Bohmian Mechanics:
A Panacea for What Ails Quantum Mechanics, or a Different and Problematic Theory?

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The popular impression of Bohmian mechanics is that it is standard quantum mechanics with the addition of some extra gadgets—exact particle positions and a guiding equation for particle trajectories—the advantages being that the gadgets pave the way for a resolution of the measurement problem that eschews state vector reduction while restoring the determinism lost in standard quantum mechanics. In fact, the Bohmian mechanics departs in significant ways from standard quantum mechanics. By itself this is not a basis for criticism; indeed, it makes Bohmian mechanics all the more interesting. But Bohmian mechanics is not, as the popular impression would have it, empirically equivalent to standard quantum mechanics in terms of probabilistic predictions for the outcomes of measurements of quantum observables. Indeed, in physically important applications to systems for which standard quantum mechanics delivers empirically well-confirmed probabilistic predictions, the sophisticated form of Bohmian mechanics designed to prove the global existence of Bohmian particle trajectories fails to deliver unequivocal predictions—of even a probabilistic variety—for the future behavior of said systems. Possible responses to this lacuna are discussed.

1 Introduction

Popular presentations of Bohmian quantum mechanics (BQM) give the impression that BQM is obtained from standard textbook quantum mechanics (SQM) by means of a subtraction and an addition: the subtraction takes the notorious von Neumann projection postulate (aka wave function collapse, state vector reduction) off the table and allows Schrödinger evolution to continue uninterrupted throughout a measurement procedure; the addition comes in the form of adjoining to SQM "hidden variables" in the form of exact particle positions together with an equation of motion for these variables, supposedly providing both for a fully deterministic interpretation of the theory and a resolution of the measurement problem.[1]

We have three comments about this popular impression and about BQM itself. First, this impression—for which the careful and scrupulous Bohmians are *not* responsible—is badly misleading. Briefly put, although BQM helps itself to the technical apparatus of Hilbert spaces, it is not a Hilbert space theory in the sense that its ideology and ontology differ markedly from that of SQM which takes the Hilbert space formalism seriously (all too seriously according to the Bohmians). Here we hasten to add our second comment: that BQM and SQM part company in significant ways in no measure diminishes the interest of BQM; indeed, to our minds it makes BQM all the more interesting. But we also hasten to add our third comment. The ways in which BQM differs from SQM make for difficult mathematical problems, one in partial differential equations (pdes) and one in ordinary differential equations (odes). The Bohmians have delivered a Beautiful Solution to this pair of problems. But a careful look at their solution reveals that the success comes at the price of introducing a new form of indeterminism or, perhaps more felicitously, a new form of indeterminacy: without supplementation, BQM offers no definite predictions, not even probabilistic predictions, about physically important systems for which SQM has provided empirically well-confirmed predictions. The purpose of this note is to develop and substantiate our third comment.[2]

We begin in Section 2 with a brief review of the pop science story of how BQM purportedly cures the indeterminism in SQM. The fuller and more interesting story is outlined in Section 3. Sections 4 and 5 discuss the motion of a particle in various singular potentials according respectively to SQM and BQM. For a particle moving in the Coulomb potential, SQM delivers deterministic evolution of the quantum state and, thus, deterministic predictions of probabilities of outcomes of future measurements given the current state; but for a particle moving strongly singular potentials it falters in failing to deliver deterministic evolution for the quantum state. By contrast, in its attempt to deliver fully deterministic predictions, the BQM treatment

[1] The focus of the present discussion is exclusively on ordinary non-relativistic QM. How the issues discussed here carry over to relativistic QFT is a topic for another occasion.

[2] For recent authoritative overviews of BQM see Dürr and Teufel (2009) and Goldstein (2013).

of particle motion in a Coulomb potential has the same faltering character as the SQM treatment of strongly singular potentials. Possible reactions on behalf of the Bohmians are discussed in Section 6. Conclusions are given in Section 7.

2 Indeterminism in SQM and the purported cure in BQM: the pop story

2.1 Determinism and indeterminism in SQM

Level 1 determinism Start with a classical mechanical system that admits a Hamiltonian formulation. Quantize in at least the minimal sense of finding an appropriate Hilbert space $\mathcal{H}$ and an appropriate operator H with dense domain $D(H) \subset \mathcal{H}$ corresponding to the classical Hamiltonian. Check whether H is essentially self-adjoint, i.e. whether it has a unique self-adjoint extension, namely, the closure $\overline{H}$ of H.[3] If so, exponentiate $\overline{H}$ to obtain a strongly continuous one-parameter unitary group $U(t) := \exp(-i\hbar\overline{H}t)$, $t \in \mathbb{R}$, which gives the deterministic Hilbert space dynamics: for any initial state vector $\psi_0 \in \mathcal{H}$ at $t = 0$, the state vector at any $t > 0$ is $\psi_t = U(t)\psi_0$. The infinitessimal version of this equation, viz.

$$i\hbar\frac{d\psi_t}{dt} = \overline{H}\psi_t \tag{1}$$

may be dubbed a Schrödinger equation, but (as will be explained shortly) it has to be carefully distinguished from the wave equation that is more commonly called the Schrödinger equation. Note that (1) is defined only for vectors in $D(\overline{H})$ whereas the relation $\psi_t = U(t)\psi_0$ is valid for all $\psi_0 \in \mathcal{H}$.

If the quantization scheme produces an H that is not essentially self-adjoint then matters become more complicated. For instance, H may still admit self-adjoint extensions—any real symmetric operator does—but the extensions may be highly non-unique. The physics of the situation may point to one of these extensions as physically natural. In that case, choose it and proceed as above. If there is no unique natural self-adjoint extension then quantum dynamics is ambiguous. We will return to this point several times with concrete examples.

[3] This check can be performed by applying von Neumann's deficiency index criterion; see Reed and Simon (1975, Sec. X.1) and the discussion below.

It is worth noting that at Level 1 quantum dynamics may be more deterministic than its classical counterpart. Classical determinism can falter for two sorts of reasons. (i) The initial value problem for the Newtonian equations of motion may not have a unique solution even locally in time, as is the case when Lipschitz continuity does not obtain.[4] (ii) The initial value problem for the Newtonian equations of motion may have a unique solution locally in time, but the solution may break down after a finite time. This behavior can occur in Newtonian mechanics of point pass particles interacting via a $1/r$ potential, either because of (a) a collision singularity or, more exotically, (b) a non-collision singularity where all the particles run off to spatial infinity in a finite time. All of these types of breakdown in classical determinism can, in some instances, be overcome at Level 1 in SQM in that the Hamiltonian operator for the quantum counterpart of the classically indeterministic system can be essentially self-adjoint so that any initial quantum state leads to a unique global evolution. Examples of this phenomenon will be given below.[5]

Level 2 determinism Needless to say, Level 1 determinism does not guarantee all of the determinism one might desire. For one thing, although Level 1 determinism gives a deterministic evolution of the quantum state, this state does not fix a unique outcome of a measurement but only a probability distribution (given by the Born rule) over the possible measurement outcomes. And even worse there are "collapse interpretations" of QM according to which the unitary evolution of the quantum state is interrupted during measurement and the state vector jumps indeterministically into an eigenstate of the operator being measured.[6] The motivation for introducing state vector collapse is to explain how quantum measurements yield determinate outcomes.

