Unreal Observables

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Abstract. This note argues that quantum observables can include not just self-adjoint operators, but any member of the class of normal operators, including those with non-real eigenvalues. Concrete experiments, statistics, and symmetries are all expressed in this more general context. However, this more general class of observables also introduces a new restriction on which sets of operators can be interpreted as observables at once. These sets are referred to here as 'sharp sets.'

1. Introduction

There is an extraordinary freedom of labeling when using language to describe physical phenomena. Consider how a primitive society might label four locations on a rock. Using language that is available to them, they might use cave drawings to identify the four poles, as in Figure 1.

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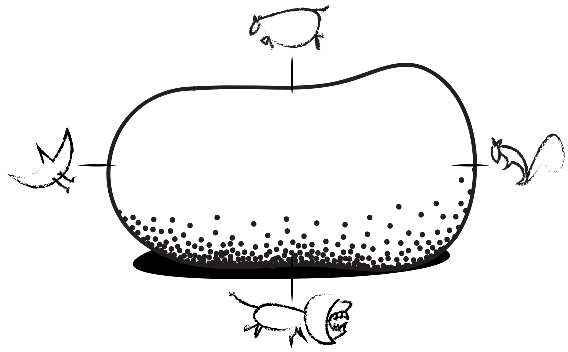

Figure 1. Labeling freedom on the surface of a rock.

More advanced societies might label the same four locations using pairs of real numbers like $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Or, they might label the locations using complex numbers, 1, i, -1 and $-i$. Nothing at this stage prevents us from labeling the rock however we please. Of course, the term 'imaginary' was introduced historically to refer to some complex numbers. However, nothing about this historical fact prevents us from using them today. Indeed, complex numbers are often used in the context of classical physics, to represent everything from concrete physical quantities like the amplitude and phase of a wave, to more abstract quantities like complex-valued classical fields.

Meanwhile, in quantum mechanics, the situation is not so egalitarian. The textbooks are nearly unanimous in declaring that it is impossible to use complex numbers to represent observable phenomena. Only the real-number eigenvalues of self-adjoint (or Hermitian) operators are permitted to represent observations — or so the orthodoxy goes. From a random sampling of textbooks, we find: "the expectation value of an observable quantity has got to be a real number (after all, it corresponds to actual measurements in the laboratory, using rulers and clocks and meters)" (Griffiths 1995, §3.3). And again, "[w]e expect on physical grounds that an observable has real eigenvalues.... That is why we talk about Hermitian observables in quantum mechanics" (Sakurai 1994, §1.3).

This orthodoxy is often repeated by philosophers. For example, David Albert introduces his 'principle (B)', that linear operators represent measurable properties, and then writes, "it's clear from principle (B) (since, of course, the values of physically measurable quantities are always real numbers) that the operators associated with measurable properties must necessarily be Hermitian operators" (Albert 1992, 40). For each example of this orthodoxy, there are many more textbooks assuring us of the same thing: observable outcomes in quantum theory must be associated with real-number eigenvalues.

The thesis of this note is that the orthodoxy is mistaken. Operators with non-real eigenvalues can play the role of observables too, in which case one might call them 'unreal observables.' Indeed, many of the very arguments that have been used to establish self-adjointness as a criterion for observables can be applied to many non-self-adjoint operators as well. In what follows, I will identify some of these arguments, as well as one limitation that restricts which sets of non-self-adjoint operators can be interpreted as observables at once.

To keep this discussion short, I will restrict attention to one tractable class of operators, the so-called 'normal operators' (to be defined below). This will allow us to see in simple terms how operators with non-real eigenvalues may play the role of observables in quantum theory. However, this is only the beginning: the non-self-adjoint operators can be generally disassembled, classified, and shown to provide fruitful ways to extend orthodox quantum theory when considered as observables (Roberts 2017). These more general classes of observables allow new philosophical interpretations, and in some cases some new physics.

