On Dedekind's Logicism

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The place of Richard Dedekind in the history of logicism is a controversial matter. From the point of view of contemporary philosophy of mathematics, what is of interest is Dedekind's contribution to the emergence of the modern mathematical style (axiomatic and structural analysis, set theory and foundations) and no doubt also his peculiar brand of structuralism understood as a philosophical position. The conception of logic incorporated in his work is certainly old-fashioned, in spite of innovative elements that would play an important role in late 19th and early 20th century discussions. Yet his understanding of logic and logicism remains of interest for the light it throws upon the development of modern logic in general, and logicistic views of the foundations of mathematics in particular. Concentration upon one or two "key figures" is not the best strategy for intellectual understanding: often we gain deeper understanding from paying attention to lesser known authors and their work sheds much light on that of the "key figures".

In a series of articles I have defended that one must distinguish two phases in the development of logicism up to the 1930s:[2] an early, triumphant phase from 1872 to the shocking discovery of the contradictions in the years 1897–1903, and a later phase of reconstruction where of course Russellian logicism based upon the theory of types was the main exponent. The 1890s was a crucial decade of explosive diffusion for this viewpoint, which found adherents in all the countries that represented mathematical powers – Germany, where the young Hilbert was among them, England, France, and Italy. The evidence suggests that German authors were the prime movers of this tendency, and I have defended the controversial claim that, as of 1900, Dedekind was still a more central reference in this respect than Frege. (The claim is of course linked with the fact that Frege did not enjoy a powerful reception before Russell, and also the Göttingen people – Hilbert, Zermelo – started to call attention to him from about 1903.)

Logicism defends two key theses, the *conceptual* thesis that the basic concepts of mathematics can be reduced to logical concepts alone, and the *doctrinal* thesis that the basic principles of mathematics can be derived from logical principles. Such was exactly Dedekind's standpoint. In his view, all of pure mathematics can be "divest[ed] … of [its] specifically arithmetic character" so as to be "subsumed under more general notions and under activities of

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[2] In English, Ferreirós (1996), (2001) and (2009). See also my book (1999), 2nd edition 2007.

the understanding *without* which no thinking is possible at all." And the universal logical principles governing such notions are such that "*with* [them] a foundation is provided for the reliability and completeness of proofs".[3] Indeed Dedekind succeeded in outlining a reduction the number system, algebra, and analysis to sets and mappings alone. Thus all of pure mathematics was proved to be an outgrowth of the pure laws of logic (modulo his conception of logic).

My claim that Dedekind must be seriously regarded as a logicist has always elicited skeptical eyebrows. Everybody accepts that he was a highly respected mathematician by 1890, and that he made most relevant contributions to the foundations of mathematics. But our understanding of "logic" is too strongly shaped by 20th century formal logic, and our conception of logicism is stamped by the mark of Frege and Russell.[4] Notice that Russell was not merely a prominent logician and logicist: through his prolific writing and influence as a philosopher, he established a standard interpretation of the recent history of logic; Frege and Peano played a key role in this interpretation, while authors such as Dedekind and Schröder were sidestepped.[5]

Placing Dedekind centre stage is perhaps difficult, since it requires us to show considerable flexibility in our conceptual reconstruction of the convoluted history of logical theory – of the very notion of logic. In my case, perhaps this was made easier by the fact that early on I was sufficiently perplexed by issues in the philosophy of logic, and very especially by the conflicting, often irreconcilable claims about logic that one can find in Frege, Wittgenstein, Russell, Carnap, and others. Let me insist that understanding Dedekind does pay off, among other things because it alerts us to the peculiarities of logical theory before 1928, and to the peculiarities of logicism before 1910 or even 1928.

We have gathered considerable evidence of a very positive reception for Dedekind's proposals in the 1890s. Ernst Schröder adopted the logicist viewpoint in explicit reference to Dedekind, and so did Hilbert in his lectures of the 1890s, all the way up to the *Grundlagen der Geometrie*. Among those less inclined to the logicist thesis, Peano's enthusiasm for Dedekind's work was evidenced in his publications from 1889, and Peirce remarked on the fuzzy borders between logic and mathematics with a reference to him (characteristically for the 1890s, no reference to Frege).[6] The relevance accorded to Dedekind's work by Frege can be measured

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[3] Quotations from the letter to Keferstein of Febr. 1890 (Dedekind 1890), 100. By "activities of the understanding" Dedekind meant set-formation and mappings; here it seems most adequate to read his terminology in Kantian spirit, and thus not at all psychologistically. See below.

[4] I must say that the critique of Dedekind's ideas in Russell's *Introduction to mathematical philosophy* (1919) is unfair most of the time, as should become clear from the analysis below. In fact, it conflicts with Russell's views in the *Principles* (1903)!

[5] Dedekind played a much more central role in Hilbert's interpretation (by which I mean his retrospective remarks, especially but not only in papers of the 1920s). Historians of science and historians of logic know well how such partisan interpretations of the past tend to be biased, and in fact they conflict with each other.

[6] For details and references, see Ferreirós (1999), 248-253.

from the fact that he discussed it both in the Preface and the Introduction to *Grundgesetze der Arithmetik*, his masterpiece (Frege 1893); the criticisms he expressed there are sharp, as usual, and off the mark in some cases. Lastly, perhaps it is relevant that Russell read Dedekind's work some years before reading Frege or even Peano.

Certainly the time was ripe for the serious proposal that pure mathematics is but a form of highly developed logic – "one of the greatest discoveries of our age", said Russell (1903, p. 5), according to whom this thesis was "very recent among mathematicians, and is almost universally denied by philosophers" (1903, xv). But my claim that Dedekind must be seriously regarded as a logicist still elicits skeptical eyebrows and critical responses. This piece is an attempt to clarify the matter by answering key criticisms and adding some more details.[7]

I: Dedekind and Mathematical Logic.

It is often written or thought that Dedekind's does not qualify as a reasonable form of logicism, because he did not develop a full system of formal logic. This in my view is the most relevant objection. Thus, Marcus Giaquinto in his recent philosophical account writes that Dedekind "did not have in mind any positive conception of logic".[8] But I shall argue for the opposite.

Frege (1893, pp. 2-3) complained that the basic principles on which Dedekind's work was based had nowhere been explicitly stated and that, in the absence of explicit formal derivations, it was impossible to check what exactly those principles were. He also said that Dedekind was able to cover a lot of ground only because none of his proofs was more than a sketch. This cannot be denied once we accept Frege's criteria and his new notion of a formal proof, and yet all other contemporaries judged it exactly in the opposite way: most of them (even Cantor) saw no need to devote so much space to explicit deduction of "obvious" results, and they had little or no understanding for Dedekind's painstaking way of following a strictly deductive paradigm.

Dedekind only entered into details concerning logic, its basic concepts, and its principles, to the limited extent he felt indispensable for the purpose at hand in *Was sind und was sollen die Zahlen?* (1888) – namely, to establish a foundation for the arithmetic of natural numbers and its extension up to the complex numbers. This limited purpose meant in particular that he did not need to discuss the elementary parts of logic having to do with propositional connectives and quantifiers; to be sure, he *uses* this logic in a clear way, but he does not theorize on it. However, as I have argued at length elsewhere, it should be obvious that what is of great relevance for

[7] I'll have particularly in mind a recent paper by Benis-Sinaceur (2008), in whose Conclusion the author makes explicit four main reasons why it would be wrong to call Dedekind a logicist. I should perhaps clarify that my disagreement with Benis-Sinaceur is only at the level of interpreting the meaning of logicism in the 1890s.

