\,\delta_{ij} \,+\, \epsilon_{ijk}\,\boldsymbol{\beta}_k\,. \tag{A12}$$

Suppose now $\mathbf{a} = a_i\,\mathbf{e}_i$ and $\mathbf{b} = b_j\,\mathbf{e}_j$ are two unit vectors in $\mathrm{I\!R}^3$, identical to those used in Eq. (A7), where the repeated indices are again summed over x, y, and z. Then the left-handed set of graded basis defined in (A12) leads to

$$\{\,a_i\,\boldsymbol{\beta}_i\,\}\,\{\,b_j\,\boldsymbol{\beta}_j\,\} \,=\, -\,a_i\,b_j\,\delta_{ij} \,+\, \epsilon_{ijk}\;a_i\,b_j\;\boldsymbol{\beta}_k\,, \tag{A13}$$

which, together with (A11) (which says that both $-\mathbf{e}_i$ and $\boldsymbol{\beta}_i$ form left-handed frames), is equivalent to the identity

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}), \tag{A14}$$

where I is the standard trivector. But this is still a right-handed identity, giving a physically incorrect result for our purposes of experimentally distinguishing the left-handed bivectors from the right-handed bivectors unambiguously.

Indeed, the geometric identities (A8) and (A14) are identical despite the fact that the corresponding bivectorial relations (A7) and (A13) are not. Thus, unlike the cross product, the geometric product between bivectors remains invariant under orientation changes if they are confined to the even (*i.e.*, bivector) and odd (*i.e.*, vector) subalgebras. But that is not at all surprising, because the relations (A3) and (A11) posit that the right-handed basis bivectors $\boldsymbol{\alpha}_i$ are dual to the right-handed basis vectors $+\mathbf{e}_i$ and the left-handed basis bivectors $\boldsymbol{\beta}_i$ are dual to the left-handed basis vectors $-\mathbf{e}_i$. However, in the EPR-Bohm type experiments one does not change coordinate systems back and forth to observe the spins. One keeps the vector basis $\{\,+\mathbf{e}_i\,\}$ fixed at both stations for the entire course of the experiment, in order to unambiguously determine whether a given spin is "up" or "down" about an experimentally fixed direction, represented by a specific vector. Therefore, to arrive at a genuinely left-handed counterpart of the identity (A8) — with all bivectors within both identities being dual to the vectors expanded in one and the same basis $\{\,+\mathbf{e}_i\,\}$, we must consider orientation changes in the entire algebra $Cl_{3,0}$ of the orthogonal directions in 3D space, by means of the basis

$$\{1,\;\, -\,\mathbf{e}_x,\, -\,\mathbf{e}_y,\, -\,\mathbf{e}_z,\;\, -\,\mathbf{e}_y\wedge\mathbf{e}_z,\, -\,\mathbf{e}_z\wedge\mathbf{e}_x,\, -\,\mathbf{e}_x\wedge\mathbf{e}_y,\;\, -\,\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{A15}$$

where the sign of every non-scalar element is now different from that in the set (A2), including the volume element I:

$$I \longrightarrow -I := -\,\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\,. \tag{A16}$$

Since the determinant of the matrix that transforms (A2) into (A15) is now $(-1)^7 = -1$, the basis defined by (A15) are genuinely left-handed, *relative* to the basis defined by both (A2) as well as (A10). The all important question then is: How do the identities (A8) and (A14) transform under the handedness transformation (A16) of the volume element? But it is not difficult to see that under (A16) all three bivectors change signs and the identity (A14) transforms into

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, I\cdot(\mathbf{a}\times\mathbf{b}). \tag{A17}$$

Evidently, this is now a genuinely left-handed bivector identity compared to the right-handed identity (A8). Crucially, there is a sign difference in the second term on the RHS of this identity, compared to the right-handed identity (A8). Consequently, in perfect harmony with how the bivector relations (A7) and (A13) transform into one another under parity transformations, the identities (A8) and (A17) now transform into one another under the transformation (A16):

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}) \xleftrightarrow{\;+I\,\longleftrightarrow\,-I\;} (\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, I\cdot(\mathbf{a}\times\mathbf{b}). \tag{A18}$$

For convenience, we can now rewrite these two alternative identities (A8) and (A17) as two hidden variable possibilities

$$(\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,+I\,)\cdot(\mathbf{a}\times\mathbf{b}) \tag{A19}$$

and

$$(\,-\,I\cdot\mathbf{a})(\,-\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,-\,I\,)\cdot(\mathbf{a}\times\mathbf{b}). \tag{A20}$$