There are other interpretations of QM that offer alternative explanations of measurement outcomes without invoking state vector collapse or a disruption of unitary evolution. In addition to BQM, there are various species of many worlds interpretations and the family of modal interpretations.[7] No

[4]Norton's (2008) dome example belongs to this class; see Sec. 4.2 below.

[5]See also Earman (2009) for additional examples.

[6]In some treatments this "state vector collapse" is taken as a primitive while in other treatments a stochastic mechanism of collapse is offered, as in the GRW interpretation (see Ghirardi 2011).

[7]For an overview of modal interpretations, see Lombardi and Dieks (2014); for the

attempt will be made here to compare the merits of the various no-collapse interpretations. BQM aims to offer a fully deterministic explanation of measurement outcomes by means of a two part strategy. The first part marks one significant departure of BQM from SQM. SQM assumes that every observable corresponds to a self-adjoint operator, with the correspondence much more than an abstract mathematical association. In particular, SQM assumes that (a) the range of possible values of an observable is given by the spectrum of the corresponding self-adjoint operator, and (b) when an experiment reveals that an observable takes on a value within a given range, what is revealed is that the state of the system is an eigenstate of the projection operator which belongs to the spectral decomposition of the corresponding self-adjoint operator and which projects onto the spectral range equal to the revealed value range of the observable. BQM rejects this picture in favor of a radically instrumentalist interpretation of the Hilbert space operator formalism, and it posits that every measurement can be reduced to a position measurement. So, for example, the measurement of the spin of a particle is not to be construed as the measurement of a spin operator but as a measurement of the position of the particle, say, as it emerges from a Stern-Gerlach apparatus, with emergence in the top channel signalling spin up and emergence in the bottom channel signalling spin down. This aspect of BQM deserves detailed scrutiny, but our focus here is on the second part of the strategy, which entails another major parting of ways with SQM.

The second part of the strategy of BQM is (i) to posit that at every moment a particle has a definite position so that a position measurement is simply the recording of the preexisting value, not the creation of a value by means of state vector collapse, and (ii) to supply a "guiding equation" for the particle positions that, coupled with the Schrödinger equation, produces a deterministic evolution for the total state consisting of the quantum state plus the particle positions. How this is supposedly achieved is sketched in the next subsection. Looking further ahead the irony will be that in its ambition to achieve Level 2 determinism BQM undermines Level 1 determinism.

2.2 Determinism in BQM: the pop story

The story is illustrated for a system of N spinless particles moving in a configuration space Ω and the Hilbert space is $\mathcal{H} = L^2(\Omega, d\mu(q))$, $q \in \Omega$.

many worlds interpretation see Vaidman (2014).

In the cases considered here $\Omega = \mathbb{R}^{Nd}$ (or some subset of $\mathbb{R}^{Nd}$), where d is the dimension of physical space and the measure $d\mu$ is Lebesgue measure. From here on reference to the measure will be dropped. Assume that the Hamiltonian operator has the Schrödinger form

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\Delta_k + V(q), \quad q \in \Omega. \tag{2}$$

For a wide variety of potentials V, an H of this form will be essentially self-adjoint. Assume for the moment that this is the case so that Level 1 determinism holds in SQM. The pop story about how BQM achieves Level 2 determinism goes as follows. Take a solution to the Schrödinger equation

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\Delta_k\psi(q,t) + V\psi(q,t) \tag{SE}$$

and plug it into the guiding equation for Bohmian particles whose positions are labeled by X_k, $k = 1, 2, ..., N$:

$$\frac{dX_k(t)}{dt} = v_k^{\psi(q,t)}|_{q=X} \tag{GE}$$

where the velocity field on the rhs of (GE) is given by

$$v_k^{\psi(q,t)} := \frac{\hbar}{m_k}\,\mathrm{Im}\,\frac{\nabla_k\psi(q,t)}{\psi(q,t)}. \tag{3}$$

Now, the pop story continues, the guiding equation is a first-order ordinary differential equation. Hence, given a solution $\psi(q,t)$ of (SE), the resulting (GE) together with the positions $X_k(0)$ of the Bohmian particles at $t = 0$ uniquely fix the future (and past) positions. And since the Schrödinger equation leads to a deterministic evolution for the quantum state, given the *total initial* state at $t = 0$—the initial quantum state $\psi(q,0)$ and the initial particle positions $X_1(0), X_2(0), ..., X_N(0)$—there is a unique joint solution $(\psi(q,t), X_1(t), X_2(t), ..., X_N(t))$ for all $t > 0$ (and all $t < 0$) of the coupled (SE)-(GE) equations. Combining this result with the Bohmian doctrine that all measurements reduce to position measurements yields Level 2 determinism and a resolution of the measurement problem.

This scheme can be generalized in various ways, but the generality of BQM is not our concern here; rather our concern is with what the pop story hides and distorts.

3 Determinism in BQM: the fuller story

3.1 The first problem

The infinitesimal version of the fundamental dynanical law of SQM given in eq. (1) and the version of the Schrödinger equation given in (SE) are not notational variants of one another but refer to different entities. The time parameter "t" in eq. (1) appears as an index on ψ to emphasize that ψ_t is not a function of t; rather ψ_t denotes a parameterized curve, each point of which is a vector in Hilbert space $\mathcal{H}$. By contrast, the $\psi(q,t)$ in (SE) is a function of t—specifically, it is a function from $\Omega \times \mathbb{R}$ (the classical spacetime over the configuration space Ω) to $\mathbb{C}$. Isn't this simply pedantic fuss over notation? Isn't a solution ψ_t of the fundamental dynamical law of SQM related to a solution $\psi(q,t)$ of (SE) by the equality (E) $\psi_t(q) = \psi(q,t)$, $q \in \Omega$, $t \in \mathbb{R}$? Yes, but only with the proper understanding of "=". Hilbert space vectors are not wave functions (square integrable functions on Ω) but equivalence classes of wave functions, where two wave functions are counted as equivalent iff they are equal except for a set of Lebesgue measure zero.[8] Choosing a value $\bar{t} \in \mathbb{R}$ gives a vector $\psi_{\bar{t}}(q)$, but setting $\psi_{\bar{t}}(q) = \psi(q,\bar{t})$ does not define a wave function $\psi(q,\bar{t})$ unambiguously for all $q \in \Omega$ but only up to equivalence. From now on when instantiations of (E) are used this caveat has be understood. This is not pedanticism since in BQM the different members of an equivalence class of wave functions can correspond to physically distinct situations that have different implications for the motion of Bohmian particles. In particular, it may be that equivalent wave functions have different "nodes" (locations where they vanish), in which case BQM says that the two wave functions differ on the possible locations for particles and

[8] To simplify the discussion consider a single spinless particle and take $\Omega = \mathbb{R}$. Then the relevant equivalence relation is defined as follows: for $\psi(x), \varphi(x) \in L^2(\mathbb{R})$, $x \in \mathbb{R}$, $\psi \approx \varphi$ iff they differ on a set of Lebesgue measure dx zero. This construal is forced by a requirement on the inner product $(\cdot,\cdot)$ for a vector space $\mathcal{V}$ and, thus, on the associated norm $||v|| := \sqrt{(v,v)}$; namely, for all $v \in \mathcal{V}$, $(v,v) = 0$ iff $v = 0$. Using the standard inner product

$$(\varphi,\psi) := \int_{\mathbb{R}} \varphi^*(x)\psi(x)dx$$

for wave functions, if the wave functions ψ and φ are equivalent and if vectors are identified with wave functions (rather than equivalence classes of them) it follows that $(\psi-\varphi,\psi-\varphi) = 0 \Longrightarrow \psi - \varphi = 0$, yielding a contradiction when ψ and φ are not equal at every point $x \in \mathbb{R}$.

on locations where the guiding equation (GE) is well-defined.

