Quantum Correlations are Weaved by the Spinors of the Euclidean Primitives

Joy Christian*
Einstein Centre for Local-Realistic Physics, 15 Thackley End, Oxford OX2 6LB, United Kingdom

The exceptional Lie group E_8 plays a prominent role both in mathematics and theoretical physics. It is the largest symmetry group connected to the most general possible normed division algebra, that of the non-associative real octonions, which — thanks to their non-associativity — form the only possible closed set of spinors that can parallelize the 7-sphere. By contrast, here we show how a similar 7-sphere also arises naturally from the algebraic interplay of the graded Euclidean primitives, such as points, lines, planes and volumes, characterizing the three-dimensional conformal geometry of the physical space, set within its eight-dimensional Clifford-algebraic representation. Remarkably, the resulting algebra remains associative, and allows us to understand the origins and strengths of all quantum correlations locally, in terms of the geometry of the compactified physical space, namely that of a quaternionic 3-sphere, S^3, with S^7 being the corresponding algebraic representation space. Every quantum correlation can thus be understood as a correlation among a set of points of this S^7, computed using manifestly local spinors in S^3, thereby setting the geometrical upper bound of $2\sqrt{2}$ on the strengths of all quantifiable correlations. We demonstrate this by first proving a comprehensive theorem about the geometrical origins of the correlations predicted by any arbitrary quantum state, and then explicitly reproducing the strong correlations predicted by the EPR-Bohm and GHZ states. The *raison d'être* of strong correlations turns out to be the twist in the Hopf bundle of S^3 within S^7.

I. MODERN PERSPECTIVE ON THE EUCLIDEAN PRIMITIVES

In physical experiments — which are usually confined to the three-dimensional physical space by necessity — we often measure relevant quantities by setting up a Cartesian coordinate system $\{x, y, z\}$ in that space. Mathematically this is equivalent to identifying the Euclidean space $\mathbb{E}^3$ with a three-fold product of the real line, $\mathrm{I\!R}^3$. In practice we sometimes even think of $\mathrm{I\!R}^3$ as *the* Euclidean space. Euclid himself, however, did not think of $\mathbb{E}^3$ in terms of such a Cartesian triple of real numbers. He defined a representation of $\mathbb{E}^3$ axiomatically, in terms of primitive geometric objects such as points and lines, together with a list of their properties, from which his theorems of geometry follow.

It is, however, not always convenient to model the physical space in the spirit of Euclid. Therefore in practice we tend to identify $\mathbb{E}^3$ with $\mathrm{I\!R}^3$ whenever possible. But there is no intrinsic way of identifying the two spaces in this manner without introducing an *unphysical* element of arbitrarily chosen coordinate system. This difficulty is relevant for understanding the origins of quantum correlations, for time and again we have learned that careless introduction of unphysical ideas in physics could lead to distorted views of the physical reality [1][2]. An intrinsic, coordinate-free representation of the Euclidean space is surely preferable, if what is at stake is the very nature of the physical reality.

Fortunately, precisely such a representation of $\mathbb{E}^3$ was proposed by Grassmann in 1844 [3][4]. In the Euclidean spirit, the basic elements of this powerful algebraic representation are not coordinate systems, but points, lines, planes, and volumes, *all treated on equal footing*. Given a set $\{\,\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z\}$ of basis vectors representing lines in $\mathbb{E}^3$, the algebra of corresponding vector space is constructed as follows. One begins by defining a volume element (or a trivector) in $\mathbb{E}^3$:

$$I_3 := \,\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,, \tag{1}$$

with $\{\mathbf{e}_i\}$ being a set of anti-commuting orthonormal vectors in $\mathrm{I\!R}^3$ such that $\mathbf{e}_j\mathbf{e}_i = -\,\mathbf{e}_i\mathbf{e}_j$ for any $i, j = x,\, y$, or z. More generally the unit vectors $\mathbf{e}_i$ satisfy the fundamental geometric or Clifford product in this *associative* algebra,

$$\mathbf{e}_i\,\mathbf{e}_j \,=\, \mathbf{e}_i\cdot\mathbf{e}_j + \mathbf{e}_i\wedge\,\mathbf{e}_j\,, \tag{2}$$

with

$$\mathbf{e}_i\cdot\mathbf{e}_j := \,\frac{1}{2}\,\{\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i\} \tag{3}$$

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*Electronic address: jjc@alum.bu.edu

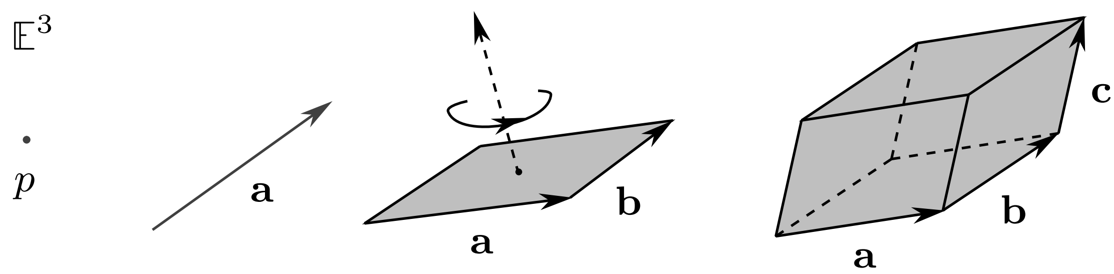

FIG. 1: Euclidean subspaces such as points (p), lines ($\mathbf{a}$), areas ($\mathbf{a} \wedge \mathbf{b}$) and volumes ($\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$) are taken as primitives of the Euclidean space $\mathbb{E}^3$ within Clifford algebra $Cl_{3,0}$, with each subspace specified by its magnitude, direction and orientation (*i.e.*, handedness), providing a unified algebraic framework across dimensions, spanned by the geometric product $\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.

being the symmetric inner product and

$$\mathbf{e}_i \wedge \mathbf{e}_j := \frac{1}{2}\{\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i\} \tag{4}$$

being the anti-symmetric outer product, implying $(\mathbf{e}_i \wedge \mathbf{e}_j)^2 = -1$. Any vector $\mathbf{x} \in \mathbb{E}^3$ is then a solution of the equation

$$I_3 \wedge \mathbf{x} = 0\,. \tag{5}$$

The normalized volume element I_3 thus represents an element of the highest grade in the corresponding algebra, namely grade-3. It is also referred to as a pseudo-scalar, dual to the scalar, which is the lowest possible grade in the algebra:

$$1 = I_3\, I_3^\dagger\,, \tag{6}$$

where the conjugate $I_3^\dagger := \mathbf{e}_z\mathbf{e}_y\mathbf{e}_x = -I_3$ is the "reverse" of I_3 implying $(I_3)^2 = -1$, and the duality relation between the elements $\widetilde{\Omega}$ and Ω of arbitrary grades is defined as

$$\widetilde{\Omega} := \Omega\, I_3^\dagger\,, \tag{7}$$

with the norm $||\;\;||$ of Ω and scalar part $\langle\;\;\rangle_s$ of the product of mixed-grade vectors $\mathbf{X}$ and $\mathbf{Y}$ of n-components defined as

$$||\,\Omega\,|| := \sqrt{\Omega \cdot \Omega^\dagger} \equiv \sqrt{\langle\,\Omega\,\Omega^\dagger\rangle_s} \quad \text{and} \quad \langle\,\mathbf{X}\,\mathbf{Y}^\dagger\rangle_s = \sum_{l=0}^{n}\langle\,\mathbf{X}_l\mathbf{Y}_l^\dagger\,\rangle_s\,. \tag{8}$$

Thus, for example, the orthonormal vectors $\mathbf{e}_k$ of grade-1 can be easily recovered from the unit bivectors $\mathbf{e}_i \wedge \mathbf{e}_j$ of grade-2 using the above duality relation:

$$\mathbf{e}_k = (\mathbf{e}_i \wedge \mathbf{e}_j)\, I_3^\dagger = (\mathbf{e}_i\mathbf{e}_j)\, I_3^\dagger\,. \tag{9}$$

In three-dimensional Euclidean space there are thus basis elements of four different grades: An identity element $\mathbf{e}_i^2 = 1$ of grade-0, three orthonormal vectors $\mathbf{e}_i$ of grade-1, three orthonormal bivectors $\mathbf{e}_j\mathbf{e}_k$ of grade-2, and a trivector $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ of grade-3. Respectively, they represent points, lines, planes and volumes in $\mathbb{E}^3$, as shown in Fig. 1. Since in $\mathrm{I\!R}^3$ there are $2^3 = 8$ ways to combine the vectors $\mathbf{e}_i$ using the geometric product (2) such that no two products are linearly dependent, the resulting algebra, $Cl_{3,0}$, is a linear vector space of $2^3 = 8$ dimensions, spanned by these graded bases:

$$Cl_{3,0} = Span\{\,1,\ \mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z,\ \mathbf{e}_x\mathbf{e}_y,\, \mathbf{e}_z\mathbf{e}_x,\, \mathbf{e}_y\mathbf{e}_z,\ \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,\}. \tag{10}$$

This algebra intrinsically characterizes the Euclidean space $\mathbb{E}^3$ without requiring a coordinate system, by the bijection

$$\mathcal{F} : \mathrm{I\!R}^3 := Span\{\,\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z\} \longrightarrow \mathrm{I\!R}^8 := Span\{\,1,\ \mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z,\ \mathbf{e}_x\mathbf{e}_y,\, \mathbf{e}_z\mathbf{e}_x,\, \mathbf{e}_y\mathbf{e}_z,\ \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,\} = Cl_{3,0}\,. \tag{11}$$

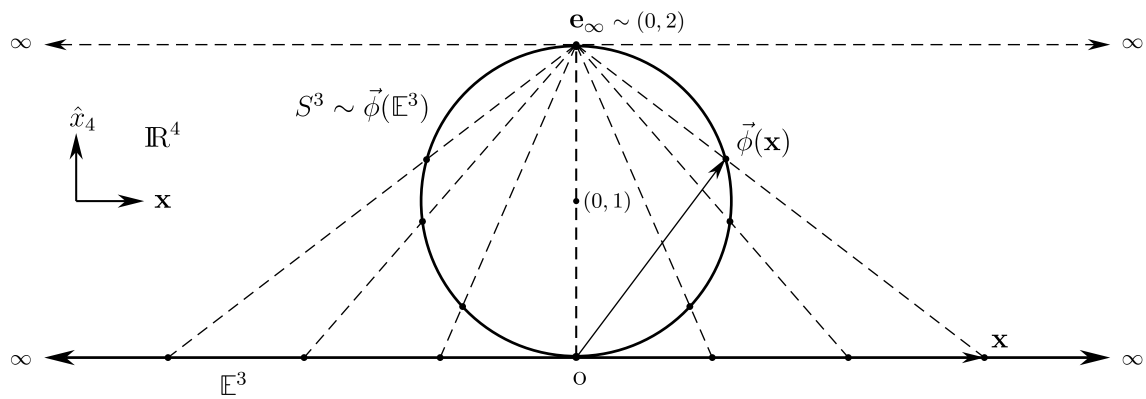

FIG. 2: One-point compactification of the Euclidean space $\mathbb{E}^3$ by means of a stereographic projection onto a unit 3-sphere in $\mathrm{I\!R}^4$.

A. One-point Compactification of the 3D Euclidean Space

The physical space represented by the above algebraic model is, however, not quite satisfactory. Stemming from an arbitrarily chosen origin, its points run off to infinity along every radial direction [4]. Moreover, there is no reason for these infinitely many infinities — which can be approached from infinitely many possible different directions — to be distinct from one another. It is therefore natural to assume that one and the same infinity is encountered along any radial direction, and identify it with a single point. One way to achieve this is by compactifying the space $\mathbb{E}^3$ by adding a single point to it at infinity. This well known procedure of one-point compactification is illustrated in Fig. 2.

Intuitively this procedure is not difficult to understand with a two-dimensional analogue of $\mathbb{E}^3$. Imagine a stretchable balloon, which is topologically a two-dimensional surface, S^2 [cf. Fig. 3]. If we surgically remove a single point from this surface and stretch the remainder out to infinity in every radial direction (like an infinite bed-sheet), then it provides an intuitive model for the two-dimensional Euclidean space, $\mathbb{E}^2$. The one-point, or Alexandroff compactification of $\mathbb{E}^2$ is an inverse of this process whereby all points at infinity from all possible radial directions in $\mathbb{E}^2$ are brought together again and identified with the previously removed point, thereby reconstructing the S^2-balloon from an $\mathbb{E}^2$-bed-sheet.

Similarly, Fig. 2 depicts an inverse stereographic projection of $\mathbb{E}^3$ onto a unit 3-sphere, S^3, by the embedding map $\vec{\phi} : \mathbb{E}^3 \to S^3$, which is given by

$$\vec{\phi}\left(\mathbf{x} \in \mathbb{E}^3\right) = \left(\frac{2}{||\mathbf{x}||^2 + 1}\right)\mathbf{x} + \left(\frac{2\,||\mathbf{x}||^2}{||\mathbf{x}||^2 + 1}\right)\hat{x}_4\,, \tag{12}$$

where two of the dimensions of $\mathbb{E}^3$ are suppressed in the figure and $\hat{x}_4$ represents the fourth dimension in the embedding space $\mathrm{I\!R}^4$. The crucial observation here is that, as an arbitrary vector $\mathbf{x} \in \mathbb{E}^3$ from the origin approaches infinity, it is mapped to the same point $\mathbf{e}_\infty$ located at $(0, 2)$, thereby closing the non-compact space $\mathbb{E}^3$ into the compact 3-sphere. By shifting the origin to $(0, 1)$ the above set of points can be inscribed by a radial 4-vector originating from $(0, 1)$ as

$$\vec{\psi}\left(\mathbf{x} \in \mathbb{E}^3\right) = \left(\frac{2}{||\mathbf{x}||^2 + 1}\right)\mathbf{x} + \left(\frac{2\,||\mathbf{x}||^2}{||\mathbf{x}||^2 + 1} - 1\right)\hat{x}_4\,. \tag{13}$$

The magnitude of this vector then confirms the unity of the radius of our conformally embedded 3-sphere within $\mathrm{I\!R}^4$:

$$1 = \left|\left|\,\vec{\psi}\left(\mathbf{x} \in \mathbb{E}^3\right)\right|\right| = \text{radius of } S^3 \hookrightarrow \mathrm{I\!R}^4. \tag{14}$$

The embedding operator $\vec{\phi}(\mathbf{x})$ [or $\vec{\psi}(\mathbf{x})$] thus transforms the entire space $\mathbb{E}^3$ into a unit 3-sphere within $\mathrm{I\!R}^4$, thereby accomplishing a one-point compactification of $\mathbb{E}^3$:

$$S^3 = \mathbb{E}^3 \cup \{\mathbf{e}_\infty\}\,. \tag{15}$$

Such a conformal mapping is angle-preserving in the sense that a small angle between two curves on S^3 projects to the same angle between the projected curves on $\mathbb{E}^3$, with a circle of any size on S^3 projecting to an exact circle on $\mathbb{E}^3$.

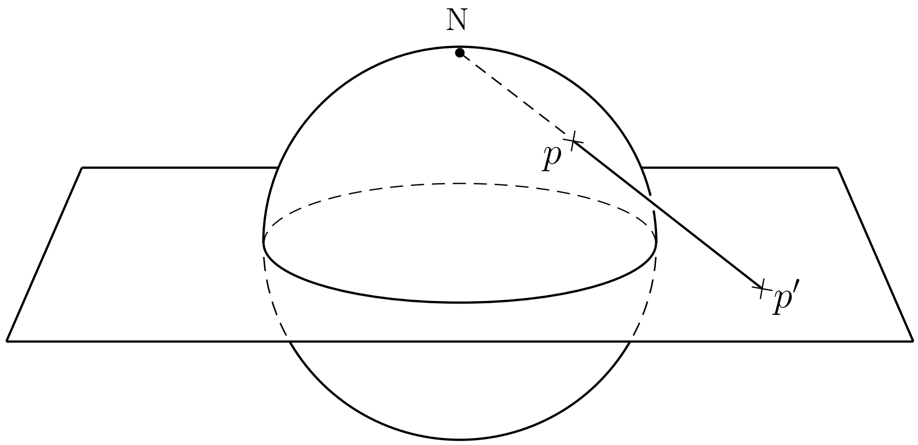

FIG. 3: Stereographic projection of S^2 onto the plane of $\mathbb{R}^2$. Both S^2 and $\mathbb{R}^2$ contain infinite number of points. Each point p of S^2 can be mapped to a point p' of $\mathbb{R}^2$, except the north pole N, which has no meaningful finite image under this projection.