[8] Giaquinto (2002), 30. But see the more careful, and historically more truthful, appraisal of the situation in p. 33.

logicism is *not* the elementary layers of logical theory, but rather the higher levels. That part of logic which is essential for logicism is the *upper layers* that must bear the weight of reducing mathematical concepts and propositions.

These upper layers concerned the theory of sets and mappings, resp., classes and relations in Russell's terminology. From the now remote perspective of the 1900s, Russell stated (1903, 11) that logic consists of *three parts*, the theory of propositions, the theory of classes, and the theory of relations. Dedekind's logicism was based on the latter two, and he contributed in important ways to developing them and laying out their foundations.

Some people are puzzled to find that the concepts of set and mapping were simply regarded as logical ones. With regards to the concept of set, this was a usual view in the late 19th century, well represented for instance with Boole, Schröder, Peano, and Russell. Logic was taken to deal centrally with concepts, and sets or classes were *extensions* of concepts; the theory of sets or classes was simply a formal theory of concepts, a part of formal logic. There is ample evidence that Dedekind regarded the theory of sets as logic from at least 1872, and later in his life he would still speak of the "logical theory of sets" (*Systemlehre der Logik*)[9].

As for the concept of mapping, Dedekind argued that it was indeed indispensable to all thought, expressing as it does "the ability of the mind to relate things to things, to make a thing correspond to another, or to represent a thing by another" (Dedekind 1888, 335–336). He seems to have been thinking of the relations between word and object, between proposition and state of affairs, but also between an algebraic equation and a geometric figure. As he stressed, this notion of a map is likewise "absolutely indispensable" for arithmetic and pure mathematics, as witnessed by the role of functions, isomorphisms, but even at the most elementary level the role of the successor mapping.[10] Writers as influential as Schröder, Peirce, or Hilbert agreed, though the logicians expressed a clear preference to reconsider Dedekind's mappings within a general theory of relations.[11]

Even Frege acknowledged the parallelism between Dedekind's system and his own: his discussion of Dedekind's work aims to establish that sets must be replaced by concepts and concept-extensions, while maps have to be replaced by relations. He writes:

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[9] See Dedekind 1932, vol. 2, 113; 'Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler' (1897). For a complete exposition of the theory of sets, he recommends Schröder's *Vorlesungen* (see p. 112) – this confirms that he did not aim to be complete in Dedekind (1888).

[10] At the same time, somewhat mysteriously, Dedekind claimed that the whole science of arithmetic must be erected "upon this single foundation" (1888, 336). I hope to return to this topic in a future paper; the natural interpretation is that he thought it possible to reduce sets to maps (indeed it is, see e.g. von Neumann's axiomatisation of set theory), but it is hard to see how he might have planned to develop the idea.

[11] Thus, Ernst Schröder devoted two of his *Vorlesungen über die Algebra der Logik*, Vol. 3 (1895) to a reelaboration of Dedekind's work on mappings and chains, and he regarded this one of the "most important objectives" of his work (*ibid* p. 346; see chapters 9 and 12).

> The same holds for the word "correlation" [*Zuordnung*] as for the word "set". Both are now used widely in mathematics, yet a deeper insight into what they are really intended to mean is largely absent. If I am right in thinking that arithmetic is a branch of pure logic, then a purely logical expression must be chosen for "correlation". I take the word "relation". Concept and relation are the foundation stones upon which I erect my edifice. (Frege 1893, 3).[12]

Notice how, perhaps unwillingly, Frege is underscoring the parallelisms (extension of) concept ≡ set, (univocal)[13] relation ≡ mapping. (Although he speaks of "correlation" and not "mapping", both the context and §38 of *Grundgesetze*, where he defines "mapping" basically in Dedekind's sense, indicate that my reading is adequate.) Notice also that the connectives and quantifiers are not even mentioned, so that Frege offers support to my view that the *upper layers* of logical theory is the essential part for logicism.

Dedekind *chose* not to present his system by axioms, since for him – in good Kantian tradition – the word "axiom" connoted a non-logical principle (see Ferreirós 1999, 247-248), but he took pains *to state* as immediate consequences of basic logical notions, *or else to prove*, all of the propositions needed.

Benis-Sinaceur (2008) has written that Dedekind regarded logic as "*le moyen* de la rigueur, non *la base* des mathématiques", the means of rigor and not the basis of mathematics. But this interpretation seems hardly compatible with Dedekind's work, especially with his statements. He emphasized that all mathematics has the notions of set and map as its basis, and he stressed that these notions are purely logical. In accordance, he wrote that arithmetic, algebra, and analysis are only a part of logic (Dedekind 1888, 335, Preface). It can be said louder and at greater length, but not clearer. And let me stress it again: this was not an idiosyncratic vision of Dedekind's, but rather in agreement with the most relevant mathematical logicians of the 1890s (excepting Frege insofar as he objected to the extensional approach to logic).

Benis-Sinaceur has also stated that Dedekind understood logic in a rather vague sense, "antérieur aux précisions apportées par la logique mathématique," but in my view this also needs to be qualified. Dedekind's understanding of logic may be judged vague, especially since he acted as a mathematician, not a philosopher, spending no ink in an explanation of some of the basic tenets of his conception. And certainly he was influenced by *traditional* (*formal*) logic, as I have tried to make explicit in (Ferreirós 1996). But he relied on the precisions proposed by mathematical logicians, and he went on to propose precisions of his own. His understanding of the theory of classes was basically in agreement with Boolean logic, as shown e.g. by the fact that he discussed union and intersection as the key operations.

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[12] Translation by Beaney (*The Frege Reader*, p. 210). The extent to which Frege's criticism is inadequate should become more and more clear in the sequel; see also Tait (1996).

[13] By "univocal" I mean an *eindeutige Zuordnung*, many-one – a functional relation.

Dedekind had an active interest in the new logical calculi, as shown by the story of his interest in Schröder's work. The manuscript of *Zahlen*, drafted in the 1870s, contains several references to Schröder's 1877 book *Der Operationskreis des Logikkalküls* (*The circle of operations of the logical calculus*), a systematic exposition of Boolean logic. In section 1 of *Zahlen* (1888) he did not develop the Boolean calculus, but this was simply because it played no essential part in his reduction of arithmetic to logic. At the same time, Dedekind was more clear and explicit than previous logicians on key points such as the objecthood of sets and the principle of extensionality for sets. Later in the 1890s Dedekind became very interested in Schröder's *Vorlesungen*, which he studied carefully, and which even inspired him to do most significant work on the theory of lattices.