It should also be kept in mind that a set of wave functions, even when formed into equivalence classes, does not constitute a Hilbert space, which must be closed in the norm induced by the inner product. But Bohmians may resist taking the norm closure; for even if one starts with a set of "good" wave functions that are sufficiently smooth so as to make (SE) and (GE) meaningful and, thereby, enabling BQM to yield the desired results for Bohmian particle trajectories, taking the norm closure can result in admitting "bad" wave functions.

One consequence of the wave function ontology of BQM should now be clear. In SQM the state space is Hilbert space, and as a result the initial value problem for state evolution is automatically solved once H is shown to be essentially self-adjoint or else H admits self-adjoint extensions and a particular self-adjoint extension is chosen; then for *any* initial state a unique solution exists for *all* times. In BQM the state space is not Hilbert space but some space of wave functions that are sufficiently smooth so that (SE) and (GE) make sense, and as a result the initial value problem for state evolution now becomes a problem in pdes: For $\psi(q, 0)$ from a suitable class of initial wave functions does a unique solution $\psi(q, t)$ of (SE) exist for all $t > 0$? And if so, is it the case that for each particular value $\bar{t}$ of t the wave function $\psi(q, \bar{t})$ belongs to the designated suitable class if $\psi(q, 0)$ does?

3.2 The second problem

The second problem for BQM is the existence and uniqueness for solutions for the guiding equation (GE). A priori this second problem might seem less difficult than the first since it concerns an ode rather than a pde; after all, standard results for odes guarantee existence and uniqueness, at least locally in time. But Bohmians are not satisfied with a determinism that falters after a finite time and, thus, they must set their sights on demonstrating global existence of Bohmian trajectories. In this regard there are three worries to deal with. First, the Bohmian trajectory may fail to exist after a finite time because Bohmian particles runs off to spatial infinity in a finite time, as can happen with Newtonian particles. The second and third worries are less exotic but even more pressing: a Bohmian particle can run into a node of the wave function or into a point where the potential V is singular and, consequently, the wave function is not differentiable; in either of these cases (GE) is not well defined. Since instantiations of the second two worries known to occur[9], a positive solution to the second problem cannot be expected in the form "For all initial particle positions ..." but at best in the form "For almost all initial particle positions ... "

The sophisticated Bohmians do shy away from problems, and they have produced an impressive simultaneous solution to the two problems that challenge BQM.

3.3 The Beautiful Solution

Here we summarize the Beautiful Solution with some of the technical niceties suppressed; none of the suppressed details affect the main points to me made below.[10]

Concentrate on Hamiltonians of the Schrödinger form (2). Let Ω_{BQM} denote the configuration space for Bohmian mechanics consisting of the standard configuration space $\Omega_{SQM} = \mathbb{R}^{Nd}$ with the points where the potential V fails to be smooth removed, e.g. with $N = 1$, $d = 3$, and V a central potential α/r^p, $p \geq 1$, the BQM configuration space is $\Omega_{BQM} = \mathbb{R}^3\backslash\{0\}$.[11] The associated Bohmian Hilbert space is $\mathcal{H}_{BQM} = L^2(\Omega_{BQM})$. This Hilbert space is used as an auxiliary device by the Bohmians since it is not considered the physical state space. A Schrödinger Hamiltonian H is a symmetric operator on the domain $C_0^\infty(\Omega_{BQM})$, which is dense in $\mathcal{H}_{BQM}$; and since H commutes with conjugation it has self-adjoint extensions (Reed and Simon 1975, Thm X.3).[12] Choose a self-adjoint extension $\widehat{H}$. Finally define $C^\infty(\widehat{H}) := \cap_n^\infty \mathcal{D}(\widehat{H}^n)$ where $\mathcal{D}(\widehat{H}^n)$ is the set of all $\psi \in \mathcal{H}_{BQM}$ for which the the expectation value of the $2n$-th power of $\widehat{H}$ is finite, i.e. $\int_{\Omega_{SQM}} \psi^* \widehat{H}^{2n} \psi dq < \infty$. This space $C^\infty(\widehat{H})$ is the state space for BQM, and it may be considered as a set of wave functions or a set of vectors (i.e. equivalence class of wave functions). The Bohmians prefer the former, but for sake of definiteness we will construe $C^\infty(\widehat{H})$ as a set of vectors.

[9] For example, the hope that Bohmian trajectories would always miss nodes was dashed by explicit counterexample; see Berndl (1996, 78).

[10] Two different proofs of the Beautiful Solution are given in Berndl et al. (1995) and Teufel and Tumulka (2005).

[11] If the singularities of the potential are not removed from the configuration space then the wave function can become non-differentiable, in which case neither (SE) nor (GE) is well-defined.

[12] $C_0^\infty(\Omega)$ denotes the the infinitely differentiable elements of $L^2(\Omega)$ with compact support on Ω.

The solution to the first problem (Section 3.1) is given in the following form. For a large class of Schrödinger type Hamiltonian operators and for any self-adjoint extension $\widehat{H}$ of the BQM Hamiltonian operator from this class, the physical state space $C^{\infty}(\widehat{H})$ is invariant under the unitary evolution $U_{\widehat{H}}(t) = \exp(-i\hbar\widehat{H}t)$, i.e. if $\psi_0 \in C^{\infty}(\widehat{H})$ then $C^{\infty}(\widehat{H}) \ni \psi_t = U_{\widehat{H}}(t)\psi_0$ for all $t \in \mathbb{R}$.[13] Furthermore, let $\psi(q,0)$ and $\psi(q,t)$ be any wave functions corresponding respectively to the vectors ψ_0 and ψ_t. Then $\psi(q,t)$ is $C^{\infty}(\Omega_{BQM}\text{x}\mathbb{R})$ almost everywhere (a.e.) relative to Lebesgue measure, and it is a.e. a solution to (SE).

The solution to the second problem (Section 3.2) is even more impressive in controlling the possible ways in which global Bohmian trajectories can fail to exist. Specifically, let $\psi(q,t)$ be the global solution to (SE) corresponding to the initial state $\psi(q,0)$ where $||\psi(q,0)|| = 1$. Substitute $\psi(q,t)$ into the guiding equation (GE). Then the set of initial particle positions at $t = 0$ which fail to give rise to globally unique solutions to (GE) is of measure zero relative to the measure $\mu_0(q) := |\psi(q,0)|^2$. Moreover, this notion of measure-zero is time-independent because $\mu_t(q) = |\psi(q,t)|^2$ for all $t \in \mathbb{R}$, where $\mu_t(q)$ denotes the image of μ_0 under the flow $q_k(0) = X_k(0) \longmapsto q_k(t) = X_k(t)$ defined by the solutions to (GE).

