Additivity Requirements in Classical and Quantum Probability

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The discussion of different principles of additivity (finite vs. countable vs. complete additivity) for probability functions has been largely focused on the personalist interpretation of probability. Very little attention has been given to additivity principles for physical probabilities. The form of additivity for quantum probabilities is determined by the algebra of observables that characterize a physical system and the type of quantum state that is realizable and preparable for that system. We assess arguments designed to show that only normal quantum states are realizable and preparable and, therefore, quantum probabilities satisfy the principle of complete additivity. We underscore the little remarked fact that unless the dimension of the Hilbert space is incredibly large, complete additivity in ordinary non-relativistic quantum mechanics (but *not* in relativistic quantum field theory) reduces to countable additivity. We then turn to ways in which knowledge of quantum probabilities may constrain rational credence about quantum events and, thereby, constrain the additivity principle satisfied by rational credence functions.

1   Introduction

The merits of different forms of additivity—finite vs. countable vs. complete additivity—for probability functions has received a good deal of attention in the annals of statistics and decision theory, with philosophers occasionally weighing in. The first impression of the literature is that the discussion is so extensive and detailed that little remains to be said, except for a quibble here

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and a footnote there. And yet on further reflection there is an obvious gap in the literature. Most of the considerations adduced are either explicitly aimed at, or are most plausibly construed as aimed at, the personalist interpretation of probability, e.g. it is shown that a clever bookie can take advantage of an agent whose degrees of belief fail to satisfy a certain form of additivity. What is largely missing is a discussion of the form of additivity satisfied by physical probabilities.

Bruno de Finetti, the patron saint of the personalist interpretation of probability, is famous for denying that there is any such thing as physical probability: "THERE ARE NO PROBABILITIES" (de Finetti 1974, Vol. 1, p. x) was his bombastic way of staking out his position. What is undeniable is that fundamental theories of physics, e.g. quantum mechanics (QM), attribute probabilities to physical systems; and while it is a nice metaphysical question as to whether or not these probabilities codify objective chances, it is undeniable that the outcome statistics in experiment after experiment correspond with great accuracy to the probabilities assigned by the theory, making it hard not to believe that the probabilities of QM reflect objective features of physical systems. But, it might be said, even granting for sake of argument that there are physical probabilities, there is nothing further to discuss: physical probabilities—and the form of additivity they satisfy—are what Nature says there are, and it is to no avail to tell Nature that she should have used some other form of additivity.

This glib response misses two points. First, there is much to be discussed in connection with the additivity properties of quantum probabilities since these properties are closely connected with the still unsettled issue of which quantum states are physically realizable. As will be seen below, this is not a straightforward empirical issue that can be decided by experimentalists. Second, the additivity properties of quantum probability have implications for the additivity properties of the credence functions of rational agents especially if, as a number of philosophers have claimed, a knowledge of objective physical probability constrains rational credence.

Our treatment of these issues begins in Section 2 with a review of the considerations adduced in the literature to motivate or justify the various forms of additivity in the context of the personalist interpretation of classical probability theory. Section 3 introduces quantum probability theory in terms of the algebraic formulation of QM. All three forms of additivity—finite, countable, and complete additivity—are exhibited by quantum probabilities depending on the algebra of observables and the nature of the allowable states

on the algebra. But we underscore the little remarked fact that unless the dimension of the Hilbert space is incredibly large, complete additivity in ordinary non-relativistic (but *not* in relativistic quantum field theory (QFT)) reduces to countable additivity. Section 4 assesses the presupposition of much of the standard practice of QM that only normal quantum states are physically realizable and, therefore, that quantum probabilities are completely additive. Section 5 discusses different ways in which quantum probabilities might constrain rational credence about quantum events and, thereby, the form of additivity a rational credence function over these events should satisfy. Conclusions are presented in Section 6.

2   Additivity requirements in classical probability

The philosophical literature on additivity requirements, as well as the statistics literature that is cited therein, is focused almost entirely on classical probability. Without any pretence at either rigor or completeness, the present section attempts to give the reader the gist of some of the main arguments pro and con regarding competing additivity principles for classical probability. Following the standard Kolmogorov axioms (Kolmogorov 1956), a classical probability space $(\Omega, \mathfrak{F}, \mathrm{Pr})$ consists of a non-empty set Ω, a set $\mathfrak{F}$ of subsets of Ω, and a map Pr from $\mathfrak{F}$ to the non-negative reals satisfying

   (A1) $\mathfrak{F}$ is a field[1] containing Ω

   (A2) $\mathrm{Pr}(\Omega) = 1$

   (A3) If $A, B \in \mathfrak{F}$ and $A \cap B = \emptyset$ then $\mathrm{Pr}(A+B) = \mathrm{Pr}(A) + \mathrm{Pr}(B)$.

The axiom (A3) is the requirement of finite additivity. It is generally agreed that any function that deserves to be called a probability function must exhibit at least these formal properties. What further properties, if any, a probability function should satisfy is a matter of controversy that turns in

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[1]To be a field $\mathfrak{F}$ must be closed under the operations of complementation, finite products (intersections) and sums (unions), where for $A, B \in \mathfrak{F}$ complementation, product, and sum are defined respectively as $\overline{A} := \Omega - A$, $AB := A \cap B$, and $A + B := A \cup B$. For the requirements of countable (respectively, complete additivity) to be meaningful, $\mathfrak{F}$ must be closed under countable (respectively, arbitrary) products and sums.

part on the interpretation of probability. We first survey arguments designed to show that degrees of belief (aka credences), if rational, should conform to axioms (A1)-(A3).

2.1 Arguments for finite additivity

The most well-known justification for (A1)-(A3) as rationality constraints on degrees of belief is the Dutch Book argument. Think of the elements A_a $\in \mathfrak{F}$, $a \in \mathcal{I}$ (index set), as standing for propositions to which some agent assigns degrees of belief codified in a credence function Cr on $\mathfrak{F}$. The idea is that Cr should serve as that agent's fair betting quotient when the stakes of the bets are small enough that considerations of risk aversion/seeking can be set aside. A bookie sets the stakes S_a for elements $A_a \in \mathfrak{F}$. An agent who bets on A_a pays P_a to the bookie, and in return receives S_a from the bookie if A_a is true and nothing if A_a is false.[2] The agent regards the bet as fair (respectively, favorable) just in case the amount P_a she has to pay for the bet is equal to (respectively, less than) the expected payoff $Cr(A_a)S_a$. Finite Dutch Book can be made against the agent if there is a finite family of bets, each of which the agent regards as fair (or favorable), but the net result of which is that the agent is guaranteed to lose money come what may (i.e. whatever the truth values of the propositions at stake). De Finetti (1974, Vol. 1, Ch 3) showed that the agent is immune to finite Dutch Book just in case Cr is a finitely additive probability function.

As interesting as this result is, de Finetti himself did not find it entirely satisfactory, both because of the issue of risk version/seeking and because he thought that the strategizing involved in the bookie-bettor negotiations should not be part of the concept of the rationality of belief (see de Finetti 1981). He therefore constructed a second argument that is independent of gambling considerations and more focused on how well a credence function scores in terms of tracking the truth. Let $\{A_a\}$, $a \in \mathcal{I}$, be a disjoint subset of $\mathfrak{F}$, i.e. for any pair $A_b, A_c \in \{A_a\}$, $A_b \cap A_c = \emptyset$ if $b \neq c$. If in addition $\sum_{a \in \mathcal{I}} A_a = \Omega$ then $\{A_a\}$ is said to be a partition of $\mathfrak{F}$. Define the indicator function t_W for possible world W by $t_W(A_a) := 1$ if A_a is true in W and 0 otherwise.[3] One way of scoring how the credence function Cr errs in tracking

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[2] If S_a is negative then the agent is selling the bet.

[3] For present purposes a possible world is any consistent assignment of truth values to elements of $\mathfrak{F}$.

the truth in a partition $\{A_a\}$ is the squared-error or Brier loss function

$$BL_W(Cr) := \sum_{a \in \mathcal{I}} (t_W(A_a) - Cr(A_a))^2.$$

Say that Cr is weakly (respectively, strongly) dominated by another credence function Cr' just in case $BL_W(Cr') \leq BL_W(Cr)$ for all W and $BL_W(Cr') < BL_W(Cr)$ for some W (respectively, $BL_W(Cr') < BL_W(Cr)$ for all W). The implicit suggestion here is that to be rational an agent who realizes she has a dominated credence function should, ceteris paribus, shift to an undominated credence function.[4] De Finetti (1974, Vol. 1, Ch. 3) showed that a credence function Cr is undominated in finite partitions just in case it is a probability function satisfying axioms (A1)-(A3). Moreover, this result is robust in the sense that it has been shown to hold for a large class of alternative scoring rules (see Joyce 1998; Schervish, Seidenfeld, and Kadane 2009).

As impressive as the Dutch Book and scoring rules arguments are they only serve at best to show that a rational credence should satisfy finite additivity as a minimum and not that rational credence need not satisfy some stronger additivity requirement. In Chapter 2 of *Foundations of the Theory of Probability* Kolmogorov (1956) considered an additional axiom of continuity that is equivalent to the requirement of countable additivity. Let $\{A_a\}$, $a \in \mathcal{I}$, be a disjoint subset of $\mathfrak{F}$ where the index set $\mathcal{I}$ is countable. Then the axiom of countable additivity demands that

(A4) $\Pr(\sum_{a \in \mathcal{I}} A_a) = \sum_{a \in \mathcal{I}} \Pr(A_a)$.

Thus, if Pr is countably additive and $\{A_a\}$ is a partition of $\mathfrak{F}$ it follows that $\sum_{a \in \mathcal{I}} \Pr(A_a) = 1$.

De Finetti was morally outraged by two consequences of the requirement of countable additivity. Consider a lottery that sells a countably infinite number of tickets. Countable additivity prevents an agent from assigning equal probabilities to each of the mutually incompatible propositions that ticket #n, $n = 1, 2, 3, ...$ will win. An agent whose credence function is merely finitely additive can make such an assignment; namely, she can assign each ticket a zero probability. Second, countable additivity is a key ingredient

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[4]But other things might not be equal: the agent might have reasons for not shifting to some particular non-dominated credence functions.

in the phenomenon of non-measurable events: every merely finitely additive probability measure on a set can always be extended to the power set of said set; but the use of the axiom of choice shows that such an extension is not always possible if the measure is countably additive. On de Finetti's personalist reading of probability the phenomenon of non-measurable events means that there are events (corresponding to subsets of Ω) to which agents are prevented by countable additivity from assigning degrees of belief, something he found repugnant.