Now the tangent bundle of S^3 happens to be trivial: $\mathrm{T}S^3 = S^3 \times \mathbb{R}^3$. This renders the tangent space at each point of S^3 to be isomorphic to $\mathbb{R}^3$. Consequently, local experiences of the experimenters within S^3 are no different from those of their counterparts within $\mathbb{E}^3$. The global topology of S^3, however, is clearly different from that of $\mathbb{R}^3$ [1][5]. In particular, the triviality of $\mathrm{T}S^3$ means that S^3 is parallelizable. As a result, a global *anholonomic* frame can be defined on S^3 that fixes each of its points uniquely. Such a frame renders S^3 diffeomorphic to the group SU(2) — *i.e.*, to the set of all unit quaternions:

$$S^3 := \left\{ \mathbf{q}(\theta,\, \mathbf{r}) := \cos\frac{\theta}{2} \,+\, \boldsymbol{\xi}(\mathbf{r})\,\sin\frac{\theta}{2} \;\middle|\; ||\,\mathbf{q}(\theta,\, \mathbf{r})\,|| = 1 \right\}, \tag{16}$$

where $\boldsymbol{\xi}(\mathbf{r})$ is a bivector rotating about $\mathbf{r} \in \mathbb{R}^3$ with the rotation angle θ in the range $0 \leq \theta < 4\pi$. In terms of the even sub-algebra of (10), the bivector $\boldsymbol{\xi}(\mathbf{r}) \in S^3$ can be parameterized by the dual vector $\mathbf{r} = r_x\,\mathbf{e}_x + r_y\,\mathbf{e}_y + r_z\,\mathbf{e}_z \in \mathbb{R}^3$ as

$$\boldsymbol{\xi}(\mathbf{r}) \,:=\, (\,I_3 \cdot \mathbf{r}\,) \,=\, r_x\,(\,I_3 \cdot \mathbf{e}_x\,) \,+\, r_y\,(\,I_3 \cdot \mathbf{e}_y\,) \,+\, r_z\,(\,I_3 \cdot \mathbf{e}_z\,) \,=\, r_x\,\mathbf{e}_y\mathbf{e}_z \,+\, r_y\,\mathbf{e}_z\mathbf{e}_x \,+\, r_z\,\mathbf{e}_x\mathbf{e}_y\,, \tag{17}$$

with $\boldsymbol{\xi}^2(\mathbf{r}) = -1$. Each configuration of any rotating rigid body can thus be represented by a quaternion $\mathbf{q}(\theta,\, \mathbf{r})$, which in turn can always be decomposed into a product of two bivectors, say $\boldsymbol{\xi}(\mathbf{u})$ and $\boldsymbol{\xi}(\mathbf{v})$, belonging to an $S^2 \subset S^3$,

$$\boldsymbol{\xi}(\mathbf{u})\,\boldsymbol{\xi}(\mathbf{v}) \,=\, \cos\frac{\theta}{2} \,+\, \boldsymbol{\xi}(\mathbf{r})\,\sin\frac{\theta}{2}\,, \tag{18}$$

in accordance with the bivector subalgebra [3]

$$\boldsymbol{\xi}_a\,\boldsymbol{\xi}_b \,=\, -\,\delta_{ab} \,-\, \sum_{c\,=\,1}^{3} \epsilon_{abc}\,\boldsymbol{\xi}_c\,, \tag{19}$$

with θ being its rotation angle from $\mathbf{q}(0,\, \mathbf{r}) = 1$. Note also that $\mathbf{q}(\theta,\, \mathbf{r})$ reduces to $\pm\,1$ as $\theta \rightarrow 2\kappa\pi$ for $\kappa \,=\, 0,\, 1,$ or 2.

B. Conformal Completion of the Euclidean Primitives

Our interest now lies in the point $\mathbf{e}_\infty$ which represents the multitude of infinities of $\mathbb{E}^3$. Within three dimensions we continue to view it as a dimensionless point and take its algebraic counterpart to be a *non-zero* vector of zero norm:

$$\mathbf{e}_\infty \neq \mathbf{0}, \;\text{ but }\; ||\mathbf{e}_\infty||^{\mathbf{2}} = \mathbf{e}_\infty \cdot \mathbf{e}_\infty \,=\, \mathbf{0} \;\Longleftrightarrow\; \mathbf{e}^{\mathbf{2}}_\infty \,=\, \mathbf{0}\,. \tag{20}$$

Such a vector that is orthogonal to itself is called a *null vector* in Conformal Geometric Algebra[1] [3][4]. It is introduced to represent both finite points in space as well as points at infinity [4]. Since points thus defined are null-dimensional or dimensionless, addition of $\mathbf{e}_\infty$ into the algebraic structure of $\mathbb{E}^3$ does not alter the latter's dimensions but only its point-set topology, rendering it diffeomorphic to a closed, compact, simply-connected 3-sphere, as we discussed above.

Equipped with $\mathbf{e}_\infty$, we are now ready to rebuild the compactified Euclidean space and its algebraic representation as follows. We begin by identifying the set $\{\,\mathbf{e}_x\mathbf{e}_y,\ \mathbf{e}_z\mathbf{e}_x,\ \mathbf{e}_y\mathbf{e}_z\}$ of bivectors as the orthonormal basis of the space $\mathbb{E}^3$:

$$\mathbb{E}^3 = \,Span\{\,\mathbf{e}_x\mathbf{e}_y,\, \mathbf{e}_z\mathbf{e}_x,\, \mathbf{e}_y\mathbf{e}_z\}. \tag{21}$$

Using the orthonormality and anti-commutativity of the vectors $\mathbf{e}_i$ the product of the basis bivectors works out to be

$$(\mathbf{e}_x\mathbf{e}_y)(\mathbf{e}_z\mathbf{e}_x)(\mathbf{e}_y\mathbf{e}_z) = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z = -1. \tag{22}$$

The associativity of geometric product then allows us to rediscover the volume form I_3 for the Euclidean space (21):

$$(\mathbf{e}_x\mathbf{e}_y)(\mathbf{e}_z\mathbf{e}_x)(\mathbf{e}_y\mathbf{e}_z) = \,(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z)(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z) = \,(\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z)^2 =: \,(I_3)^2 = -1. \tag{23}$$

As it stands, this volume form is open and has the topology of $\mathbb{R}^3$. But we can now close it with the null vector $\mathbf{e}_\infty$:

$$I_c := \,I_3\,\mathbf{e}_\infty = \,\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\mathbf{e}_\infty\,, \tag{24}$$

where we have used the subscript c on I_c to indicate that it is a volume element of the compact 3-sphere, S^3. As we noted earlier, in the Euclidean space the reverse of I_3 is $I_3^\dagger = -I_3$. Likewise in the conformal space the reverse of I_c is

$$I_c^\dagger = \,I_3^\dagger\mathbf{e}_\infty = \,-I_3\,\mathbf{e}_\infty = \,-I_c\,. \tag{25}$$

As a result, in the conformal space the general duality operation between elements $\widetilde{\Omega}$ and Ω of any grade is given by

$$\widetilde{\Omega}_c := \Omega\,I_c^\dagger = \,\Omega\,I_3^\dagger\,\mathbf{e}_\infty\,. \tag{26}$$

This allows us, in particular, to work out the dual elements of all of the basis bivectors in (21) in the conformal space:

$$\mathbf{e}_x\mathbf{e}_y I_3^\dagger\mathbf{e}_\infty = \,\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\mathbf{e}_y\mathbf{e}_x\mathbf{e}_\infty = \,\mathbf{e}_z\mathbf{e}_\infty\,, \tag{27}$$

$$\mathbf{e}_z\mathbf{e}_x I_3^\dagger\mathbf{e}_\infty = \,\mathbf{e}_z\mathbf{e}_x\mathbf{e}_z\mathbf{e}_y\mathbf{e}_x\mathbf{e}_\infty = \,\mathbf{e}_y\mathbf{e}_\infty\,, \tag{28}$$

$$\text{and}\ \ \mathbf{e}_y\mathbf{e}_z I_3^\dagger\mathbf{e}_\infty = \,\mathbf{e}_y\mathbf{e}_z\mathbf{e}_z\mathbf{e}_y\mathbf{e}_x\mathbf{e}_\infty = \,\mathbf{e}_x\mathbf{e}_\infty\,. \tag{29}$$

Moreover, analogous to how the dual of $+1$ in $\mathbb{E}^3$ is $-I_3$, the dual of $+1$ in the conformal space also works out to be

$$(+1)\,I_c^\dagger = -I_3\,\mathbf{e}_\infty = -I_c\,. \tag{30}$$

We have thus worked out the conformal counterparts of all of the basis elements appearing in the algebraic vector space (10). Putting them together we can now formalize the desired algebraic representation of our conformal space:

$$\mathcal{K}^- = \,Span\{\,1,\,\mathbf{e}_x\mathbf{e}_y,\,\mathbf{e}_z\mathbf{e}_x,\,\mathbf{e}_y\mathbf{e}_z,\,\mathbf{e}_x\mathbf{e}_\infty,\,\mathbf{e}_y\mathbf{e}_\infty,\,\mathbf{e}_z\mathbf{e}_\infty,\,-I_3\mathbf{e}_\infty\,\}. \tag{31}$$

Evidently, not unlike (10), this vector space too is eight-dimensional. Unlike (10), however, it is closed and compact. The three-dimensional physical space — *i.e.*, the 3-sphere discussed above — can now be viewed as embedded in the four-dimensional ambient space, $\mathbb{R}^4$, as depicted in Fig. 2. In this higher dimensional space $\mathbf{e}_\infty$ is then a *unit* vector,

$$||\mathbf{e}_\infty||^2 = \,\mathbf{e}_\infty\cdot\mathbf{e}_\infty = \,1 \iff \,\mathbf{e}_\infty^2 = \,1\,, \tag{32}$$

and the corresponding algebraic representation space (31) is nothing but the eight-dimensional *even* sub-algebra of the $2^4 = 16$-dimensional Clifford algebra $Cl_{4,0}$. Thus a one-dimensional subspace, represented by the unit vector $\mathbf{e}_\infty$ in the ambient space $\mathbb{R}^4$, represents a *null*-dimensional space – *i.e.*, the infinite point of $\mathbb{E}^3$ – in the physical space S^3.

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[1] The conformal space we are considering is an *in*-homogeneous version of the space usually studied in Conformal Geometric Algebra [4]. It can be viewed as an 8-dimensional subspace of the 32-dimensional representation space postulated in Conformal Geometric Algebra. The larger representation space results from a homogeneous freedom of the origin within $\mathbb{E}^3$, which is neither required nor useful here.

C. Orientation of the Representation Space as a Binary Degree of Freedom

Before we explore the properties of the above vector space, let us endow it with one more degree of freedom without which it is unjustifiably restrictive. To that end, we first define what is meant by an orientation of a vector space [5][6]:

Definition of Orientation: An orientation of a finite dimensional vector space $\mathcal{V}_n$ is an equivalence class of ordered basis, say $\{b_1, \dots, b_n\}$, which determines the same orientation of $\mathcal{V}_n$ as the basis $\{b'_1, \dots, b'_n\}$ if $b'_i = \omega_{ij}\, b_j$ holds with $det(\omega_{ij}) > 0$, and the opposite orientation of $\mathcal{V}_n$ as the basis $\{b'_1, \dots, b'_n\}$ if $b'_i = \omega_{ij}\, b_j$ holds with $det(\omega_{ij}) < 0$.

Thus each positive dimensional real vector space has precisely two possible orientations, which we shall denote as $\lambda = +1$ or $\lambda = -1$. More generally an oriented smooth manifold consists of that manifold together with a choice of orientation for each of its tangent spaces. It is worth stressing that orientation is a *relative* concept. In particular, the orientation of a tangent space $\mathcal{V}_n$ of a manifold defined by the equivalence class of ordered basis such as $\{b_1, \dots, b_n\}$ is meaningful only with respect to that defined by the equivalence class of ordered basis $\{b'_1, \dots, b'_n\}$, and vice versa.

Now in geometric algebra the choice of the sign of the unit pseudoscalar amounts to choosing an orientation of the space [3][4][5]. In our three-dimensional Euclidean space defined in (21) with an orthonormal set of unit bivector basis, $I_3 = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ picks out the right-handed orientation for $\mathbb{E}^3$. The convention usually is to assume such a right-handed set of basis bivectors (or vectors) *ab initio*. But the algebra itself does not fix the handedness of the basis. In our presentation above we could have equally well started out with a left-handed set of bivectors in (21) by letting $-I_3$ instead of $+I_3$ select the basis. Instead of the representation space (31) we would have then ended up with the space

$$\mathcal{K}^+ = \mathit{Span}\{\,1,\, +\mathbf{e}_x\mathbf{e}_y,\, +\mathbf{e}_z\mathbf{e}_x,\, +\mathbf{e}_y\mathbf{e}_z,\, +\mathbf{e}_x\mathbf{e}_\infty,\, +\mathbf{e}_y\mathbf{e}_\infty,\, +\mathbf{e}_z\mathbf{e}_\infty,\, +I_3\mathbf{e}_\infty\,\}. \tag{33}$$

On the other hand, in the light of the above definition of orientation, the representation space (31) can be written as

$$\mathcal{K}^- = \mathit{Span}\{\,1,\, -\mathbf{e}_x\mathbf{e}_y,\, -\mathbf{e}_z\mathbf{e}_x,\, -\mathbf{e}_y\mathbf{e}_z,\, -\mathbf{e}_x\mathbf{e}_\infty,\, -\mathbf{e}_y\mathbf{e}_\infty,\, -\mathbf{e}_z\mathbf{e}_\infty,\, -I_3\mathbf{e}_\infty\,\}. \tag{34}$$

It is easy to verify that the bases of $\mathcal{K}^+$ and $\mathcal{K}^-$ are indeed related by an 8×8 diagonal matrix whose determinant is $(-1)^7 < 0$. Consequently, $\mathcal{K}^+$ and $\mathcal{K}^-$ indeed represent right-oriented and left-oriented vector spaces, respectively, in accordance with our definition of orientation. We can therefore leave the orientation unspecified and write $\mathcal{K}^\pm$ as

$$\mathcal{K}^\lambda = \mathit{Span}\{\,1,\, \lambda\mathbf{e}_x\mathbf{e}_y,\, \lambda\mathbf{e}_z\mathbf{e}_x,\, \lambda\mathbf{e}_y\mathbf{e}_z,\, \lambda\mathbf{e}_x\mathbf{e}_\infty,\, \lambda\mathbf{e}_y\mathbf{e}_\infty,\, \lambda\mathbf{e}_z\mathbf{e}_\infty,\, \lambda I_3\mathbf{e}_\infty\,\},\;\; \lambda^2 = 1 \iff \lambda = \pm 1. \tag{35}$$

D. Representation Space $\mathcal{K}^\lambda$ is Closed Under Multiplication

As an eight-dimensional linear vector space, $\mathcal{K}^\lambda$ has some remarkable properties. To begin with, $\mathcal{K}^\lambda$ is *closed* under multiplication. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two unit vectors in $\mathcal{K}^\lambda$. Then $\mathbf{X}$ and $\mathbf{Y}$ can be expanded in the basis of $\mathcal{K}^\lambda$ as

$$\mathbf{X} = X_0 + X_1\,\lambda\mathbf{e}_x\mathbf{e}_y + X_2\,\lambda\mathbf{e}_z\mathbf{e}_x + X_3\,\lambda\mathbf{e}_y\mathbf{e}_z + X_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + X_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + X_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + X_7\,\lambda I_3\mathbf{e}_\infty\,, \tag{36}$$

$$\text{and}\;\; \mathbf{Y} = Y_0 + Y_1\,\lambda\mathbf{e}_x\mathbf{e}_y + Y_2\,\lambda\mathbf{e}_z\mathbf{e}_x + Y_3\,\lambda\mathbf{e}_y\mathbf{e}_z + Y_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + Y_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + Y_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + Y_7\,\lambda I_3\mathbf{e}_\infty\,, \tag{37}$$

and using (8) they can be normalized as

$$||\mathbf{X}||^2 = \sum_{\mu=0}^{7} X_\mu^2 \,=\, 1 \quad\text{and}\quad ||\mathbf{Y}||^2 = \sum_{\nu=0}^{7} Y_\nu^2 \,=\, 1\,. \tag{38}$$

Now it is evident from the multiplication table below (Table I) that if $\mathbf{X}, \mathbf{Y} \in \mathcal{K}^\lambda$, then so is their product $\mathbf{Z} = \mathbf{XY}$:

$$\mathbf{Z} = Z_0 + Z_1\,\lambda\mathbf{e}_x\mathbf{e}_y + Z_2\,\lambda\mathbf{e}_z\mathbf{e}_x + Z_3\,\lambda\mathbf{e}_y\mathbf{e}_z + Z_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + Z_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + Z_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + Z_7\,\lambda I_3\mathbf{e}_\infty = \mathbf{XY}. \tag{39}$$

More importantly, we shall soon see that for vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{K}^\lambda$ (not necessarily unit) the following relation holds:

$$||\mathbf{XY}|| \,=\, ||\mathbf{X}||\;||\mathbf{Y}||\,. \tag{40}$$

In particular, this means that for any two unit vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{K}^\lambda$ with the geometric product $\mathbf{Z} = \mathbf{XY}$ we have

$$||\,\mathbf{Z}\,||^2 = \sum_{\rho=0}^{7} Z_\rho^2 \,=\, 1\,. \tag{41}$$

$*$	1	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$
1	1	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	-1	$\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_z$	-1	$\mathbf{e}_x\mathbf{e}_y$	$\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_x\mathbf{e}_y$	-1	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	-1	$-\mathbf{e}_x\mathbf{e}_y$	$\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_z$
$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$\mathbf{e}_z\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_y$	-1	$-\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$
$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_x$	$\mathbf{e}_y\mathbf{e}_z$	-1	$-\mathbf{e}_x\mathbf{e}_y$
$\lambda\,I_3\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_x\mathbf{e}_y$	1

TABLE I: Multiplication Table for a "Conformal Geometric Algebra" of $\mathbb{E}^3$. Here $I_3 = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$, $\mathbf{e}_\infty^2 = +1$, and $\lambda = \pm 1$.