Actually, we have overstated the second part of the solution. It is conjectured that for any self-adjoint extension $\widehat{H}$ of the BQM Hamiltonian the solution $\psi(q,t) = U_{\widehat{H}}(t)\psi(q,0)$ gives rise to global Bohmian trajectories (see Berndl et al. 1995, 669-670), but the proof given does not show this. We will return to this point later.

3.4 Comments

The Beautiful Solution is achieved with the help of what amounts to the postulation of a new law of physics to the effect that only wave functions that are C^{∞} in the Hamiltonian are physically realizable.[14] The Bohmians admit that

[13] Hunziker (1966) showed that smooth solutions to (SE) are generated if the potential is bounded and smooth and the initial wave function belongs to Schwartz space (smooth functions whose derivatives vanish faster than any power of $|x|$ as $x \to \infty$). The Bohmians have improved on this result in that boundedness of the potential is not required in their result.

[14] Why do the Bohmians need such a postulate? Lipshitz continuity for the velocity field $v_k^{\psi} = \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \psi(q,t)}{\psi(q,t)}$ is sufficient for proving *local* uniqueness for solutions to the guiding equation. But having smooth ($C^{\infty}(\mathbb{R}^3\text{x}\mathbb{R})$) solutions of (SE) plays a crucial role in the

this is a strong restriction since although this set of wave functions is dense in the Hilbert space it is "most likely not a residual set in the norm topology of $[\mathcal{H}_{BQM}]$" (Berndl et al. 1995, 669).[15] But, nevertheless, they justify it on the grounds that only those wave functions satisfying said restriction can result from physically possible preparation procedures: "No physicist believes that a generic L^2-wave function (in the residual sense) results from the 'collapsed' wave function from a preparation procedure" (Berndl et al. 1995, 669-670). Of course, the real issue is not about what physicists believe or even about what they can in fact produce by a preparation procedure but rather about what Nature can present us with.[16] These matters call for further discussion, but we will not pause to provide it here.

Our main concern is with the price that apparently has to be paid for achieving the Beautiful Solution: for the Coulomb potential—surely one of the most important applications of ordinary non-relativistic QM—either BQM must be supplemented if the Level 1 determinism achieved by SQM, and along with it the successful probabilistic predictions of SQM, are not to be abandoned. In short, the story that, in exchange for the addition of few extra gadgets to SQM, BQM restores determinism is belied—or at the very least, the story needs to be much more complicated.

4 Singular potentials in SQM

In order to appreciate how the vaulting ambitions of BQM lead to the loss of Level 1 determinism in cases where SQM delivers it, it is helpful to contrast how the two theories deal with singular potentials. The successes and failures in SQM in this regard are reviewed in the present Section, and subsequently in Section 5 are contrasted with the situation in BQM.

proof of the *global* existence of Bohmian trajectories, and something akin to the condition that the initial wave function is C^∞ in the Hamiltonian seems to be necessary to ensure that the corresponding solution is smooth.

[15] A residual set is a countable intersection of open dense sets.

[16] From the point of view of SQM, what a preparation procedure produces is a quantum state, not a wave function. From this point of view the claim at issue would be that a preparation procedure can produce a state of a system only if the equivalence class of wave functions that constitutes said state contains an element that is C^∞ in the Hamiltonian of the system.

4.1 The Coulomb potential in SQM

Atomic physics and chemistry are based largely on the Coulomb interaction. This is fortunate for QM (and for us) since if Nature had chosen a different form of interaction SQM might well be stymied. The point is brought out by contrasting how SQM handles the Coulomb potential vs. more strongly singular potentials.

For simplicity of presentation attention will be focused on a single particle moving in a central Coulomb potential in $d = 3$ space, but similar conclusions hold for N particles moving under pairwise Coulomb potentials. Units are chosen so that the particle has mass 1/2 and the Hamiltonian operator has the form $H = -\Delta + V(r)$. In this subsection we will be concerned with the Coulomb potential $V(r) = -1/r$. The configuration space in SQM is $\Omega_{SQM} = \mathbb{R}^3$, and the Hilbert space is $L^2(\mathbb{R}^3)$. The Coulomb Hamiltonian operator acting on the dense domain $C_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$ is essentially self-adjoint (see Reed and Simon 1975, 186, Ex. 1). Thus, all of the apparatus of SQM can be applied; in particular, for any initial state vector $\psi_0 \in L^2(\mathbb{R}^3)$ at $t = 0$ SQM gives an unambiguous prediction at any $t > 0$ for the probabilities of outcomes of the measurement of any observable at said time.

In outline form the proof of the essential self-adjointness of the Coulomb Hamiltonian operator proceeds in three steps. First, it is shown that $-\Delta$ by itself is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Next is shown that $V(r) = -1/r$ can be regarded as a "small perturbation" of $-\Delta$ in the following sense:

> *Def.* Let A and B be densely defined operators with $D(A) \subset D(B)$. Then B is said to be A-bounded with relative bound a iff for some $a, b \in \mathbb{R}$ and all $\phi \in D(A)$
>
> $$||B\phi||^2 \le a||A\phi||^2 + b||\phi||^2$$
>
> or equivalently for some $\widetilde{a}, \widetilde{b} \in \mathbb{R}$
>
> $$||B\phi|| \le \widetilde{a}||A\phi|| + \widetilde{b}||\phi||$$

Finally, the following theorem is invoked:

Kato-Rellich Theorem (Reed and Simon 1975, Thm X.12). Suppose that A acting on $D(A)$ is self-adjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $D(A)$ and essentially self-adjoint on a core of A.[17]

Similar techniques can be applied to multiparticle systems with pairwise Coulomb interactions in order to prove that the Hamiltonian operator is essentially self-adjoint.

4.2 Repulsive potentials in SQM

Consider central potentials that have the form $V(r) = \alpha r^\beta$ with $\alpha > 0$ and $\beta > 0$. All of the Schrödinger Hamiltonian operators with these potentials are essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ (Simon 1973a). And this is so even though for some values of $\beta > 0$ the derivative of the potential is discontinuous or fails to exist at $r = 0$. In such cases the Newtonian equations of motion may admit multiple solutions for a particle initially stationed at the origin (as in Norton's dome example where $\beta = 3/2$), and the Bohmian guiding equation (GE) is mute about the fate of such a particle. But according to SQM there is a unique evolution for any wave packet, e.g. a symmetric wave packet centered at $r = 0$ at $t = 0$ will diffuse symmetrically away from the origin.

4.3 Strongly singular potentials in SQM

So far SQM has shrugged off singularities in the potential to deliver Level 1 determinism. But now consider potentials of the form $V(r) = -\alpha/r^p$ with $p \geq 2$ and $\alpha > 0$. An heuristic treatment of such potentials was given by Case (1950), who found that when $p > 2$ or else $p = 2$ and $\alpha > 1/4$ the generalized eigenfunctions of the Schrödinger H are "overcomplete" in the sense that eigenfunctions corresponding to different energies are not necessarily orthogonal unless a phase factor in the eigenfunctions is held fixed across the different energies.[18] In 1950 Case did not have available the now standard

[17] If A is a closed operator then a subset $C \subset D(A)$ is a core for A if $\overline{A|_C} = A$.