Moral outrage aside, are there other considerations in favor of sticking with finite additivity and not moving to countable additivity? Here are three. First, there are some versions of decision theory, such as that of Savage (1972), that operate with mere finite additivity. Second, Kadane, Schervish, and Seidenfeld (1986) argue that various anomalies and paradoxes of Bayesian statistics can be resolved by the use of so-called improper priors and that such priors can be interpreted in terms of finitely additive probabilities. Third, the scoring rule argument does not justify moving from finite to countable additivity; for it is not the case that any merely finitely additive credence function is weakly dominated in terms of Brier score by some countably additive credence function. As an example let $\{A_a\}$, $a \in \mathcal{I}$, be a countable atomic partition of $\mathfrak{F}$, and let Cr_f be merely finitely additive credence function that assigns 0 to each atom A_a of the partition.[5] Then $BL_W(Cr_f) = 1$ for all W but (as is easily checked) there is no countably additive Cr_c such that $BL_W(Cr_c) \leq 1$ for all W.[6] Yet more considerations

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[5]That A_a is an atom means that there is no $A' \in \mathfrak{F}$ such that A' is a proper subset of A_a. Where does the non-zero credence of the Cr_f in question reside? By finite additivity, any finite union of the atoms in the atomic partition gets zero Cr_f credence. Credence is, therefore, diffuse, i.e. non-zero credence resides only on infinite unions of atoms. It is this diffuseness that allows Cr_f to escape domination on the square-error criterion. One might take this example to show that immunity from domination on the squared-error criterion is not much of a merit badge.

[6]Note that

$$BL_W(Cr_c) = \sum_{a \neq a_W} (Cr_c(A_a))^2 + (1 - Cr_c(A_{a_W}))^2$$
$$= \sum_a (Cr_c(A_a))^2 + 1 - 2Cr_c(A_{a_W})$$

where a_W is the number of the element A_{a_W} of the partition that is true in W. For this loss to be less than or equal to 1 in W it must be the case that

in favor of mere finite additivity are to be found in Seidenfeld (2001).

2.2 Arguments for countable additivity

As mentioned above, Kolmogorov (1956) considered an axiom of continuity. Let $A_a \in \mathfrak{F}$, $a = 1, 2, 3, ...$, be a countable sequence descending to $\emptyset$, i.e. $A_1 \supseteq A_2 \supseteq ...$ and $\cap_{a=1}^{\infty} A_a = \emptyset$. Continuity at $\emptyset$ requires that for any such sequence

(A5) $\lim_{a \to \infty} \Pr(A_a) = 0.$

Axiom (A5) is provably equivalent to countable additivity (see, for example, Halmos 1950, Theorem 9F). Thus, considerations in favor of continuity are ipso facto considerations in favor of countable additivity. One way to argue for continuity and, thus, countable additivity is to maintain that a failure of continuity is an indication that the probability space is defective in that it should be regarded as a subset of some larger space (see Kingman 1967). The argument would have to be implemented by showing that the non-continuous Pr is extendible to a continuous $\Pr'$ on the field $\mathfrak{F}'$ of a larger space Ω' of events and, just as importantly, that the elements of $\mathfrak{F}' \backslash \mathfrak{F}$ are genuine events to which probability should be assigned and are not mere mathematical constructs. While one can think of circumstances where such an argument scheme can be implemented on the personalist interpretation of probability, there does not seem to be any reason to think that it offers a general justification for the personalist to adopt countable additivity. Physical probability is a different matter, and the effectiveness of continuity considerations for such probabilities depend on the details of the physics that grounds the probabilities. The case for continuity of quantum probabilities will be examined below in Section 4.2.

$$Cr_c(A_{a_W}) \geq \sigma > 0, \ \ \sigma := \frac{\sum_a (Cr_c(A_a))^2}{2}.$$

If we try to suppose that this can hold for all W a contradiction results. For the number of each element of the partition "comes up" in some W so the inequality must hold for each element in the partition. Summing over the partition it follows from countable additivity that $\sum_a Cr_c(A_a) = 1$ is greater than or equal to to $n\sigma$ for any value of n. Whether or not a finitely additive credence function can escape domination for other scoring rules remains to be seen.

Perhaps more effective from the personalist perspective is the Dutch Book argument which can be generalized to cover a countably infinite family of bets. Agree that to be realistic the stakes in such an infinite betting scheme must be such that the total amount of money $\sum_{a \in \mathcal{I}} |Cr(A_a)S_a|$, $\mathcal{I}$ a countable set, that initially changes hands must be finite. Even with this proviso in effect it is provable that within such infinite betting schemes an agent escapes Dutch Book just in case her credence function obeys (A1)-(A4). De Finetti was aware of this fact, but in some places he seems to opine that using infinite betting schemes to justify countable additivity is question begging (see de Finetti 1972, p. 91), while in other places he seems to deprecate infinite betting schemes on verificationist/operationalist grounds (see Mura 2008, pp. 114-117).[7]

Arntzenius, Elga, and Hawthorne (2004) side with de Finetti in holding that Dutch Book arguments have no force for infinite bets and, therefore, do not provide a justification for countable additivity. Their complaint centers on what they call the Bet Agglomeration Principle (BAP) which requires that if an agent judges each bet in a package of bets to be acceptable then the agent is committed to judging the whole package to be acceptable. By rehearsing cautionary tales about fallacies that result when trying to carry reasoning about finite collections over to infinite collections, Arntzenius et al. attempt to convince the reader that (BAP) lacks plausibility for an infinite package of bets. While caution in moving from reasoning about the finite to reasoning about the infinite is well advised, we feel that whatever force the original Dutch Book argument has in the finite case is not lost in the passage to the infinite case. In both cases the an agent judges whether a bet is fair (respectively, favorable or unfavorable) according as her expected value for the bet is 0 (respectively, > 0 or < 0).[8] A sure loss for a finite package of bets (leading to a dominance argument against buying the package) even though each bet in the package has 0 (or > 0) expected value for the agent is supposed to signal that the agent's credences are incoherent; and if axioms (A1) and (A2) hold for the agent's credence function, then the blame for the incoherence lies squarely with the failure of finite additivity. If this analysis has force for finite case it seems to have equal for the infinite case: a sure loss

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[7]See Howson (2008) for an overall assessment of de Finetti's position on coherence and Dutch Book.

[8]That the agent's value or utility function is linear for small stakes is assumed in the Dutch Book construction.

for a countably infinite package of bets (leading to a dominance argument against buying the package) even though each bet in the package has 0 (or > 0) expected value for the agent signals that the agent's credences are incoherent; and if axioms (A1)-(A3) hold for the agent's credence function, the blame for the incoherence lies squarely with the failure of countable additivity. As far as we can discern, there is nothing to indicate that the incoherence in the infinite case is due to peculiarities of infinities rather than to the violation of countable additivity.

A different but related line of justification for countable additivity comes from the concept of conglomerability. A fundamental result of classical probability is that any probability function Pr satisfying (A1)-(A3) is extendible to full conditional probability. A full conditional $\widetilde{\mathrm{Pr}}$ is a real valued function on $\mathfrak{F}\mathrm{x}\mathfrak{F}^0$, with $\mathfrak{F}^0$ consisting of the non-null elements of $\mathfrak{F}$, satisfying

(i) for $A \in \mathfrak{F}^0$, $\widetilde{\mathrm{Pr}}(\bullet/A)$ is a probability on $\mathfrak{F}$ and $\widetilde{\mathrm{Pr}}(A/A) = 1$

(ii) for $A \subset B \subset C \subset \Omega$, $B \in \mathfrak{F}^0$, $\widetilde{\mathrm{Pr}}(A/C) = \widetilde{\mathrm{Pr}}(A/B)\widetilde{\mathrm{Pr}}(B/C)$.

That $\widetilde{\mathrm{Pr}}$ extends Pr means that $\mathrm{Pr}(A) = \widetilde{\mathrm{Pr}}(A/\Omega)$ for all $A \in \mathfrak{F}$. If Pr is merely finitely additive the extension may not be unique. It is easy to verify that if $\widetilde{\mathrm{Pr}}$ extends Pr and $\mathrm{Pr}(B) \neq 0$ then $\widetilde{\mathrm{Pr}}(A/B) = \mathrm{Pr}(A/B) :=$ $\mathrm{Pr}(A{\cap}B)/\mathrm{Pr}(B)$, per the standard definition of conditional probability. If $\widetilde{\mathrm{Pr}}$ is a full conditional probability that extends Pr and $\mathfrak{P} = \{A_a\}$ is a partition of $\mathfrak{F}$ then $\widetilde{\mathrm{Pr}}$ is said to be conglomerable with respect to $\mathfrak{P}$ just in case for all $B \in \mathfrak{F}$

$$\mathrm{sup}_{A_a} \widetilde{\mathrm{Pr}}(B/A_a) \geq \mathrm{Pr}(A) \geq \mathrm{inf}_{A_a} \widetilde{\mathrm{Pr}}(B/A_a).$$

A $\widetilde{\mathrm{Pr}}$ that is conglomerable with respect to every partition is said to be conglomerable simpliciter.

Consider the situation of an agent with a non-conglomerable credence function Cr satisfying (A1)-(A3). There is an extension of Cr to a full conditional $\widetilde{Cr}$, a partition $\mathfrak{P} = \{A_a\}$, and a $B \in \mathfrak{F}$ such that $Cr(B) < \widetilde{Cr}(B/A_a)$ for all $A_a \in \mathfrak{P}$. The agent knows that exactly one element of the partition will eventuate, and he judges the conditional probability of B on each and every eventuality to be greater than the unconditional probability. This strikes many commentators as bizarre if not outright irrational. In addition there are pragmatic reasons for demanding conglomerability for credence

functions; namely, a violation of conglomerability makes an agent vulnerable to money pump constructions (see Seidenfeld and Schervich 1983): the agent is sure to lose if he makes an unconditional bet on B with $Cr(B)$ as fair betting quotient as well as a package of conditional bets on B for each $A_a \in \mathfrak{P}$ with $\widetilde{Cr}(B/A_a)$ as fair betting quotient.[9]

Now comes the plug for countable additivity: a Pr satisfying (A1)-(A3) is countably additive just in case it is conglomerable with respect to every countable partition (see Hill and Lane 1985; Schervish, Seidenfeld, and Kadane 1984); so satisfying countable additivity makes an agent proof against non-conglomerability in countable partitions while violating it guarantees that there exists an extension to a full conditional probability and a partition on which conglomerability fails. Nevertheless, de Finetti, who first investigated the phenomenon of conglomerability, did not think that conglomerability is an essential requirement for personal probability. The status of the requirement remains unsettled in the statistics literature, while some philosophers (e.g. Arntzenius, Elga, and Hawthorne 2004) advocate tolerance for violations.