The important observation here is that, without loss of generality, we can restrict our representation space to a set of *unit* vectors in $\mathcal{K}^\lambda$. We are then dealing with a unit 7-sphere as an algebraic representation of the Euclidean space. If, for convenience, we now identify the basis elements of $\mathcal{K}^\lambda$ (in order) with the ordered elements of the following set

$$\{\,\boldsymbol{\zeta}_0,\,\boldsymbol{\zeta}_1,\,\boldsymbol{\zeta}_2,\,\boldsymbol{\zeta}_3,\,\boldsymbol{\zeta}_4,\,\boldsymbol{\zeta}_5,\,\boldsymbol{\zeta}_6,\,\boldsymbol{\zeta}_7\,\}\,, \tag{42}$$

then the algebra generated by them – which has been explicitly displayed in Table I – can be succinctly rewritten as

$$\boldsymbol{\zeta}_\mu\,\boldsymbol{\zeta}_\nu \;=\; \{-1\}^{\delta_{\mu 7}}\,\{\,-\,\delta_{\mu\nu}\}\;+\;\lambda\sum_{\rho\,=\,1}^{7}\Big[\,f_{\mu\nu\rho}\;+\;\{-1\}^{\delta_{\rho 7}}\;l_{\mu\nu\rho}\Big]\,\boldsymbol{\zeta}_\rho\,,\qquad \mu,\,\nu\;=\;1,\,2,\,\ldots,\,7, \tag{43}$$

where $f_{\mu\nu\rho}$ is a totally anti-symmetric permutation tensor with its only non-vanishing independent components being

$$f_{123}\;=\;f_{246}\;=\;f_{365}\;=\;f_{415}\;=\;+1\,, \tag{44}$$

and similarly $l_{\mu\nu\rho}$ is a totally *symmetric* permutation tensor with only non-vanishing independent components being

$$l_{176}\;=\;l_{257}\;=\;l_{347}\;=\;-1\,. \tag{45}$$

The eight-dimensional multi-vectors $\mathbf{X}$ and $\mathbf{Y}$ within $\mathcal{K}^\lambda$ can now be expanded more conveniently in the basis (42) as

$$\mathbf{X}\;=\;\sum_{\mu\,=\,0}^{7}X_\mu\,\boldsymbol{\zeta}_\mu\qquad\text{and}\qquad\mathbf{Y}\;=\;\sum_{\nu\,=\,0}^{7}Y_\nu\,\boldsymbol{\zeta}_\nu\,. \tag{46}$$

E. Representation Space $\mathcal{K}^\lambda$ as a Set of Orthogonal Pairs of Quaternions

In his seminal works Clifford introduced the concept of dual numbers, z, analogous to complex numbers, as follows:

$$z = r + d\,\varepsilon,\;\;\text{where}\;\varepsilon \neq 0\;\text{ but }\;\varepsilon^2 = 0\,, \tag{47}$$

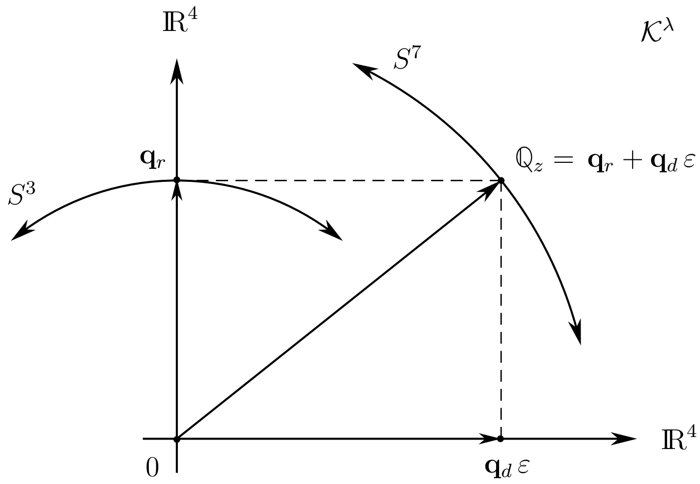

FIG. 4: An illustration of the 8D plane of $\mathcal{K}^\lambda$, which may be interpreted as an Argand-type diagram for a pair of quaternions.

where ε is the dual operator, r is the real part, and d is the dual part [3][4]. Similar to how the "imaginary" operator i is introduced in the complex number theory to distinguish the "real" and "imaginary" parts of a complex number, Clifford introduced the dual operator ε to distinguish the "real" and "dual" parts of a dual number. The dual number theory can be extended to numbers of higher grades, including to numbers of composite grades, such as quaternions:

$$\mathbb{Q}_z = \mathbf{q}_r + \mathbf{q}_d\,\varepsilon\,, \tag{48}$$

where $\mathbf{q}_r$ and $\mathbf{q}_d$ are quaternions and $\mathbb{Q}_z$ is a dual-quaternion (or in Clifford's terminology, a bi-quaternion). Recall that, as defined in (16), the set of all quaternions is a 3-sphere, which can be normalized to radius ϱ and rewritten as

$$S^3 = \left\{ \mathbf{q}_r := q_0 + q_1\,\lambda\,\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\,\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\,\mathbf{e}_y\mathbf{e}_z \;\middle|\; ||\mathbf{q}_r|| = \varrho \right\}. \tag{49}$$

Consider now a second, dual copy of the set of quaternions within $\mathcal{K}^\lambda$, corresponding to the fixed orientation $\lambda = +1$:

$$S^3 = \left\{ \mathbf{q}_d := -q_7 + q_6\,\mathbf{e}_x\mathbf{e}_y + q_5\,\mathbf{e}_z\mathbf{e}_x + q_4\,\mathbf{e}_y\mathbf{e}_z \;\middle|\; ||\mathbf{q}_d|| = \varrho \right\}. \tag{50}$$

If we now identify ε with the duality operator $I_c^\dagger = -\lambda\, I_3\mathbf{e}_\infty$ used in (26), then (in the reverse additive order) we have

$$\varepsilon \equiv -\lambda\, I_3\mathbf{e}_\infty \;\text{ with }\; \varepsilon^2 = +1 \;\text{ (since } \mathbf{e}_\infty \text{ is a unit vector within } \mathcal{K}^\lambda) \tag{51}$$

$$\text{and }\; \mathbf{q}_d\,\varepsilon \equiv -\mathbf{q}_d\,\lambda\, I_3\mathbf{e}_\infty = q_4\,\lambda\,\mathbf{e}_x\mathbf{e}_\infty + q_5\,\lambda\,\mathbf{e}_y\mathbf{e}_\infty + q_6\,\lambda\,\mathbf{e}_z\mathbf{e}_\infty + q_7\,\lambda\, I_3\mathbf{e}_\infty\,, \tag{52}$$

which is a multi-vector "dual" to the quaternion $\mathbf{q}_d$ at infinity. Note that we continue to write ε as if it were a scalar because it commutes with $\mathbf{q}_d$. Comparing (50) and (52) with (35) we can now rewrite $\mathcal{K}^\lambda$ as a set of paired quaternions:

$$\mathcal{K}^\lambda = \left\{ \mathbb{Q}_z := \mathbf{q}_r + \mathbf{q}_d\,\varepsilon \;\middle|\; ||\mathbb{Q}_z|| = \sqrt{2}\,\varrho \right\}. \tag{53}$$

Now the normalization of $\mathbb{Q}_z$ in fact necessitates that every $\mathbf{q}_r$ be orthogonal to its dual $\mathbf{q}_d$:

$$||\mathbb{Q}_z|| = \sqrt{2}\,\varrho \implies \mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0\,, \;\text{ or equivalently}, \;\langle\, \mathbf{q}_r\,\mathbf{q}_d^\dagger \,\rangle_s = 0 \;\;(\textit{i.e.}, \mathbf{q}_r\,\mathbf{q}_d^\dagger \text{ is a pure quaternion}). \tag{54}$$

We can see this by working out the product of $\mathbb{Q}_z$ with $\mathbb{Q}_z^\dagger$ while using $\varepsilon^2 = +1$, which gives

$$\mathbb{Q}_z\,\mathbb{Q}_z^\dagger = \left(\mathbf{q}_r\,\mathbf{q}_r^\dagger + \mathbf{q}_d\,\mathbf{q}_d^\dagger\right) + \left(\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger\right)\varepsilon\,. \tag{55}$$

Now, using the definition of $\mathbf{q}$ in (16), it is not difficult to see that $\mathbf{q}_r\,\mathbf{q}_r^\dagger = \mathbf{q}_d\,\mathbf{q}_d^\dagger = \varrho^2$, reducing the above product to

$$\mathbb{Q}_z\,\mathbb{Q}_z^\dagger = 2\,\varrho^2 + \left(\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger\right)\varepsilon\,. \tag{56}$$

It is thus clear that for $\mathbb{Q}_z\,\mathbb{Q}_z^\dagger$ to be a scalar $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger$ must vanish, or equivalently $\mathbf{q}_r$ must be orthogonal to $\mathbf{q}_d$.

But there is more to the normalization condition $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0$ then meets the eye. It also leads to the crucial norm relation (40), which is at the very heart of the only possible four normed division algebras associated with the four parallelizable spheres S^0, S^1, S^3 and S^7. To verify it, consider a product of two different members of the set $\mathcal{K}^\lambda$,

$$\mathbb{Q}_{z1}\,\mathbb{Q}_{z2} \,=\, (\mathbf{q}_{r1}\,\mathbf{q}_{r2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}) \,+\, (\mathbf{q}_{r1}\,\mathbf{q}_{d2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2})\,\varepsilon\,, \tag{57}$$

together with their individual definitions

$$\mathbb{Q}_{z1} \,=\, \mathbf{q}_{r1} + \mathbf{q}_{d1}\,\varepsilon \quad\text{and}\quad \mathbb{Q}_{z2} \,=\, \mathbf{q}_{r2} + \mathbf{q}_{d2}\,\varepsilon\,. \tag{58}$$

If we now work out the products $\mathbb{Q}_{z1}\mathbb{Q}_{z1}^\dagger$, $\mathbb{Q}_{z2}\mathbb{Q}_{z2}^\dagger$ and $(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger$, then, thanks to the orthogonality condition $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0$, the norm relation is not difficult to verify:

$$||\mathbb{Q}_{z1}\,\mathbb{Q}_{z2}|| \,=\, ||\mathbb{Q}_{z1}||\,||\mathbb{Q}_{z2}||\,. \tag{59}$$

Without loss of generality we can now restrict our algebraic representation space to a unit 7-sphere by simply setting the radius ϱ of S^3 to $\frac{1}{\sqrt{2}}$. In what follows S^7 will provide the conformal counterpart of the algebra $Cl_{3,0}$ given in (10):

$$\mathcal{K}^\lambda \supset S^7 := \left\{ \mathbb{Q}_z := \mathbf{q}_r + \mathbf{q}_d\,\varepsilon \;\middle|\; ||\mathbb{Q}_z|| = 1 \text{ and } \mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0 \right\}, \tag{60}$$

where $\varepsilon = -\lambda\,I_3\mathbf{e}_\infty\,,\ \varepsilon^2 = \mathbf{e}_\infty^2 = +1\,,$

$$\mathbf{q}_r \,=\, q_0 + q_1\,\lambda\,\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\,\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\,\mathbf{e}_y\mathbf{e}_z\,, \quad\text{and}\quad \mathbf{q}_d \,=\, -q_7 + q_6\,\mathbf{e}_x\mathbf{e}_y + q_5\,\mathbf{e}_z\mathbf{e}_x + q_4\,\mathbf{e}_y\mathbf{e}_z\,, \tag{61}$$

so that

$$\mathbb{Q}_z \,=\, q_0 + q_1\,\lambda\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\mathbf{e}_y\mathbf{e}_z + q_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + q_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + q_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + q_7\,\lambda I_3\mathbf{e}_\infty\,. \tag{62}$$

Thus, to summarize this section, we started out with the observation that the correct model of the physical space is provided by the algebra of Euclidean primitives, such as points, lines, planes and volumes, as discovered by Grassmann and Clifford in the 19^{th} century. We then recognized the need to "close" the Euclidean space with a non-zero null vector $\mathbf{e}_\infty$ representing its infinities, thereby compactifying $\mathbb{E}^3$ to a 3-sphere, S^3. The corresponding algebraic representation space of $\mathbb{E}^3$ then turned out to be a unit 7-sphere, S^7. It is quite remarkable that S^3 and S^7, which are the two spheres associated with the only two non-trivially possible normed division algebras, namely the quaternionic and octonionic algebras [1], emerge in this manner from the elementary algebraic properties of the Euclidean primitives. Unlike the non-associative octonionic algebra, however, the 7-sphere we have arrived at corresponds to an *associative* Clifford (or geometric) algebra [2][4]. And yet, as we shall see, it is sufficient to explain the origins of *all* quantum correlations.

II. DERIVATION OF QUANTUM CORRELATIONS FROM EUCLIDEAN PRIMITIVES

A. Constructing Measurement Functions in the Manner of Bell

In order to derive quantum correlations predicted by arbitrary quantum states, our first task is to construct a set of measurement functions of the form:

$$\pm\,1 \,=\, \mathscr{N}(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^7 \hookrightarrow \mathrm{I\!R}^8. \tag{63}$$

These functions describe *local* detections of binary measurement results, $\mathscr{N}(\mathbf{n},\,\lambda) = \pm 1$, by some analyzers fixed along freely chosen directions $\mathbf{n}$. They are of the same deterministic form as that considered by Bell[2] [7][8], except for their co-domain, which we have taken to be the algebraic representation space S^7 constructed above, embedded in $\mathrm{I\!R}^8$. For an explicit construction of the functions $\mathscr{N}(\mathbf{n},\,\lambda)$, let us consider the following multi-vector in $\mathrm{I\!R}^8$ analogous to (62):

$$\mathbb{N}_z = n_0 + \{n_1\,\lambda\mathbf{e}_x\mathbf{e}_y + n_2\,\lambda\mathbf{e}_z\mathbf{e}_x + n_3\,\lambda\mathbf{e}_y\mathbf{e}_z\} + \{n_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + n_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + n_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty\} + n_7\,\lambda I_3\mathbf{e}_\infty \tag{64}$$

$$\equiv\, n_0 + \lambda\,\boldsymbol{\xi}(\mathbf{n}_r) \,+\, \lambda\,\boldsymbol{\xi}(\mathbf{n}_d)\,\varepsilon_+ \,-\, \lambda\,n_7\,\varepsilon_+ \tag{65}$$

$$\equiv\, n_0 + \lambda\,\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7)\,, \tag{66}$$

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[2] Readers not familiar with Bell's locally causal framework are urged to review it from the appendix of Ref. [9] before proceeding further.

where

$$\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7)\;:=\;\boldsymbol{\xi}(\mathbf{n}_r)\,+\,\boldsymbol{\xi}(\mathbf{n}_d)\,\varepsilon_+\,-\,n_7\,\varepsilon_+\,, \tag{67}$$

$$S^3\ni\text{bivector}\;\boldsymbol{\xi}(\mathbf{n}_r)\;:=\;n_1\,\mathbf{e}_x\mathbf{e}_y+n_2\,\mathbf{e}_z\mathbf{e}_x+n_3\,\mathbf{e}_y\mathbf{e}_z\;\equiv\;I_3\cdot\mathbf{n}_r\,, \tag{68}$$

$$S^3\ni\text{bivector}\;\boldsymbol{\xi}(\mathbf{n}_d)\;:=\;n_6\,\mathbf{e}_x\mathbf{e}_y+n_5\,\mathbf{e}_z\mathbf{e}_x+n_4\,\mathbf{e}_y\mathbf{e}_z\;\equiv\;I_3\cdot\mathbf{n}_d\,, \tag{69}$$

$$\text{pseudoscalar}\;\varepsilon_+\;:=\;-\,I_3\mathbf{e}_\infty\,, \tag{70}$$

$$\text{3D vector}\;\;\mathbf{n}_r\;:=\;n_3\,\mathbf{e}_x+n_2\,\mathbf{e}_y+n_1\,\mathbf{e}_z\,,\quad\text{with}\quad||\mathbf{n}_r||=\sqrt{n_1^2+n_2^2+n_3^2}\;=\;\frac{1}{\sqrt{2}}\,, \tag{71}$$

$$\text{and}\quad\text{3D vector}\;\;\mathbf{n}_d\;:=\;n_4\,\mathbf{e}_x+n_5\,\mathbf{e}_y+n_6\,\mathbf{e}_z\,,\quad\text{with}\quad||\mathbf{n}_d||=\sqrt{n_4^2+n_5^2+n_6^2}\;=\;\frac{1}{\sqrt{2}}\,. \tag{72}$$

Next, consider the non-scalar part $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7,\,\lambda)$ of the above S^7-vector $\mathbb{N}_z\,=\,n_0\,+\,\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7,\,\lambda)$, so that

$$\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7,\,\lambda)\;=\;\lambda\,\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7)\quad\Longleftrightarrow\quad\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7)\;=\;\lambda\,\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,-n_7,\,\lambda)\,,\quad\text{since}\;\;\lambda^2\,=\,1\,. \tag{73}$$