More original, of course, was his work on mappings. At the time, the theory of relations was a hot topic among the pioneers of mathematical logic, but its connections with advanced mathematics via functional relations had not been underscored, let alone analyzed in detail. Little wonder that Schröder was left in admiration of the "epoch-making" contribution made by Dedekind, given the difficulty of the advances he introduced into the calculus of logic in order to establish conclusively its connection with arithmetic (Schröder 1895, 346-352). The exposition of mapping theory in section 2 of (1888) is certainly a model, as seen e.g. by the treatment of the composition of maps;[14] also very neat and mature is the treatment of equipollence as an equivalence relation. But the theory of chains is just wonderful, in effect the most original and profound theoretical development in Dedekind's booklet. The concept of *chain of a subset* $A \subset S$ *under a map* $\varphi: S \rightarrow S$ was obtained by analyzing and generalizing the conditions that an internal mapping must satisfy in order to make proofs by induction possible and conclusive. As for its reach, let me remind you that in 1887 Dedekind was able to use this theory to prove the Cantor-Bernstein theorem; in the booklet he preferred to leave a (somewhat generalized) version of the crucial lemma as an exercise for readers.[15]

All of the above means that, without in the least denying the great contribution made by Frege (1879; 1893) with his analysis and formalization of elementary logic, especially quantification theory, we can regard Dedekind's as a perfectly reasonable form of logicism. Indeed, Schröder, Hilbert, and others regarded it as a most noteworthy logicistic proposal. It could even be said, comparing Dedekind's version of logicism with Russell's in the *Principles*

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[14] The only shortcoming is that Dedekind did not differentiate clearly between injectivity and bijectivity: he defined the first concept but in fact he frequently employed the second. In all likelihood, Dedekind found it trivial and innocuous to restrict the codomain (final set) to the image.

[15] See Ferreirós (1993).

of 1903, that the former was a more explicit logicistic proposal to the extent that the latter adhered to so-called "if-thenism."[16]

To summarize some key elements of Dedekind's contribution to set theory, we should emphasize the following: the terminology of *Dinge* and *Systeme*, taken up by Hilbert (in 1897, 1899 and 1900), and the statement that sets are "Dinge," objects; the axiom of extensionality; the conceptual analysis behind Zermelo's axiom of infinity; reliance on the unstated (and contradictory) principle of comprehension; the algebra of union and intersection. And his contribution to mapping theory: isolating the general notion itself (in the wake of previous work by Dirichlet and Riemann), dealing with injective and bijective mappings; the powerful theory of chains, the theorem of Cantor-Bernstein; and not least, in his algebraic studies, work with homomorphisms, isomorphisms, and automorphisms. The original theoretical developments for which he should be remembered include also the introduction of set-theoretic structures and the development of some crucial methods for their study (substructures, lattices, morphisms). His logicist ideas regarding mathematics led him to a view of mathematical existence that essentially prefigures that of Hilbert. And also regarding the deductive, axiomatic structuration of mathematical theories he is a key initiator.

Dedekind indeed had very positive ideas and conceptions to propose. In the end however, judging retrospectively, his contributions belong more to modern mathematics and algebra than to mathematical logic narrowly construed. But one should keep in mind that the golden age of logicism was characterized by the fact that those new ideas and methods seemed to link mathematics and logic indissolubly. Only insofar as the antinomies established that set theory goes well beyond logic, did this understanding change. This is just one of the ways in which placing Dedekind's work more centrally is helpful to understand the history of logicism.

II: A Role for Transcendental Logic?

Some aspects of Dedekind's language have led readers to think that his approach was mentalistic or even psychologistic to the point that one cannot identify his position as close in any way to Frege's logical thought. In his rhetoric we hear about "our thinking", about "acts of the understanding", and even about "my thought-world". Far from Frege's objectualism, he consistently presented the numbers as "free creations" of the human mind, from his earliest related piece in 1854 to his very last publication in 1913. What should we make of all this? Should we interpret his "logic" as mentalistic or not?

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[16] I thank Andrew Arana for this remark.

Dedekind went to University in the 1850s and attended among others lectures in philosophy by Hermann R. Lotze. At that time idealism was no doubt on the retreat in Germany, scientism, mechanism and materialism were on the rise, but the influence of Romantic thought and writings still endured. In my view Dedekind adopted then, early in life, some favorite ideas and forms of expression that he would keep throughout his life, e.g. the idea that numbers are "creations of the mind." But he would deprive such ideas more and more of any kind of idealistic or subjectivistic connotation. His logical ideas were objectivistic, even though some of his rhetoric preserved forms of expression that seem to contradict such a reading.

It seems very likely that Dedekind, like his friend Riemann, read at early age writings of Kant and of Leibniz;[17] and I find it hard to interpret some of his more characteristic views without bringing in at least a few elements of Kantian thought. Dedekind characterized logic as the doctrine of the "*Denkgesetze*," the laws of thought, which was just the mainstream view in the 19th century. In the Kantian view, such laws govern how we *should* think, not the psychology of how we actually think; pure logic is independent of psychology. This distinction did not originate with Frege, but was characteristic of the Kantian treatment of logic, which put forward an objectivistic conception of logic – even when Kant and followers spoke about the Self and its sphere of thinking.[18]

We shall consider below Dedekind's infamous proposition 66 in *Zahlen* (1888, 357). Thus it begins:

> 66. *S a t z*. Es gibt unendliche Systeme.
> Beweis. Meine Gedankenwelt, d. h. die Gesamtheit *S* aller Dinge, welche Gegenstand meines Denkens sein können, ist unendlich. …
>
> 66. *Theorem*. There are infinite sets.
> Proof. My thought-world, i.e. the totality *S* of all things which can be an object of my thought, is infinite. …

Not only is there talk of "my" thought and of an apparently psychological realm of thoughts, but in the proof a special role is played by a particular object: *mein eigenes Ich*, "my own Self." Could things be worse? Well, indeed they are not so bad. Let me remind you that Kant, in the *Critique of pure reason*, Transcendental Deduction, says things as the following:

> The "I think" must *be capable* of accompanying all my representations, for otherwise something would be represented in me which could not be thought; … All the diversity or manifold content

[17] Lotze's philosophy combined some elements of idealism and strong Leibnizian influences with post-Kantianism, or at least with mechanistic scientism. That German philosopher also expressed views close to a vague form of logicism and has been regarded as an important influence on Frege by at least one author.

[18] Indeed the Kantian tradition, which included authors such as Herbart and Drobisch, was at the time identified as a strictly "formal" conception of logic and contrasted with the views of the idealists (see Ueberweg 1882, 47-53; also Ferreirós 1996, sect. II).

> of intuition has, therefore, a necessary relation to the "I think," in the subject in which this diversity is found. But this representation, "I think," is an act of *spontaneity*; that is to say, it cannot be regarded as belonging to mere sensibility. I call it *pure apperception* … because it is self-consciousness which, whilst it gives birth to the representation "I think," must necessarily be capable of accompanying all our representations… (Kant, *Kritik* B 132)

Elsewhere he adds that the apperception "is the simple representation of the Self" (*ist die einfache Vorstellung des Ich*, B 68). This, I submit, is the Self to which Dedekind referred.

Now, of course Kant's *Ich* is not psychological, but "transcendental" as he liked to say. Already in formal logic, as we have noticed, Kant and followers emphasized the need to avoid psychologism. And the idea of transcendental logic is not to analyse some psychological mechanism, but rather a scheme that applies to any possible thinking subject. Kant's *Ich* is not a particular subject, but a necessary component of any process of thought and representation, intuitive or not.[19] In Dedekind's expression "meine Gedankenwelt," my thought-world, we must understand that "me" is the transcendental Self: one might just as well read *the* thought-world.