Suppose now that, unlike de Finetti, you are persuaded of the need to embrace countable additivity. But why stop there? Suppose that $\mathfrak{F}$ admits uncountable partitions. That the summation formula $\Pr(\sum_{a \in \mathcal{I}} A_a) = \sum_{a \in \mathcal{I}} \Pr(A_a)$ of (A4) holds for all disjoint subsets $\{A_a\}$ of $\mathfrak{F}$ with $card(\mathcal{I}) = \kappa$ $(\kappa > \aleph_0)$ is the requirement of κ-additivity.[10] Why not adopt κ-additivity? Indeed, why not go all the way to complete additivity, the requirement that the summation formula holds for all disjoint subsets of whatever cardinality?

2.3 Arguments for complete additivity

Continuity arguments for countable additivity can be generalized to arguments for complete additivity. But by the same token the effectiveness of the latter when applied to personalist probabilities is subject to the same qualms and limitations as the former; and correspondingly the effectiveness of the latter when applied to physical probabilities depends on the details of the

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[9] A bet on B conditional on A_a is called off if A_a is false but proceeds as an ordinary bet on B if A_a is true.

[10] The sum on the rhs is understood as the supremum over all finite subsets $\mathcal{F} \subset \mathcal{I}$ of $\sum_{a \in \mathcal{F}} \Pr(A_a)$. It is easy to verify that at most a countable infinity of the A_a can be assigned positive probability.

physical theory in which these probabilities are embedded. Looking ahead, quantum theory does provide a physical basis for the continuity underlying completely additive quantum probabilities (see Section 4.2).

Sticking to the case of classical probabilities, insofar as conglomerability provides a motivation for countable additivity it also provides a motivation for complete additivity for personalist probabilities. If $\mathfrak{F}$ admits uncountable partitions then in general a countably but not completely additive Pr will exhibit non-conglomerability on uncountable partitions (see Kadane, Schervish, and Seidenfeld 1986). The proof that countable additivity entails conglomerability on every countable partition generalizes to show that complete additivity entails conglomerability on all partitions. It seems reasonable to conjecture that the converse implication also holds.

The application of the Dutch Book argument to the uncountable case is more delicate. The generalized Dutch Book argument in the preceding subsection applied to a countable family of bets applies equally to an uncountable family *if* it is required that the agent stands ready to accept any bet whose price she regards as exactly fair. But it does not apply to an uncountable family of bets if the agent is only required to accept bets that she regards as strictly favorable (see Skyrms 1992 and Easwaran 2013). The reason is that subtracting a sweetener $p_a > 0$ from the amount $P_a = Cr(A_a)S_a$ the bettor regards as an exactly fair price for the bet in order to make it favorable results in a violation of the condition $\sum_{a \in \mathcal{I}} |P_a - p_a| < \infty$ for the initial exchange of money to be finite when $\mathcal{I}$ is uncountable. The issue then turns on the fair vs. favorable form of bets as a means of eliciting degrees of belief, which seems more a question of psychology than a question of rationality of belief.

Easwaran (2013) offers a principle which is supposed to support countable but not complete additivity:

> (E) If $\mathfrak{P} = \{A_a\}$, $a \in \mathcal{I}$, is a partition of $\mathfrak{F}$ and Pr_1 and Pr_2 are two probability functions then it cannot be the case that $\mathrm{Pr}_2(A_a) > \mathrm{Pr}_1(A_a)$ for all $A_a \in \mathfrak{P}$.

As an example of how (E) operates, suppose that $\mathcal{I} = \mathbb{N}^+$ and consider two real valued set functions: the merely finitely additive Pr_1 that assigns $\mathrm{Pr}_1(A_n) = 0$ for all $n \in \mathbb{N}^+$ and the countably additive set function Pr_2 that assigns $\mathrm{Pr}_2(A_n) = \dfrac{1}{2n}$ for $n \in \mathbb{N}^+$. (E) is violated if both Pr_1 and Pr_2 count

as probability functions. Then the idea is to demonstrate that this example can be generalized to show that if Pr_1 is any merely finitely additive set function then there is a countably additive set function Pr_2 such that (E) is violated if both Pr_1 and Pr_2 are probability functions. Easwaran takes it as a given that a countably additive set function is a probability function and, thus, he concludes from (E) that a merely finitely additive set function is not a probability function. And he goes on to argue that (E) does not serve to undergird complete additivity because it does not follow that if Pr_1 is any merely countably additive set function then there is a completely additive additive set function Pr_2 such that (E) is violated if both Pr_1 and Pr_2 are probability functions.

The force of this line of argumentation is unclear since it is unclear what should be included in the concept of a probability function other than a real valued set function that satisfies some form of additivity, which form being open to dispute. As regards personal probability one wants guidance about what form of additivity an agent should adopt either by appeal to prudential considerations or by appeal to constraints on rationality of belief. Principle (E) does not provide such guidance. As applied to physical probabilities, principle (E) amounts to an a priori metaphysical postulate about what chances can be realized in nature. Here Easwaran shares Pruss's (2012) intuition that Pr_1 and Pr_2 violating (E) cannot both give possible chances for some lottery:

> [S]urely there cannot be a lottery with the same tickets as [another] lottery, and yet still with every ticket being much more likely to win (Easwaran 2013, p. 58).

Nature has a way of upsetting a prioristic metaphysics. As will be seen below, if non-normal as well as normal quantum states are physically realizable then both lotteries in which each of a countably infinite number of tickets has a zero chance of winning and lotteries in which each ticket has a non-zero chance of winning are physically possible. On the other hand, if only normal quantum states are physically realizable then the intuition and principle (E) are safe; but this is because the probability measures induced by these states are completely additive, upsetting Easwaran's goal of supporting countable but not complete additivity.

Switching back to personal probability Easwaran opines that

[I]f we think of the partition as defining a set of incompatible [and exhaustive] scientific hypotheses we are uncertain about, rather than as lottery tickets, then a violation of this principle [(E)] would mean that an update from Pr_1 to Pr_2 would confirm *every* alternative (Easwaran 2013, p. 58),[11]

which is taken to be problematic. But there is a more fundamental problem at work here. Getting from the above Pr_1 to Pr_2 by updating in the form of Bayesian conditionalization is impossible since such updating will not convert a merely finitely additive set function to a countably additive one or vice versa. Analogs of this problem will come up below in the context of quantum probability.

We close by noting that while the debate over countable vs. complete additivity is instructive, it can be an empty debate even when partitions of the set of measurable events are uncountable. Recall that κ is a measurable cardinal means that there is probability space $(\Omega, \mathfrak{F}, \mathrm{Pr})$ where $card(\Omega) = \kappa$, $\mathfrak{F} = P(\Omega)$ (the power set of Ω), and $\mathrm{Pr}(\{\omega\}) = 0$ for all $\omega \in \Omega$. It is easy to establish

> *Lemma 1.* Consider a probability space $(\Omega, \mathfrak{F}, \mathrm{Pr})$ where $\mathfrak{F} = P(\Omega)$. Countable additivity for Pr implies complete additivity just in case $card(\Omega)$ is less than the least measurable cardinal.

Proof: Suppose that $card(\Omega)$ is a measurable cardinal. Then by definition there is a countably additive Pr on $P(\Omega)$ such that $\mathrm{Pr}(\{\omega\}) = 0$ for all $\omega \in \Omega$. If Pr were completely additive then a contradiction would result since $1 = \mathrm{Pr}(\Omega) = \mathrm{Pr}(\cup_{\omega \in \Omega}\{\omega\}) = \sum_{\omega} \mathrm{Pr}(\{\omega\}) = 0$. Conversely, suppose that $card(\Omega)$ is less than the first measurable cardinal and that Pr is a countably additive probabilty on $P(\Omega)$. By hypothesis $\mathrm{Pr}(\{\omega\}) = 0$ cannot hold for all $\omega \in \Omega$. But only a countable number of singletons can have positive probability. Let C be the set of all $\omega \in \Omega$ such that $\mathrm{Pr}(\{\omega\}) > 0$. Then C is a countable set such that $\mathrm{Pr}(C) > 0$. If $\mathrm{Pr}(C) = 1$ then intersecting C with arbitrary subsets of Ω quickly leads to the conclusion that Pr is completely additive. So suppose that $\mathrm{Pr}(C) < 1$ and, thus, $\mathrm{Pr}(\overline{C}) > 0$. Define $\widehat{\mathrm{Pr}}(\bullet) := \mathrm{Pr}(\bullet \cap \overline{C})/\mathrm{Pr}(\overline{C})$. Note that $\widehat{\mathrm{Pr}}$ is a countably additive

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[11]Italics in original. We have altered Easwaran's notation to conform to ours.

probability on $P(\Omega)$ and that $\widehat{\Pr}(\{\omega\}) = 0$ for all $\omega \in \Omega$, contradicting the assumption that $card(\Omega)$ is less than the least measurable cardinal.[12]

Ulam (1930) showed that measurable cardinals are inaccessible, lying beyond the cardinals of the familiar Cantorian infinities. Lemma 1 will allow us to conclude that for ordinary QM countable additivity coincides with complete additivity if the dimension if the Hilbert space is not a large cardinal (see Section 3.3). But in general the move from countable to complete additivity relies on the non-existence of non-measurable sets. For example, Lebesgue measure on an interval $\mathbb{I} \subset \mathbb{R}$ cannot be completely additive since it assigns zero to each $\{r\}$, $r \in \mathbb{I}$. It follows from Lemma 1 that if the continuum hypothesis (asserting that $\aleph_1$ is the cardinality of reals $\mathbb{R}$) is true then Lebesgue measure cannot be extended to a measure on all subsets of $\mathbb{I}$.

2.4 Summary

Even on the above cursory review of the literature one cannot help being struck by how various and delicate are the considerations, both pro and con, for the different forms of additivity. And it is hard to escape the feeling that the personalist interpretation of probability does not provide enough constraints to single out the correct form of additivity; indeed, one begins to doubt that there is any such thing as *the* correct form of additivity for personal probabilities.

Initially, discussions of additivity principles for personal probabilities should be kept separate from discussions of additivity principles for physical probabilities since the relevant considerations are so different in the two cases. To start with the most obvious, Nature doesn't fear Dutch Book since She doesn't engage in bets, and it seems a category mistake to try to apply the concept of rationality to Her probability assignments. Leibniz and other philosophers have offered arguments to show that Nature must of necessity obey a principle of continuity. But the modern attitude is that only empirical investigation can reveal whether Nature conforms to the expectations of philosophers. This is not to say that the issue of what additivity principle physical probabilities obey can be settled by a straightforward experimental inquiry, a point that will richly illustrated below for quantum probabilities (see Section 4).