Exploiting the natural freedom of choice in characterizing the orientation of 3-sphere by either $+\,I$ or $-\,I$, we can now combine the identities (A19) and (A20) into a single hidden variable equation (at least for the computational purposes):

$$(\,\lambda\,I\cdot\mathbf{a})(\,\lambda\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,\lambda\,I\,)\cdot(\mathbf{a}\times\mathbf{b}), \tag{A21}$$

where $\lambda = \pm 1$ now specifies the orientation of the 3-sphere. It is important to keep in mind here that the combined equation (A21) is simply a convenient shortcut for representing two completely independent initial states of the physical system, one corresponding to the counterclockwise orientation of the 3-sphere and the other corresponding to the clockwise orientation of the 3-sphere. Moreover, at no time are these two alternative possibilities mixed during the course of an experiment. They represent two independent physical scenarios, corresponding to two independent runs of the experiment. Next, if we employ the notation $\boldsymbol{\mu} = \lambda\, I$, then the combined identity (A21) takes the convenient form

$$(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \;=\; -\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}). \tag{A22}$$

This is the central equation of my local model. I have used it in various forms and notations since 2007 [**A**2]. It is simply an isomorphic representation of the familiar identity from quantum mechanics, with the correspondence $\boldsymbol{\mu} \longleftrightarrow i\boldsymbol{\sigma}$:

$$(i\boldsymbol{\sigma}\cdot\mathbf{a})(i\boldsymbol{\sigma}\cdot\mathbf{b}) \;=\; -\,\mathbf{a}\cdot\mathbf{b}\,\mathbb{1} \,-\, i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}). \tag{A23}$$

In Ref. [**A**1] I have combined the alternative bivector relations (A7) and (A13) into a single hidden variable equation

$$L_\mu(\lambda)\,L_\nu(\lambda) \;=\; -\,\delta_{\mu\nu} \,-\, \sum_\rho \epsilon_{\mu\nu\rho}\,L_\rho(\lambda)\,, \tag{A24}$$

together with $L_\mu(\lambda) := \lambda\,D_\mu$, with alternative choices $\lambda = \pm 1$ for the orientation of S^3. Contracting this equation on both sides with the components a^μ and b^ν of arbitrary unit vectors $\mathbf{a}$ and $\mathbf{b}$ then gives the combined bivector identity

$$\mathbf{L}(\mathbf{a},\,\lambda)\,\mathbf{L}(\mathbf{b},\,\lambda) \;=\; -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda)\,, \tag{A25}$$

which is a convenient notation for the identity (A22). It simply combines the two alternative identities (A8) and (A17) into a single identity, rendering the unit bivector $\mathbf{L}(\mathbf{n},\,\lambda^k)$ a random variable *relative* to the freely chosen fixed detector bivector $\mathbf{D}(\mathbf{n})$, for a given run k:

$$\mathbf{L}(\mathbf{n},\,\lambda^k) \;=\; \lambda^k\,\mathbf{D}(\mathbf{n}) \;\Longleftrightarrow\; \mathbf{D}(\mathbf{n}) \;=\; \lambda^k\,\mathbf{L}(\mathbf{n},\,\lambda^k)\,. \tag{A26}$$

The expectation value of simultaneous outcomes $\mathscr{A}(\mathbf{a},\,\lambda^k) = \pm 1$ and $\mathscr{B}(\mathbf{b},\,\lambda^k) = \pm 1$ in S^3 then works out as follows:

$$\begin{aligned}\mathcal{E}(\mathbf{a},\,\mathbf{b}) \;&=\; \lim_{n\,\to\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^k)\;\mathscr{B}(\mathbf{b},\,\lambda^k)\right] \;=\; \lim_{n\,\to\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathbf{L}(\mathbf{a},\,\lambda^k)\,\mathbf{L}(\mathbf{b},\,\lambda^k)\right] \\ &=\frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k = +1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = +1)\Big] \,+\, \frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k = -1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = -1)\Big],\end{aligned} \tag{A27}$$

where the last simplification occurs because λ^k is a fair coin. Using the relations (A25) and (A26) the above sum can now be evaluated by noting that the spins in the right and left oriented S^3 satisfy the following geometrical relations:

$$\begin{aligned}\mathbf{L}(\mathbf{a},\,\lambda^k = +1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = +1) \;&=\; -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda^k = +1) \\ &=\; -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{D}(\mathbf{a}\times\mathbf{b}) \\ &=\; \mathbf{D}(\mathbf{a})\;\mathbf{D}(\mathbf{b}) \;=\; (\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b})\end{aligned} \tag{A28}$$

and

$$\begin{aligned}\mathbf{L}(\mathbf{a},\,\lambda^k = -1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = -1) \;&=\; -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda^k = -1) \\ &=\; -\,\mathbf{a}\cdot\mathbf{b} \,+\, \mathbf{D}(\mathbf{a}\times\mathbf{b}) \\ &=\; -\,\mathbf{b}\cdot\mathbf{a} \,-\, \mathbf{D}(\mathbf{b}\times\mathbf{a}) \\ &=\; \mathbf{D}(\mathbf{b})\;\mathbf{D}(\mathbf{a}) \;=\; (\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a}).\end{aligned} \tag{A29}$$

In other words, when λ^k happens to be equal to $+1$, $\mathbf{L}(\mathbf{a},\,\lambda^k)\;\mathbf{L}(\mathbf{b},\,\lambda^k) = (\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b})$, and when λ^k happens to be equal to -1, $\mathbf{L}(\mathbf{a},\,\lambda^k)\;\mathbf{L}(\mathbf{b},\,\lambda^k) = (\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a})$. Consequently, the expectation value (A27) reduces at once to

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \;=\; \frac{1}{2}(\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b}) \,+\, \frac{1}{2}(\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a}) \;=\; -\,\frac{1}{2}\,\{\mathbf{ab} \,+\, \mathbf{ba}\} = \,-\,\mathbf{a}\cdot\mathbf{b} \,+\, 0\,, \tag{A30}$$

because the orientation λ^k of S^3 is a fair coin. Here the last equality follows from the definition of the inner product.

References

[**A**1] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003), page 33.
[**A**2] J. Christian, *Disproof of Bell's Theorem by Clifford Algebra Valued Local Variables*, arXiv:quant-ph/0703179 (2007).
[**A**1] J. Christian, *Local Causality in a Friedmann-Robertson-Walker Spacetime*, `http://arXiv.org/abs/1405.2355` (2014).

Appendix B: Ordering Relation between the Spins and the Detectors as a Local Hidden Variable

It is instructive to refute the arguments by Richard Gill somewhat differently. In the end the 3-sphere model for the EPR-Bohm correlation is a local hidden variable model. As such, we can simply define the hidden variable $\lambda = \pm 1$ of the model as the ordering relation between the spin bivectors $\mathbf{L}(\mathbf{a},\,\lambda)$ and $\mathbf{L}(\mathbf{b},\,\lambda)$ and the detector bivectors $\mathbf{D}(\mathbf{a})$ and $\mathbf{D}(\mathbf{b})$, and only subsequently identify it with the orientation of the 3-sphere, as an equivalent version of this definition.

To this end, we again begin with the bivector subalgebra (A24) in the notation of Ref. [**B**1] as our central equation,

$$L_{\mu}(\lambda)\,L_{\nu}(\lambda) \,=\, -\,\delta_{\mu\nu} \,-\, \sum_{\rho} \epsilon_{\mu\nu\rho}\,L_{\rho}(\lambda)\,, \tag{B1}$$

which, upon contraction of both sides with the components a^{μ} and b^{ν} of unit vectors $\mathbf{a}$ and $\mathbf{b}$ gives the bivector identity

$$\mathbf{L}(\mathbf{a},\,\lambda)\,\mathbf{L}(\mathbf{b},\,\lambda) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda)\,. \tag{B2}$$

Next, instead of assuming the hidden variable $\lambda = \pm 1$ to be an orientation of S^3, we define $\lambda = \pm 1$ to be the ordering relation between the spin bivectors $\mathbf{L}(\mathbf{a},\,\lambda)$ and $\mathbf{L}(\mathbf{b},\,\lambda)$ and the detector bivectors $\mathbf{D}(\mathbf{a})$ and $\mathbf{D}(\mathbf{b})$, with 50/50 chance:

$$\mathbf{L}(\mathbf{a},\,\lambda = +1)\;\mathbf{L}(\mathbf{b},\,\lambda = +1) \,=\, \mathbf{D}(\mathbf{a})\;\mathbf{D}(\mathbf{b}) \tag{B3}$$

or

$$\mathbf{L}(\mathbf{a},\,\lambda = -1)\;\mathbf{L}(\mathbf{b},\,\lambda = -1) \,=\, \mathbf{D}(\mathbf{b})\;\mathbf{D}(\mathbf{a}). \tag{B4}$$