I should add that my reconstruction finds support in Hilbert's views on the matter expressed in 1905. Before saying that he cannot accept Dedekind's approach to the foundations of arithmetic because the notion of a totality of all things involves an unavoidable contradiction, Hilbert remarks that he "would characterize his method as *transcendental* insofar as he proves the existence of the infinite following a path whose fundamental idea is used in a similar way by philosophers" (Hilbert 1905, 175).

More generally, I believe that all of Dedekind's declarations regarding logic, thinking, acts of the understanding, the thought-world, numbers, and mathematics, must be interpreted in an objectivistic sense. Once we do this, the distance between his position and Frege's does not seem so unbridgeable. And this seems to be the way Dedekind himself saw it. In the second Preface to *Zahlen*, 1893, he emphasized the "points of very close contact" with Frege, the fact that they "stand upon the same ground" in crucial points of the analysis of number (as in particular mathematical induction; 1888, 342).

In the proof of theorem 66, Dedekind defines a mapping $\varphi: S \rightarrow S$ such that the image of a certain thought t is the thought "t can be an object of my thought". I call the reader's attention to the characteristic modal element, which Dedekind emphasizes: we are dealing with something that *can be* an object of thought, not with actual but with possible thoughts. This is interestingly close to Kant's statements like the one quoted above, and the idea that logic delineates the realm of the possible was not uncommon (from Leibniz to Wittgenstein). Plus, of course Dedekind

[19] The idea is very Cartesian, of course, but perhaps it can also be linked with the well-known truth of first-order logic with identity: $\exists x\ (x = x)$.

accepted that our actual thoughts, the realm of what is present to the psychological ego at any given time, do not form an actual infinity. This alone might suffice to establish that his position is not psychologistic: at stake is what belongs to the *Gedankenwelt* by right, not in any actual psychological experience.

Readers who may still not be convinced might probably object that Dedekind's view of numbers as "free creations of the mind" sets him in the antipodes of logicism. But, firstly, one must interpret creations of the transcendental or the logical mind – creations without the least mark of subjectivism. Second, I would like to call your attention to the fact that Dedekind's appeals to creation (*Schöpfung*) became more and more rigid, more and more strictly regulated by the general laws of logic. In 1854, he had in mind the way in which "we" successively create or generate the series 1, 2, 3, …, and also the ulterior "creation" of new kinds of number in response to a requirement of closure under the inverse operations. Already at that early point (he was 23 years old) Dedekind emphasized the lack of arbitrariness of these successive acts of "introduction" of new numbers, operations and functions in mathematics.[20] Later, in 1872, he proposed the creation of new individuals, the irrational numbers, as images of the irrational cuts on the set of rational numbers. But he stressed that no property of the new individuals could be ascribed arbitrarily; rather, everything depends strictly on the behaviour of the cuts, regulated by the laws of the logic of sets. To hammer this point, let me quote a letter where he emphasizes that nothing essential depends on the creation of new numbers:

> if one does not want to introduce any new numbers, I have nothing against it; the theorem I have proved ([1872,] § 5, IV) reads then as follows: the set of all cuts in the domain of the rational numbers, which itself is discontinuous, forms a continuous manifold (letter to Lipschitz, June 1876; Dedekind 1932, vol. 3, 471).

Some years later, in 1888, the "creation" that we are talking about becomes truly Pickwickian: suppose given a simply infinite set N, ordered by a mapping φ; "if in the consideration of" such a set

> we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the ordering mapping φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. In consideration of this freeing the elements from any other content (abstraction) we are justified in calling numbers a free creation of the human mind (*eine freie Schöpfung des menschlichen Geistes*; 1888, 360).

Now it is not even a matter of creating new individuals, but merely of regarding *given* individuals merely from the point of view of an axiomatically characterized structure to which

[20] See Dedekind (1854), especially p. 430: "diese Erweiterungen der Definitionen lassen der Willkür keinen Raum mehr, sondern sie folgen mit zwingender Notwendigkeit…".

they belong, abstracting from any other of their properties. Taking into account the purely logical character ascribed by Dedekind to the "general laws" reigning over sets and mappings, and the fact that the "creation" of numbers is strictly regulated by such laws, it seems to me that his talk of numbers as "free creations of the mind" does not cause any tension with a crisp form of logicism.

It is interesting to consider how these matters are linked with Dedekind's views on existence proofs and consistency (in the modern sense, *Widerspruchsfreiheit*, absence of contradiction). In another letter to Lipschitz, July 1876, Dedekind goes on to emphasize that

> nothing is more dangerous in mathematics, than to *presuppose* existences without sufficient proof… How shall the admissible assumptions of existence be distinguished from the innumerably many inadmissible ones …? Shall this be made to depend only on success, on the casual awareness of an internal contradiction?"[21]

This general theme was clearly very present in his mind between 1876 and 1888, and it led to the proofs for the existence of infinite sets and simply infinite sets. In the 1870s draft for *Zahlen* one finds the problem expressed, e.g. in its part II (out of V), when Dedekind formulates and proves the

> *Theorem of complete induction*. A concept (proposition), into which an arbitrary number n enters, is completely defined (proven) when it is defined (proven) for $n = 1$, and when a rule is given according to which, from the definition [*sic*] of the correctness of the proposition for an arbitrary n the definition (the truth) for n' can always be derived.

Dedekind gives a still rudimentary proof, but subsequently he inserts a side remark:

> The proof of the correctness of the proof-method from n to $n+1$ is right; by contrast the proof (completeness) of the concept-definition by the method from n to $n+1$ is at this point not yet sufficient; the existence (free from contradiction) of the concept remains dubious. This will first be possible through the *distinctness*, through consideration of the set [*n*]!!!!!! Foundation.[22]

(At this point, [*n*] denotes an initial segment of *N*; "distinct" is the technical term for an injective mapping, but also for bijectively related, i.e. equipollent sets.) Once more it is unfortunate that Dedekind devoted almost no space in *Zahlen* to an explanation of the methodological criteria he

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[21] Ded 1932, vol. 3, 477. Here is the full text: "da die stetige Vollständigkeit der reellen Zahlen für den wissenschaftlichen Aufbau der Arithmetik unentbehrlich ist, so ist unerläßlich von vorneherein genau zu *wissen*, wie vollständig das Gebiet der Grössen ist, weil Nichts in der Mathematik gefährlicher ist, als ohne genügenden Beweis Existenzen *anzunehmen* und zwar erst dann, wenn die Noth, das augenbickliche Bedürfniß es gebeut. Woran sollen die erlaubten Existenz-Annahmen erkannt und von den unzähligen unerlaubten unterschieden werden… ? Soll dies nur von dem Erfolge, von dem zufälligen Gewahrwerden eines inneren Widerspruchs abhängig gemacht werden?"