For our purposes it is suffice to represent the detectors with the special case of this non-scalar part for which $n_7\equiv 0$:

$$\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)\;=\;\lambda\,\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\quad\Longleftrightarrow\quad\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\;=\;\lambda\,\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)\,, \tag{74}$$

where

$$\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\;:=\;\boldsymbol{\xi}(\mathbf{n}_r)\,+\,\boldsymbol{\xi}(\mathbf{n}_d)\,\varepsilon_+\;=\;(I_3\cdot\mathbf{n}_r)\,+\,(I_3\cdot\mathbf{n}_d)\,\varepsilon_+\;=\;I_3\cdot\{\,\mathbf{n}_r\,+\,\mathbf{n}_d\,\varepsilon_+\}\,. \tag{75}$$

Next recall that, although global topology of S^3 is different from that of $\mathrm{I\!R}^3$, local experiences of experimenters within S^3 are no different from those of their counterparts within $\mathrm{I\!R}^3$, not the least because the tangent space at any point of S^3 is isomorphic to $\mathrm{I\!R}^3$. With this in mind, we identify the counterparts of measurement directions $\mathbf{n}$ within $\mathbb{E}^3$ with the dual vectors $\mathbf{n}_r\,+\,\mathbf{n}_d\,\varepsilon_+$ within its algebraic representation space S^7. Then $\mathbf{n}$ relates to $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ as

$$S^3\supset S^2\ni\boldsymbol{\xi}(\mathbf{n})\;=\;I_3\cdot\mathbf{n}\;\longleftrightarrow\;\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\in S^5\subset S^7. \tag{76}$$

This allows us to identity the anti-symmetric part $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ in (64) as a detector of the physical system represented by $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)$, originating in the initial state λ and producing the measurement results $\mathscr{N}(\mathbf{n},\,\lambda)=\pm 1$ along freely chosen unit directions $\mathbf{n}\longleftrightarrow\mathbf{n}_r+\mathbf{n}_d\,\varepsilon_+$ within $\mathrm{I\!R}^3$. Indeed, using the definitions (66) to (75) it is easy to verify that

$$\mathbf{N}^2(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)\;=\;\lambda^2\,\mathbf{D}^2(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\;=\;\mathbf{D}^2(\mathbf{n}_r,\,\mathbf{n}_d,\,0)\;=\;-1\,. \tag{77}$$

In general, for two vectors $\mathbf{a}$ and $\mathbf{b}$ the product $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda)\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda)$ is highly non-trivial, as we saw in (43):

$$\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda)\;=\;-\,\mathbf{a}_r\cdot\mathbf{b}_r-\mathbf{a}_d\cdot\mathbf{b}_d-\mathbf{N}(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,\mathbf{a}_r\cdot\mathbf{b}_d+\mathbf{a}_d\cdot\mathbf{b}_r,\,\lambda)\,. \tag{78}$$

Unlike the general case, however, since we wish to identify the external vectors $\mathbf{a}\leftrightarrow\mathbf{a}_r+\mathbf{a}_d\,\varepsilon_+$ and $\mathbf{b}\leftrightarrow\mathbf{b}_r+\mathbf{b}_d\,\varepsilon_+$ with the measurement directions within $\mathbb{E}^3$, the following constraints induced by their scalar product naturally hold:

$$\mathbf{a}\cdot\mathbf{b}\;:=\;\frac{1}{2}\{\mathbf{ab}\,+\,\mathbf{ba}\}\;=\;(\mathbf{a}_r\cdot\mathbf{b}_r\,+\,\mathbf{a}_d\cdot\mathbf{b}_d)\,+\,(\mathbf{a}_r\cdot\mathbf{b}_d\,+\,\mathbf{a}_d\cdot\mathbf{b}_r)\,\varepsilon_+\;\Longrightarrow\;\begin{cases}\mathbf{a}_r\cdot\mathbf{b}_d\;=\;\mathbf{a}_d\cdot\mathbf{b}_r\;=\;0\\ \qquad\text{and}\\ \mathbf{a}_r\cdot\mathbf{b}_r\;=\;\mathbf{a}_d\cdot\mathbf{b}_d\;=\;\frac{1}{2}\cos\theta_{\mathbf{ab}}\,,\end{cases} \tag{79}$$

which are consistent with $\mathbf{a}\cdot\mathbf{b}=1$ for the special case $\mathbf{a}=\mathbf{b}$ and the normalization conditions for $\mathbf{a}_r$ and $\mathbf{b}_d$, giving

$$\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda)\;=\;-\,\mathbf{a}_r\cdot\mathbf{b}_r\,-\,\mathbf{a}_d\cdot\mathbf{b}_d\,-\,\mathbf{N}\,(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda)\,. \tag{80}$$

Labeling the experimental trials with index k, we can now define the measurement functions (63) as maps of the form

$$S^7\ni\pm 1\;=\;\mathscr{N}(\mathbf{n},\,\lambda^k):\mathrm{I\!R}^3\times\{\,\lambda^k\}\longrightarrow S^7\hookrightarrow\mathrm{I\!R}^8. \tag{81}$$

These maps can be realized for the freely chosen measurement directions, specified by the vectors such as $\mathbf{a}$ and $\mathbf{b}$, as

$$S^7\ni\mathscr{A}(\mathbf{a}\,,\,\lambda^k):=\lim_{\substack{\mathbf{s}_{r1}\,\to\,\mathbf{a}_r\\ \mathbf{s}_{d1}\,\to\,\mathbf{a}_d}}\left\{\pm\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\right\}=\begin{Bmatrix}\mp 1 & \text{if} & \lambda^k\,=\,+1\\ \pm 1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\;\text{and}\;\Big\langle\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\Big\rangle\;=\;0 \tag{82}$$

and

$$S^7\ni\mathscr{B}(\mathbf{b}\,,\,\lambda^k):=\lim_{\substack{\mathbf{s}_{r2}\,\to\,\mathbf{b}_r\\ \mathbf{s}_{d2}\,\to\,\mathbf{b}_d}}\left\{\mp\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\,\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,\right\}=\begin{Bmatrix}\pm 1 & \text{if} & \lambda^k\,=\,+1\\ \mp 1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\;\text{and}\;\Big\langle\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\,\Big\rangle\;=\;0\,. \tag{83}$$

Here we have assumed that orientation $\lambda = \pm 1$ of S^7 is a fair coin. Evidently, the functions $\mathscr{A}(\mathbf{a}\,,\,\lambda^k)$ and $\mathscr{B}(\mathbf{b}\,,\,\lambda^k)$ define local, realistic, and deterministically determined measurement outcomes [1][5][9]. Apart from the common cause λ^k originating in the overlap of the backward lightcones of $\mathscr{A}(\mathbf{a}\,,\,\lambda^k)$ and $\mathscr{B}(\mathbf{b}\,,\,\lambda^k)$, the event $\mathscr{A} = \pm 1$ depends only on a freely chosen measurement direction $\mathbf{a}$. And likewise, apart from the common cause λ^k, the event $\mathscr{B} = \pm 1$ depends only on a freely chosen measurement direction $\mathbf{b}$. In particular, the function $\mathscr{A}(\mathbf{a}\,,\,\lambda^k)$ does not depend on either $\mathbf{b}$ or $\mathscr{B}$, and the function $\mathscr{B}(\mathbf{b}\,,\,\lambda^k)$ does not depend on either $\mathbf{a}$ or $\mathscr{A}$. This leads us to the following remarkable theorem.

B. Quantum Correlations from the Algebra of Euclidean Primitives

Theorem: *Every quantum mechanical correlation can be understood as a classical, local, realistic, and deterministic correlation among a set of points of S^7 constructed above, represented by maps of the form defined in (82) and (83).*

Proof: Recall that – as von Neumann recognized in his classic analysis [10] – regardless of the model of physics one is concerned with – whether it is the quantum mechanical model or a hidden variable model – it is sufficient to consider expectation values of the observables measured in possible states of the physical systems, since probabilities are but expectation values of the indicator random variables. Thus, probability $P(E)$ of event E is expectation value $\mathcal{E}(\mathbb{1}_E)$,

$$P(E) = \mathcal{E}(\mathbb{1}_E)\,, \tag{84}$$

of the indicator random variable $\mathbb{1}_E$ defined as

$$\mathbb{1}_E := \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{otherwise}\,. \end{cases} \tag{85}$$

Conversely, the expectation value of $\mathbb{1}_E$ is

$$\mathcal{E}(\mathbb{1}_E) \,=\, \frac{1 \times P(E) \,+\, 0 \times \{1 - P(E)\}}{P(E) \,+\, \{1 - P(E)\}} \,=\, P(E)\,. \tag{86}$$

Thus every statement involving probabilities can be translated into a statement involving expectation values, and vice versa. In what follows we shall therefore work exclusively with expectation values, because our primary goal here is to trace the origins of the quantum correlations to the algebraic and geometrical properties of the Euclidean primitives.

To that end, consider an arbitrary quantum state $|\Psi\rangle \in \mathcal{H}$ of a system, where $\mathcal{H}$ is a Hilbert space of arbitrary dimensions – not necessarily finite. Apart from their usual quantum mechanical meanings, we impose no restrictions on either $|\Psi\rangle$ or $\mathcal{H}$. In particular, the state $|\Psi\rangle$ can be as entangled as one may wish [1]. Next, consider a self-adjoint operator $\widehat{\mathcal{O}}(\mathbf{n}^1,\,\mathbf{n}^2,\,\mathbf{n}^3,\,\mathbf{n}^4,\,\mathbf{n}^5,\,\dots)$ on this Hilbert space, parameterized by arbitrary number of local contexts $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$, $\mathbf{n}^4$, $\mathbf{n}^5$, *etc.* The quantum mechanical expectation value of this observable in the state $|\Psi\rangle$ is then defined by:

$$\mathcal{E}_{_{Q.M.}}(\mathbf{n}^1,\,\mathbf{n}^2,\,\mathbf{n}^3,\,\mathbf{n}^4,\,\mathbf{n}^5,\,\dots) \,=\, \langle\Psi|\,\widehat{\mathcal{O}}(\mathbf{n}^1,\,\mathbf{n}^2,\,\mathbf{n}^3,\,\mathbf{n}^4,\,\mathbf{n}^5,\,\dots)\,|\Psi\rangle\,. \tag{87}$$

More generally, if the system is in a mixed state, then its quantum mechanical expectation value can be expressed as

$$\mathcal{E}_{_{Q.M.}}(\mathbf{n}^1,\,\mathbf{n}^2,\,\mathbf{n}^3,\,\mathbf{n}^4,\,\mathbf{n}^5,\,\dots) \,=\, \text{Tr}\left\{\widehat{\mathcal{W}}\;\widehat{\mathcal{O}}(\mathbf{n}^1,\,\mathbf{n}^2,\,\mathbf{n}^3,\,\mathbf{n}^4,\,\mathbf{n}^5,\,\dots)\right\}, \tag{88}$$

where $\widehat{\mathcal{W}}$ is a statistical operator of unit trace representing the state of the system. Setting $\mathbf{n}^1 = \mathbf{a} \longleftrightarrow \mathbf{a}_r + \mathbf{a}_d\,\varepsilon_+$, $\mathbf{n}^2 = \mathbf{b} \longleftrightarrow \mathbf{b}_r + \mathbf{b}_d\,\varepsilon_+$, *etc.*, the corresponding local-realistic expectation value for the same system can be written as

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) \,=\, \int_{\Lambda} \mathscr{A}(\mathbf{a}\,,\,\lambda)\;\mathscr{B}(\mathbf{b}\,,\,\lambda)\,\mathscr{C}(\mathbf{c}\,,\,\lambda)\;\mathscr{D}(\mathbf{d}\,,\,\lambda)\;\dots\;\rho(\lambda)\,d\lambda\,, \tag{89}$$

where the binary measurement functions $\mathscr{N}(\mathbf{n}\,,\,\lambda^k)$ are defined in Eq. (81) and the overall probability distribution $\rho(\lambda)$,

$$\text{with} \quad \int_{\Lambda} \rho(\lambda)\;d\lambda \,=\, 1 \quad \text{for all} \quad \lambda \in \Lambda\,, \tag{90}$$

is in general a continuous function of λ. Since in our framework $\lambda = \pm 1$ is a fair coin, the above integral simplifies to

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) \,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\;\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\;\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\;\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\;\dots\right]. \tag{91}$$

We shall soon prove, however, that – thanks to the definitions like (82) – this average is geometrically equivalent to

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) = \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)\,\dots\right]. \tag{92}$$

Moreover, since as we saw in subsection I D the representation space $\mathcal{K}^\lambda$ defined in (60), with or without the constraints in (79), remains closed under multiplication, the product appearing in the expectation (92) is equivalent to the product

$$\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda) \,=\, -\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\,\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda\right), \tag{93}$$

for some vectors $\mathbf{x}$ and $\mathbf{y}$, depending in general on the measurement directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, *etc.* Consequently we have

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) = \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\,\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\,\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\,\dots\right] \tag{94}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)\,\dots\right] \tag{95}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\right] \tag{96}$$

$$= -\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda\right)\right] \tag{97}$$

$$= -\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\lambda^k\right]\mathbf{D}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0\right) \tag{98}$$

$$= -\,\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,-\,0\,, \tag{99}$$

because λ^k, as in (74), is a fair coin. We can now identify this local-realistic expectation with its quantum counterpart:

$$\langle\Psi|\,\widehat{\mathcal{O}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,|\Psi\rangle \,=\, \mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) \,=\, -\,\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,. \tag{100}$$

This identification proves our main theorem: Every quantum mechanical correlation can be understood as a classical, local, deterministic and realistic correlation among a set of points of the representation space $S^7\subset\mathcal{K}^\lambda$ described above.

It is instructive to evaluate the sum in Eq. (96) somewhat differently to bring out the fundamental role played by the orientation λ^k in the derivation of the strong correlations (99). Instead of assuming $\lambda^k=\pm1$ to be an orientation of S^7 as our starting point, we may view it as specifying the ordering relation between $\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=\pm1)$ and $\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\;0,\,\lambda^k=\pm1)$ and the corresponding detectors $\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)$ and $\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)$ with 50/50 chance of occurring, and only subsequently identify it with the orientation of S^7. Then, using the relations (74) and (93), the sum in Eq. (96) can be evaluated directly by recognizing that in the right and left oriented S^7 the following geometrical relations hold:

$$\begin{aligned}
&\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=+1)\;\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\;0,\,\lambda^k=+1)\\
&\qquad=\,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\,\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda^k=+1\right)\\
&\qquad=\,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\,\mathbf{D}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0\right)\\
&\qquad=\,\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)\;\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)
\end{aligned} \tag{101}$$

and

$$\begin{aligned}
&\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=-1)\;\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\;0,\,\lambda^k=-1)\\
&\qquad=\,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\,\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda^k=-1\right)\\
&\qquad=\,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,+\,\mathbf{D}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0\right)\\
&\qquad=\,-\,\mathbf{y}_r\cdot\mathbf{x}_r\,-\,\mathbf{y}_d\cdot\mathbf{x}_d\,-\,\mathbf{D}\left(\mathbf{y}_r\times\mathbf{x}_r+\mathbf{y}_d\times\mathbf{x}_d,\,\mathbf{y}_r\times\mathbf{x}_d+\mathbf{y}_d\times\mathbf{x}_r,\,0\right)\\
&\qquad=\,\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)\;\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0).
\end{aligned} \tag{102}$$

Changes in the orientation λ^k thus alternates the algebraic order of $\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=\pm1)$ and $\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k=\pm1)$ *relative* to the algebraic order of the detectors $\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)$ and $\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)$. Consequently, the sum (96) reduces to

$$\begin{aligned}\mathscr{E}_{L.R.}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) &= \lim_{m\to\infty}\left[\frac{1}{m}\sum_{k=1}^{m}\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\right]\\ &= \frac{1}{2}\{\,\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=+1)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k=+1)\}\,+\,\frac{1}{2}\{\,\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=-1)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k=-1)\}\\ &= \frac{1}{2}\{\,\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)\,\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)\}\,+\,\frac{1}{2}\{\,\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)\,\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)\}\\ &= -\,\frac{1}{2}\{\mathbf{x}_r\mathbf{y}_r\,+\,\mathbf{y}_r\mathbf{x}_r\}\,-\,\frac{1}{2}\{\mathbf{x}_d\mathbf{y}_d\,+\,\mathbf{y}_d\mathbf{x}_d\}\,=\,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,=\,-\,\mathbf{x}\cdot\mathbf{y}\\ &= -\,\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,,\end{aligned} \tag{103}$$

because the orientation λ^k of S^7 is a fair coin. Here $\mathbf{x}\cdot\mathbf{y}=\frac{1}{2}\{\mathbf{xy}+\mathbf{yx}\}$ is the standard definition of the inner product.