Eventually, however, the discussions of additivity principles for personal

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[12]This proof is based on Kanamori (2003, 2.2 Exercise, p. 23).

probabilities must include reference to additivity principles for physical probabilities; at least this is so if, as a number of philosophers have maintained, a knowledge of objective chance constrains rational credence. In Section 5 we will pose but not try to settle the issue of whether these constraints can produce an override of the arguments reviewed above for one or another form of additivity for personal probabilities.

3   Quantum probabilities and forms of additivity

In this section we employ an approach to quantum probabilities that makes pellucid the interconnections among additivity principles for quantum probabilities, the structure of the algebra of quantum observables, and the realizability of quantum states.

3.1   Quantum algebras

The treatment of quantum probabilities adopted here goes through the algebraic approach to quantum theory.[13] In this approach a quantum system is characterized by an algebra of observables and a set of physically possible states on the algebra. For present purposes the algebra is assumed to be a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$, which may be separable or non-separable.[14] By definition, $\mathfrak{N}$ is an algebra of bounded operators, closed in the weak operator topology[15] or, equivalently (by von Neumann's double commutant theorem), $\mathfrak{N} = \mathfrak{N}'' := (\mathfrak{N}')'$, where $\mathfrak{X}'$ denotes the set of bounded operators that commute with $\mathfrak{X}$.

A projection $E \in \mathfrak{N}$ is a self-adjoint element such that $E^2 = E$. Projections E_1 and E_2 are said to be orthogonal just in case $E_1 E_2 = E_2 E_1 = 0$. If $\mathfrak{N}$ acts on a separable (respectively, non-separable) $\mathcal{H}$ then $\mathfrak{N}$ is σ-finite (respectively σ-non-finite), i.e. any family of mutually orthogonal projections from $\mathfrak{N}$ has a countable number of members (respectively, some families of

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[13]The relevant mathematical background is to be found in Bratteli and Robinson (1987) and Kadison and Ringrose (1991).

[14]A separable (respectively, non-separable) $\mathcal{H}$ has a countable (respectively, non-countable) basis.

[15]A sequence of bounded operators A_n on $\mathcal{H}$ converges to A in the weak operator topology just in case $|\langle\psi|A_n - A|\phi\rangle| \to 0$ as $n \to \infty$ for any $|\phi\rangle, |\psi\rangle \in \mathcal{H}$.

mutually orthogonal projectors have an uncountable number of members). The collection of projections $\mathcal{P}(\mathfrak{N})$ of a von Neumann $\mathfrak{N}$ has a natural lattice structure. First, $\mathcal{P}(\mathfrak{N})$ is equipped with a partial order whereby $E_1 \leq E_2$ iff $Range(E_1) \subseteq Range(E_2)$; in this case E_1 is said to be a subprojection of E_2. The meet $E_1 \wedge E_2$ and join $E_1 \vee E_2$ are defined respectively the greatest lower bound and the least upper bound, and they are respectively the projections corresponding to $Range(E_1) \cap Range(E_2)$ and the closure of $Range(E_1) \cup Range(E_2)$. The above definition of the orthogonality of E_1 and E_2 is equivalent to requiring $E_1 \leq E_2^{\perp}$, where $E^{\perp} := I - E$. When E_1 and E_2 are orthogonal, $E_1 \wedge E_2 = E_1 E_2 = E_2 E_1 = E_2 \wedge E_1 = 0$ and $E_1 \vee E_2 = E_1 + E_2$. The elements of the projection lattice $\mathcal{P}(\mathfrak{N})$ are variously referred to as quantum events, yes-no questions, or quantum propositions.

Type I algebras have minimal projections;[16] Type II algebras have no minimal projections but do have finite dimensional projections;[17];while Type III algebras have only infinite dimensional projections. In ordinary non-relativistic QM sans superselection rules it is typically assumed that $\mathfrak{N}$ is the Type I factor $\mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on $\mathcal{H}$.[18] When superselection rules are present in ordinary QM $\mathfrak{N}$ is also Type I but a non-factor. In relativistic QFT much more exotic Type III algebras are encountered, an example of which will be given in Section 4.3.

3.2   Quantum states

A quantum state ω on $\mathfrak{N}$ is a normed positive linear functional mapping elements of $\mathfrak{N}$ to $\mathbb{C}$. There are many way to classify quantum states, but for our purposes perhaps the most important distinction is that between normal and non-normal states. Important features of normal states are given by:

> *Theorem 1.* The following conditions are equivalent for a state ω on a von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$:

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[16]Minimal projections are those whose only subprojections are themselves and the null projection. Minimal projections are atoms (cf. f.n. 5) in the projection lattice.

[17]Projections E_2 and E_1 are equivalent just in case there's a partial isometry between their ranges; a projection E_1 is infinite dimensional if and only if it has a proper subprojection to which it is equivalent.

[18]A factor algebra is one whose intersection with its own commutant consists of multiples of the identity.

(i) ω is completely additive, i.e. $\omega(\sum_a E_a) = \sum_a \omega(E_a)$ for any family $\{E_a\}$ of mutually orthogonal projections in $\mathfrak{N}$,[19]

(ii) there is a density operator ϱ, i.e., a trace class operator on $\mathcal{H}$ with $Tr(\varrho) = 1$, such that $\omega(A) = Tr(\varrho A)$ for all $A \in \mathfrak{N}$,

(iii) $\omega = \sum_{n=1}^{\infty} \omega_{|\psi_n\rangle}$, where $\{|\psi_n\rangle\}$ is an orthogonal family of vectors in $\mathcal{H}$ such that $\sum_{n=1}^{\infty} |\langle \psi_n | \psi_n \rangle| = 1$, and $\omega_{|\psi_n\rangle}$ is the (not necessarily normed) linear functional on $\mathfrak{N}$ defined by $\omega_{|\psi_n\rangle}(A) = \langle \psi_n | A | \psi_n \rangle$ for all $A \in \mathfrak{N}$

(iv) ω is weak-operator continuous on the unit ball of $\mathfrak{N}$,[20]

(v) ω is ultra-weakly continuous on $\mathfrak{N}$.[21]

When the linear functionals $\{\omega_{|\psi_n\rangle}\}$ appearing in condition (iii) are normed, they qualify as *vector states*. In general, a vector state ω is a state such that there is a vector $|\psi\rangle \in \mathcal{H}$ with $\omega(A) = \langle \psi | A | \psi \rangle$ for all $A \in \mathfrak{N}$; such states are normal.[22] A mixed or impure state ω is a state that admits a non-trivial decomposition into a convex linear combination of states, i.e. there are distinct states φ_1 and φ_2 such that $\omega = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$ with $0 \leq \lambda_1, \lambda_2 \leq 1$ and $\lambda_1 + \lambda_2 = 1$. A non-mixed state is said to be pure. In the case where $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ the pure states coincide with the vector states, but when superselection rules are present the coincidence is broken since vector states can be impure. For Type III algebras all normal states are vector states, but there are no normal pure states.

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[19] The sum $\sum_a E_a$ is understood in terms of convergence in the weak-operator topology.

[20] The unit ball of $\mathfrak{N}$ consists of all $A \in \mathfrak{N}$ such that $||A|| \leq 1$.

[21] A sequence of bounded operators A_n on $\mathcal{H}$ converges to A in the ultra-weak topology just in case $|\varrho(A_n - A)| \to 0$ as $n \to \infty$ for all density operators on $\mathcal{H}$. Convergence in the weak operator topology implies convergence in the ultra-weak topology. For proofs of the equivalence of (i)-(v) see Kadison and Ringrose (1991, Vol. 2, Theorem 7.1.12) and Bratteli and Robinson (1987, Theorem 2.4.21).

[22] Of course, vectors belonging to the same ray generate the same state (= expectation value functional).

3.3 From quantum states to quantum probabilities; different forms of additivity for quantum probabilities

Every quantum state ω on a von Neumann algebra $\mathfrak{N}$ generates a function Pr^{ω} on the lattice $\mathcal{P}(\mathfrak{N})$ of quantum propositions: $\mathrm{Pr}^{\omega}(E) := \omega(E)$ for $E \in \mathcal{P}(\mathfrak{N})$. Such a Pr^{ω} deserves to be called a quantum probability function because is takes values in the non-negative reals, $\mathrm{Pr}^{\omega}(I) = 1$ (where I is the identity operator), and $\mathrm{Pr}^{\omega}(E_1 + E_2) = \mathrm{Pr}^{\omega}(E_1) + \mathrm{Pr}^{\omega}(E_2)$ for orthogonal $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$. In short, Pr^{ω} satisfies the quantum analogs of the axioms (A1)-(A3) for classical probability. Quantum probability theory can be construed as the study of probability functions on $\mathcal{P}(\mathfrak{N})$ (see Hamhalter 2003).

The different forms of additivity for quantum probabilities arise from different choices for algebras and for states on the algebras. If $\mathfrak{N}$ acts on a finite dimensional $\mathcal{H}$ then all states are normal and there is no distinction between finite, countable and complete additivity. If $\mathfrak{N}$ acts on an infinite dimensional but separable $\mathcal{H}$ then there is no distinction between countable and complete additivity since $\mathcal{P}(\mathfrak{N})$ is σ-finite, but the distinction between finite and countable additivity arises when $\mathfrak{N}$ admits a countable infinity of of mutually orthogonal projections. Then normal states induce countably (= completely) additive probabilities whereas non-normal states induce merely finitely additive probabilities. Most of the applications of the Hilbert space apparatus to physical systems, including the systems studied in relativistic QFT, use separable Hilbert spaces. But there are conceptually possible (if idealized) systems whose descriptions require a non-separable $\mathcal{H}$, e.g. a system whose Hilbert space is an infinite tensor product of spaces of dimension 2 or greater, as would arise in the description of a spin chain with a countable infinity of spin sites. For a single non-relativistic particle moving in one dimension, a non-separable Hilbert space representation of the CCRs is required to accommodate a continuum of point-valued position eigenstates (Halvorson 2001). Non-separable Hilbert spaces also crop up in the more exotic (and arguably less idealized!) setting of quantum gravity: they host the polymer representations of Loop Quantum Gravity, useful for taking semi-classical limits (Ashetekhar et al. 2003). Like the position representation Halvorson discusses, polymer representations are set in non-separable Hilbert spaces with the dimensionality of the continuum.