[22] This draft has been published in Dugac (1976), Appendix LVI; see p. 300. "Der Beweis der Richtigkeit der Beweismethode von n auf $n+1$ ist richtig; dagegen ist der Beweis (Vollständigkeit) der Begriffserklärung durch die Methode von n auf $n+1$ auf dieser Stelle noch nicht genügend; die Existenz (widerspruchsfrei) des Begriffs bleibt zweifelhaft. Dies wird erst möglich durch die *Deutlichkeit*, durch die Betrachtung des Systems [n]!!!!!! Fundament."

was setting for himself. But the question found expression shortly thereafter in the famous letter to Keferstein, Febr. 1890. Talking about theorems 66 and 72, Dedekind remarks:

> After the essential nature of the simply infinite system, whose abstract type is the number sequence *N*, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system *exist* at all in our realm of thought? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay). (van Heijenoort 1967, 101)

As I have defended elsewhere, Hilbert was embracing exactly the same viewpoint in his famous statements regarding consistency proofs expressed in the year 1900 – only later, from 1903, would he make the shift towards the notion of a syntactic proof of consistency (Ferreirós 2009).

Let me come back to Theorems 66 and 72 in *Zahlen* (1888, 357 & 359). Some readers find them rather mysterious, to the point of even finding difficulty to distinguish their contents, since superficially they seem to repeat the same argument. But this is not at all the case: Th. 66 is a proof that *the thought-world V* is Dedekind-infinite, hence there exist infinite sets; Th. 72 proves that there exists a *simply infinite* set. In order to prove 66, one must identify an injective mapping φ: $V \rightarrow V$ and one object not in the image $\varphi(V)$; we have seen that the mapping assigns to a certain thought t the image $\varphi(t)$: "*t* can be an object of my thought". This does not depend on any empirical aspect of concrete thought. As for the object that does not belong to $\varphi(V)$, Dedekind chooses "e.g." the *Ich*, "my own Self"; once again, what he chooses does not depend on any empirical, given characteristic of reality, but is (according to Kantian doctrine) a necessary accompaniment of any possible representation or thought. It seems to me that, all in all, Dedekind is trying hard to stick to the principle that logic does not depend on what there is, concretely given in actual reality, but has to do only with general laws that apply to any possible reality.

Theorem 72 assumes given any Dedekind-infinite set I (in particular one may consider V, in view of Th. 66, but also any other example). Since I is infinite, according to Dedekind's definition of infinity, it must come with (at least) one distinguished element e and an injective mapping ψ, such that $e \notin \psi(I)$. From this assumption it is proven that there is a subset of I – the chain $\psi_0(e)$ or minimal closure of the unitary set $\{e\}$[23] under ψ – which is a simply infinite set. It could hardly be more different than Th. 66. (Of course, if we apply the argument starting from

[23] It is well known that Dedekind's sign-language exploited a confusion of the thing e and the corresponding unitary set. Bad symbolism, but there was no conceptual misunderstanding; this is shown by several manuscripts, including one of 1889 where he denotes the unitary set by "[*e*]" (see also 'Gefahren der Systemlehre', where he shows that the ambiguity can be used to derive a contradiction).

V, the distinguished element "my own Self" and the mapping φ: $V \rightarrow V$ will be as above; but the difference remains exactly the same.)

III: Formal Logic, the Principle of Comprehension.

I have acknowledged as the most relevant objection that Dedekind did not develop a full system of formal logic. In effect, he did not elaborate a theory of deduction, and even less did he aim to reduce deduction to a calculus. One will look in vain for explicit treatment of the elementary rules of inference he was employing, of the propositional connectives or the quantifiers in his work (the latter is no doubt the most significant lacuna, and after all the quantifiers could have been regarded as set-operators). He did not have a system comparable to that of Peano, not to mention the much more rigorous one of Frege; even worse, Dedekind was not very interested in formal languages, and one should say that his "logic" was abstract and not formal.

This is linked with some mysterious ingredients of his viewpoint. The mysterious bit comes into play when one tries to go deeper into the question, how Dedekind's views and his methods relate to formal logic and the emergence of the new logical calculi. To make a long story short, his understanding of logic belongs to a tradition that falls outside van Heijenoort's famous distinction between logic-as-language and logic-as-calculus, a distinction thus shown not to be all inclusive.[24] Let me call the third way *logic-as-thought*, and say summarily that the early Hilbert of the 1900s, the Hungarian G. König in 1914, and Zermelo even as late as 1930, all belong to it. This tradition has remained inspirational to many mathematical logicians, I believe, even though it causes tension with the strictly formal orientation that is most characteristic of 20th century mathematical logic.

Let me now explain. Although Dedekind had some interest in the new logical calculi, to the point that in the late 1890s his style was influenced by Schröder's (e.g., he worked more explicitly formally with the axioms he set up for lattice theory), the overall orientation of his work was to move mathematics away from symbolic forms, and in particular to move algebra away from its traditional concentration of equations, polynomials and other "forms of representation". He has always been regarded as an exponent of the "conceptual turn" of German mathematics, insisting throughout his life on Riemann as a model and on the avoidance of symbolic forms of representation as a key guiding principle (see below).

This was not mere blurb, since that principle can be seen at work behind the most characteristic new methods that Dedekind developed. Interestingly, the issue has ramifications in what concerns the link between set theory and formal logic. In previous work I have called

[24] See van Heijenoort (1967*a*).

attention to the intriguing fact that the principle of comprehension is clearer in Dedekind's work of the 1870s than in *Zahlen* (1888). This makes little sense at first sight. *Zahlen* is a more explicit presentation of the basic notions and principles of Dedekind's foundational work, and one should expect to find the principle of comprehension here.

For those who may doubt that Dedekind was in favor of Comprehension, let me offer a few quotations. Particularly relevant is the "First draft 1872-1878" of *Zahlen*, edited by P. Dugac. In a text that must be dated 1872 or at most a couple years later, we find the clear statement:

> To possess a property E means to belong to a set S, namely the set S of all things that possess the property E.[25] (Dugac 1976, Appendix LVI, 296)

Also a bit earlier we find the following:

> A *set* or *collection* [or manifold] S of things is determined when for every thing it can be judged whether it belongs to the set or not. … All those things *possessing a common property* are usually treated, insofar as distinguishing between them is not important, as a new thing by contrast to the other things. This is called a set or collection of all those things.[26] (Dugac 1976, 293, emphasis added)

As one can see, there is a quite clear formulation and endorsement of Comprehension in the first draft of Dedekind's booklet.

Other mathematicians were also aware of the fact that Dedekind favored the principle of comprehension, and with it a logicistic conception of the foundations of set theory, hence "naïve" set theory. Here is what Cantor wrote to Hilbert in Nov. 1899:

> … the key point of his [Dedekind's] researches must be seen in the naïve assumption that *all well-defined collections, or systems*, are likewise «*consistent systems*». *You have convinced yourself that the aforementioned assumption is erroneous* [this remark refers to Hilbert (1900)]… (Purkert & Ilgauds 1987, 154)

Cantor is referring to the antinomies of the ordinals and of the alephs (usually named Burali-Forti's and Cantor's paradox) and to his proposed distinction between «consistent systems» or sets and «inconsistent systems», i.e., multiplicities that do not form a unity. He also emphasizes that Dedekind's approach is "*in diametrical opposition*" to Cantor's understanding of the foundations of set theory. Now, the principle that "all well-defined collections" are sets is but the principle of comprehension: for a set to be given, it is sufficient to determine it through a well-defined concept.[27]

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[25] In all cases I translate the German "System" by "set", just like Dedekind translated it into "ensemble" in French. No significant misrepresentation is caused by this.