The remainder of the present note proceeds as follows. After introducing the mathematics of self-adjoint and normal operators in Section 2, we turn in Section 3 to one of the earliest historical arguments for self-adjointness, Dirac's 'simultaneous measurability argument', showing that it is actually an argument for unreal observables as well. Section 4 shows how the statistical predictions of quantum theory work with unreal observables, and Section 5 observes that symmetries make sense in this context as well. However, there can be too much of a good thing: in Section 6, we turn to a limitation of this perspective, by identifying a restriction on which sets of normal operators can be interpreted as observables at once, called 'sharp sets'.

2. Mathematics of Normal Operators

We begin with a few mathematical preliminaries. Let $\mathcal{H}$ be a Hilbert space with a countable basis. In finite dimensions, a linear operator A satisfying $A\psi = A^{*}\psi$ for all vectors ψ is called *self-adjoint*, or sometimes *Hermitian*. In infinite dimensions, an additional condition is needed to guarantee self-adjointness: A and A^{*} must have the same domain (though nothing about the present discussion will turn on the complications arising from infinite dimensions). A *normal* operator $A : \mathcal{H} \to \mathcal{H}$ is a linear operator that commutes with its adjoint, $AA^{*} = A^{*}A$. So, every self-adjoint operator is normal, but not conversely. For example, a unitary operator is normal but not self-adjoint.

A normal operator can always be written in terms of an 'independent pair' of self-adjoint operators, in the following sense. Every linear operator A can be written $A = B + iC$, with B and C self-adjoint, by defining $B := \frac{A^{*}+A}{2}$ and $C := \frac{i(A^{*}-A)}{2}$. A simple calculation then shows that, when written in this form, A satisfies the condition, $AA^{*} - A^{*}A = 2i(CB - BC)$. If the left-hand-side of this equation is zero, which is what being normal means, then B and C commute, and vice versa. So, $A = B + iC$ is normal if and only if $BC = CB$. In quantum theory, commuting operators describe observables that are statistically independent; thus, one can view a normal operator $A = B + iC$ as consisting in a pair of independent self-adjoint operators B and C.

Given a linear operator $A : \mathcal{H} \to \mathcal{H}$, the set of complex numbers λ such that $(A - \lambda I)$ has no inverse is called the *spectrum* of A. Those λ that can be written in the form $A\psi = \lambda\psi$ for some ψ are called *eigenvalues*. A *discrete* or *pure point* spectrum operator is one whose spectrum consists entirely of eigenvalues, which is the case whenever $\mathcal{H}$ has finite dimensions. It is a simple exercise to show that a self-adjoint operator has a spectrum consisting entirely of real numbers.[1] However, the central fact of interest for us is that for normal operators, the converse is also true:

Fact. *A normal operator has an entirely real spectrum if and only if it is self-adjoint (Rudin 1991, Thm. 12.26).*

This means that every non-self-adjoint normal operator has a spectrum that is not entirely real. For this reason, we dub such operators 'unreal'. Our aim is now to understand the extent to which they can be observables, too.

3. Dirac's Simultaneous Measurability Argument

We will soon turn to a practical discussion of experimental statistics for unreal observables. But it is instructive to begin with an interesting early argument for self-adjointness, due to Paul Dirac[2]. In the 1930 First Edition of his influential textbook, he called a quantum observable "analogous to the value of a variable at a particular instant of time" in classical physics. However, after introducing the algebraic properties of observables, Dirac goes on to suggest that observables in quantum mechanics involve an extension that goes beyond this classical analogy:

> It is convenient to count sums and products of any observables as other observables. This involves, as we shall see shortly, an extension of the meaning of an observable to include the analogues of complex functions of classical dynamical variables.... An observable is thus not necessarily a quantity capable of direct measurement by a single observation, but is a theoretical generalization of such a quantity. (Dirac 1930, 27-28)

[1]Let $A = A^*$ and let ψ be a (non-zero) eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$. Then $\lambda\langle\psi,\psi\rangle = \langle\psi,\lambda\psi\rangle = \langle\psi,A\psi\rangle = \langle A\psi,\psi\rangle = \langle\lambda\psi,\psi\rangle = \lambda^*\langle\psi,\psi\rangle$. The inner product is non-degenerate, so $\langle\psi,\psi\rangle \neq 0$. Therefore, $\lambda = \lambda^*$.