Evidently the above method of calculating suggests that a given initial state λ of the physical system can indeed be viewed as specifying an ordering relation between $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)$ and the detectors $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ that measure it:

$$\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=+1)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k=+1)\,=\,\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0)\,\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0) \tag{104}$$

or

$$\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k=-1)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k=-1)\,=\,\mathbf{D}(\mathbf{y}_r,\,\mathbf{y}_d,\,0)\,\mathbf{D}(\mathbf{x}_r,\,\mathbf{x}_d,\,0). \tag{105}$$

Then, using the right-hand sides of the Eqs. (101) and (102), the above pair can be reduced to the combined relation

$$\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda\right)\,=\,\lambda\,\mathbf{D}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0\right), \tag{106}$$

which is identical to the relation (74) for normalized vectors. We have thus proved that the ordering relations (104) and (105) between $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)$ and $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ are equivalent to the alternatively possible orientations of S^7.

1. *Special Case of a Two-level System Entangled in a Singlet State*

Now, to complete the above proof we must prove the step from Eq. (91) to Eq. (92). To that end, let us first consider observations of the spins of only two spin-$\frac{1}{2}$ particles produced in a decay of a single spinless particle as shown in Fig. 5. After the decay the two emerging spin-$\frac{1}{2}$ particles move freely in opposite directions, subject to spin measurements along freely chosen unit directions $\mathbf{a}$ and $\mathbf{b}$, which may be located at a spacelike distance from one another [9]. Since initially the emerging pair has zero net spin, its quantum mechanical state is described by the entangled singlet state

$$|\Psi_{\mathbf{z}}\rangle=\frac{1}{\sqrt{2}}\Big\{|\mathbf{z},\,+\rangle_1\otimes|\mathbf{z},\,-\rangle_2\,-\,|\mathbf{z},\,-\rangle_1\otimes|\mathbf{z},\,+\rangle_2\Big\}, \tag{107}$$

with $\boldsymbol{\sigma}\cdot\mathbf{z}\,|\mathbf{z},\,\pm\rangle\,=\,\pm\,|\mathbf{z},\,\pm\rangle$ describing the eigenstates of the Pauli spin "vector" $\boldsymbol{\sigma}$ in which the particles have spin "up" or "down" along $\mathbf{z}$-axis, in the units of $\hbar=2$. Our interest lies in comparing the quantum mechanical predictions,

$$\mathscr{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{a},\,\mathbf{b})\,=\,\langle\Psi_{\mathbf{z}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\,\otimes\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{z}}\rangle\,=\,-\,\cos\theta_{\mathbf{ab}}\,, \tag{108}$$

together with

$$\mathscr{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{a})\,=\,\langle\Psi_{\mathbf{z}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\mathbb{1}\,|\Psi_{\mathbf{z}}\rangle\,=\,0\quad\text{and}\quad\mathscr{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{b})\,=\,\langle\Psi_{\mathbf{z}}|\,\mathbb{1}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{z}}\rangle\,=\,0\,, \tag{109}$$

of spin correlations between the two subsystems, with those derived within our locally causal framework, regardless of the relative distance between the two remote locations represented by the unit detection vectors $\mathbf{a}$ and $\mathbf{b}$. Here $\mathbb{1}$ is the identity matrix. The corresponding locally causal description of this system within our framework thus involves only two contexts, $\mathbf{n}^1=\mathbf{a}\longleftrightarrow\mathbf{a}_r+\mathbf{a}_d\,\varepsilon_+$ and $\mathbf{n}^2=\mathbf{b}\longleftrightarrow\mathbf{b}_r+\mathbf{b}_d\,\varepsilon_+$, with measurement results defined by the functions

$$S^7\ni\mathscr{A}(\mathbf{a}\,,\,\lambda^k):=\lim_{\substack{\mathbf{s}_{r1}\,\to\,\mathbf{a}_r\\ \mathbf{s}_{d1}\,\to\,\mathbf{a}_d}}\big\{-\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\big\}=\begin{Bmatrix}+1 & \text{if} & \lambda^k\,=\,+1\\ -1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\ \text{with}\ \Big\langle\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\Big\rangle\,=\,0 \tag{110}$$

and

$$S^7 \ni \mathscr{B}(\mathbf{b}\,,\,\lambda^k) := \lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r \\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}} \left\{ +\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0) \right\} = \begin{Bmatrix} -1 & \text{if} & \lambda^k \,=\, +1 \\ +1 & \text{if} & \lambda^k \,=\, -1 \end{Bmatrix} \text{ with } \Big\langle \mathscr{B}(\mathbf{b}\,,\,\lambda^k) \Big\rangle \,=\, 0\,, \tag{111}$$

where $\mathbf{s}_1 \longleftrightarrow \mathbf{s}_{r1} + \mathbf{s}_{d1}\,\varepsilon_+$ and $\mathbf{s}_2 \longleftrightarrow \mathbf{s}_{r2} + \mathbf{s}_{d2}\,\varepsilon_+$ represent the directions of the two spins emerging from the source.

Next, recalling that physically all bivectors $\boldsymbol{\xi}(\mathbf{n}) \in S^2 \subset S^3$ represent spins [1][5], we require that the total spin-zero angular momentum for the initial or "complete" state associated with the above measurement functions is conserved,

$$\text{total real spin} \;:=\; -\,\lambda\,\boldsymbol{\xi}(\mathbf{s}_{r1}) \,+\, \lambda\,\boldsymbol{\xi}(\mathbf{s}_{r2}) \,=\, 0 \;\;\Longleftrightarrow\;\; \mathbf{s}_{r1} \,=\, \mathbf{s}_{r2} \,\equiv\, \mathbf{s}_r \tag{112}$$

and

$$\text{total dual spin} \;:=\; -\,\lambda\,\boldsymbol{\xi}(\mathbf{s}_{d1}) \,+\, \lambda\,\boldsymbol{\xi}(\mathbf{s}_{d2}) \,=\, 0 \;\;\Longleftrightarrow\;\; \mathbf{s}_{d1} \,=\, \mathbf{s}_{d2} \,\equiv\, \mathbf{s}_d\,, \tag{113}$$

just as it is in the EPR-Bell type experiment depicted in Fig. 5. For $\mathbf{N}(\mathbf{s}_r,\,\mathbf{s}_d,\,0,\,\lambda^k)$ this is equivalent to the condition

$$-\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k) \,+\, \mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k) \,=\, 0 \;\;\Longleftrightarrow\;\; \mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k) \,=\, \mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,. \tag{114}$$

In the light of the product rule (80) for anti-symmetric elements, the above condition is also equivalent to the condition

$$\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k) \,=\, \big\{\mathbf{N}(\mathbf{s}_r,\,\mathbf{s}_d,\,0,\,\lambda^k)\big\}^2 \,=\, \mathbf{N}^2(\mathbf{s}_r,\,\mathbf{s}_d,\,0,\,\lambda^k) \,=\, -1\,. \tag{115}$$

In the next subsection we will derive this condition geometrically as a natural consequence of the twist in the Hopf bundle of S^3. Here it leads to the following statistical equivalence, which can be viewed also as a geometrical identity:

$$\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\;\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\right] \,\equiv\, \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\right]. \tag{116}$$

Given the definitions (110) and (111), there are more than one ways to prove this identity. In the following we will use one such way. But it can also be proved by simply taking the limits in (110) and (111) while maintaining (114), and then using Eq. (74). Then the computation of correlations between $\mathscr{A}(\mathbf{a},\,\lambda^k) = \pm 1$ and $\mathscr{B}(\mathbf{b},\,\lambda^k) = \pm 1$ works out as

$$\mathcal{E}^{\rm EPR}_{{}_{L.R.}}(\mathbf{a},\,\mathbf{b}) = \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\;\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\right] \tag{117}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\left\{\lim_{\substack{\mathbf{s}_{r1}\,\rightarrow\,\mathbf{a}_r \\ \mathbf{s}_{d1}\,\rightarrow\,\mathbf{a}_d}}\Big[-\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\Big]\right\}\left\{\lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r \\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}}\Big[\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\Big]\right\}\right] \tag{118}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\left\{\lim_{\substack{\mathbf{s}_{r1}\,\rightarrow\,\mathbf{a}_r \\ \mathbf{s}_{d1}\,\rightarrow\,\mathbf{a}_d}}\;\lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r \\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}}\Big[-\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\big\{\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,\big\}\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\Big]\right\}\right] \tag{119}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\left\{\lim_{\substack{\mathbf{s}_{r1}\,\rightarrow\,\mathbf{a}_r \\ \mathbf{s}_{d1}\,\rightarrow\,\mathbf{a}_d}}\;\lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r \\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}}\Big[-\,\lambda^k\,\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\;\{-1\}\;\lambda^k\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\Big]\right\}\right] \tag{120}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\left\{\lim_{\substack{\mathbf{s}_{r1}\,\rightarrow\,\mathbf{a}_r \\ \mathbf{s}_{d1}\,\rightarrow\,\mathbf{a}_d}}\;\lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r \\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}}\Big[+\big(\lambda^k\big)^2\;\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\Big]\right\}\right] \tag{121}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\right] \tag{122}$$

$$=\,-\,\mathbf{a}_r\cdot\mathbf{b}_r \,-\, \mathbf{a}_d\cdot\mathbf{b}_d \,-\, \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathbf{N}\left(\mathbf{a}_r\times\mathbf{b}_r + \mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d + \mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda^k\right)\right] \tag{123}$$

$$=\,-\,\mathbf{a}_r\cdot\mathbf{b}_r \,-\, \mathbf{a}_d\cdot\mathbf{b}_d \,-\, \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\lambda^k\right]\mathbf{D}\left(\mathbf{a}_r\times\mathbf{b}_r + \mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d + \mathbf{a}_d\times\mathbf{b}_r,\,0\right) \tag{124}$$

$$=\,-\,\cos\theta_{\mathbf{ab}} \,-\, 0\,. \tag{125}$$

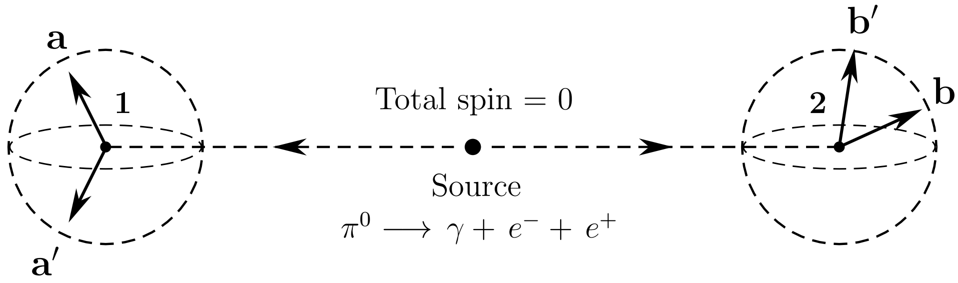

FIG. 5: A spin-less neutral pion decays into an electron-positron pair. Measurements of spin components on each separated fermion are performed at remote stations **1** and **2**, providing binary outcomes (respectively) along arbitrary directions **a** and **b**.

Here Eq. (118) follows from Eq. (117) by substituting the functions $\mathscr{A}(\mathbf{a},\,\lambda^k)$ and $\mathscr{B}(\mathbf{b},\,\lambda^k)$ from their definitions (110) and (111); Eq. (119) follows from Eq. (118) by using the "product of limits equal to limits of product" rule [which can be verified by noting that the same multivector results from the limits in Eqs. (118) and (119)]; Eq. (120) follows from Eq. (119) by using (i) the relations (74) [thus rewriting all anti-symmetric elements in the same bases], (ii) the associativity of the geometric product, and (iii) the consequence (115) of the conservation of the spin angular momenta in S^3; Eq. (121) follows from Eq. (120) by recalling that scalars λ^k commute with the elements of all grades; Eq. (122) follows from Eq. (121) because $\lambda^2 = +1$, and by removing the superfluous limit operations; Eq. (123) follows from Eq. (122) by using the geometric product (80); Eq. (124) follows from Eq. (123) by using the relations (74); and finally Eq. (125) follows from Eq. (124) by using Eq. (79) and because the scalar coefficient of $\mathbf{D}$ vanishes in the $m \to \infty$ limit since λ^k is a fair coin. This proves that singlet correlations are correlations among the scalar points of a quaternionic S^3.

As we did above for the general case, let us again evaluate the sum in Eq. (122) somewhat differently to bring out the crucial role played by λ^k in the derivation of the correlations (125). Using the relations (74) and (80), the sum (122) can be evaluated directly by recognizing that in the right and left oriented S^7 the following geometrical relations hold:

$$\begin{aligned}
&\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=+1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=+1)\\
&\qquad= -\mathbf{a}_r\cdot\mathbf{b}_r - \mathbf{a}_d\cdot\mathbf{b}_d - \mathbf{N}\big(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda^k=+1\big)\\
&\qquad= -\mathbf{a}_r\cdot\mathbf{b}_r - \mathbf{a}_d\cdot\mathbf{b}_d - \mathbf{D}\left(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0\right)\\
&\qquad= \mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)
\end{aligned}\tag{126}$$

and

$$\begin{aligned}
&\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=-1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=-1)\\
&\qquad= -\mathbf{a}_r\cdot\mathbf{b}_r - \mathbf{a}_d\cdot\mathbf{b}_d - \mathbf{N}\big(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda^k=-1\big)\\
&\qquad= -\mathbf{a}_r\cdot\mathbf{b}_r - \mathbf{a}_d\cdot\mathbf{b}_d + \mathbf{D}\left(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0\right)\\
&\qquad= -\mathbf{b}_r\cdot\mathbf{a}_r - \mathbf{b}_d\cdot\mathbf{a}_d - \mathbf{D}\left(\mathbf{b}_r\times\mathbf{a}_r+\mathbf{b}_d\times\mathbf{a}_d,\,\mathbf{b}_r\times\mathbf{a}_d+\mathbf{b}_d\times\mathbf{a}_r,\,0\right)\\
&\qquad= \mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0).
\end{aligned}\tag{127}$$

Changes in λ^k thus alternates the *relative* order of $\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)$. As a result, the sum (122) reduces to

$$\begin{aligned}
\mathcal{E}^{\rm EPR}_{_{L.R.}}(\mathbf{a},\mathbf{b}) &= \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\right]\\
&= \frac{1}{2}\{\,\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=+1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=+1)\} + \frac{1}{2}\{\,\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=-1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=-1)\}\\
&= \frac{1}{2}\{\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\} + \frac{1}{2}\{\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\}\\
&= -\frac{1}{2}\{\mathbf{a}_r\mathbf{b}_r + \mathbf{b}_r\mathbf{a}_r\} - \frac{1}{2}\{\mathbf{a}_d\mathbf{b}_d + \mathbf{b}_d\mathbf{a}_d\} = -\,\mathbf{a}_r\cdot\mathbf{b}_r - \mathbf{a}_d\cdot\mathbf{b}_d = -\,\mathbf{a}\cdot\mathbf{b} = -\cos\theta_{\mathbf{ab}}\,,
\end{aligned}\tag{128}$$

because the orientation λ^k of S^7 is a fair coin. Here $\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}\{\mathbf{ab}+\mathbf{ba}\}$ is the standard definition of the inner product.

The above method of calculating the correlations suggests that a given initial state λ of the physical system can be viewed also as specifying an ordering relation between $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)$ and the detectors $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ that measure it:

$$\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=+1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=+1)\,=\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0) \tag{129}$$

or

$$\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k=-1)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k=-1)\,=\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0). \tag{130}$$

Then, using the right-hand sides of the Eqs. (126) and (127), the above pair can be reduced to the combined relation

$$\mathbf{N}\left(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda\right)\,=\,\lambda\,\mathbf{D}\left(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0\right), \tag{131}$$

which is equivalent to the relation (74) for normalized vectors. We have thus proved that the ordering relations (129) and (130) between $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda)$ and $\mathbf{D}(\mathbf{n}_r,\,\mathbf{n}_d,\,0)$ are equivalent to the alternatively possible orientations of S^7.