[18] The rigorous discussion of generalized eigenfunctions requires rigged Hilbert spaces since these functions are not elements of the original Hilbert space. A central result is that for a self-adjoint operator A on $\mathcal{H}$ there is a rigging of $\mathcal{H}$ in which there is complete orthonormal set of generalized eigenfunctions of A.

von Neumann analysis of self-adjointness, but in hindsight what he discovered is this. The Hamiltonian operators in question are symmetric operators on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ which is dense in $L^2(\mathbb{R}^3\backslash\{0\})$, and since they commute with complex conjugation they admit self-adjoint extensions.[19] In the von Neumann nomenclature, this means that the deficiency indices $< n, m >$ (where n and m are integers) of the operators are equal.[20] If $n = m = 0$ then the operator is essentially self-adjoint, whereas if $n = m > 0$ then the operator has an n-parameter family of self-adjoint extensions. What Case found is that for the potentials in question $n = m \geq 1$ with one of the parameters in the family of self-adjoint extensions corresponding to Case's phase factor.[21]

The different self-adjoint extensions of the Schrödinger Hamiltonians for these strongly singular potentials produce different physics; in particular, the energy eigenvalues and the unitary dynamics are both different for the different extensions. The latter means that, until a particular extension is chosen, SQM fails to provide unambiguous probabilistic predictions about the future behavior of a system in a specified initial state if the interactions in the system are mediated by strongly singular potentials. By means of heuristic arguments Landau and Lifshitz (1977, Secs. 18 and 35) conclude that for the potentials in question the energy is not bounded below and, therefore, the particle moving in these potentials can "fall to the center." Of course, whether the particle will literally fall to the center—or execute any other motion—cannot be treated quantum mechanically until a self-adjoint extension of the Hamiltonian operator is chosen. But once the choice is made, one knows in advance that even if a "fall to the center" takes place it will not involve a catastrophe that prevents the solution from continuing after some finite time.

Three broad options are available to deal with the failure of Level 1 determinism for these strongly singular potentials. Option 1 is to add additional principles or laws to SQM to single out or at least narrow down the infinite class of self-adjoint extensions. Some physicists have a favorite method of obtaining a specific extension, e.g. Schechter (2002, Sec. 4.2) favors the form

[19] The initial domain cannot be taken to be the test functions $C_0^\infty(\mathbb{R}^3)$ since the potentials at issue are not $L^2_{loc}(\mathbb{R}^3)$ and the operator defined by $-\Delta\varphi + V\varphi$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, is not a densely defined operator in $L^2(\mathbb{R}^3)$.

[20] A specific example of the computation of deficiency indices, deficiency, spaces, etc. will be given in the following section and the Appendix.

[21] For a rigorous proof that the Schrödinger Hamiltonian operator with $V(r) = -\alpha/r^p$ is not essentially self-adjoint when $\alpha = 1$ and $p = 3$ see Berezin and Schubin (1991, 157-159).

sum extension, which is given by a specific and seemingly natural construction.[22] For for semi-bounded operators the Friedrichs extension[23], which is maximal among the semi-bounded self-adjoint extensions, also has its champions; but it is unavailable for the potentials now under consideration where the Hamiltonian operator is not bounded below. In any case, no convincing argument is given to support the idea that Nature herself must or does in fact favor the extension given by one or another of these techniques, and it is not easy to see what form such an argument would take. Option 2 is to admit that it is a matter of purely contingent fact which self-adjoint extension Nature chooses. Experiments and inductive inference are thus required to settle the matter, and only when it is settled is Level 1 determinism restored. A fall-back Option 3 is to look for commonalities among all of the self-adjoint extensions—this much the theory does unequivocally predict without having to exercise either of the first two Options. But such commonalities may be disappointingly meager.

Since none of these options is attractive, SQM would be in a bit of a quandary if Nature were to mediate interactions via strongly singular potentials. The Bohmiams are responsible for an analogous quandary of their own making.

5 Singular potentials in BQM

5.1 The Coulomb potential in BQM

The treatment of the Coulomb interaction in BQM creates a situation that SQM is forced to confront for strongly singular potentials. The Hamiltonian operator $-\Delta - \frac{1}{r}$ is symmetric but not essentially self-adjoint on the domain $D(-\Delta - \frac{1}{r}) := C_0^\infty(\mathbb{R}^3 \backslash \{0\})$; indeed, even $-\Delta$ by itself is not essentially

[22] In rough outline, the construction goes as follows. Associated with a symmetric operator S with domain $D(S)$ is a hermitian two-form $h(\phi, \psi) := (S\phi, \psi)$, $\phi, \psi \in D(S)$, which in turn defines a norm $||\xi||_h := \sqrt{h(\xi, \xi)}$, $\xi \in D(S)$. Associated with the norm closure of the two-form is a self-adjoint operator T, the form extension of S. If h_1 and h_2 are closed two-forms then so is $h_{1,2} := h_1 + h_2$, and the operator $T_{1,2}$ associated with $h_{1,2}$ is the form sum of the operators T_1 and T_2 associated respectively with h_1 and h_2.

[23] The Friedrichs extension of a positive symmetric S with domain $D(S)$ is the form extension with respect to the form $h_F(\phi, \psi) := ((S + I)\phi, \psi)$, $\phi, \psi \in D(S)$. For more details about form sum and Friedrichs extensions see Faris (1975).

self-adjoint on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$.[24] To see this consider the case of zero angular momentum where

$$\Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \tag{4}$$

On a suitable domain $\Delta^* = \Delta$, and with this ansatz we seek to find the deficiency spaces by finding the solutions of

$$-\Delta\psi_\pm(r) = \mp i\psi_\pm(r), \ \ \psi_\pm(r) \in L^2(\mathbb{R}^3\backslash\{0\}). \tag{5}$$

There are two linearly independent solutions

$$\psi_\pm(r) = \frac{\exp(\frac{-(1\pm i)r}{\sqrt{2}})}{r} \tag{6}$$

An integration by parts argument shows that $\psi_\pm(r) \in D(\Delta^*)$ (see Dürr and Teufel 2005, Sec. 14.3). These solutions span the von Neumann deficiency spaces $\kappa_\pm$ where $\dim(\kappa_+) = \dim(\kappa_-) = 1$, so there is a one-parameter infinity of self-adjoint extensions. The isometries $U_\gamma : \kappa_+ \to \kappa_-$ are given by $U_\gamma\psi_+ = \gamma\psi_-$, $|\gamma| = 1$. Thus, the domains of the self-adjoint extensions Δ_γ of Δ are (see Reed and Simon 1975, Thm X.2)

$$\begin{aligned} D(\Delta_\gamma) &= \{\varphi + \beta\psi_+ + \beta U_\gamma\psi_+ | \varphi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}), |\beta| = 1\} \\ &= \{\varphi + \beta\psi_+ + \beta\gamma\psi_- | \varphi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}), |\beta| = 1\} \end{aligned} \tag{7}$$

and the action of $-\Delta_\gamma$ on $D(-\Delta_\gamma)$ is given by

$$\begin{aligned} -\Delta_\gamma(\varphi + \beta\psi_+ + \beta\gamma\psi_-) &= -\Delta\varphi + i\beta\psi_+ - i\beta\gamma\psi_- \\ &= -\Delta\varphi + i\beta\Big(\frac{\exp(\frac{-(1+i)r}{\sqrt{2}})}{r} - \gamma\frac{\exp(\frac{-(1-i)r}{\sqrt{2}})}{r}\Big), \\ \varphi &\in C_0^\infty(\mathbb{R}^3\backslash\{0\}). \end{aligned} \tag{8}$$

[24] However, $-\Delta$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ when $d \geq 4$. The reader is also cautioned that the fact an operator A is not essentially self-adjoint on a domain does not necessarily mean that $A + B$ is not essentially self-adjoint on that domain (see following subsection). But in the present case neither $-\Delta$ alone nor $-\Delta - \frac{1}{r}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$.