However, to get an example in ordinary QM where there is a distinction

between countably additive and completely additive quantum probabilities requires a non-separable Hilbert space with an outlandishly large dimension. Suppose, as is customary in ordinary non-relativistic QM, that $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$. Whether separable or not, $\mathcal{H}$ has an orthonormal basis. Elements of the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ correspond one-one to closed subspaces of $\mathcal{H}$, which in turn correspond one-one to (the closure of spans of) subsets of an orthonormal basis of $\mathcal{H}$. Thus, a quantum probability function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ corresponds to a probability measure on the power set of an orthonormal basis, and Lemma 1 can be adapted to conclude that

> *Lemma 2* (Eilers and Horst 1975; Drish 1979). Every countably additive probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is completely additive if and only if $\dim(\mathcal{H})$ is less than the first measurable cardinal.

So except in the fevered imagination of mathematical physicists who dream of applications requiring a Hilbert space of outlandishly large dimension, the discussion of additivity requirements for probabilities in ordinary QM boils down to finite vs. countable additivity.

Note, however, that Lemma 2 does not generalize to the case where $\mathfrak{N}$ is Type III as, for example, with the local algebras encountered in relativistic QFT. The projections in such an $\mathfrak{N}$ correspond to infinite dimensional subspaces of $\mathcal{H}$ and, thus, a probability measure on $\mathcal{P}(\mathfrak{N})$ does not determine a probability measure on the all subsets of an orthonormal basis of $\mathcal{H}$.

3.4 From quantum probabilities to quantum states

As noted already, quantum states generate quantum probability functions. But are there quantum probabilities that are not generated by quantum states? The generalized Gleason theorem shows that, under mild technical restrictions, the answer is No.

> *Theorem 2.* Let $\mathfrak{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, separable or non-separable. Suppose that $\mathfrak{N}$ does not contain any direct summands of Type I_2. Then for any quantum probability function Pr on $\mathcal{P}(\mathfrak{N})$ there is a unique extension of Pr to a quantum state ω^{Pr} on $\mathfrak{N}$. Further, if Pr is completely

additive (respectively, merely finitely additive or merely countably additive) then ω^{Pr} is a normal (respectively, non-normal) quantum state.[23]

In the case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ the technical restriction in Theorem 2 amounts to the requirement that $\dim(\mathcal{H}) \geq 3$.[24] Of course, when $\mathcal{H}$ is separable the requirement of complete additivity reduces to countable additivity; and in view of Lemma 2, this is also the case even when $\mathcal{H}$ is non-separable but $\dim(\mathcal{H})$ is not a measurable cardinal and $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$.

In sum, technical restriction aside, quantum probability functions and quantum states are in one-one correspondence. Our discussion below is largely based on the ansatz that quantum states correspond to objective features of quantum systems. It follows that the probability function a quantum state induces deserves to be called a physical probability, and this is so independently of the fraught issue of whether various no-go results on hidden variable interpretations succeed in showing that quantum probabilities are irreducible in the sense of not arising from the ignorance of values of variables neglected in the conventional quantum description. Thus, under the ansatz, the question of what form of additivity these quantum physical probabilities obey reduces to the question of what quantum states are physically realizable.

Before proceeding to this question, however, we note for the record that our ansatz is vehemently rejected by the quantum Bayesians (QBians as they style themselves) who seek to apply de Finetti's personalist interpretation of probabilities to quantum probabilities.[25] QBism is able to get a foothold because the one-one correspondence between probability functions on $\mathcal{P}(\mathfrak{N})$

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[23] See Hamhalter (2003) and Maeda (1990) detailed treatments of this crucial theorem. A Type I$_2$ summand has the form $E_1 + E_2 = I$ where E_1 and E_2 are orthogonal projections.

[24] The original version of the Gleason theorem is equivalent to the statement:

> Let $\mathcal{H}$ be a separable Hilbert space with $\dim(\mathcal{H}) > 2$ and let Pr be a quantum probability function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Then there is a unique extension of Pr to a quantum state ω^{Pr} on $\mathfrak{B}(\mathcal{H})$. Further, if Pr is countably additive (respectively, merely finitely additive) then ω^{Pr} is a normal (respectively, non-normal) quantum state.

The real work in proving Theorem 2 lies in the generalization to non-Type I algebras.

[25] A presentation of QBism intended for a philosophy of science audience is to be found in Caves, Fuchs, and Schack (2007).

and quantum states on $\mathfrak{N}$ allows quantum states to be construed as devices for representing quantum probability functions. And if, per QBism, a probability function on $\mathcal{P}(\mathfrak{N})$ is construed as the credence function of a Bayesian agent, quantum states are reduced to the status of book keeping devices for tracking personal probabilities. The problems and prospects for QBism is a fascinating and complex topic that cannot be pursued here.[26]

Finally, we note that in quantum probability theory there is a coincidence between two senses of "measurable event," i.e. projectors acting on the Hilbert space $\mathcal{H}$ that are assigned a probability vs. projectors that correspond to genuine observables and, thus, can in principle be measured by a physical experiment that returns a yes or no answer. This coincidence provides an explanation of probabilistic non-measurability, viz. some mathematically possible events (projectors on the Hilbert space on which the observable algebra $\mathfrak{N}$ acts) do not deserve to be assigned a probability because they are not in $\mathfrak{N}$. When $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ the projection lattice $\mathcal{P}(\mathfrak{N})$ contains every projector and, thus, probability is assigned to every mathematically possible event. When $\mathfrak{N}$ is Type III the projection lattice $\mathcal{P}(\mathfrak{N})$ contains only infinite dimensional projectors and, thus, events corresponding to finite dimensional projectors receive no probability.

4   The realizability of quantum states

In this section we assess considerations pro and con for the widely held notion that only normal quantum states are physically realizable. In view of the importance of the issue, not just for the issue of additivity principles but for the foundations of QM in general, there is surprisingly much presumption and surprisingly little convincing argumentation in the literature. In what follows we point to the presumptions and try to sharpen the argumentation.

4.1   Presupposition and grumbling

Virtually all of the QM textbooks used to train physics students at both the undergraduate and graduate levels presuppose that quantum states are normal; for they presuppose that quantum probabilities are to be calculated via the "Born rule" (aka trace prescription), i.e. there is a density operator

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[26]For critical reviews of QBism, see Timpson (2008), Bacciagaluppi (2013), and Earman (2016a).

ϱ such that the probability of $E \in \mathcal{P}(\mathfrak{N})$ is $Tr(E\varrho)$, which is to say that the state defined by $\omega_\varrho(\bullet) := Tr(\bullet\varrho)$ is a normal state in conformity with condition (ii) of Theorem 1.

Furthermore, central topics in quantum theory are pursued under this presupposition, and many core theorems are not valid without it. A good example is quantum entanglement, a topic which is crucial to both quantum computing and quantum information theory. Consider a composite system algebra $\mathfrak{N}_{12}$ generated by subsystem algebras $\mathfrak{N}_1$ and $\mathfrak{N}_2$ in the sense that $\mathfrak{N}_{12} = \mathfrak{N}_1 \vee \mathfrak{N}_2$ (the smallest von Neumann algebra containing both $\mathfrak{N}_1$ and $\mathfrak{N}_2$). Restrict attention to cases where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are mutually commuting, a minimal condition for the subsystems to be considered independent subsystems.[27] A composite system state ω_{12} on $\mathfrak{N}_{12}$ is said to be entangled over $\mathfrak{N}_1$ and $\mathfrak{N}_2$ if and only if it is not a product state, i.e. there are no states ω_1 and ω_2 on $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively such that $\omega_{12}(E_1 E_2) = \omega_{12}(E_2 E_1) = \omega_1(E_1)\omega_2(E_2)$ for all $E_1 \in \mathfrak{N}_1$ and $E_2 \in \mathfrak{N}_2$. Of course, even in the case of classical (i.e. abelian) subsystem algebras entanglement can occur since it amounts to no more than the existence of correlations between the subsystem observables. Genuine quantum entanglement, however, requires more than mere correlation; it requires that the composite system state ω_{12} cannot be approximated by mixtures (positive linear combinations) of product states; for otherwise the entanglement could result from using a classical randomizing device to set the mixture weights, in which case the correlations between the subsystems would be due entirely to our ignorance of which product state the randomizer produced.

But the crucial result linking quantum entanglement to the non-abelian character of quantum observables requires the use of normal states. Consider the case where the systems are independent not only in the sense that the subsystem algebras $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are mutually commuting but also in the stronger sense that they generate a tensor product composite algebra, i.e. $\mathfrak{N}_1 \vee \mathfrak{N}_2 \simeq \mathfrak{N}_1 \overline{\otimes} \mathfrak{N}_2$.[28] In the case of ordinary QM this condition is automatically satisfied since $\mathfrak{B}(\mathcal{H}_1) \vee \mathfrak{B}(\mathcal{H}_2) \simeq \mathfrak{B}(\mathcal{H}_1)\overline{\otimes}\mathfrak{B}(\mathcal{H}_2) \simeq \mathfrak{B}(\mathcal{H}_1) \otimes \mathfrak{B}(\mathcal{H}_2)$.

Theorem 3 (Raggio 1988). Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be mutually commut-

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[27] By standard quantum doctrine the mutual commutativity of the subsystem algebras means that pairs of observables, one from each subsystem, are co-measurable.

[28] The overbar indicates the von Neumann tensor product, that is, the von Neumann algebra generated by taking the weak closure of $\mathfrak{N}_1 \otimes \mathfrak{N}_2$. See Kadison and Ringrose 1991, Vol. 2, §11.2.

ing von Neumann algebras acting on a common Hilbert space and suppose that $\mathfrak{N}_1 \vee \mathfrak{N}_2 = \mathfrak{N}_1 \overline{\otimes} \mathfrak{N}_2$. Then the following two conditions are equivalent:

(i) $\mathfrak{N}_1 \vee \mathfrak{N}_2$ can display quantum entanglement in the sense that it admits normal states ω_{12} that cannot be approximated by mixtures of normal product states; that is, ω_{12} does not lie in the norm closure[29] of the hull of convex linear combinations of normal product states,

(ii) At least one of the subsystem algebras $\mathfrak{N}_1$ and $\mathfrak{N}_2$ is non-abelian.

This result, which relies on using normal states to define quantum entanglement, makes clear why quantum entanglement is a new phenomenon that arises in the transition from the classical to the quantum understood as the transition from abelian to non-abelian algebras of observables.