2. *Conservation of Spin-0 from the Twist in the Hopf Bundle of* S^3

Note that, apart from the initial state λ^k, the only other assumption used in this derivation is that of the conservation of spin angular momentum (115). These two assumptions are necessary and sufficient to dictate the singlet correlations:

$$\mathcal{E}^{\rm EPR}_{_{L.R.}}(\mathbf{a},\,\mathbf{b})\,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\right]\,=\,-\,\cos\theta_{\mathbf{ab}}\,. \tag{132}$$

The conservation of spin, however, can be understood in terms of the twist in the Hopf bundle of $S^3\cong{\rm SU}(2)$. Recall that locally (in the topological sense) S^3 can be written as a product $S^2\times S^1$, but globally it has no cross-section [11]. It can be viewed also as a principal U(1) bundle over S^2, with the points of its base space S^2 being the elements of the Lie algebra su(2), which are pure quaternions, or bivectors [1][9][12]. The product of two such bivectors are in general non-pure quaternions of the form (18), and are elements of the group SU(2) itself. That is to say, they are points of the bundle space S^3, whose elements are the preimages of the points of the base space S^2 [11]. These preimages are 1-spheres, S^1, called Hopf circles, or Clifford parallels [13]. Since these 1-spheres are the fibers of the bundle, they do not share a single point in common. Each circle threads through every other circle in the bundle, making them linked together in a highly non-trivial configuration, which can be quantified by the following relation among the fibers [12]:

$$e^{i\psi_-}\,=\,e^{i\phi}\,e^{i\psi_+}\,, \tag{133}$$

where $e^{i\psi_-}$ and $e^{i\psi_+}$, respectively, are the U(1) fiber coordinates above the two hemispheres H_- and H_+ of the base space S^2, with spherical coordinates $(0\leqslant\theta<\pi,\;0\leqslant\phi<2\pi)$; ϕ is the angle parameterizing a thin strip $H_-\cap H_+$ around the equator of S^2 $[\theta\sim\frac{\pi}{2}]$; and $e^{i\phi}$ is the transition function that glues the two sections H_- and H_+ together, thus constituting the 3-sphere. It is evident from Eq. (133) that the fibers match perfectly at the angle $\phi=0$ (modulo 2π), but differ from each other at all intermediate angles ϕ. For example, $e^{i\psi_-}$ and $e^{i\psi_+}$ differ by a minus sign at the angle $\phi=\pi$. Now to derive the conservation of spin (115), we rewrite the exponential relation (133) in our notation as

$$\{-\,\boldsymbol{\xi}(\mathbf{a}_r)\,\boldsymbol{\xi}(\mathbf{s}_{r1})\}\,=\,\{\boldsymbol{\xi}(\mathbf{a}_r)\,\boldsymbol{\xi}(\mathbf{b}_r)\}\;\{\boldsymbol{\xi}(\mathbf{s}_{r2})\,\boldsymbol{\xi}(\mathbf{b}_r)\} \tag{134}$$

by identifying the angles $\eta_{\mathbf{a}_r\mathbf{s}_{r1}}$ and $\eta_{\mathbf{s}_{r2}\mathbf{b}_r}$ between $\mathbf{a}_r$ and $\mathbf{s}_{r1}$ and $\mathbf{s}_{r2}$ and $\mathbf{b}_r$ with the fibers ψ_- and ψ_+, and the angle $\eta_{\mathbf{a}_r\mathbf{b}_r}$ between $\mathbf{a}_r$ and $\mathbf{b}_r$ with the generator of the transition function $e^{i\phi}$ on the equator of S^2. Here we have used the sign conventions to match the sign conventions in our definitions (110) and (111) and the correlations (125). The above representation of Eq.(133) is not as unusual as it may appear at first sight once we recall that geometric products of the bivectors appearing in it are all non-pure quaternions, which can be parameterized to take the exponential form

$$-\,\boldsymbol{\xi}(\mathbf{u})\,\boldsymbol{\xi}(\mathbf{v})\,=\,-\,(\lambda\,I\cdot\mathbf{u})\,(\lambda\,I\cdot\mathbf{v})\,=\,\cos(\,\eta_{\mathbf{uv}})\,+\,\frac{\mathbf{u}\wedge\mathbf{v}}{||\mathbf{u}\wedge\mathbf{v}||}\,\sin(\,\eta_{\mathbf{uv}})\,=\,\exp\left\{\frac{\mathbf{u}\wedge\mathbf{v}}{||\mathbf{u}\wedge\mathbf{v}||}\,\eta_{\mathbf{uv}}\right\}. \tag{135}$$

Multiplying both sides of Eq. (134) from the left with $\boldsymbol{\xi}(\mathbf{a}_r)$ and noting that all unit bivectors square to -1, we obtain

$$\boldsymbol{\xi}(\mathbf{s}_{r1})\,=\,-\,\boldsymbol{\xi}(\mathbf{b}_r)\,\boldsymbol{\xi}(\mathbf{s}_{r2})\,\boldsymbol{\xi}(\mathbf{b}_r)\,. \tag{136}$$

Multiplying the numerator and denominator on the RHS of this similarity relation with $-\,\boldsymbol{\xi}(\mathbf{b}_r)$ from the right and $\boldsymbol{\xi}(\mathbf{b}_r)$ from the left then leads to the conservation of the spin angular momentum, just as we have specified in Eq. (112):

$$\lambda\,\boldsymbol{\xi}(\mathbf{s}_{r1})\,=\,\lambda\,\boldsymbol{\xi}(\mathbf{s}_{r2})\;\Longleftrightarrow\;\mathbf{s}_{r1}\,=\,\mathbf{s}_{r2}\,. \tag{137}$$

Similarly, we can derive analogous conservation law for the zero spin within the dual 3-sphere, as specified in Eq. (113):

$$\lambda\,\boldsymbol{\xi}(\mathbf{s}_{d1}) \,=\, \lambda\,\boldsymbol{\xi}(\mathbf{s}_{d2}) \iff \mathbf{s}_{d1} \,=\, \mathbf{s}_{d2}\,. \tag{138}$$

Given the conservation laws derived in Eqs. (137) and (138), we can combine them to arrive at the net condition (115):

$$\mathbf{N}(\mathbf{s}_{r1},\, \mathbf{s}_{d1},\, 0,\, \lambda^k)\,\mathbf{N}(\mathbf{s}_{r2},\, \mathbf{s}_{d2},\, 0,\, \lambda^k) \,=\, -1\,, \tag{139}$$

which was used in Eq. (120) to derive the strong correlations (125). We have thus shown that the conservation of spin angular momentum is not an additional assumption, but follows from the very geometry and topology of the 3-sphere.

In fact it is not difficult to see from the twist in the Hopf bundle of S^3, captured in Eq. (134), that if we set $\mathbf{a}_r = \mathbf{b}_r$ (or equivalently $\eta_{\mathbf{a}_r\mathbf{b}_r} = 0$) for all fibers, then S^3 reduces to the trivial bundle $S^2 \times S^1$, since then the fiber coordinates $\eta_{\mathbf{a}_r\mathbf{s}_{r1}}$ and $\eta_{\mathbf{s}_{r2}\mathbf{b}_r}$ would match up exactly on the equator of S^2 $[\theta \sim \frac{\pi}{2}]$. In general, however, for $\mathbf{a}_r \neq \mathbf{b}_r$, $S^3 \neq S^2 \times S^1$. For example, when $\mathbf{a}_r = -\,\mathbf{b}_r$ (or equivalently when $\eta_{\mathbf{a}_r\mathbf{b}_r} = \pi$) there will be a sign difference between the fibers at that point of the equator [11][12]. That in turn would produce a twist in the bundle analogous to the twist in a Möbius strip. It is this non-trivial twist in the S^3 bundle that is responsible for the observed sign flips in the product $\mathscr{A}\mathscr{B}$ of measurement results, from $\mathscr{A}\mathscr{B} = -1$ for $\mathbf{a}_r = \mathbf{b}_r$ to $\mathscr{A}\mathscr{B} = +1$ for $\mathbf{a}_r = -\,\mathbf{b}_r$, as evident from the correlations (125). In the appendix of the first chapter of Ref. [1] this is illustrated in a toy model of Alice and Bob in a Möbius world.

3. *The General Case of Arbitrarily Entangled Quantum State*

We now proceed to generalize the above two-particle case to the general case of arbitrary quantum state considered in (87). To this end, let us consider arbitrary number of measurement results corresponding to those in (87) and (88):

$$\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\;\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\;\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\;\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\;\mathscr{E}(\mathbf{e}\,,\,\lambda^k)\;\mathscr{F}(\mathbf{f}\,,\,\lambda^k)\;\mathscr{G}(\mathbf{g}\,,\,\lambda^k)\;\dots\,, \tag{140}$$

with each pair such as $(\mathscr{C},\,\mathscr{D})$ defined for the contexts such as $\mathbf{n}^3 = \mathbf{c} \longleftrightarrow \mathbf{c}_r + \mathbf{c}_d\,\varepsilon_+$ and $\mathbf{n}^4 = \mathbf{d} \longleftrightarrow \mathbf{d}_r + \mathbf{d}_d\,\varepsilon_+$:

$$S^7 \ni \mathscr{C}(\mathbf{c}\,,\,\lambda^k) := \lim_{\substack{\mathbf{t}_{r1}\,\to\,\mathbf{c}_r \\ \mathbf{t}_{d1}\,\to\,\mathbf{c}_d}} \left\{ -\,\mathbf{D}(\mathbf{c}_r,\,\mathbf{c}_d,\,0)\,\mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k) \right\} = \begin{Bmatrix} +1 & \text{if} & \lambda^k \,=\, +1 \\ -1 & \text{if} & \lambda^k \,=\, -1 \end{Bmatrix} \text{ with } \langle\, \mathscr{C}(\mathbf{c}\,,\,\lambda^k)\,\rangle \,=\, 0 \tag{141}$$

and

$$S^7 \ni \mathscr{D}(\mathbf{d}\,,\,\lambda^k) := \lim_{\substack{\mathbf{t}_{r2}\,\to\,\mathbf{d}_r \\ \mathbf{t}_{d2}\,\to\,\mathbf{d}_d}} \left\{ +\,\mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k)\,\mathbf{D}(\mathbf{d}_r,\,\mathbf{d}_d,\,0) \right\} = \begin{Bmatrix} -1 & \text{if} & \lambda^k \,=\, +1 \\ +1 & \text{if} & \lambda^k \,=\, -1 \end{Bmatrix} \text{ with } \langle\, \mathscr{D}(\mathbf{d}\,,\,\lambda^k)\,\rangle \,=\, 0\,. \tag{142}$$

If the number of measurement results happens to be odd instead of even, then the product of an even number of results can be first evaluated, and then that factor can be paired with the remaining result, as done in Eq. (180) below.

It is important to recall here the elementary fact that any experiment of any kind in physics can always be reduced to a series of questions with "yes" / "no" answers, represented by binary measurement outcomes of the form (140) to (142). Therefore the measurement framework we have developed here is completely general and applicable to any experiment.

Now, as in the EPR-Bohm type experiment with a singlet state discussed above [cf. Fig. 5 and Eqs. (114), (115) and (139)], for each pair of measurement outcomes such as (142) the twist in the Hopf bundle of S^3 dictates the condition

$$-\,\mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k) \,+\, \mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k) \,=\, 0 \iff \mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k) \,=\, \mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k)\,, \tag{143}$$

or equivalently the condition

$$\mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k) \,=\, \left\{\mathbf{N}(\mathbf{t}_r,\,\mathbf{t}_d,\,0,\,\lambda^k)\right\}^2 \,=\, \mathbf{N}^2(\mathbf{t}_r,\,\mathbf{t}_d,\,0,\,\lambda^k) \,=\, -1\,. \tag{144}$$

Consequently, by following the steps analogous to those in Eqs. (110) to (122), we arrive at the geometrical equivalence

$$\lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\;\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\right] \,\equiv\, \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)\right] \tag{145}$$

for each pair $(\mathscr{C},\,\mathscr{D})$ of measurement outcomes. As a result, the correlations among the outcomes (140) take the form

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) = \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\,\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\,\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\,\dots\right] \tag{146}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)\,\dots\right] \tag{147}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\right] \tag{148}$$

$$= \,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0,\,\lambda\right)\right] \tag{149}$$

$$= \,-\,\mathbf{x}_r\cdot\mathbf{y}_r\,-\,\mathbf{x}_d\cdot\mathbf{y}_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\lambda^k\right]\mathbf{D}\left(\mathbf{x}_r\times\mathbf{y}_r+\mathbf{x}_d\times\mathbf{y}_d,\,\mathbf{x}_r\times\mathbf{y}_d+\mathbf{x}_d\times\mathbf{y}_r,\,0\right) \tag{150}$$

$$= \,-\,\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,-\,0\,, \tag{151}$$

because λ^k is a fair coin. We can now identify this locally causal expectation with its quantum mechanical counterpart:

$$\langle\Psi|\,\widehat{\mathcal{O}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,|\Psi\rangle\,=\,\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,=\,-\,\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,. \tag{152}$$

This completes the proof of the theorem for the general quantum state stated at the beginning of the subsection II B.

C. Derivation of Tsirel'son's Bounds on the Strengths of the Correlations

Let us now investigate the bounds on the strengths of the correlations (151) by deriving Tsirel'son's bounds [5] for arbitrary quantum states [1]. To this end, instead of (140) consider an alternative set of measurement results such as

$$\mathscr{A}(\mathbf{a}'\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}'\,,\,\lambda^k)\,\mathscr{C}(\mathbf{c}'\,,\,\lambda^k)\,\mathscr{D}(\mathbf{d}'\,,\,\lambda^k)\,\mathscr{E}(\mathbf{e}'\,,\,\lambda^k)\,\mathscr{F}(\mathbf{f}'\,,\,\lambda^k)\,\mathscr{G}(\mathbf{g}'\,,\,\lambda^k)\,\dots\,, \tag{153}$$

with each pair such as $(\mathscr{C},\,\mathscr{D})$ defined for contexts such as $\mathbf{n'}^3=\mathbf{c}'\longleftrightarrow\mathbf{c}'_r+\mathbf{c}'_d\,\varepsilon_+$ and $\mathbf{n'}^4=\mathbf{d}'\longleftrightarrow\mathbf{d}'_r+\mathbf{d}'_d\,\varepsilon_+$. The correlation between these results can then be derived following steps analogous to those in the previous subsection:

$$\mathcal{E}_{_{L.R.}}(\mathbf{a}',\,\mathbf{b}',\,\mathbf{c}',\,\mathbf{d}',\,\dots)\,= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\mathscr{A}(\mathbf{a}'\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}'\,,\,\lambda^k)\,\mathscr{C}(\mathbf{c}'\,,\,\lambda^k)\,\mathscr{D}(\mathbf{d}'\,,\,\lambda^k)\,\dots\right] \tag{154}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{a}'_r,\,\mathbf{a}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}'_r,\,\mathbf{b}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}'_r,\,\mathbf{c}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}'_r,\,\mathbf{d}'_d,\,0,\,\lambda^k)\,\dots\right] \tag{155}$$

$$= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\right] \tag{156}$$

$$= \,-\,\mathbf{x}'_r\cdot\mathbf{y}'_r\,-\,\mathbf{x}'_d\cdot\mathbf{y}'_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}'_r\times\mathbf{y}'_r+\mathbf{x}'_d\times\mathbf{y}'_d,\,\mathbf{x}'_r\times\mathbf{y}'_d+\mathbf{x}'_d\times\mathbf{y}'_r,\,0,\,\lambda)\right] \tag{157}$$

$$= \,-\,\mathbf{x}'_r\cdot\mathbf{y}'_r\,-\,\mathbf{x}'_d\cdot\mathbf{y}'_d\,-\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\lambda^k\right]\mathbf{D}(\mathbf{x}'_r\times\mathbf{y}'_r+\mathbf{x}'_d\times\mathbf{y}'_d,\,\mathbf{x}'_r\times\mathbf{y}'_d+\mathbf{x}'_d\times\mathbf{y}'_r,\,0) \tag{158}$$

$$= \,-\,\cos\theta_{\mathbf{x'y'}}(\mathbf{a}',\,\mathbf{b}',\,\mathbf{c}',\,\mathbf{d}',\,\dots)\,-\,0\,. \tag{159}$$

In particular, in Eq. (156) we then have the relation

$$\mathcal{E}_{_{L.R.}}(\mathbf{a}',\,\mathbf{b}',\,\mathbf{c}',\,\mathbf{d}',\,\dots)\,=\,\mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')\,= \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\right]. \tag{160}$$

Needless to say, we are free to choose the contexts different from the primed and unprimed ones chosen in (153) and (140), as well as any combinations and/or mixtures of them, such as ($\mathbf{a}$, $\mathbf{b}'$, $\mathbf{c}''$, $\mathbf{d}'''$, $\mathbf{e}''''$, ...). Consequently, we may consider the following four relations corresponding to some alternative combinations of measurement contexts so that

$$\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}) \,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\right], \tag{161}$$

$$\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}') \,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\right], \tag{162}$$

$$\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}) \,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\right], \tag{163}$$

$$\text{and}\quad \mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}') \,=\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\right]. \tag{164}$$

Using the above four expressions the corresponding Bell-CHSH string of expectation values [5], namely the coefficient

$$\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y})\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}')\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y})\,-\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}') \tag{165}$$

corresponding to this fully general case of arbitrary number of contexts and measurement results, can be written as

$$\begin{aligned}\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y})\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}')\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y})\,-\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}') \,=\, \lim_{m\,\rightarrow\,\infty}\Bigg[\frac{1}{m}\sum_{k\,=\,1}^{m}\Big\{ &\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\\ &+\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\\ &+\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\\ &-\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k)\Big\}\Bigg].\end{aligned} \tag{166}$$

But since $\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k)$ and $\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)$ represent two independent equatorial points of an S^6 within S^7, we take them to belong to two disconnected "sections" of the bundle $S^5 \times S^1$ (*i.e.*, two disconnected $S^5 \subset S^6$), satisfying

$$\big[\,\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k),\;\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\,\big]\,=\,0\quad\forall\;\mathbf{x}_r\;\text{ and }\;\mathbf{y}_d\,\in\,\mathrm{I\!R}^3, \tag{167}$$

which is equivalent to anticipating null outcomes along the directions $\mathbf{x}_r \times \mathbf{y}_d$ exclusive to both $\mathbf{x}_r$ and $\mathbf{y}_d$. If we now square the integrand of equation (166), use the above commutation relations, and use the fact that all $\mathbf{N}(\mathbf{n}_r,\,\mathbf{n}_d,\,0,\,\lambda^k)$ square to -1, then the absolute value of the above Bell-CHSH string (165) leads to the following variance inequality:

$$|\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y})\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}')\,+\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y})\,-\,\mathscr{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')|\,\leqslant\sqrt{\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\,\Big\{\,4\,+\,4\,\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k)\,\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k)\,\Big\}\right]}, \tag{168}$$

where the classical commutators

$$\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k) := \frac{1}{2}\,\big[\,\mathbf{N}(\mathbf{x}_r,\,\mathbf{x}_d,\,0,\,\lambda^k),\;\mathbf{N}(\mathbf{x}'_r,\,\mathbf{x}'_d,\,0,\,\lambda^k)\,\big] \,=\, -\,\mathbf{N}\left(\mathbf{x}_r\times\mathbf{x}'_r+\mathbf{x}_d\times\mathbf{x}'_d,\;\mathbf{x}_r\times\mathbf{x}'_d+\mathbf{x}_d\times\mathbf{x}'_r,\;0,\;\lambda^k\right) \tag{169}$$

and

$$\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k) := \frac{1}{2}\,\big[\,\mathbf{N}(\mathbf{y}'_r,\,\mathbf{y}'_d,\,0,\,\lambda^k),\;\mathbf{N}(\mathbf{y}_r,\,\mathbf{y}_d,\,0,\,\lambda^k)\big] \,=\, -\,\mathbf{N}\left(\mathbf{y}'_r\times\mathbf{y}_r+\mathbf{y}'_d\times\mathbf{y}_d,\;\mathbf{y}'_r\times\mathbf{y}_d+\mathbf{y}'_d\times\mathbf{y}_r,\;0,\;\lambda^k\right) \tag{170}$$

are the geometric measures of the torsion within S^7 [1][5]. Thus, it is the non-vanishing torsion $\mathscr{T}$ within S^7 — the parallelizing torsion which makes the Riemann curvature of this representation space vanish — that is responsible for the stronger-than-linear correlations. We can see this from Eq. (168) by setting $\mathscr{T}=0$, and in more detail as follows.