It is tempting to think that the self-adjoint extensions of the Bohmian Hamiltonian operator with Coulomb potential can now be obtained by applying the Kato-Rellich theorem (recall Section 4.1) to the previously found self-adjoint extensions Δ_γ of Δ. However, in this instance the potential is not a "small perturbation" of Δ_γ since $D(\Delta_\gamma) \subsetneq D(V) = \{\varphi \in \mathcal{H} : \int_0^\infty \frac{1}{r^2}\varphi^*(r)\varphi(r)r^2dr = \int_0^\infty \varphi^*(r)\varphi(r)dr < \infty\}$. The operators Δ and $\frac{1}{r}$ have a common dense domain $D(\Delta) \cap D(\frac{1}{r})$, which is in fact equal to $D(\Delta)$, but $D(\Delta_\gamma)$ is not contained in $D(\frac{1}{r})$.

So to get the self-adjoint extensions of $-\Delta - \frac{1}{r}$ with initial domain $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ we need to employ a frontal attack and to find the deficiency spaces by finding the solutions of

$$[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} - \frac{1}{r}]\phi_\pm = \mp i\phi_\pm, \quad \phi_\pm \in \mathcal{H} = L^2(\mathbb{R}^3\backslash\{0\})). \tag{9}$$

The general solutions take the form

$$\begin{aligned} \phi_\pm(r) \;&=\; c_1 \exp(\frac{-(1\pm i)r}{\sqrt{2}})M(1 - \frac{1\mp i}{2\sqrt{2}}, 2, \sqrt{2}(1\pm i)r)) \\ &\quad + c_2 \exp(\frac{-(1\pm i)r}{\sqrt{2}})U(1 - \frac{1\mp i}{2\sqrt{2}}, 2, \sqrt{2}(1\pm i)r)) \end{aligned} \tag{10}$$

where $M(a,b,z)$ and $U(a,b,z)$ are respectively the confluent hypergeometric functions of the first and second kind (see Appendix) and c_1 and c_2 are arbitrary constants. Since M and U are linearly independent the deficiency spaces are two-dimensional so that the isometries between the spaces are no longer the trivial multiplication by a phase factor, and the extended domains become correspondingly more complicated than in the case of $-\Delta$ alone.

The singularity structure of the potential for a multiparticle system with pairwise Coulomb interactions is yet more complicated, and correspondingly the BQM configuration space and the self-adjoint extensions of the BQM Hamiltonian operator for such a system are even more complicated.

5.2 Repulsive potentials in BQM

For central potentials that have the form $V(r) = \alpha r^{\beta}$ with $\alpha > 0$ and with $\beta > 0$ the Schrödinger Hamiltonian operator is not essentially self-adjoint on $C_0^{\infty}(\mathbb{R}^3 \backslash \{0\})$ but admits multiple self-adjoint extensions. Finding explicit expressions for the self-adjoint extensions will not be attempted here. When the exponent β is chosen so that $V(r)$ is not smooth, the application of the Beautiful Result requires the use of the domain $C_0^{\infty}(\mathbb{R}^3 \backslash \{0\})$ in place of $C_0^{\infty}(\mathbb{R}^3)$.

It should be noted that mutilating the configuration space in the way required by BQM does not always wreck essential self-adjointness. For example, the Schrödinger Hamiltonian operators for some singular repulsive potentials, e.g. α/r^2 with $\alpha \geq 3/4$, are essentially self-adjoint on $C_0^{\infty}(\mathbb{R}^3 \backslash \{0\})$ (Simon 1973b)—intuitively, the strong repulsive force near the origin is sufficient to prevent the particle from reaching the origin and, thus, to prevent probability from leaking away.

5.3 Comments

A separate section on strongly singular potentials in BQM is not needed since the treatment is substantially the same as in SQM.

One is immediately struck by how much extra work is needed in the BQM treatment of the Coulomb potential and how complicated the results are. But Bohmians are happy to take on the extra work and to deal with the complications. What they should not be happy with is a strong disanalogy with SQM. For Coulomb interactions SQM gives unequivocal probabilistic predictions for the future behavior of a system once the initial state is specified. By contrast BQM yields a treatment of Coulomb interactions that is similar in crucial aspects to the SQM (and BQM) treatment of strongly singular potentials: since the different self-adjoint extensions of the BQM Coulomb Hamiltonian operator give rise to different dynamics, the upshot is that, even with the initial state specified, BQM gives no unequivocal predictions for the future behavior of the systems that are bread and butter of atomic physics and chemistry.

An instructive toy example of the difference in dynamics for different self-adjoint extensions of a non-essentially self-adjoint Hamiltonian operator is the motion of a free particle on the positive half-line $\mathbb{R}^+ = (0, +\infty)$ (Reed and Simon 1975, 144-145). The Hamiltonian operator $-\Delta_x = -d^2/dx^2$ is symmetric on the domain $C_0^\infty(\mathbb{R}^+)$ which is dense in $L^2(\mathbb{R}^+)$. There is a one-parameter family of self-adjoint extensions $-\Delta_{x,a}$ with the parameter $a \in \mathbb{R} \cup \{\infty\}$. An incoming wave function that can be approximated near the origin by a plane wave e^{-kx} with momentum $k > 0$ gets reflected at the origin and becomes an outgoing plane wave δe^{kx} with a change of phase $\delta(a,k) = (ik-a)/(ik+a)$. For $a = \infty$ the phase change $\delta(\infty, k) = -1$ for all k, as if the particle makes an elastic collision at $x = 0$.

We cannot supply such a neat example of how the different self-adjoint extensions of the Hamiltonian operator in the BQM treatment of the Coulomb potential make for differences in scattering, but we can respond to the question of whether or not the different dynamics will result in different Bohmian particle trajectories. For the dynamics generated by two different self-adjoint extensions the same initial state $\psi(q,0)$ results in different solutions $\psi_1(q,t)$ and $\psi_2(q,t)$ of (SE). The question is then whether the same initial Bohmian particle positions $X_k(0)$ will lead to different Bohmian trajectories when the two solutions $\psi_1(q,t)$ and $\psi_2(q,t)$ are inserted into (GE), the answer being affirmative iff the terms on the rhs of eq. (3) differ for the two solutions, i.e. $\operatorname{Im} \dfrac{\nabla\psi_1(q,t)}{\psi_1(q,t)} \neq \operatorname{Im} \dfrac{\nabla\psi_2(q,t)}{\psi_2(q,t)}$ for some $t > 0$. The affirmative could be demonstrated by direct calculation in concrete examples, but one can see in the abstract that Bohmian trajectories must in general be different if (as the Bohmians desire) global trajectories exist in said self-adjoint extensions for almost any initial particle positions; for then, if the Bohmian trajectories were the same for both extensions, the time independence of the equal initial measures $\mu_0^1(q) := |\psi_1(q,0)|^2$ and $\mu_0^2(q) := |\psi_2(q,0)|^2$ under the configuration space flow $q_k(0) = X_k(0) \longmapsto q_k(t) = X_k(t)$ generated by the Bohmian trajectories would mean that $\mu_t^1(q) = \mu_t^2(q)$ and, thus, that $|\psi_1(q,t)|^2 = |\psi_2(q,t)|^2$ for all $t \in \mathbb{R}$. In sum, the different self-adjoint extensions produce different physics for both the ontology of SQM and the ontology of BQM.