Since the spins emerging from the source are oblivious to the detectors located at remote stations, we represent the spins with a trivector $\boldsymbol{\mu}$ and detectros with a trivector I, respectively, without assuming any relation between them:

$$\mathbf{L}(\mathbf{n},\,\lambda) \,=\, \boldsymbol{\mu}\cdot\mathbf{n} \tag{B5}$$

and

$$\mathbf{D}(\mathbf{n}) \,=\, I\cdot\mathbf{n}\,, \tag{B6}$$

for any given dual vector $\mathbf{n}$. Our intent now is to find the relationship between $\boldsymbol{\mu}$ and I using the identities (B2) and

$$\mathbf{D}(\mathbf{a})\,\mathbf{D}(\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{D}(\mathbf{a}\times\mathbf{b})\,. \tag{B7}$$

Substituting the right-hand sides of these identities into the ordering relations (B3) and (B4) reduces the relations to

$$-\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda = +1) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{D}(\mathbf{a}\times\mathbf{b}) \tag{B8}$$

or

$$-\,\mathbf{a}\cdot\mathbf{b} \,-\, \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda = -1) \,=\, -\,\mathbf{b}\cdot\mathbf{a} \,-\, \mathbf{D}(\mathbf{b}\times\mathbf{a}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, \mathbf{D}(\mathbf{a}\times\mathbf{b})\,, \tag{B9}$$

which, after canceling the scalar factor $-\,\mathbf{a}\cdot\mathbf{b}$ and using $\lambda = \pm 1$ and the definitions (B5) and (B6), further reduces to

$$\mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda) \,=\, \lambda\,\mathbf{D}(\mathbf{a}\times\mathbf{b}) \tag{B10}$$

$$\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}) \,=\, \lambda\,I\cdot(\mathbf{a}\times\mathbf{b}) \tag{B11}$$

$$\boldsymbol{\mu} \,=\, \lambda\,I\,. \tag{B12}$$

We have thus proved that the ordering relations (B3) and (B4) between the spin bivectors $\mathbf{L}(\mathbf{a},\,\lambda)$ and $\mathbf{L}(\mathbf{b},\,\lambda)$ and the detector bivectors $\mathbf{D}(\mathbf{a})$ and $\mathbf{D}(\mathbf{b})$ are equivalent to our hypothesis that the orientation of the 3-sphere is a fair coin.

The expectation value of simultaneous outcomes $\mathscr{A}(\mathbf{a},\,\lambda^k) = \pm 1$ and $\mathscr{B}(\mathbf{b},\,\lambda^k) = \pm 1$ in S^3 then works out as before:

$$\begin{aligned}\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,&=\, \lim_{n\,\rightarrow\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^k)\;\mathscr{B}(\mathbf{b},\,\lambda^k)\right] \,=\, \lim_{n\,\rightarrow\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathbf{L}(\mathbf{a},\,\lambda^k)\,\mathbf{L}(\mathbf{b},\,\lambda^k)\right] \\ &=\, \frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k = +1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = +1)\Big] \,+\, \frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k = -1)\;\mathbf{L}(\mathbf{b},\,\lambda^k = -1)\Big] \\ &=\, \frac{1}{2}\,\{D(\mathbf{a})\,D(\mathbf{b})\} + \frac{1}{2}\,\{D(\mathbf{b})\,D(\mathbf{a})\} \,=\, \frac{1}{2}(\,+\,I\cdot\mathbf{a})(\,+\,I\cdot\mathbf{b}) \,+\, \frac{1}{2}(\,+\,I\cdot\mathbf{b})(\,+\,I\cdot\mathbf{a}) \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,,\end{aligned} \tag{B13}$$

because the ordering alternatives (B3) and (B4) are assumed to be a fair coin, and $\frac{1}{2}\,\{\mathbf{ab} + \mathbf{ba}\} = \mathbf{a}\cdot\mathbf{b}$ by definition.

References

[**B**1] J. Christian, *Local Causality in a Friedmann-Robertson-Walker Spacetime*, `http://arXiv.org/abs/1405.2355` (2014).