When it is not required that the system algebra has a tensor product structure, a striking example of the difference between normal and non-normal states makes for entanglement is given the algebraic treatment of the Klein-Gordon field. The algebraic version of relativistic QFT assumes that there is a net $\mathcal{O} \mapsto \mathfrak{N}(\mathcal{O})$ of local von Neumann algebras $\mathfrak{N}(\mathcal{O})$ associated with open bounded regions $\mathcal{O}$ of Minkowski spacetime $\mathcal{M}$ and that this net satisfies the isotony property: if $\mathcal{O} \subset \mathcal{O}'$ then $\mathfrak{N}(\mathcal{O}) \subset \mathfrak{N}(\mathcal{O}')$. The quasi-local global algebra $\mathfrak{M}$ is the von Neumann algebra generated by the local algebras $\mathfrak{N}(\mathcal{O})$ as $\mathcal{O}$ ranges over all of the open bounded regions of $\mathcal{M}$. Now focus on the algebras $\mathfrak{N}(\mathcal{O}_L)$ and $\mathfrak{N}(\mathcal{O}_R)$ associated respectively with the interiors $\mathcal{O}_L$ and $\mathcal{O}_R$ of the left and right Rindler wedge regions of Minkowski spacetime.[30] Since $\mathcal{O}_L$ and $\mathcal{O}_R$ are relatively spacelike, the requirement of micro-causality (which is satisfied by the Klein-Gordon field) entails that $\mathfrak{N}(\mathcal{O}_L)$ and $\mathfrak{N}(\mathcal{O}_R)$ commute. It is a fact that these subsystems are intrinsically entangled in the sense that *every normal state* on $\mathfrak{N}(\mathcal{O}_L) \vee \mathfrak{N}(\mathcal{O}_R)$ is quantum entangled over $\mathfrak{N}(\mathcal{O}_L)$ and $\mathfrak{N}(\mathcal{O}_R)$ and, as a consequence, $\mathfrak{N}(\mathcal{O}_L) \vee \mathfrak{N}(\mathcal{O}_R) \ncong \mathfrak{N}_L \overline{\otimes} \mathfrak{N}_R$ (see Buchholz 1974). But there are plenty of non-normal product states on $\mathfrak{N}(\mathcal{O}_L) \vee \mathfrak{N}(\mathcal{O}_R)$; but, as noted, these states are generally ignored.

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[29] The set of normal states is closed in the norm topology.

[30] Let (x, y, z, t) be an inertial coordinate system for Minkowski spacetime. Then the right Rindler wedge with vertex at the origin consists of those points $\infty > x > 0$, $x^2 - t^2 > 0$. Reflecting about the origin gives the left Rindler wedge.

Occasionally there are grumblings in the literature of mathematical physics and philosophy to the effect that the standard practice is too hidebound and that non-normal states should be considered as a means of understanding some puzzling features of QM. For example, Srinvas (1980) argues that non-normal states are needed to define the joint probabilities associated with successive measurements of quantum observables with continuous spectra. Relatedly, Halvorson (2001) considers the use of non-normal states as a means of assigning sharp values to the position of a particle and, thus, as providing a basis for interpreting the modulus square $|\psi(q)|^2$ of the wave function (in the usual L^2 realization) of a spinless particle as the probability density that the particle has exact position q. Non-normal states also provide a means to overcome no-go results on the existence of conditional expectations for normal states such as that of Takesaki (1972) showing that a faithful normal state ω on a von Neumann algebra $\mathfrak{N}$ admits a conditional expectation with respect to a subalgebra $\mathfrak{M} \subset \mathfrak{N}$ just in case $\mathfrak{M}$ is invariant under the modular automorphism group associated with ω.[31]

But neither the presuppositions of standard practice nor the desire to overcome perceived conundra in QM constitute arguments, one way or the other, as to why only normal states should be deemed to be physically realizable and, thus, as to why quantum probabilities are completely additive.

4.2   Some arguments for normality

For QBians, for whom quantum states are merely devices for representing credence functions of Bayesian agents, arguments for normality of states would have to take the form of the arguments reviewed in Section 2 for the countable or complete additivity of personal probabilities. It is unclear to us, however, how some of the considerations developed in the context of classical probability carry over to quantum probability. For example, considerations of conglomerability depend on the fact that any classical probability function can be extended to a full conditional probability function. We do not know whether this fact has an analog for quantum probabilities in the sense that any additive real valued function on a the projection lattice $\mathcal{P}(\mathfrak{N})$ of a non-abelian von Neumann algebra $\mathfrak{N}$ has an extension to a full conditional

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[31]A faithful state ω on $\mathfrak{N}$ is such that $\omega(A) = 0$ for $A \in \mathfrak{N}$ implies $A = 0$. In the classical case where $\mathfrak{N}$ is abelian the modular automorphism group is trivial so conditional expectations always exist. Treatments of Tomita-Takesaki modular theory can be found in Kadison and Ringrose (1991, Vol. 2) and Bratteli and Robinson (1987).

probability.[32] While there are a number of intriguing issues to be investigated in the attempt to give a personalist interpretation of quantum probabilities, we will not pursue them here. We return to thinking of quantum states as corresponding to objective features of quantum systems and as being ontologically prior to the probabilities they induce. And we search for arguments to the effect that physically realizable states should be normal.

Von Neumann algebras are closed in the weak operator topology and, thus, in the ultra-weak topology. So by Theorem 1 if a state ω on $\mathfrak{N}$ is non-normal there is a sequence of elements A_n in the unit ball of $\mathfrak{N}$ (respectively, $\mathfrak{N}$) that converge in the weak operator topology (respectively, the ultra-weak topology) to an element A of the unit ball of $\mathfrak{N}$ (respectively, in $\mathfrak{N}$) such that $\omega(A_n)$ does not converge to $\omega(A)$. This is awkward, but the awkwardness does not provide an argument for normality unless accompanied by the premise that Nature is never awkward.

A possible argument for continuity of quantum states and, thus, for complete additivity of quantum probabilities, would start from the observation that there is a canonical procedure, called the GNS construction, by which a non-normal/non-continuous state ω for a von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$ can be extended to a normal/continuous state $\widetilde{\omega}$ on a larger von Neumann algebra $\widetilde{\mathfrak{N}} \supset \mathfrak{N}$ acting on a larger $\widetilde{\mathcal{H}} \supset \mathcal{H}$.[33] Thus, if the self-adjoint elements of $\widetilde{\mathfrak{N}}$ that are not also elements of $\mathfrak{N}$ have physical significance, it can be argued that the failure normality/continuity of ω is due to focusing too narrowly on a proper subset of all the genuine observables. But there is no a priori guarantee that the GNS construction always lends itself to this interpretation; the evaluation has to be made on a case-by-case basis.

Reflection on such examples and the possible uses for non-normal states mentioned in the preceding subsection together lend plausibility to Ruetsche's (2011ab) conclusion that a decision on which states to count as physically realizable cannot be made in isolation of from considerations of what phenomena are to be modeled, what explanations are to be sustained, and what lawlike relations are deemed to be important. And such considerations are sensitive to the theoretical, interpretative, and pragmatic contexts in which they arise. Nevertheless, we will attempt below a kind of transcendental argument for normality. But before turning to this ambitious project we note

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[32]Of course, to even formulate a precise version of this question a quantum analog of classical conditionalization needs to be specified. This issue will be addressed below.

[33]See Kadison and Ringrose (1991, Vol. 1, Sec. 4.5) for the GNS construction.

that even if there is no effective global argument for normality of states, one may still hope to find persuasive arguments in local, constrained contexts. We give an example from algebraic QFT.

4.3   Local normality of states in relativistic QFT

Recall from Section 4.1 the net $\mathcal{O} \mapsto \mathfrak{N}(\mathcal{O})$ of local von Neumann algebras $\mathfrak{N}(\mathcal{O})$ associated with open bounded regions $\mathcal{O}$ of Minkowski spacetime $\mathcal{M}$ and the quasi-local global algebra $\mathfrak{M}$ generated by the $\mathfrak{N}(\mathcal{O})$ as $\mathcal{O}$ ranges over all of the open bounded regions of $\mathcal{M}$. Since Minkowski spacetime is based on Hausdorff manifold, for any spacetime point $p \in \mathcal{M}$ there is a sequence $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open regions such that $\mathcal{O}_{n+1} \subset \mathcal{O}_n$ for all $n \in \mathbb{N}$ and $\cap_{n=1}^{\infty} \mathcal{O}_n = \{p\}$. The property of local definiteness is designed to capture the notion that there are no observables "at a point" so that physically realizable states become indistinguishable in sufficiently small regions: technically, the requirement is that for any physically realizable states ω and φ on $\mathfrak{M}$, $||(\omega - \varphi)|_{\mathfrak{N}(\mathcal{O}_n)}|| \to 0$ as $n \to \infty$ (see Haag 1992, p. 131). The only other assumption needed is the unobjectionable posit that the physically realizable states on $\mathfrak{M}$ contain at least one normal state φ. It follows that all physically realizable states ω are locally normal, i.e. for any $p \in \mathcal{M}$ there is a neighborhood $\tilde{\mathcal{O}}$ of p such that $\omega|_{\mathfrak{N}(\tilde{\mathcal{O}})}$ is normal. To see this, consider a sequence $\{\mathcal{O}_n\}_{n=1}^{\infty}$ descending to p. By local definiteness there must be a sufficiently large $n_0 \in \mathbb{N}$ such that $||(\omega - \varphi)|_{\mathfrak{N}(\mathcal{O}_{n_0})}|| < 2$. This implies that $\omega|_{\mathfrak{N}(\mathcal{O}_{n_0})}$ and $\varphi|_{\mathfrak{N}(\mathcal{O}_{n_0})}$ belong to the same folium, and since $\varphi|_{\mathfrak{N}(\mathcal{O}_{n_0})}$ is normal so is $\omega|_{\mathfrak{N}(\mathcal{O}_{n_0})}$. To complete the proof set $\tilde{\mathcal{O}} = \mathcal{O}_{n_0}$.

This result is modest, and even such as it is its effectiveness turns on the plausibility of the property of local definiteness. Given that *non*-normal states have been promoted as a way to make sense of a particle being strictly located at a point (Halvorson 2001), we might expect at least some proponents of non-normal states to resist the demand for local definiteness. Be that as it may, the result nevertheless serves as a useful illustration of how rigorous arguments for normality can be constructed in a piecemeal way.