[2]The original arguments for self-adjointness were stated by Born, Jordan and Heisenberg in the 1920's; see Roberts (2017) for a more detailed discussion of the history.

Dirac is speaking here about an “algebra of observables” in a way that remains common today, and which includes both Hermitian and non-Hermitian operators. However, he is quick to add (as also remains common) that when it comes to the physical interpretation, observables “must be understood to be all real observables”.

This hedge against non-Hermitian or “complex” operators became considerably more positivistic in the Second Edition of 1935, where Dirac began to refer to non-Hermitian or “complex” operators as having “no meaning” as observables: “Such a complex function may, of course, be considered formally as a complex observable, but since no meaning can be attached to the measurement of a complex observable, it is preferable to restrict the word ‘observable’ to refer to real functions of dynamical variables and to introduce a corresponding restriction on the linear operators that represent observables” (Dirac 1935, 28-29). His argument for this was given in a brief footnote[3] in the Second Edition of 1935, which is expanded in the Third Edition of 1947 into the main text. Having already discussed the fact that any operator can be written in the form $A = B + iC$ with B and C self-adjoint, Dirac writes the following.

> One might think one could measure a complex dynamical variable by measuring its real and pure imaginary parts. But this would involve two measurements or two observations, which would be all right in classical mechanics, but would not do in quantum mechanics, where two observations in general interfere with one another.... We therefore have to restrict the dynamical variables that we can measure to be real (Dirac 1947, 35)

In other words, a linear operator $A = B + iC$ cannot in general be observed via a single direct measurement, because the joint observation of the two Hermitian components B and C might not itself be an observable. Dirac thus concludes that we must restrict observables to the Hermitian operators.

[3]Dirac’s footnote reads: “It would not do to measure separately the real and pure imaginary parts, because this would mean two measurements, which in general interfere with one another” (Dirac 1935, 29)

However, if two operators commute, then they can be 'jointly observed' in exactly the sense required by Dirac, in that there exists a common set of eigenvectors (or more generally a spectral decomposition) for the two operators. This leaves open a possibility: Dirac's concern can be satisfied whenever $A = B + iC$ is such that B and C commute. As we have seen, this turns out to be equivalent to the condition that A is normal. Thus, what Dirac's argument actually shows is that all normal operators, and not just the self-adjoint ones, are candidates for observables in quantum theory.

4. Experiments with Unreal Observables

Let us now turn to a more concrete discussion of experiments and statistics using unreal observables. In quantum theory, observables associate experimental states with symbols. Like the practice of labeling a rock, the symbols we use do not need to be real numbers; the relevant quantitative information can be expressed by complex numbers as well. For example, when a fermion deflects spin-up or spin-down along the z-axis after passing through the Stern-Gerlach apparatus, we conventionally label the outcomes $+1$ and -1, the eigenvalues of the Pauli matrix $\sigma_z = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$. But we could equally have labeled those outcomes $+i$ and $-i$, the eigenvalues of the anti-Hermitian matrix $i\sigma_z$. This matrix has the same eigenvectors as σ_z, and can be associated with commutation relations that look very similar to the usual ones. The principal difference is that $i\sigma_z$ has complex eigenvalues: it is an unreal observable. But this is no difference of physical interest, since these are just alternative labels for the same physical experiment, illustrated in Figure 2.