Returning to the toy example of the motion of a particle on the half-line, it is known that for every self-adjoint extension $-\Delta_{x,a}$ of the free particle Hamiltonian the resulting dynamics supports the existence of global Bohmian particle trajectories (Berndl et al. 1995). Can the same be said of all of the self-adjoint extensions found in the preceding subsection for the BQM Schrödinger Hamiltonian operator with a Coulomb potential? The Bohmians have conjectured a positive answer but the proof of global existence for Bohmian trajectories has only been given for the form sum extension.

The toy example of the motion of a free particle on the half-line also raises another issue. The Hilbert space $L^2(\mathbb{R}^+)$ for the motion on the half-line has a natural embedding into the Hilbert space $L^2(\mathbb{R})$ for the motion of a particle on the full real line: since $C_0^\infty(\mathbb{R}^+)$ and $C_0^\infty(\mathbb{R})$ are dense respectively in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R})$ the inclusion map $\iota : C_0^\infty(\mathbb{R}^+) \to C_0^\infty(\mathbb{R})$ extends by linearity and continuity to an isometric map of $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R})$.[25] Given this natural identification of $L^2(\mathbb{R}^+)$ as a subspace of $L^2(\mathbb{R})$, one can ask whether some self-adjoint extension $-\Delta_{x,a}$ of $-\Delta_x = -d^2/dx^2$ acting on $C_0^\infty(\mathbb{R}^+)$ produces a unitary dynamics on $L^2(\mathbb{R}^+)$ that agrees with the dynamics induced by the $L^2(\mathbb{R})$ unitary dynamics generated by the full-line Hamiltonian Δ_x^f (the unique self-adjoint extension of $-\Delta_x = -d^2/dx^2$ acting on $C_0^\infty(\mathbb{R})$). Obviously not; indeed, the latter dynamics does not even preserve $L^2(\mathbb{R}^+)$—for any initial wave function $\psi(x, 0)$ whose support lies entirely in $\mathbb{R}^+$, $\psi(x, t) = U_{\Delta_x^f}(t)\psi(x, 0)$ has support in $\mathbb{R}^- = (-\infty, 0)$ for any $t > 0$, and for some $\psi(x, 0)$ whose support lies entirely in $\mathbb{R}^+$, $\psi(x, t)$ has as small a tail as you like in $\mathbb{R}^+$ for large values of t. The analogous question for the BQM vs. the SQM treatment of a particle moving in a central Coulomb potential has a happier answer. Since $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ is dense in $L^2(\mathbb{R}^3)$, the inclusion map $\iota : C_0^\infty(\mathbb{R}^3\backslash\{0\}) \to C_0^\infty(\mathbb{R}^3)$ extends by linearity and continuity to an isomorphism of $L^2(\mathbb{R}^3\backslash\{0\})$ onto $L^2(\mathbb{R}^3)$. Under this identification some member $\widehat{H}_{BQM}$ of the family of self-adjoint extensions for the BQM Coulomb Hamiltonian operator $-\Delta - \dfrac{1}{r}$ acting on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ will have a domain $D(\widehat{H}_{BQM}) \subset L^2(\mathbb{R}^3)$ that includes $C_0^\infty(\mathbb{R}^3)$, and since if two self-adjoint operators are equal if they agree when restricted to a core of one of them, $\widehat{H}_{BQM}$ equals the unique self-adjoint extension $\overline{H}_{SQM}$ of the SQM Hamiltonian operator acting on $C_0^\infty(\mathbb{R}^3)$.

6 Reactions

The sophisticated Bohmians are, of course, well aware of the situation described in the preceding section. They have a forthright and characteristically bold two-fold response. First, they aver that "In Bohmian mechanics there is no a priori reason to demand self-adjointness of the Hamiltonian" (Berndl et al. 1994, 434) and "An axiom, or dogma, of self-adjointness of

[25] Of course $L^2(\mathbb{R}^+)$ are $L^2(\mathbb{R})$ isomorphic since all infinite dimensional and separable Hilbert spaces are isomorphic. But an arbitrary isomorphism will not agree with the inclusion map of $C_0^\infty(\mathbb{R}^+)$ into $C_0^\infty(\mathbb{R})$.

the Hamiltonian (or equivalently unitarity of the wave function evolution) appears quite inappropriate from the Bohmian perspective" (Berndl et al. 1995, 669). We agree completely that the (essential) self-adjointness of the Hamiltonian operator cannot be taken as an a priori axiom, whether in BQM or SQM—the example of strongly singular potentials shows this for SQM (recall Section 4.3). But the point is that in SQM there is a natural sense in which the Schrödinger Hamiltonian operator for a Coulomb potential *is* essentially self-adjoint whereas this is *not* the case in BQM.

This leads to the second part of the Bohmian response: "It is now [in the presence of multiple self-adjoint extensions for the BQM Hamiltonian operator] a matter of physics of the system being described to choose the right one" (Berndl et al. 1994, 434) and "the choice of the right self-adjoint extension is a matter of physics" (Berndl 1996, 79).[26] We do not know what to make of the 'It's a matter of physics' mantra except as a tacit appeal to one of the Options (outlined in Sec. 4.3) that arose in SQM in reaction to the failure of Level 1 determinism for strongly singular potentials.

Option 1 for BQM (add new principles/laws) is interesting in the light of the fact that the global existence of Bohmian trajectories has not been proved for all self-adjoint extensions of BQM Hamiltonian operators for the Coulomb potential and other singular potentials, although as noted Bohmians have conjectured that global trajectories exist for all self-adjoint extensions. If the conjecture fails then the existence of global trajectories could be used as a selection principle to identify the physically permissible extensions. Such a move strikes us as using circular reasoning that would not appeal to anyone not convinced of the need for BQM in the first place. If the conjecture is at least partially correct and multiple self-adjoint extensions support global Bohmian trajectories then implementing Option 1 would require the introduction of additional physical principles or laws—over and above the requirement that global Bohmian particle trajectories exist—to single out a unique self-adjoint extension. Bohmians have shown a willingness to postulate new laws restricting the permissible initial wave functions (recall Section 3), but what new laws would suffice for current purposes and how to justify their law status remains to be seen.

Option 2 for BQM (claim that Nature makes the choice of self-adjoint

[26] Speaking of the fact that $-\Delta$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ Dürr and Teufel (2005, 295) write: "There is of course no problem with that. It is a matter of physics to select the correct 'physical' extension."

extension a purely contingent matter to be settled by experiment and inductive inference) would be particularly interesting if there were a serious possibility that Nature might choose one of the many self-adjoint extensions of the Bohmian Hamiltonian operator (say, for the Coulomb potential) that produces a dynamics that differs from the dynamics delivered by SQM. Then BQM would be vindicated as an alternative to SQM. But if, as a straight induction from all past experience indicates, Nature always chooses the self-adjoint extension that agrees with the dictates of SQM, do the Bohmians then say with a straight face, 'That's just the way things turned out,' or do they harbor some doubts to the effect that 'Carrying out our program has led to the illusion that there are live physical possibilities (corresponding to the infinity of self-adjoint extensions generated by our treatment) that have to be adjudicated either by experiment or by new physical principles not recognized by SQM'?