Using the above expressions for the intrinsic torsions $\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k)$ and $\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k)$ and defining the unnormalized vectors

$$\mathbf{u}_r := (\mathbf{x}_r \times \mathbf{x'}_r + \mathbf{x}_d \times \mathbf{x'}_d) \quad \text{and} \quad \mathbf{u}_d := (\mathbf{x}_r \times \mathbf{x'}_d + \mathbf{x}_d \times \mathbf{x'}_r) \tag{171}$$

and

$$\mathbf{v}_r := (\mathbf{y'}_r \times \mathbf{y}_r + \mathbf{y'}_d \times \mathbf{y}_d) \quad \text{and} \quad \mathbf{v}_d := (\mathbf{y'}_r \times \mathbf{y}_d + \mathbf{y'}_d \times \mathbf{y}_r)\,, \tag{172}$$

together with $\mathbf{u}\cdot\mathbf{v} := \mathbf{u}_r\cdot\mathbf{v}_r \,+\, \mathbf{u}_d\cdot\mathbf{v}_d$ analogous to $\mathbf{a}\cdot\mathbf{b} := \mathbf{a}_r\cdot\mathbf{b}_r \,+\, \mathbf{a}_d\cdot\mathbf{b}_d$ given in Eq. (79), we have the product

$$\begin{aligned}\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k)\,\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k) \,&=\, -\,\mathbf{u}_r\cdot\mathbf{v}_r \,-\, \mathbf{u}_d\cdot\mathbf{v}_d \,-\, \mathbf{N}\left(\mathbf{u}_r\times\mathbf{v}_r + \mathbf{u}_d\times\mathbf{v}_d,\, \mathbf{u}_r\times\mathbf{v}_d + \mathbf{u}_d\times\mathbf{v}_r,\, 0,\, \lambda^k\right) \\ &=\, -\,\mathbf{u}\cdot\mathbf{v} \,-\, \mathbf{N}\left(\mathbf{u}_r\times\mathbf{v}_r + \mathbf{u}_d\times\mathbf{v}_d,\, \mathbf{u}_r\times\mathbf{v}_d + \mathbf{u}_d\times\mathbf{v}_r,\, 0,\, \lambda^k\right).\end{aligned} \tag{173}$$

As a result, we have

$$\begin{aligned}\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k)\,\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k)\right] &= -\,\mathbf{u}\cdot\mathbf{v} \,-\, \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathbf{N}\left(\mathbf{u}_r\times\mathbf{v}_r + \mathbf{u}_d\times\mathbf{v}_d,\, \mathbf{u}_r\times\mathbf{v}_d + \mathbf{u}_d\times\mathbf{v}_r,\, 0,\, \lambda^k\right)\right] \\ &= -\,\mathbf{u}\cdot\mathbf{v} \,-\, \lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\lambda^k\right]\mathbf{D}\left(\mathbf{u}_r\times\mathbf{v}_r + \mathbf{u}_d\times\mathbf{v}_d,\, \mathbf{u}_r\times\mathbf{v}_d + \mathbf{u}_d\times\mathbf{v}_r,\, 0\right) \\ &= -\,\mathbf{u}\cdot\mathbf{v} \,-\, 0 \,=\, -\,\mathbf{u}_r\cdot\mathbf{v}_r \,-\, \mathbf{u}_d\cdot\mathbf{v}_d\,,\end{aligned} \tag{174}$$

where $\mathbf{u}$ and $\mathbf{v}$ are unnormalized vectors. Using the constraints analogous to those expressed in Eq. (79), we then have

$$\lim_{m\,\rightarrow\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{T}_{\mathbf{x}\,\mathbf{x}'}(\lambda^k)\,\mathscr{T}_{\mathbf{y}'\,\mathbf{y}}(\lambda^k)\right] = \,-\,\mathbf{u}_r\cdot\mathbf{v}_r \,-\, \mathbf{u}_d\cdot\mathbf{v}_d \,=\, -\,(\mathbf{x}\times\mathbf{x}')\cdot(\mathbf{y}'\times\mathbf{y})\,, \tag{175}$$

upon using a vector identity. Consequently, substituting the above value in the variance inequality (168), it simplifies to

$$|\,\mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}) \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}') \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}) \,-\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')\,| \,\leqslant\, 2\sqrt{1 \,-\, (\mathbf{x}\times\mathbf{x}')\cdot(\mathbf{y}'\times\mathbf{y})}\,. \tag{176}$$

Finally, since trigonometry dictates the geometrical bounds $-1 \leqslant\, (\mathbf{x}\times\mathbf{x}')\cdot(\mathbf{y}'\times\mathbf{y}) \,\leqslant +1$, this inequality reduces to

$$|\,\mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}) \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}') \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}) \,-\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')\,| \,\leqslant\, 2\sqrt{2}\,, \tag{177}$$

exhibiting the bounds on all possible correlations. This result can also be derived directly from the correlations (151):

$$|\,\mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}) \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}') \,+\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}) \,-\, \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')\,| \,=\, |-\cos\theta_{\mathbf{x}\mathbf{y}} \,-\, \cos\theta_{\mathbf{x}\mathbf{y}'} \,-\, \cos\theta_{\mathbf{x}'\mathbf{y}} \,+\, \cos\theta_{\mathbf{x}'\mathbf{y}'}\,| \,\leqslant\, 2\sqrt{2}\,. \tag{178}$$

Let us stress again that these bounds are completely general, valid for any quantum state, such as the one in Eq. (88).

D. Fragility of the Strong Correlations Increases with the Number of Contexts

As we saw in Eq. (125), in the case of two contexts the scalar part of the product $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)$ is

$$-\,\cos\theta_{\mathbf{x}\mathbf{y}}(\mathbf{a},\,\mathbf{b}) \,=\, -\,\mathbf{a}_r\cdot\mathbf{b}_r \,-\, \mathbf{a}_d\cdot\mathbf{b}_d \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,=\, -\,\cos\theta_{\mathbf{a}\mathbf{b}}\,. \tag{179}$$

And it is this scalar part that captures the pattern of strong correlations exhibited by the singlet system. Analogously, for three contexts the scalar part of the product $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)$ works out to give

$$-\,\cos\theta_{\mathbf{x}\mathbf{y}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c}) \,=\, \mathbf{a}_r\cdot\{(\mathbf{b}_r\times\mathbf{c}_r) \,+\, (\mathbf{b}_d\times\mathbf{c}_d)\} \,+\, \mathbf{a}_d\cdot\{(\mathbf{b}_r\times\mathbf{c}_d) \,+\, (\mathbf{b}_d\times\mathbf{c}_r)\}\,, \tag{180}$$

with the geometric complexity of the scalar part now increased considerably. And for four contexts the scalar part of the product $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)$ works out to be even more intricate:

$$\begin{aligned}-\,\cos\theta_{\mathbf{x}\mathbf{y}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\, &(\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_d\cdot\mathbf{d}_d) \\ &-(\mathbf{a}_r\times\mathbf{b}_r + \mathbf{a}_d\times\mathbf{b}_d)\cdot(\mathbf{c}_r\times\mathbf{d}_r + \mathbf{c}_d\times\mathbf{d}_d) \,-\, (\mathbf{a}_r\times\mathbf{b}_d + \mathbf{a}_d\times\mathbf{b}_r)\cdot(\mathbf{c}_r\times\mathbf{d}_d + \mathbf{c}_d\times\mathbf{d}_r)\,,\end{aligned} \tag{181}$$

because

$$\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda)\;=\;-\,\mathbf{a}_r\cdot\mathbf{b}_r\;-\;\mathbf{a}_d\cdot\mathbf{b}_d\;-\;\mathbf{N}\left(\mathbf{a}_r\times\mathbf{b}_r+\mathbf{a}_d\times\mathbf{b}_d,\,\mathbf{a}_r\times\mathbf{b}_d+\mathbf{a}_d\times\mathbf{b}_r,\,0,\,\lambda\right) \tag{182}$$

and

$$\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda)\;=\;-\,\mathbf{c}_r\cdot\mathbf{d}_r\;-\;\mathbf{c}_d\cdot\mathbf{d}_d\;-\;\mathbf{N}\left(\mathbf{c}_r\times\mathbf{d}_r+\mathbf{c}_d\times\mathbf{d}_d,\,\mathbf{c}_r\times\mathbf{d}_d+\mathbf{c}_d\times\mathbf{d}_r,\,0,\,\lambda\right). \tag{183}$$

Needless to say, this pattern of increased geometrical complexity continues with the addition of each new context. As a result, the fragility of the strong correlations also increases rapidly with the number of contexts. This is easy to see already from the above scalar part for just four contexts. It is easy to see that even a slight change, such as $\mathbf{a}_r \pm \Delta\mathbf{a}_r$, in only one of the four contexts $\mathbf{a}_r + \mathbf{a}_d\,\varepsilon_+$ would lead to a dramatic change in the pattern of the corresponding correlation.

E. Reproducing the Strong Correlations Exhibited by the Four-Particle GHZ State

Now, as a second example of strong correlations, consider the four-particle Greenberger-Horne-Zeilinger state [8]:

$$|\Psi_{\mathbf{z}}\rangle\;=\;\frac{1}{\sqrt{2}}\Big\{|\mathbf{z},\,+\rangle_1\otimes|\mathbf{z},\,+\rangle_2\otimes|\mathbf{z},\,-\rangle_3\otimes|\mathbf{z},\,-\rangle_4\;-\;|\mathbf{z},\,-\rangle_1\otimes|\mathbf{z},\,-\rangle_2\otimes|\mathbf{z},\,+\rangle_3\otimes|\mathbf{z},\,+\rangle_4\Big\}. \tag{184}$$

Unlike the singlet state, this entangled state is not rotationally invariant [8]. There is a privileged direction, and it is taken to be the $\mathbf{z}$-direction of the experimental setup [8]. The $\mathbf{z}$-direction thus represents the axis of anisotropy of the system. The quantum mechanical expectation value of the product of the four outcomes of the spin components in this state – namely, the products of finding the spin of particle 1 along $\mathbf{a}$, the spin of particle 2 along $\mathbf{b}$, *etc.* – is given by

$$\mathcal{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d})\;:=\;\langle\Psi_{\mathbf{z}}|\,\boldsymbol{\sigma}\cdot\mathbf{a}\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{b}\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{c}\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{d}\,|\Psi_{\mathbf{z}}\rangle. \tag{185}$$

This expectation value has been calculated in the Appendix F of Ref. [8]. In the spherical coordinates – with angles such as $\theta_{\mathbf{a}}$ and $\phi_{\mathbf{a}}$ representing the polar and azimuthal angles, respectively, of the direction $\mathbf{a}$, $\mathbf{b}$, *etc.* – it works out to be

$$\mathcal{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d})\;=\;\cos\theta_{\mathbf{a}}\,\cos\theta_{\mathbf{b}}\,\cos\theta_{\mathbf{c}}\,\cos\theta_{\mathbf{d}}\;-\;\sin\theta_{\mathbf{a}}\,\sin\theta_{\mathbf{b}}\,\sin\theta_{\mathbf{c}}\,\sin\theta_{\mathbf{d}}\,\cos\left(\,\phi_{\mathbf{a}}\,+\,\phi_{\mathbf{b}}\,-\,\phi_{\mathbf{c}}\,-\,\phi_{\mathbf{d}}\,\right). \tag{186}$$

Our goal now is to reproduce this result within our locally causal framework described above. To this end, we note that the state (184) represents, not a two-level, but a four-level system. Each of the two pairs of the spin-1/2 particles it represents has four alternatives available to it. These alternatives can be represented by a state-vector of the form

$$|\psi\rangle\;=\;\gamma_1\,|\,+\,+\,\rangle\,+\,\gamma_2\,|\,+\,-\,\rangle\,+\,\gamma_3\,|\,-\,+\,\rangle\,+\,\gamma_4\,|\,-\,-\,\rangle\,, \tag{187}$$

where γ_1, γ_2, γ_3, and γ_4 are complex numbers satisfying the normalization $|\,\gamma_1\,|^2+|\,\gamma_2\,|^2+|\,\gamma_3\,|^2+|\,\gamma_4\,|^2=1$, which is equivalent to defining a unit 7-sphere, with $|\,\gamma_1\,|^2$, $|\,\gamma_2\,|^2$, $|\,\gamma_3\,|^2$, and $|\,\gamma_4\,|^2$ being the probabilities of actualizing the states $|\,+\,+\,\rangle$, $|\,+\,-\,\rangle$, $|\,-\,+\,\rangle$, and $|\,-\,-\,\rangle$, respectively. Therefore we may begin with four local maps of the form

$$S^7\ni\mathscr{A}(\mathbf{a}\,,\,\lambda^k):=\lim_{\substack{\mathbf{s}_{r1}\,\rightarrow\,\mathbf{a}_r\\ \mathbf{s}_{d1}\,\rightarrow\,\mathbf{a}_d}}\big\{-\,\mathbf{D}(\mathbf{a}_r,\,\mathbf{a}_d,\,0)\,\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\big\}=\begin{Bmatrix}+1 & \text{if} & \lambda^k\,=\,+1\\ -1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\text{ with }\Big\langle\,\mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\Big\rangle\;=\;0\,, \tag{188}$$

$$S^7\ni\mathscr{B}(\mathbf{b}\,,\,\lambda^k):=\lim_{\substack{\mathbf{s}_{r2}\,\rightarrow\,\mathbf{b}_r\\ \mathbf{s}_{d2}\,\rightarrow\,\mathbf{b}_d}}\big\{+\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k)\,\mathbf{D}(\mathbf{b}_r,\,\mathbf{b}_d,\,0)\,\big\}=\begin{Bmatrix}-1 & \text{if} & \lambda^k\,=\,+1\\ +1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\text{ with }\Big\langle\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\,\Big\rangle\;=\;0\,, \tag{189}$$

$$S^7\ni\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\;:=\lim_{\substack{\mathbf{t}_{r1}\,\rightarrow\,\mathbf{c}_r\\ \mathbf{t}_{d1}\,\rightarrow\,\mathbf{c}_d}}\big\{-\,\mathbf{D}(\mathbf{c}_r,\,\mathbf{c}_d,\,0)\,\mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k)\,\big\}=\begin{Bmatrix}+1 & \text{if} & \lambda^k\,=\,+1\\ -1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\text{ with }\Big\langle\,\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\,\Big\rangle\;=\;0\,, \tag{190}$$

and

$$S^7\ni\mathscr{D}(\mathbf{d}\,,\,\lambda^k):=\lim_{\substack{\mathbf{t}_{r2}\,\rightarrow\,\mathbf{d}_r\\ \mathbf{t}_{d2}\,\rightarrow\,\mathbf{d}_d}}\big\{+\mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k)\,\mathbf{D}(\mathbf{d}_r,\,\mathbf{d}_d,\,0)\,\big\}=\begin{Bmatrix}-1 & \text{if} & \lambda^k\,=\,+1\\ +1 & \text{if} & \lambda^k\,=\,-1\end{Bmatrix}\text{ with }\Big\langle\,\mathscr{D}(\mathbf{d}\,,\,\lambda^k)\,\Big\rangle\;=\;0\,, \tag{191}$$

together with their product

$$(\mathscr{A}_{\mathbf{a}}\,\mathscr{B}_{\mathbf{b}}\,\mathscr{C}_{\mathbf{c}}\,\mathscr{D}_{\mathbf{d}})(\lambda^k) \,=\, \mathscr{A}(\mathbf{a}\,,\,\lambda^k)\,\mathscr{B}(\mathbf{b}\,,\,\lambda^k)\,\mathscr{C}(\mathbf{c}\,,\,\lambda^k)\,\mathscr{D}(\mathbf{d}\,,\,\lambda^k) \,=\, \pm 1 \,\in\, S^7, \tag{192}$$

and the corresponding conservation laws

$$\mathbf{N}(\mathbf{s}_{r1},\,\mathbf{s}_{d1},\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{s}_{r2},\,\mathbf{s}_{d2},\,0,\,\lambda^k) \,=\, \left\{\mathbf{N}(\mathbf{s}_r,\,\mathbf{s}_d,\,0,\,\lambda^k)\right\}^2 \,=\, \mathbf{N}^2(\mathbf{s}_r,\,\mathbf{s}_d,\,0,\,\lambda^k) \,=\, -1 \tag{193}$$

and

$$\mathbf{N}(\mathbf{t}_{r1},\,\mathbf{t}_{d1},\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{t}_{r2},\,\mathbf{t}_{d2},\,0,\,\lambda^k) \,=\, \left\{\mathbf{N}(\mathbf{t}_r,\,\mathbf{t}_d,\,0,\,\lambda^k)\right\}^2 \,=\, \mathbf{N}^2(\mathbf{t}_r,\,\mathbf{t}_d,\,0,\,\lambda^k) \,=\, -1\,. \tag{194}$$