The statistics for normal operators like $i\sigma_z$ works out the same as for self-adjoint ones, because the statistics depends only on the eigenvectors and not the eigenvalues. Let A be an operator with complex eigenvalue λ and corresponding eigenstate φ. Then the transition probability from an arbitrary state ψ to φ is given by the usual Born rule, $|\langle\varphi,\psi\rangle|^2$. In particular, since $i\sigma_z$ has the same eigenvectors as σ_z, it displays the same statistics. Moreover, if the eigenvectors of any observable A form an orthonormal basis,

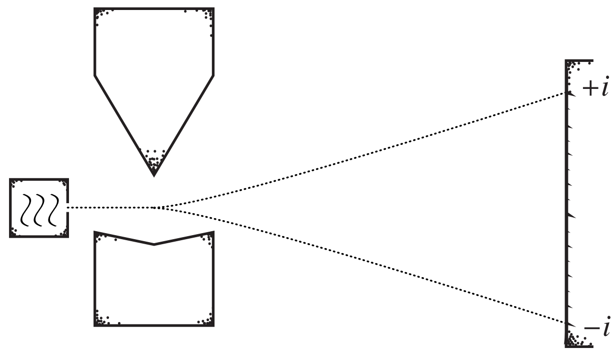

FIGURE 2. Spin-up and spin-down labeled with complex numbers

then its expectation value when the state ψ is prepared will still be given by $\langle\psi, A\psi\rangle = \sum_{i=1}^{n} \lambda_i |\langle\varphi_i, \psi\rangle|^2$. Such an expectation value is generally a complex number when A is normal. But this is to be expected because the eigenvalues are complex.

Normal operators conveniently allow the ordinary quantum statistical rules because of the *spectral theorem*[4] applies to them. In finite dimensions, the spectral theorem implies that given a self-adjoint operator A, there exists an orthonormal basis of eigenvectors $\varphi_1, \varphi_2, \ldots, \varphi_n$ of A. That fact is what allowed Born to view a vector ψ as defining a probability distribution $p_\psi(\varphi_i) := |\langle\varphi_i, \psi\rangle|^2$, since it implies $\sum_i^n p_\psi(\varphi_i) = 1$. In other words, the physical significance of the spectral theorem is that it allows us to view a state as defining a probability distribution on definite experimental outcomes.

Although it is not often emphasised in quantum theory textbooks, the spectral theorem holds not only for self-adjoint operators, but for normal operators too. So, the statistical foundation for quantum theory is guaranteed for all normal operators, even when they do not have a real spectrum.

In his classic textbook on quantum mechanics, Messiah used the spectral theorem as an argument that we should treat self-adjoint (Hermitian) operators as observables, writing, "[a]ll... operators do not possess a complete, orthonormal set of eigenfunctions.

[4]A precise statement and proof can be found in many textbooks, such as Conway (1990, Theorem X.4.11). For a more detailed discussion in the context of non-self-adjoint observables, see Roberts (2017).

However, the Hermitian operators capable of representing physical quantities possess such a set. For this reason we give the name 'observable' to such operators" (Messiah 1999, §V.9). Like Dirac, it turns out that Messiah's argument actually implies that we should give the name 'observable' to the much larger class of normal operators as well, since all these operators possess a 'complete, orthonormal set of eigenfunctions' in the required sense.

5. Symmetries

If we allow normal operators without a real spectrum to be observables, can we still make sense of symmetry in quantum theory? One might take the following to be a concern. There is a fundamental relationship between self-adjoint operators and continuous groups of symmetries in quantum theory, which is given by Stone's theorem.[5] This says that, if U_s is a strongly continuous one-parameter unitary group that satisfies $U_r U_s = U_{r+s}$ for all $r, s \in \mathbb{R}$, then there exists a unique self-adjoint operator A such that $U_s = e^{isA}$ for all s. A notable feature of the symmetry group U_s is that it commutes with A. This has the consequence that A is conserved along the unitary group parameter s, in that $A(s) := U_s^* A U_s = A$.

This result, reminiscent of Noether's theorem for symmetries of a Lagrangian in variational theories, plays an important role in physical reasoning. Translation symmetries are associated with a measurable quantity known as momentum that is conserved along the translations; rotational symmetries have angular momentum which is conserved along rotations; and so on. One can imagine an interlocutor arguing that this is a characteristic property of observables: an observable generates a canonical group of symmetries along which it is conserved. This would allow one to conclude that observables must be self-adjoint.