4.4   A more ambitious argument for normality

We turn from piecemeal arguments to the more ambitious attempt to show that confining physically realizability of states to normal states is a precondition for state preparation; or more precisely, a precondition for state prepa-

ration that is describable within QM itself, as must be the case if QM is a complete theory.[34] We confine attention to ordinary QM where the algebra of observables is $\mathfrak{B}(\mathcal{H})$, and we take it for granted that experimentalists can and do succeed in preparing normal pure states on $\mathfrak{B}(\mathcal{H})$, e.g. eigenstates of spin or energy. The first part of the argument shows that if non-normal states are physically realizable then state preparation is not generally possible. The second part of the argument shows that if only normal states are physically realizable then preparation of pure states is not only possible but is also explicable within the language of the theory.

First part: Suppose that a non-normal state ω is realized. There are two conceivable routes from ω to some normal state φ the experimentalist wishes to prepare; but both routes are blocked. One route is via Schrödinger evolution, as described by a strongly continuous one-parameter unitary group $U_t = \exp(-iHt)$, $t \in \mathbb{R}$, where H is the Hamiltonian of the system.[35] In the algebraic setting, Schrödinger evolution results in a change of state from the initial state ω at $t = 0$ to a new state ω_t at $t > 0$ given by $\omega \mapsto \omega_t(\bullet) := \omega(U_t \bullet U_t^{-1})$. But since U_t is an automorphism of the algebra observables, if ω is non-normal (respectively, normal) then so is $\omega_t(\bullet)$ for all t—whatever the Hamiltonian, the system can evolve undisturbed as long as pleased but the state does not change its colors.

The second possible route from a non-normal to a normal state invokes the von Neumann projection postulate, which in the algebraic setting comes to this: suppose that some projector $E \in \mathfrak{B}(\mathcal{H})$ is measured and the measurement yields a positive response; then the pre-measurement state ω becomes the post-measurement state $\omega_E(\bullet) := \omega(E \bullet E)/\omega(E)$. This assumes that $\omega(E) \neq 0$; but if $\omega(E) = 0$ a positive result of the measurement of E should not have been obtained. To complete the argument all that is needed is the fact that if ω is a non-normal (respectively, normal) state then so is $\omega_E(\bullet)$ for any projector E such that $\omega(E) \neq 0$.

Second part: Now suppose that only normal states are physically realizable. To describe in the formalism of QM how the preparation of a normal pure state place consider again such a target state φ. The support projector S_φ for this state is the orthogonal complement of the union of all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\varphi(E) = 0$. The support projector S_φ for φ is unique (see Kadison and Ringrose 1991, p. 468), and it serves as a filter for φ within

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[34]Ruetsche 2011a develops a similar argument, but takes its force to be piecemeal.

[35]This assumes that the Hamiltonian operator H is essentially self-adjoint.

the class of all normal states (pure or not); that is, for any normal state ω (pure or mixed) such that $\omega(S_\varphi) \neq 0$, $\omega_{S_\varphi}(\bullet) := \omega(S_\varphi \bullet S_\varphi)/\omega(S_\varphi) = \varphi(\bullet)$. Thus, a positive outcome of a measurement of S_φ converts, per the von Neumann projection postulate, any pre-measurement state ω such that $\omega(S_\varphi) \neq 0$ to the target state φ. Note that this argument does not extend to normal impure states since such states do not have filters (see Ruetsche and Earman 2011).

We brand the argument "transcendental" because it presents normality of quantum states as a condition on the very possibility of a central sort of quantum experience: the experience of state preparation. Those uncomfortable with the branding can move the argument into a less continental register: given the argument, our long history of success at state preparation is good inductive evidence that non-normal states, which would derail our attempts at state preparation, don't occur in nature. Good but not dispositive: our long history of success at state preparation is not a history of *perfect* success. A fan of non-normal states could well observer that, so long as *most* laboratory systems occupy normal states, our usual story about state preparation applies most of the time; and since detectors are never 100% efficient, that is all the success we need to make sense of laboratory practice.[36]

Much of the formalism of this transcendental argument against the realizability of non-normal states applies not just to the case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ but to more general von Neumann algebras. However, Type II and Type III algebras do not admit any normal pure states, and impure states do not have filters in these algebras. This raises issues about state preparation in QFT where the local algebras associated with open bounded regions of spacetime are generically Type III, but we will not attempt to deal with that issue here.

4.5 Can there be an infinite fair lottery?

In this subsection we discuss the implications of the normality of realizable quantum states for an issue that has received a fair amount of attention in the literature; namely, can there be a lottery that has an infinite number of mutually exclusive outcomes, each of which is—in some appropriate sense—equally probable?[37] Different interpretations of what counts as the

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[36]We are grateful to Gordon Belot for raising this point.

[37]One might want to require more of an infinite fair lottery. For example, in a lottery that sells tickets numbered $1, 2, 3, ...$, one might want to require that the probability is distributed "uniformly" over the natural numbers. For various interpretations of this

appropriate sense of probability lead to different senses of the question.

Before proceeding further it is necessary to correct a misimpression which is abroad in the literature. It is often said that, regardless of the interpretation of probability, countable additivity makes an infinite fair lottery impossible. This is so if the set of mutually exclusive and exhaustive lottery outcomes is countable; but when the set of mutually exclusive and exhaustive lottery outcomes is uncountable, a countably additive but not completely additive probability function can assign equal probability—namely, zero—to each of the outcomes (think of Lebesgue measure on the real line). However, as noted above (Lemma 2), there is no gap between countable and complete additivity for the probabilities of ordinary QM when the dimension of the Hilbert space is less than the first measurable cardinal.

The de Finetti school would relativize the question at issue to rational agents and would take fair-for-an-agent to mean that the agent assigns equal degrees of belief to each of the lottery's infinite number of outcomes. De Finetti's advocacy of finite additivity was, in part, due to his desire to give a positive answer to the question in the case where the outcomes of the lottery are countable and, thus, countable additivity precludes fairness. As discussed above, the difficulty for de Finetti's answer to our question is that various considerations militate in favor of countable additivity as a necessary condition for rationality of degrees of belief.

Here we focus on another interpretation of the question which assumes, contra de Finetti, that there are objective chances; namely, Is it physically possible to implement a lottery with an infinite number of mutually exclusive outcomes in such a way that each and every outcome has the same objective chance of occurring? We take physical possibility to be judged by compatibility with fundamental physics. If that physics is classical physics then intuition might suggest a positive answer, at least if some idealizations are permitted. Consider, for example, an infinitely sharp dart thrown at a region of space isomorphic to the unit interval of the real line. Intuition would suggest that nothing prevents the dart being thrown "at random" so that the probability of hitting any given point in the interval is zero, while the probability of hitting a finite subinterval is proportional to the length of the interval, as in Lebesgue measure. But what is wanted is not intuition mongering but demonstration. If the fundamental physics is taken to be QM

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notion, see Kadane and O'Hagen (1995). For present purposes, however, it is enough to focus on the implication that fairness entails a failure of complete additivity.

then we have makings of a demonstration of a no-go result. If normal states are the only physically realizable quantum states then fair infinite quantum lotteries are precluded since normal states induce completely additive probabilities. And the discussion of the preceding subsections indicates that the weight of the evidence lies is in favor of affirming the antecedent and, thus, in a negative answer to the question at issue. But, as noted, the case for affirming the antecedent is not air tight.

With remaining doubt set to the side, another way to state the conclusion is that Quantum Mother Nature always plays favorites in cases of an infinite number of mutually exclusive outcomes; and this favoritism manifests itself not just in hypothetical lotteries but in practical physical applications. Consider, for example, a system with a purely discrete energy spectrum that is bounded from below but not from above. Normal quantum states necessarily favor lower energy levels and disfavor the far upper reaches of the spectrum: for any normal state and any $\varepsilon > 0$ there is a finite N such that the probability is at least $1 - \varepsilon$ that the energy of the system lies below the Nth eigenlevel. Or in the case of a particle moving in infinite space, normality favors localization within a finite region over dispersion towards spatial infinity. Normality of quantum states thus contributes part of the explanation of some of the general features of the world we observe around us, and without it the world could have a different appearance from what we have grown to expect.

Other avenues beckon for exploring the question at issue. For example, by allowing probabilities to take on infinitesimal values some commentators claim that there can be fair infinite lotteries in which each of the outcomes has the same non-zero value, albeit a value less than any finite number (see Wenmachers and Horsten 2013); but such a proposal is subject to various difficulties (see Pruss 2012, 2014). Other commentators propose to explore the possibility of building a lottery machine delivering equal real-valued chances for an infinity of outcomes by expanding the space of possibilities beyond those allowed by standard QM (see Norton 2016). Such explorations can be valuable if they are conducted with an eye to illuminating the foundations of QM; but otherwise they run the danger of degenerating into idle speculation about distant possible worlds. Rather than pursuing such speculation it seems to us more productive to try to come to grips with the facts that our world is quantum mechanical and that, therefore, life is a series of quantum lotteries which are never fair.

4.6 Summary

The presumption that only normal states are physically realizable, which seems to be baked into the standard practice of QM, is more than mere presumption since it can be motivated in various ways, both local and global. But since a knock-down argument for normality is lacking, the possibility that non-normal states are realized for actual quantum systems has to be kept in mind. We emphasize that, apart from the light it casts on the foundations of quantum theory, there are at least two other reasons to be interested in the issue of the realizability of non-normal quantum states. First, if such states are realizable then there is an interesting sense in which there can be an infinite fair lottery, giving equal objective chance (of zero) to each of an infinite number of outcomes. Second, as we will point out in the next section, if non-normal states are realizable then rational agents would find themselves in an embarrassing predicament.

5 Implications for rational credence for quantum events

We now want to examine the implications of additivity principles satisfied by quantum probabilities for rational credence functions defined over quantum events. One avenue for establishing a connection goes through David Lewis' Principal Principle (PP), which is a purported principle of rationality linking objective chance and rational credence (see Lewis 1980). Another avenue might appeal to those of the de Finetti school who want to be able to speak with the vulgar by treating so-called objective chance as objectified credence. Still a third avenue relies on the empirical success of QM.