[26] I should remind the reader that "thing" is a technical term of Dedekind's logic. A "thing" is a thought-object, "any object of our thinking" (Dugac 1976, 293; Dedekind 1888, 344) such as 1, my own self, or – say – Leibniz.

[27] Hilbert himself shared this conviction of Dedekind's up to 1897: "The collection of all alephs can be conceived as a well-defined and concrete set, for given any thing one can always determine whether it is

Coming back to *Zahlen*, Comprehension is still clearly visible in remark 60, which comments upon the (generalized) Theorem of complete induction, which Dedekind has just proven (1888, 355). But if one compares the published text with the first draft, it is interesting to see that the Theorem of complete induction receives a more abstract and purely set-theoretic formulation, while back in 1872 it was formulated dually for "a concept $f(n)$" and for a "*proposition* in which an indeterminate thing n of the set (1) appears".[28] But the main question is, why did Dedekind avoid to make the principle more explicit? Why did he not formulate it as clearly as around 1875?

The proper place would have been point 2. of the booklet, where he preferred the vague formulation: "It is very often the case that different things *a, b, c …* are conceived under a common viewpoint for any reason, and set together in the mind; one then says that they form a *set…*" (Dedekind 1888, 344). (By contrast, the principle of extensionality was emphasized; see 1888, 345.)

After a long puzzlement over this little mystery, I have come to the following conclusion. I believe that it was *fully intentionally* that Dedekind avoided setting the principle of comprehension as the basis for set theory. The reason was not that he somehow anticipated that Comprehension was contradictory – he did not foresee that. The reason had to do (1) with deep-seated convictions regarding the methodology of pure mathematics, and (2) with a possible shortcoming of that approach.

(1) From very early on, Dedekind became an admirer of Riemann's work and a partisan of the so-called "conceptual turn". He emphasized that mathematical concepts and theories ought not to be based on "particular forms of representation", but on "invariant characteristic properties" of a more abstract kind. Quoting Gauss, who had written that "such truths should be extracted from concepts rather than notations," he called this a "decision for the inner in contrast to the outer".[29] As an example, Dedekind regarded his definition of an ideal (as a set of integers closed under sum and difference, and under product by any integer in the ring) as based on a "characteristic inner property", in contrast to the definition, employed by Hurwitz, as the set of all numbers representable by

$$\eta_0\alpha_0 + \eta_1\alpha_1 + \ldots + \eta_r\alpha_r$$

an aleph or not; and nothing else can be required from a well-defined set." (quoted verbatim by Cantor in his letter of Oct. 2, 1897, from Hilbert's previous letter. See Purkert & Ilgauds 1987, 226).

[28] See Dugac 1976, 295. The set (1) is just **N**, the $\varphi_0(1)$ of (1888), i.e. the φ-chain of the unitary set {1}. To quote fully this noteworthy text: "If a concept $f(n)$ in which an indeterminate thing n of the set (1) appears (a function of n) is *defined* for $n = 1$, and if a general rule is given to derive from the determination of the concept for thing n the determination of the concept for its image n', then the concept is *completely* defined." A "parallel theorem" replaces proposition for concept, proven for defined.

[29] For details see Ferreirós (1999), 100-103, and 28-31.

with $\alpha_0, \alpha_1, \ldots, \alpha_r$ given integers, and $\eta_0, \eta_1, \ldots, \eta_r$ arbitrary integers in the ring. He regarded Hurwitz's definition as objectionable since it was based on an "outer form of representation" which could be replaced by infinitely many others (changing the basis $\alpha_0, \alpha_1, \ldots, \alpha_r$), hence did not make explicit the "invariance" of the defined concept (Dedekind 1895, 55).

The tendency was thus to decrease formalization in mathematics, while the development of formal calculi could be seen as increasing it.[30] Judged from this standpoint, reliance on Comprehension as the basis of set theory could make the realm of sets depend on "arbitrary" particular traits of the language under consideration. Sets would be defined through particular representations, thus disguising their essential invariance, and the same set might be definable in two different ways (e.g. the famous case of the unfeathered biped). Worse, it was even conceivable that this way of approaching the matter might compromise the richness of the realm of sets. This is related with the potential shortcoming:

(2) There is evidence that Dedekind was distrustful of linguistic representations for the reason that no language could be rich enough to allow explicit expressions for all mathematical objects. He arrived at this conclusion by reflecting on the definition or "creation" of the real numbers, and obviously he was convinced that no symbolic language (*Zeichensprache*) was rich enough to designate each one of them. I cannot judge when he came to this reflection, but it could well have been between 1875 and 1887, and related to his distancing from the principle of comprehension. In a letter to his good friend H. Weber, Jan. 1888, he is emphasizing the pedagogic convenience of introducing new individuals corresponding to the irrational cuts, and "the right we have to arrogate to ourselves such a creative faculty (*Schöpfungskraft*)"; and he writes:

> Whether the symbolic language suffices to designate each one of the new individuals [the irrationals] that are to be created, does not carry any weight; it always suffices to designate the individuals that turn up in any (limited) investigation.[31]

As one can see, he defends the possibility of "creating" non-denumerably many new individuals, on the basis of fully general logical laws, on the face of the impossibility for our

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[30] As Tappenden (2006) has emphasized, Frege too was influenced by Riemann and his "conceptual" style. There might seem to exist a tension between his and Dedekind's Riemannian background, but I do not see this as a necessary conclusion: the "particular forms of representation" that Riemann and Dedekind wished to avoid were arithmetico-analytic formulas (like power series), so their objections to such formulas do not affect the symbolic representation of thoughts in Frege's conceptography. It is interesting and certainly relevant to compare the letters exchanged between Frege and Hilbert in 1895 (Frege 1980, 32; notice how close Hilbert's first remark was to Dedekind's views). However, we cannot pursue this matter any further, at least here.

[31] Dedekind 1932, vol. 3, 490: "Ob ferner die Zeichensprache ausreicht, um alle neu zu schaffenden Individuen einzeln zu bezeichnen, fällt nicht ins Gewicht; sie reicht immer dazu aus, um die in irgend einer (begrenzten) Untersuchung autretenden Individuen zu bezeichnen."

sign-languages to determine them explicitly.[32] This may have become a strong reason to avoid defining the realm of sets as the counterpart of any given system of linguistic expressions.

The idea would then be that the principle "To possess a property E means to belong to a set S, namely the set S of all things that possess the property E" (see above) is only valid in one direction. Possessing a property means belonging to a set, but belonging to a set does not mean possessing a property that can be represented in a given symbolic language. You might try to object and refer to the property "$x \in$ S", but remember that the sets cannot be designated one by one, our sign-languages only suffice to designate the sets that turn up in limited investigations.

It seems to me quite likely that this lack of coherence (1) between the principle of comprehension and the methodological principle of avoiding forms of representation, plus (2) the potential shortcoming of the formal approach, were resolved by Dedekind in his decision to relegate Comprehension in *Zahlen*.