[5]See, e.g., (Blank et al. 2008, Theorem 5.9.2).

However, such a requirement is difficult to justify in a principled way. The argument makes no appeal to 'observation', but rather to the nature of continuous symmetries; as a consequence, it is difficult to justify using it to define observables. And after all, the generator of the unitary group $U_s = e^{isA}$ is not strictly speaking self-adjoint. It is the anti-Hermitian operator iA. So, as an argument for self-adjoint observables this observation carries little force.

Moreover, even if one did find such an argument convincing, there is a generalisation of Stone's theorem that holds for normal operators as well. Suppose that one is mainly convinced that an observable should generate a group along which the observable is conserved, whether or not that turns out to be a group of symmetries. That property turns out to hold of all normal operators. In particular, every bounded normal operator A generates a strongly continuous one-parameter semigroup $S_b = e^{bA}$ of bounded normal operators along which it is conserved; and conversely, every strongly continuous semigroup of bounded normal operators S_a has a bounded normal generator A, in the sense that $S_b = e^{bA}$ (Rudin 1991, Theorem 13.38).

Thus, the argument from Stone's theorem for self-adjoint observables is difficult to motivate, and at any rate, there are interesting generalisations of Stone's theorems that hold for all normal operators as well.

6. The limitation of sharp sets

The discussion above shows that a normal operator can be treated as an observable, just as a self-adjoint operator can. This sort of egalitarianism led Roger Penrose to write that that, "I shall demand only that my quantum 'observables' be normal linear operators, rather than the stronger conventional requirement that they be Hermitian" (Penrose 2004, 539). Lévy-Leblond (1976) adopted a similar position, observing that since a self-adjoint operator has spectral decomposition $A = \sum_i \lambda_i E_i$, every Borel function f of a self-adjoint operator does too.

However, the suggestion that all of the normal operators may be observables may go too far. There is an important sense in which not all sets of normal operators can be interpreted as observables at once.

To see why, let us return to the Stern-Gerlach apparatus. Our discussion above showed that self-adjoint operators like the Pauli spin matrices can be treated as observables in much the same sense as their anti-Hermitian, normal operator counterparts. However, we should not go so far as to assume that we may choose any set of normal operators we like to represent the outcomes of spin experiments in the same interpretation. The problem arises out of the facts that,

$$\sigma_y\sigma_z = i\sigma_x, \qquad \sigma_z\sigma_x = i\sigma_y, \qquad \sigma_x\sigma_y = i\sigma_z.$$

Suppose we adopt an interpretation in which the ordinary Pauli matrices σ_x, σ_y, and σ_z are observables. Then the products $\sigma_y\sigma_z$, $\sigma_z\sigma_x$ and $\sigma_x\sigma_y$ would each represent the 'joint' observation of spin in two orthogonal directions. But the joint observation of spin in orthogonal directions is not possible. Famously, a spin eigenstate in any given direction is a superposition of eigenstates in the orthogonal directions. As a consequence, if σ_x, σ_y, and σ_z are observables, then it follows that $i\sigma_x = \sigma_y\sigma_z$, $i\sigma_y = \sigma_z\sigma_x$, and $i\sigma_z = \sigma_x\sigma_y$ are not.

On the other hand, suppose we adopt an interpretation in which the non-self-adjoint operators $i\sigma_x$ and $i\sigma_y$ are observables, in addition to σ_z. These three operators are mutually incompatible observables, so their products do not correspond to any physical observations. This implies by the relations,

$$(i\sigma_y)(i\sigma_x) = i\sigma_z \qquad \sigma_z(i\sigma_y) = \sigma_x \qquad (i\sigma_x)\sigma_z = \sigma_y,$$

that $i\sigma_z$, σ_x and σ_y are not observables. So, we have a choice available to us. We have the freedom to interpret the set $\{\sigma_x, \sigma_y, \sigma_z\}$ as observables, or the set $\{i\sigma_x, i\sigma_y, \sigma_z\}$ as observables. This is our prerogative in choosing how to label the outcomes of the Stern-Gerlach experiment. However, the result is not a free for all: if we take the first set as observables, then the second is not, and vice versa.