5.1 Lewis' Principal Principle for quantum probabilities

The discussion of PP in the philosophical literature focuses mainly on classical probabilities, and it typically assumes that rational agents adopt credence functions that satisfy the standard Kolmogorov probability axioms (A1)-(A3) and also that rational agents update their credence by classical Bayesian conditionalization. The later assumption means that if the agent starts with an initial credence function Cr and then has a learning experience, the content

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of which is that a proposition E is learned for certain, then the agent updates her credence function to $Cr'(\bullet) := Cr(\bullet/E)$. The idea behind Lewis' PP was that if objective chance operates in the world then there is an additional constraint that rational credence functions must satisfy: roughly expressed, if a rational agent learns what the objective chance of an event is, then updating her credence function on that knowledge should produce a credence in said event that is equal to its objective chance. Disputes abound on how to give precise expression to PP; we will not attempt to review this literature here since the disputes are largely irrelevant to the main points we wish to make.[38]

If a suitable version of PP could be formulated for quantum probabilities and if such a principle could be justified as a principle of rationality then our (tentative) conclusion that quantum chances are completely additive would entail that rational credence functions on the projection lattice of quantum propositions must be completely additive. However, no satisfactory justification exists for PP as an additional principle of rationality in the classical probability setting, and there is no reason to think that one will emerge in the quantum setting. The only systematic attempt at a justification for PP in the classical setting is due to Pettigrew (2012). His argument relies on an ingenious reworking of de Finetti's scoring rule argument designed to show that rational degrees of belief should satisfy the Kolmogorov axioms of probability (A1)-(A3) (recall Section 2.1). Pettigrew gives a dominance argument for PP-like principles using a class of scoring rules that measure how well credence functions track objective chance. Needless to say, those who like de Finetti reject the very notion of objective chance would be completely unmoved; and those who accept the notion of objective chance but who harbor initial doubts about the normative force of Lewis' PP are not apt to accept that tracking objective chance serves as a criterion of rationality.

An alternative route to justification uses the idea that rational credence must be construed as an expression of epistemic uncertainly about objective chance and, therefore, when the uncertainty is resolved, rational credence necessarily aligns with chance. But this line of justification lacks polemical force for the same reasons as before. More promising is the idea that rational credence can be represented *as if* it were an expression of epistemic uncertainty about objective chance, and that is still sufficient to ensure that

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[38]For a sampling of the literature see Arntzenius and Hall (2003); Bigelow, Collins, and Pargeter (1993); Black (1998); Haddock (2011); Hall (1994, 2004); Ismael (2008); Meacham (2010); Pettigrew (2012); Roberts (2001, 2013); Strevens (1995); Thau (1994); Vranus (2002, 2004).

rational credence aligns with chance when the uncertainty is resolved. But the implementation of this idea requires the proof of representation theorems, which do not come for free but require substantive assumptions. Quantum theory provides these assumptions; but they lead to a point of view on PP-like principles that is quite different from that intended by Lewis and most of the commentators.

To simplify the discussion we will confine attention to ordinary QM where $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, and relying on the discussion of Section 4 we will assume that that normal pure states on $\mathfrak{B}(\mathcal{H})$ induce probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that qualify as objective chances. To proceed further, two threshold issues need to be addressed. The first is what updating rule to adopt for probability functions on the lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ of quantum propositions. The widely accepted answer is that classical Bayesian conditionalization is replaced by Lüders conditionalization, whereby for $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\Pr(F) \neq 0$, $\Pr(\bullet) \mapsto \Pr(\bullet//F) := \Pr(F \bullet F)/\Pr(F)$. The adoption of Lüders rule is justified by the fact that, like its classical counterpart, it is the unique quantum conditionalization rule with a desirable property; namely, if $\Pr(F) \neq 0$ then $\Pr(\bullet//F)$ is the unique probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that for any $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, if $E \leq F$ then $\Pr(E//F) = \Pr(E)/\Pr(F)$, i.e. $\Pr(\bullet//F)$ is the renormalization of $\Pr(\bullet)$ to $\Pr'(\bullet)$ where $\Pr'(F) = 1$ (see Bub 1977). The second issue is what element of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ (if any) can be construed as asserting that the chances are those induced on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ by the quantum state φ? Again relying on the discussion of Section 4 we will presume that when φ is a normal pure state the support projector S_φ of φ fulfills the role of the desired proposition.

With these preliminaries in hand, it is easy to prove a result representing quantum probabilities as mixtures of quantum chance functions (see Earman 2016b). But the following lemma is sufficient for present purposes:

> *Lemma 3.* Let Pr be a completely additive probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\dim(\mathcal{H}) > 2$, and let φ be a normal pure state on $\mathfrak{B}(\mathcal{H})$. Then $\Pr(E//S_\varphi) = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, provided that $\Pr(S_\varphi) \neq 0$.

By Gleason's theorem Pr extends uniquely to a normal state ω on $\mathfrak{B}(\mathcal{H})$. If $\Pr(S_\varphi) \neq 0$ then $\omega(S_\varphi) \neq 0$, and $\Pr(E//S_\varphi) = \dfrac{\Pr(S_\varphi E S_\varphi)}{\Pr(S_\varphi)} = \dfrac{\omega(S_\varphi E S_\varphi)}{\omega(S_\varphi)} = \varphi(E)$, where the last equality follows by the filter property of S_φ.

With Pr interpreted as the credence function of an agent, Lemma 3 is the quantum analog of what is called Miller's Principle (see Pettigrew 2012), a special case of Lewis' PP. In words, provided that $\Pr(S_\varphi) \neq 0$, learning that the chances are given by the normal pure state φ brings the Lüders updated Pr-credence for any $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ into line with the φ-chances of E. The upshot is that for agents whose credence functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are completely additive and who update by Lüders conditionalization, knowledge of quantum chance leads to the alignment of credence with chance as a theorem of quantum probability theory[39]; no additional principle of rationality, such as Lewis' PP, is needed to cudgel such agents to align their credences with chance. On the other hand, assuming again that chances are induced by normal states, agents whose credence functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ fail to be completely additive cannot possibly satisfy a PP-like principle. Thus, in the context of quantum probabilities Lewis' PP reduces to the claim that rationality of belief demands that credence functions obey complete additivity. This claim may be correct, but it requires justification. If lines of justifications discussed above (recall Section 2.3) succeed, Lewis' PP is superfluous; if they fail, Lewis' PP doesn't help.

5.2   Speaking with the vulgar

The above considerations provide a reason why even those Bayesian agents who follow de Finetti's lead in rejecting objective chance may, nevertheless, wish to adopt complete additivity for credence over quantum events quantum events. Let Pr and Pr$'$ be any completely additive probability functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. When $\dim(\mathcal{H}) > 2$ it follows from Lemma 3 that if φ is any normal pure state such that $\Pr(S_\varphi) \neq 0$ and $\Pr'(S_\varphi) \neq 0$ then $\Pr(E//S_\varphi) = \Pr'(E//S_\varphi) = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. This is the probabilistic counterpart of state preparation. With Pr and Pr$'$ interpreted as credence functions, Bayesian personalists who adopt complete additivity can reinterpret this result in terms of the merger of opinion induced by Lüders conditionalization on the information that a measurement of S_φ has yielded a yes result. This allows personalists to to speak with the vulgar and recognize objective quantum chance in the form of objectified (= merged) credence.

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[39]At least this is so for $\dim(\mathcal{H}) > 2$. This restriction is needed for the validity of Gleason's theorem which is used in the proof of Lemmas 3 (see Earman 2016b).

5.3 Don't bet against QM

Bayesians who break with the de Finetti camp and are willing to entertain the notion of objective chance may find in the empirical success of QM a reason not only to speak with the vulgar but to learn from them. Here is an example of what we mean by empirical success. As in the preceding subsection, restrict attention to a system of ordinary QM with observable algebra $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, and for sake of concreteness let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Prepare the system in a normal pure state φ (see Section 4.2). Then perform a measurement of some projector $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, and record whether the result is positive or negative. Reset the system (or an identical system) in the same state φ, and again perform a measurement of the same E. Repeat again, and again, and again, and ... . In each trial the quantum probability of a positive outcome is $\varphi(E)$ regardless of the outcome of the other trials, which is to say that the trials are independent and identically distributed. Thus, the strong law of large numbers can be applied to conclude that, as the number of trials is increased without limit, the relative frequency of positive outcomes for E converges almost surely (as judged by quantum probability) to the value $\varphi(E)$. It is a well established fact that in experiment after experiment rapid (apparent) convergence to the value predicted by QM is observed; indeed, the evidence of the predictive success of QM is so massive that only a trained philosophical skeptic could doubt it. Those who have shed doubts about the success of QM and who know that the system is prepared in state φ will find it prudent not to bet against the odds set by φ since they know that such betting will lead almost surely (as judged by quantum probability) to ruin in the long run. This prudence implies that an agent should operate with a credence function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that is countably additive since otherwise there will be an $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that the credence assigned to E is different from $\varphi(E)$. Of course, Bayesian agents who follow this path will find themselves subject to the slings and arrows hurled by those who think that rational credence should operate with mere finite additivity; but they can respond that the prudence of not betting against the odds set by QM demands that they tolerate the abuse.

Although we think that this line of argumentation has some appeal, we note that it can lead to unpleasant consequences should it turn out that quantum systems can be prepared in non-normal states. Repeat the above argumentation with a non-normal state $\overline{\varphi}$ in place of the normal state φ. Supposing that empirical success attends $\overline{\varphi}$, the same prudential considera-

tions as before now indicate that a rational agent who accepts the success of QM and knows that the system is prepared in state $\overline{\varphi}$ should adopt a credence function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that is merely finitely additive since otherwise the agent would be betting against the odds set by $\overline{\varphi}$. But now the slings and arrows of those who think that rationality demands countable additivity draw real blood since an agent who operates with a merely finitely additive credence function can be Dutch booked (recall Section 2.2). The spectre of such a quandary for agents, who face sure loss by Dutch book if they are merely finitely additive or almost sure loss by betting against QM if they are countably additive, would be banished by arguments that put normality of physically realizable quantum states beyond reasonable doubt; such arguments are devoutly to be desired.

6 Conclusion

We confess that when we were learning probability theory we thought that the issue of what form of additivity a probability function should satisfy is a merely technical issue that is important, if at all, only at the margins. On the contrary, the issue is crucial to understanding the nature of rational credence, the nature of quantum probability, and the relation between the two.

Our contribution to the discussion of these matters is modest. We pointed out that if quantum states are regarded as ontologically prior to the probabilities they induce, then the question of what form of additivity quantum probabilities satisfy reduces to the question of what types of states are to be countenanced in the practice of quantum mechanics. The answer to this question is obviously constrained by empirical considerations, but it is not a question that can be settled by experiment alone. Our answer is hedged: for most purposes normal quantum states suffice and, moreover, some aspects of quantum practice would be inexplicable if non-normal states were allowed; but we see no water tight global argument that only normal states are physically realizable.

The restriction to normal states means that quantum probabilities are completely additive; but except for applications of QM requiring a Hilbert space of outlandishly large dimension, one can say that quantum probabilities are countably additive since complete additivity coincides with countable additivity. And while the relation between rational credence and physical

probability may not be as straightforward as suggested by Lewis' Principal Principle, prudential considerations do suggest that adopting a credence function over the lattice of quantum propositions that is not countably additive is foolish behavior.

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