Option 3 (make do with the commonalities among the self-adjoint extensions) is just as unattractive in the BQM treatment of Coulomb interactions as it was in the SQM treatment of strongly singular potentials since the commonalities are sparse.

If the Bohmians do not want to exercise any of these Options they can seek to provide a different approach to proving the global existence of Bohmian particle trajectories that avoids the problem that necessitates resort to the unattractive Options. It is fruitless to speculate on the chances of success of such an endeavour until a specific proposal is provided. Alternatively, Bohmians could abandon the ambition of proving the global existence of Bohmian particle trajectories and settle for local existence. But since in BQM all measurements are supposed to reduce to position measurements, such an abandonment would conjure the specter of a pervasive breakdown in determinism for times which the local existence and uniqueness theorems fail to reach. No doubt there are still other avenues the Bohmians can pursue, and we would not be so rash as to predict that none of them will lead to an attractive form of BQM. We do insist, however, that the case of the Coulomb potential indicates that, without further tweaking, BQM in its present incarnation is not an attractive alternative to SQM.

An outside observer who is free of both Bohmian and anti-Bohmian prejudices might want to call a plague on both houses on the grounds that debate about the correct treatment of Coulomb interaction is based on an illicit idealization.

> In any physical problem in which we express the interaction between two systems by means of a potential which becomes infinite when the distance between them becomes zero, we are dealing with an idealization. Thus, the Coulomb interaction between the electron and the nucleus is not strictly proportional to $1/r$ all the way down to $r = 0$. The finite size of the nucleus sets one limit. Even for a single proton there is the probable finite radius of the proton. (Case 1950, 797)

While we are cognizant of the danger that the use of idealizations can produce "effects" that are merely artifacts of the idealizations, we do not share Case's worries about the present instance. When physicists speak of the radius of the proton they do not mean to indicate there evidence that there is a hard core of the proton in the sense of a cut off point $r = \epsilon > 0$ beyond which the wave function of the electron cannot penetrate the proton. But if there were such a cutoff one would need a modification of the Coulomb law to include a repulsive potential that becomes infinitely high at $r = \epsilon$, and depending on the form of the repulsive potential there could be problems for BQM similar to the ones discussed above. Our main point, however, is that if the point particles—assumed in *both* SQM and BQM—and the Coulomb potential are idealizations then they are extraordinarily successful ones, at least as deployed in SQM; indeed, since no gain in empirical accuracy promises to be forthcoming in the non-relativistic regime by removing the alleged idealizations, it could be argued that they are not, after all, idealizations in the context of ordinary quantum mechanics.[27]

7 Conclusion

Nothing in what we have said shows that BQM in not a viable alternative to SQM in the non-relativistic regime. But it does show how complicated and subtle the deployment of BQM is—far more nuances are required than what the pop science presentations would lead one to believe. In particular, we want to underscore the main irony uncovered above: BQM aims to give a fully deterministic interpretation of QM by inflating the ontology of

[27] Of course, these issues take on a different cast in relativistic QFT; but going in this direction opens the topic of a Bohmian account of QFT, a matter that we cannot even begin to treat here.

SQM to include precise particle trajectories and assigning distinct physical significance to wave functions that correspond to the same Hilbert space vector; but implementing the BQM strategy in this inflated ontology seems to undermine the Level 1 probabilistic determinism of SQM. For example, without the help of supplementary physical principles BQM doesn't give any non-ambiguous prescription for the probabilities of future measurement outcomes for a particle moving in a central Coulomb potential even when the total Bohmian initial state (initial SQM state plus initial particle positions) is specified. SQM does give unambiguous probabilistic predictions in this case, and only an unregenerate inductive skeptic would deny that the massive evidence in favor of the empirical adequacy of SQM leaves little room to doubt its predictions.

Acknowledgment: We are grateful to Anthony Anderson and Laura Ruetsche for helpful suggestions. Needless to say, we alone are responsible for errors and mistakes.

Appendix

In solving eq. (10) consider the ϕ_+ case omitting for brevity the subscript:

$$[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r}]\phi = i\phi \tag{A1}$$

For $r \to \infty$ the inverse powers of r become negligible. So for "large values" of r the solutions of (A1) will approximate the solutions to

$$\frac{d^2\phi_\infty}{dr^2} = i\phi_\infty \tag{A2}$$

which are given by

$$\phi_\infty(r) = \exp(\frac{-(1+i)r}{\sqrt{2}}) \tag{A3}$$

Now substitute $\varphi(r) = \phi_\infty(r)\xi(r)$ into (A1), and find that $\xi(r)$ must satisfy the equation

$$\frac{d^2\xi}{dr^2} + [\frac{2}{r} - \sqrt{2}(1+i)]\frac{d\xi}{dr} + [(1 - \sqrt{2} + i\sqrt{2})\frac{1}{r}]\xi = 0 \tag{A4}$$

Since we are only interested in solutions with $r \in (0, \infty)$ eq. (A4) can be rewritten in the form

$$r\frac{d^2\xi}{dr^2} + [2 - \sqrt{2}(1+i)r]\frac{d\xi}{dr} + [1 - \sqrt{2} + i\sqrt{2}]\xi = 0 \tag{A5}$$

With $u := \sqrt{2}(1+i)r$ one has

$$\frac{d\xi}{dr} = \sqrt{2}(1+i)\frac{d\xi}{du} \quad \text{and} \quad \frac{d^2\xi}{dr^2} = 4i\frac{d^2\xi}{du^2} \tag{A6}$$

so that (A5) becomes

$$u\frac{d^2\xi}{du^2} + (2-u)\frac{d\xi}{du} - (1 - \frac{1-i}{2\sqrt{2}})\xi = 0 \tag{A7}$$

This last equation has the form

$$z\frac{d^2w}{dz^2} + (b-z)\frac{d\xi}{du} - aw = 0, \tag{A8}$$

which is known as Kummer's equation or the confluent hypergeometric differential equation. It has two linearly independent solutions $M(a, b, z)$, confluent hypergeometric functions of the $1st$ kind or Kummer's function, and $U(a, b, z)$, confluent hypergeometric functions of the $2nd$ kind (see Abramowitz and Stegun, 1972, 504). In our case $a = 1 - \dfrac{1-i}{2\sqrt{2}}$ and $b = 2$. So the general solution to (A7) is

$$\xi(u) = c_1 M(1 - \frac{1-i}{2\sqrt{2}}, 2, u) + c_2 U(1 - \frac{1-i}{2\sqrt{2}}, 2, u) \tag{A9}$$

where c_1 and c_2 are arbitrary constants. Thus, the general solution to (A1) is

$$\begin{aligned} \phi(r) \quad = \quad & c_1 \exp(\frac{-(1+i)r}{\sqrt{2}}) M(1 - \frac{1-i}{2\sqrt{2}}, 2, \sqrt{2}(1+i)r)) \\ & c_2 \exp(\frac{-(1+i)r}{\sqrt{2}}) U(1 - \frac{1-i}{2\sqrt{2}}, 2, \sqrt{2}(1+i)r)) \end{aligned} \tag{A10}$$

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