If not all sets of normal operators can be interpreted as observables at once, which ones can? Here is a proposal. When two normal operators fail to commute, they are called *incompatible*. Incompatible observables do not have a common basis of eigenvectors (or more generally a common spectral resolution). As a consequence, if A and B are incompatible observables, their 'joint observation' AB is not defined. So, a product of incompatible observables may be viewed as an 'unobservable' in quantum theory. Incompatible observables are an essential aspect of quantum theory, and cannot be eliminated. Instead, the proposal here is that an adequate set of observables cannot contain any products of incompatibles. This is captured by the following definition.

Definition. A *sharp set* $\mathcal{S}$ of normal operators on a Hilbert space is one such that, if $A = BC$ for some A, B and C in $\mathcal{S}$, then $BC = CB$. A sharp set is *maximal* in an operator algebra $\mathcal{A}$ if and only if, whenever a new normal operator $N \in \mathcal{A}$ is added to $\mathcal{S}$, the result is no longer a sharp set.

Our proposed restriction is that, if we wish to interpret multiple normal operators as observables at once, then those operators must form a sharp set. Any given normal operator may be an observable, but a collection of them must be sharp. Moreover, if an operator algebra $\mathcal{A}$ (such as the set of matrices on a 2-dimensional Hilbert space) restricts the operators of interest, then the set of all observables must be a maximal sharp set in $\mathcal{A}$.

An few observations about sharp sets: every set of self-adjoint operators is sharp. For, given any self-adjoint operators A, B and C such that $A = BC$, we have that,

$$BC = A = A^* = (BC)^* = C^*B^* = CB,$$

so the set is sharp. In particular, $\mathcal{S} = \{\sigma_x, \sigma_y, \sigma_z\}$ is a sharp set. However, there are also many sharp sets containing non-self-adjoint operators, such as $\mathcal{S}' = \{i\sigma_x, i\sigma_y, \sigma_z\}$. In contrast, the union $\mathcal{S} \cup \mathcal{S}'$ is not a sharp set, since $i\sigma_x = \sigma_y\sigma_z$ is a product of non-commuting operators, and all three are normal operators in the set.

There is a great deal that remains to be understood about the structure of the maximal sharp sets apart from these simple facts. For example, one would like to be able to identify a maximal sharp set containing $\mathcal{S}' = \{i\sigma_x, i\sigma_y, \sigma_z\}$ among the 2×2 matrices, and to understand the relationship between this set and the maximal sharp set of self-adjoint operators. For now, these remain open problems. Given the discussion above, it seems to be a reasonable hypothesis that transforming between two such sets would be a symmetry of quantum theory, in that it would preserve the predictive structure up to a relabeling. Notably, such a transformation would not generally be unitary, since unitary transformations preserve the spectra of operators. Sharp sets thus introduce an apparently new kind of symmetry into quantum theory.

7. Conclusion

Unreal observables, taken to be normal operators that are not self-adjoint, serve just as well to represent observations in quantum theory as self-adjoint observables do. We have seen that nothing is lost in the predictive structure of quantum theory by making use of these more general observables, and that the standard reasons given for ignoring unreal observables fall short. Real numbers are a red herring; the quantum statistical algorithm includes unreal observables; and quantum symmetries make sense in the context of unreal observables as well. There is however a new limitation introduced by opening up observables to all normal operators, which is that products of incompatible observables are not themselves observable. This leads us to introduce the new concept of a sharp set. Sharp sets avoid this problem and introduce an interesting new notion of symmetry into quantum theoroy. But this is not a reason to avoid unreal observables; if anything, it is a reason for further study of their rich structure.

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