Before moving on, let me just indicate that all of this does not, in the least, mean that Dedekind was far from modern axiomatic thinking. There is a common misunderstanding that axiomatic thinking in mathematics is and must be formal, symbolic. But the mainstream tradition of axiomatic work in the 20th century has not been strictly formalistic – it is a somewhat "informal" style of work, based on a relatively naïve understanding of set theory. This mainstream style of work can be found in Dedekind's algebraic and number theoretic work, and also in Hilbert's *Grundlagen der Geometrie* – contrasting with his formalistic style of work in the 1920s.[33]

IV: A Reconstruction of Dedekind's Dichotomy Conception.

Even distancing oneself from Frege's excessive zeal (section I), it is obvious that Dedekind's work could be improved. His concepts of set and map were precise for all practical purposes, but the basic principles on which the system rested should have been laid out more clearly. Let me briefly indicate a possible reconstruction of Dedekind's mature viewpoint in set theory. This would be a version of the *dichotomy conception*, which holds that a set is determined top-down, by partitioning the class of everything into two parts,[34] but expressed in an abstract form. Although my reconstruction is more explicit than what we find in *Zahlen*, I have strived to make this proposal entirely consistent with the spirit of Dedekind's methods in the 1880s and 1890s.

[32] A set is fully determined when for every thing it is determinate whether it belongs to S or not, but the general laws of set theory (logic) do not depend at all on how this determination is brought about, or even on whether we are in a position to decide (*ob wir einen Weg kennen, um hierüber zu entscheiden*). It is not justified (*berechtigt*) to limit the "free conceptual formation in mathematics" (*freien Begriffsbildung in der Mathematik*) in the manner of Kronecker (Dedekind 1888, 345).

[33] See among other works Corry (1996), Sieg & Schlimm (2005), Ferreirós (1999) and (2009).

[34] See Gödel 1947, 180.

His theory of sets would be based on two “basic laws”, which he regarded as key ingredients of an *abstract logic* dealing with the theory of sets:

(A) *Subsets*: Any subcollection of a set S is itself a set, a “thing” S’ $\subseteq$ S.

(One must consider here not only subcollections definable by expressions in a symbolic language or *Zeichensprache*, but any arbitrary collection of elements of S.[35])

(B) *Universal Set*: The totality of all things is another thing, a set V, the “thought-world” (*Gedankenwelt*) or Universal set.

Let me remind you that “thing” is a technical term which applies to any object of thought, and the criterion of identity for things is the Leibnizian one (Dedekind 1888, 344).

The principle of comprehension can now be proven as a theorem, an easy consequence of (A) and (B):

Theorem of Comprehension. Given any property $E(x)$ there exists the set S = $\{x: E(x)\}$.

Proof. All those things that satisfy $E(x)$ are elements of V, which is a set by (B). Therefore, by (A), the subcollection of V which consists in $\{x: E(x)\}$ is itself a thing, a set S. Q.E.D.

It is well known that, conversely, Comprehension suffices to prove principle (B), but one can argue – as established in the discussion above – that, as soon as we require a formal version of Comprehension based on an explicit symbolic language, it is *not enough* to prove (A). This principle (A) is strictly stronger, and it entails Choice.

The interesting trait of this reconstruction is the abstract nature of the presupposed logic. As I indicated above, Dedekind does not belong to Frege’s tradition of logic-as-language, nor to Schröder’s of logic-as-calculus, but rather to a third tradition of logic-as-thought, abstract thought.

One might think that two indispensable basic principles escaped Dedekind’s attention: the principle of Powerset, which was needed in order to develop the theory of real numbers from the natural numbers; and the principle of Choice, implicitly used in some of his previous work

[35] This principle of subsets is thus meant to include but transcend the Separation principle of first-order ZFC.

and also towards the end of his booklet.[36] Let me add that I used to think this way, too. However, under the reconstruction presented above this is not correct.

Choice would be a somewhat obvious application of principles in the theory of maps or sets: given a family of sets F, one has a mapping which assigns to every $x \in$ F an element $e_x \in x$ (indeed there are infinitely many mappings, as soon as one of the $x \in$ F is infinite); alternatively, given the family F one has the union set $\bigcup$F = $\{x \colon \exists y\ (x \in y \wedge y \in \mathrm{F})\}$, which exists by the Theorem of Comprehension, and by principle (A) of Subsets there is a choice set G $\subset \bigcup$F which fulfils the desired condition.

As for the Powerset principle, one should note that every subcollection of S is a set by principle (A), and that the powerset $\wp$(S) is simply $\{x \colon x \subseteq \mathrm{S}\}$, which exists by the Theorem of Comprehension. (Dedekind did not formulate Powerset explicitly, and it will only appear in print with Russell (1903), but his reliance on the set of *all* cuts of **Q** for the definition of the real numbers was a crucial step in that direction.[37])

A somewhat related matter perhaps deserves to be mentioned here. In other work I have tried to make clear that Dedekind's methods led him to develop a "set-theoretic" style of axiomatic analysis that is quite different from the work of Peano on the natural numbers, or that of Hilbert on geometry and the real numbers. I have come to the conclusion that Peano himself saw such differences between his axiomatisation of 1889 and Dedekind's of 1888 (see Ferreirós 2005). To clarify the matter, let me remind you of Dedekind's axioms: A *simply infinite* set N has a distinguished element e and an ordering mapping φ such that

1. $\varphi(N) \subseteq N$ 2. $e \notin \varphi(N)$
3. $N = \varphi_0(e)$, i.e. N is the φ-chain of the unitary set $\{e\}$
4. φ is an injective mapping from N to N, i.e. if $\varphi(a) = \varphi(b)$ then $a = b$.

Leaving aside axioms 2. and 4., which are more easy to assimilate to Peano's axioms, the other two axioms are characteristically set-theoretic in the intended sense, and not elementary as most of Peano's and Hilbert's axioms.

Peano tended to impose conditions on the behaviour of his individuals, the natural numbers, and the operations on them. These are elementary conditions which most often are amenable to formalization within first-order logic. Dedekind establishes structural conditions on subsets of the (structured) sets he is defining, on the behaviour of relevant maps, or both things at a time. Axiom 1. says that N is closed under the map φ, axiom 3. says that N is the *minimal* closure of

[36] According to Heck (1998), this principle did not escape Frege's attention, who noticed it in the course of analyzing Dedekind's proofs; and it was also noticed by Peano and his associates (Bettazzi, B. Levi). See Moore (1982), chap. 1.
[37] Cantor considered and rejected it in letters to Hilbert of Oct. 1898; see Ferreirós 1999, 447-448.

the unitary set $\{e\}$ under φ. Such axioms are non-elementary and tend to require second-order logic for their formalisation, as happens with axiom 3 (however, 1. is easy to formalize in first order). This feature of his methodology becomes clearer still in Dedekind's algebraic work, for instance his use of module theory for establishing the theory of ideals, and his map-theoretic approach to Galois theory.

IV: Dedekind *vs*. Frege: Philosophical Differences.

There is of course no denying the differences between Dedekind and Frege: we find some strong similarities between their versions of logicism, but there are also key disparities. I do not wish to deny that the *style or methodology* of their ways of treating the foundations of arithmetic is quite different; however, it is my conviction that *philosophical* differences in their understanding of logic and mathematics were more important than any difference having to do with formal logic. (One can say that Frege tends to work more formally and, as far as possible, closer to an elementary logical treatment, while Dedekind works in an explicitly set-theoretic style. This is shown e.g. by the contrast between Dedekind's heavy use of chain theory in his development of the laws of arithmetic,[38] and Frege's development which relies mainly on the second-order system that has the Cantor-Hume principle as its sole axiom (Fregean arithmetic). The contrast is reflected in a difference that Frege himself emphasized, namely that Dedekind first defines the infinite and then takes the finite to be non-infinite, while Frege first defined the finite, after which the infinite appears as the non-finite.[39] It would be desirable that someone undertakes a full study of these methodological differences.)