As we saw above, the expected value of the product of the outcomes $\mathscr{A}(\mathbf{a}\,,\,\lambda^k)$, $\mathscr{B}(\mathbf{b}\,,\,\lambda^k)$, $\mathscr{C}(\mathbf{c}\,,\,\lambda^k)$ and $\mathscr{D}(\mathbf{d}\,,\,\lambda^k)$ then works out to be the scalar part of the product $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)$, as spelled out in Eq.(181). Using a simple vector identity this expected value can be further simplified to take the form

$$\begin{aligned}\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\,\;& (\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_d\cdot\mathbf{d}_d)\\
&-\, (\mathbf{a}_r\cdot\mathbf{c}_r)(\mathbf{b}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_r)(\mathbf{a}_r\cdot\mathbf{d}_r) \,-\, (\mathbf{a}_r\cdot\mathbf{c}_d)(\mathbf{b}_r\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_d)(\mathbf{a}_r\cdot\mathbf{d}_d)\\
&-\, (\mathbf{a}_d\cdot\mathbf{c}_r)(\mathbf{b}_d\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_r)(\mathbf{a}_d\cdot\mathbf{d}_r) \,-\, (\mathbf{a}_d\cdot\mathbf{c}_d)(\mathbf{b}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_d)(\mathbf{a}_d\cdot\mathbf{d}_d)\\
&-\, (\mathbf{a}_r\cdot\mathbf{c}_r)(\mathbf{b}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_r)(\mathbf{a}_r\cdot\mathbf{d}_d) \,-\, (\mathbf{a}_r\cdot\mathbf{c}_d)(\mathbf{b}_d\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_d)(\mathbf{a}_r\cdot\mathbf{d}_r)\\
&-\, (\mathbf{a}_d\cdot\mathbf{c}_r)(\mathbf{b}_r\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_r)(\mathbf{a}_d\cdot\mathbf{d}_d) \,-\, (\mathbf{a}_d\cdot\mathbf{c}_d)(\mathbf{b}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_d)(\mathbf{a}_d\cdot\mathbf{d}_r).\end{aligned} \tag{195}$$

Upon using the constraints in Eq. (79) to set the terms involving $\mathbf{a}_r\cdot\mathbf{c}_d$ *etc.* to zero, this expected value reduces to

$$\begin{aligned}\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\,\;& (\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_d\cdot\mathbf{d}_d)\\
&-\, (\mathbf{a}_r\cdot\mathbf{c}_r)(\mathbf{b}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_r)(\mathbf{a}_r\cdot\mathbf{d}_r) \,-\, (\mathbf{a}_d\cdot\mathbf{c}_d)(\mathbf{b}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_d)(\mathbf{a}_d\cdot\mathbf{d}_d)\\
&-\, (\mathbf{a}_r\cdot\mathbf{c}_r)(\mathbf{b}_d\cdot\mathbf{d}_d) \,+\, (\mathbf{b}_d\cdot\mathbf{c}_d)(\mathbf{a}_r\cdot\mathbf{d}_r) \,+\, (\mathbf{b}_r\cdot\mathbf{c}_r)(\mathbf{a}_d\cdot\mathbf{d}_d) \,-\, (\mathbf{a}_d\cdot\mathbf{c}_d)(\mathbf{b}_r\cdot\mathbf{d}_r).\end{aligned} \tag{196}$$

Then, again using the constraints in Eq. (79) to identify $\mathbf{a}_r\cdot\mathbf{b}_r$ with $\mathbf{a}_d\cdot\mathbf{b}_d$, *etc.*, the expected value takes the form

$$\begin{aligned}\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\,\;& 2(\mathbf{a}_r\cdot\mathbf{b}_r)(\mathbf{c}_r\cdot\mathbf{d}_r) \,+\, 2(\mathbf{a}_d\cdot\mathbf{b}_d)(\mathbf{c}_d\cdot\mathbf{d}_d) \,-\, 2(\mathbf{a}_r\cdot\mathbf{c}_r)(\mathbf{b}_r\cdot\mathbf{d}_r)\\
&+\, 2(\mathbf{b}_r\cdot\mathbf{c}_r)(\mathbf{a}_r\cdot\mathbf{d}_r) \,+\, 2(\mathbf{b}_d\cdot\mathbf{c}_d)(\mathbf{a}_d\cdot\mathbf{d}_d) \,-\, 2(\mathbf{a}_d\cdot\mathbf{c}_d)(\mathbf{b}_d\cdot\mathbf{d}_d).\end{aligned} \tag{197}$$

Next, in order to satisfy the above constraints, we relate the external measurement directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}\in\mathrm{I\!R}^{\mathbf{3}}$, chosen freely by the experimenters, with the directions $\mathbf{a}_r$, $\mathbf{a}_d$, *etc.* within our representation space $S^7\subset\mathcal{K}^\lambda$, as follows:

$$\mathscr{A}(a_x,\,a_y,\,a_z,\,\lambda^k) \,=\, \pm 1 \,\in\, S^7 \text{ to be detected by } \mathbf{D}(\mathbf{a}_r;\,\mathbf{a}_d;\,0) \,=\, \mathbf{D}\left(-\frac{a_x}{\sqrt[4]{2}},\,+\frac{a_y}{\sqrt[4]{2}},\,0;\;\;0,\,0,\,-\frac{a_z}{\sqrt[4]{2}};\;\;0\right), \tag{198}$$

$$\mathscr{B}(b_x,\,b_y,\,b_z,\,\lambda^k) \,=\, \pm 1 \,\in\, S^7 \text{ to be detected by } \mathbf{D}(\mathbf{b}_r;\,\mathbf{b}_d;\,0) \,=\, \mathbf{D}\left(+\frac{b_x}{\sqrt[4]{2}},\,+\frac{b_y}{\sqrt[4]{2}},\,0;\;\;0,\,0,\,+\frac{b_z}{\sqrt[4]{2}};\;\;0\right), \tag{199}$$

$$\mathscr{C}(c_x,\,c_y,\,c_z,\,\lambda^k) \,=\, \pm 1 \,\in\, S^7 \text{ to be detected by } \mathbf{D}(\mathbf{c}_r;\,\mathbf{c}_d;\,0) \,=\, \mathbf{D}\left(+\frac{c_x}{\sqrt[4]{2}},\,+\frac{c_y}{\sqrt[4]{2}},\,0;\;\;0,\,0,\,+\frac{c_z}{\sqrt[4]{2}};\;\;0\right), \tag{200}$$

$$\mathscr{D}(d_x,\,d_y,\,d_z,\,\lambda^k) \,=\, \pm 1 \,\in\, S^7 \text{ to be detected by } \mathbf{D}(\mathbf{d}_r;\,\mathbf{d}_d;\,0) \,=\, \mathbf{D}\left(+\frac{d_x}{\sqrt[4]{2}},\,-\frac{d_y}{\sqrt[4]{2}},\,0;\;\;0,\,0,\,-\frac{d_z}{\sqrt[4]{2}};\;\;0\right). \tag{201}$$

Here the 4$^{\rm th}$ roots of 2 in the denominators of $\mathbf{D}$ $\left[\right.$instead of $\sqrt{2}$ as in Eq. (71)$\left.\right]$ arise because the product of four factors, $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{c}_r,\,\mathbf{c}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{d}_r,\,\mathbf{d}_d,\,0,\,\lambda^k)$, instead of two, $\mathbf{N}(\mathbf{a}_r,\,\mathbf{a}_d,\,0,\,\lambda^k)\,\mathbf{N}(\mathbf{b}_r,\,\mathbf{b}_d,\,0,\,\lambda^k)$, is involved in the calculation (147) of the correlation, while maintaining the unity of the radius of S^7. Note also that components of only external vectors are involved in the definitions of the four detectors. And they do not mix with each other, so that Bell's condition of local causality, or parameter independence [7], is strictly respected throughout. Substituting these coordinate values into the remaining vectors in the expected value (197) then reduces that value to

$$\begin{aligned}\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\, &+a_z\,b_z\,c_z\,d_z \,-\, a_y\,b_y\,c_y\,d_y \,-\, a_x\,b_y\,c_x\,d_y \,-\, a_y\,b_x\,c_y\,d_x \,-\, a_x\,b_x\,c_x\,d_x\\
&+\, a_x\,b_x\,c_y\,d_y \,-\, a_x\,b_y\,c_y\,d_x \,-\, a_y\,b_x\,c_x\,d_y \,+\, a_y\,b_y\,c_x\,d_x\,.\end{aligned} \tag{202}$$

In the spherical coordinates – with angles $\theta_{\mathbf{a}}$ and $\phi_{\mathbf{a}}$ representing respectively the polar and azimuthal angles of the direction $\mathbf{a}$, *etc.*, for all four measurement directions – this expression of the expected value can be further simplified to

$$\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) \,=\, \cos\theta_{\mathbf{a}}\,\cos\theta_{\mathbf{b}}\,\cos\theta_{\mathbf{c}}\,\cos\theta_{\mathbf{d}} \,-\, \sin\theta_{\mathbf{a}}\,\sin\theta_{\mathbf{b}}\,\sin\theta_{\mathbf{c}}\,\sin\theta_{\mathbf{d}}\,\cos\left(\,\phi_{\mathbf{a}} \,+\, \phi_{\mathbf{b}} \,-\, \phi_{\mathbf{c}} \,-\, \phi_{\mathbf{d}}\,\right). \tag{203}$$

This is exactly the quantum mechanical prediction (186) for the four-particle GHZ state (184). We have derived this prediction, however, as purely geometric effects within our locally causal framework. The GHZ correlations thus simply exhibit the classical, deterministic, local, and realistic correlations among four points of our representation space S^7.

III. CONCLUDING REMARKS

Any experiment in physics can be reduced to a series of elementary questions with possible "yes" or "no" answers. These answers in turn may be observed as "clicks" of event-detectors, as is usually done in the EPR-Bohm type correlation experiments [10]. When we compare such answers – possibly recorded by remotely located observers – we find that they are correlated in a remarkably disciplined manner, with the strength of the correlations exceeding the expectations based on Bell's theorem [7][14]. The natural question then is: Why are these answers correlated in such a disciplined manner when in quantum mechanics there appears to be no predetermined cause dictating the correlations? In this paper we have shown that the discipline and strength exhibited in the correlation experiments are natural consequences of the fact that the three-dimensional physical space in which all experiments are conducted respects the symmetries of a Clifford-algebraic 7-sphere, which arises from an associative interplay of the graded Euclidean primitives, such as points, lines, planes and volumes. These primitives provide the basis for the conformal geometry of the physical space, namely that of a quaternionic 3-sphere, S^3, embedded in an eight-dimensional Clifford-algebraic manifold, $\mathcal{K}^\lambda$. They allow us to understand the origins and strengths of *all* quantum correlations locally, as aspects of the geometry of the compactified physical space S^3, with $S^7 \subset \mathcal{K}^\lambda$ being its algebraic representation space. Thus every quantum correlation can be understood as a correlation among a set of points of this S^7. We have demonstrated this by proving a comprehensive theorem about the geometric origins of the correlations predicted by arbitrary quantum states:

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) = \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{A}(\mathbf{a},\,\lambda^k)\,\mathscr{B}(\mathbf{b},\,\lambda^k)\,\mathscr{C}(\mathbf{c},\,\lambda^k)\,\mathscr{D}(\mathbf{d},\,\lambda^k)\,\dots\right] = -\cos\theta_{\mathbf{xy}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots). \tag{204}$$

We have also proved within our framework that the strengths of these correlations are bounded by Tsirel'son's bounds:

$$|\,\mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}) + \mathcal{E}_{_{L.R.}}(\mathbf{x},\,\mathbf{y}') + \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}) - \mathcal{E}_{_{L.R.}}(\mathbf{x}',\,\mathbf{y}')\,| \leqslant 2\sqrt{1-(\mathbf{x}\times\mathbf{x}')\cdot(\mathbf{y}'\times\mathbf{y})} \leqslant 2\sqrt{2}\,. \tag{205}$$

We have then explicitly reproduced the strong correlations predicted by the EPR-Bohm state within our framework,

$$\mathcal{E}^{\rm EPR}_{_{L.R.}}(\mathbf{a},\,\mathbf{b}) = \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{A}(\mathbf{a},\,\lambda^k)\,\mathscr{B}(\mathbf{b},\,\lambda^k)\right] = -\cos\theta_{\mathbf{ab}}\;\;\text{with}\;\;\mathcal{E}^{\rm EPR}_{_{L.R.}}(\mathbf{n}) = \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{N}(\mathbf{n},\,\lambda^k)\right] = 0\,, \tag{206}$$

as well as explicitly reproduced the strong correlations predicted by the 4-particle Greenberger-Horne-Zeilinger state:

$$\begin{aligned}\mathcal{E}^{\rm GHZ}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d}) &= \lim_{m\,\to\,\infty}\left[\frac{1}{m}\sum_{k\,=\,1}^{m}\mathscr{A}(\mathbf{a},\,\lambda^k)\,\mathscr{B}(\mathbf{b},\,\lambda^k)\,\mathscr{C}(\mathbf{c},\,\lambda^k)\,\mathscr{D}(\mathbf{d},\,\lambda^k)\right]\quad\Big(\text{together with}\,\Big\langle\mathscr{N}(\mathbf{n},\,\lambda^k)\Big\rangle = 0\Big)\\ &= \cos\theta_{\mathbf{a}}\cos\theta_{\mathbf{b}}\cos\theta_{\mathbf{c}}\cos\theta_{\mathbf{d}} - \sin\theta_{\mathbf{a}}\sin\theta_{\mathbf{b}}\sin\theta_{\mathbf{c}}\sin\theta_{\mathbf{d}}\cos(\phi_{\mathbf{a}}+\phi_{\mathbf{b}}-\phi_{\mathbf{c}}-\phi_{\mathbf{d}})\,.\end{aligned} \tag{207}$$

The comprehensive theorem we have proved dictates that — at least in principle — it is always possible to locally reproduce the strong correlations predicted by any arbitrary quantum state. The *raison d'être* for the strength of the correlations turns out to be the non-trivial twist in the Hopf bundle of S^3 [1][5], or in its algebraic representation space S^7. Given the fact that we started out our analysis with the most primitive elements of the physical space in the spirit of Euclid's elements for geometry, our demonstration suggests that the quantum correlations observed in Nature are best viewed as consequences of spacetime, rather than spacetime as an emergent property of quantum entanglement.

[1] J. Christian, *Disproof of Bell's Theorem: Illuminating the Illusion of Entanglement*, Second Edition (Brwonwalker Press, Boca Raton, Florida, 2014); See also arXiv:0904.4259 (2009), arXiv:1106.0748 (2011) and arXiv:1203.2529 (2012).
[2] P-P. Dechant, *The Birth of E_8 Out of the Spinors of the Icosahedron*, Proc. R. Soc. A **472**, 20150504 (2016).
[3] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).
[4] L. Dorst, D. Fontijne and S. Mann, *Geometric Algebra for Computer Science* (Elsevier, Amsterdam, 2007).
[5] J. Christian, *Macroscopic Observability of Spinorial Sign Changes under 2π Rotations*, International Journal of Theoretical Physics, DOI 10.1007/s10773-014-2412-2; See also the last two appendices of arXiv:1211.0784 as well as arXiv:1501.03393.
[6] J. W. Milnor, *Topology from the Differentiable Viewpoint* (Princeton University Press, Princeton, New Jersey, 1997).
[7] J. S. Bell, Physics **1**, 195 (1964); *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1987).
[8] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[9] J. Christian, *Local Causality in a Friedmann-Robertson-Walker Spacetime*, `http://arXiv.org/abs/1405.2355` (2014-2017).
[10] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955).
[11] L. H. Ryder, J. Phys. A **13**, 437 (1980); D. W. Lyons, Mathematical Magazine **76**, 87 (2003).
[12] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Physics Reports **66**, No. 6, 213 (1980) [cf. page 272, Eq. (4.21)].
[13] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (Jonathan Cape, London, 2004).
[14] J. Christian, *On a Surprising Oversight by John S. Bell in the Proof of his Famous Theorem*, arXiv:1704.02876 (2017).