Thus, I believe it was mainly philosophical differences – coupled with some incorrect and uncharitative interpretations that we have already discussed – that led Frege to insist on the idea that Dedekind's work "hardly contributes to its confirmation [of his logicistic viewpoint]" (1893, 3). We can summarize the key disparities between them as follows:

(i) the lack of a formal symbolic logic and above all formal derivations in Dedekind;

(ii) the intensional approach favored by Frege and Russell, contrasting with Dedekind's preference for the extensional; and

(iii) the structuralist approach to mathematical objects adopted by Dedekind, which contrasts with Frege's and Russell's "objectualism" or "singularism".[40]

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[38] Given σ: $S \rightarrow S$, and a subset $A \subseteq S$, the *chain of A* is the closure of A under σ in S, which Dedekind defines as an intersection of sets closed under σ (denoted $\sigma_0(A)$ or A_0). In the case of interest, where σ is the successor function, chains are infinite structures and so Dedekind's approach is heavily "infinitistic."

[39] See Frege's review of Cantor's *Zur Lehre vom Transfiniten*, in (Frege 1984, 180) and also Heck (1998), which compares Dedekind's with Frege's treatment.

[40] It is interesting that, while Dedekind is "less modern" as to (i), he was much closer to modern mathematics as to (ii) and (iii).

Dedekind's position was specific and peculiar compared to Frege, Russell, Peano, etcetera, and perhaps it was closer to Schröder's. But there is no reason to tie logicism so strongly with Frege's philosophical views that any approach deviating from his in a single trait immediately fails to be logicistic.

That however happens somewhat often. Benis-Sinaceur (2008) argues that another reason why Dedekind's views do not qualify as logicism is because he did not have a theory of *a priori* objects. This last is true, actually Dedekind's views tended to be incompatible with the idea that logic or mathematics has to do with *a priori* objects, if by that we mean objects that are fully and totally independent of thought. But the crux of logicism, the proposal of a reduction of mathematics to logic, is not necessarily linked with this philosophical understanding of logic. Logic is not necessarily a theory of *a priori* objects, but rather a theory of inference, of consequence relations. To use the typical 19th century phrase, logic is a theory of the (normative, not psychological) "laws of thought", or, if you prefer to follow the mature Frege, a theory of the "laws of being true" – which by no means necessarily require the existence of an object Truth à la Frege.

The contrast between an intensional and an extensional approach is central to understanding Frege's negative reaction to the notion of set. Instead of taking as a basis the extensional notion of set, he opted emphatically for the intensional notion of a concept.[41] Frege (1893, pp. 2-3) complained that Dedekind lacked a precise concept of set, and even that this concept of "system" or "class" favored by some mathematicians of his time had *nothing* to do with logic. This was especially because it was an extensional concept, not an intensional one as Frege preferred,[42] and because mathematicians failed to tie it with the logical notion of a concept. He even argued that the idea of set employed by his contemporaries is nothing but the idea of a concrete group of physical objects, e.g. a pile of stones; otherwise put, he insisted that a class or a set *is not* a concept-extension, that a class is and can only be a *class-as-many* (Russell 1903). As applied to Dedekind (1888) this is a totally unfair interpretation, which led Frege to incorrectly identify the reasons why Dedekind "preferred" to avoid the introduction of an empty set.[43]

[41] A concept is defined by its attributes or properties, and two concepts may be different although they apply to the same objects. This is what "intensional" signifies. Every featherless biped is a human being, and vice versa, but nobody would claim that the two concepts are identical; their extensions are, by Frege's Law V.

[42] Needless to say, the evaluation of this matter offered by a Quinean logician would be exactly the opposite of Frege's.

[43] Manuscripts written both years before 1888, and also immediately after, make it clear that Dedekind accepted the empty set on the very same grounds as Frege (as the extension of a contradictory concept), and also that he understood perfectly well the distinction between a and $\{a\}$.

But the great logician acknowledged explicitly that in his approach "concept" plays a role analogous to "set" in Dedekind's (Frege 1893, 3). And his theory of numbers depended crucially upon concept-extensions (*Begriffsumfänge*),[44] which in the more general form of values-ranges constituted the most important addition to his *Begriffsschrift* in the period from 1879 to the *Grundgesetze* (1893). Taking this into account, and given that Frege accepted the principle of comprehension,[45] the extreme differences alleged by him diminish very greatly, coming down (for the most part) to a question of philosophical interpretation of logic.

As we have seen, Dedekind's development from 1872 to 1888 was actually away from basing set theory on Comprehension, and thus away from a focus on concepts or intensional notions. His approach to mappings was also decidedly extensional, and his mathematical methods were becoming more and more abstract and structural. One might argue that this marks a clear methodological difference with respect to logicists such as Frege and Russell, who were insisting on the intensional. Indeed, the contrast intensional/extensional seems to be behind the contrasts in style and methodology that we remarked above.

But once again, although differences become apparent, if we take into account the history of logic and logicism broadly construed, the argument that such differences turn Dedekind into something else than a logicist is far from convincing. Schröder too was clearly in favor of an extensional approach to logic, and in Dedekind's time, under his influence, Schröder became a logicist. Something similar holds also for Hilbert (Ferreirós 2009). Or if we look at the 1920s, Frank Ramsey influentially tried to amend Russell's logicism, and in order to do so he abandoned the intensional emphasis of his predecessors and opted for an extensional viewpoint. Carnap's views in 1928 were also of this kind.

As significant as the intensional/extensional difference, and perhaps more important for present-day discussions, is the contrast between Frege's *singularism* and Dedekind's structuralism. Frege emphasized that numbers are and must be concrete, uniquely determined objects, each number being a single *particular* objective thing which is given *a priori* and which is "by its nature" suited to play the role it plays. Thus, the cardinal numbers (1, 2, 3, $\aleph_0$) are

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[44] See Frege's own statements in a letter to Russell (28 july 1902): "the question is, How do we apprehend logical objects? And I have found no other answer to it than this, as value-ranges of functions. I have always been aware that there were difficulties with this, and your discovery of the contradiction has added to them; but what other way is there?" (cited by Heck in Demopoulos 1995, p. 286). See also *Grundgesetze* (1893), vii and ix—x. Incidentally, it has been suggested by Sundholm (2001) that Frege adopted value-ranges in 1889/90 as a reaction to Dedekind's *Zahlen*, on which he lectured that year at Jena. On the need for value-ranges in order to analyze Dedekind's work within Frege's *Begriffsschrift*, see Heck (1998).

[45] Let me add that Frege's famous Law V, which he himself regarded as dubious in (1893, vii) and later blamed for giving rise to antinomies, is not a statement of Comprehension, but rather of the principle of Extensionality for concept-extensions (Ferreirós 1999, 303-304). Comprehension is inbuilt into Frege's symbolism, which allows unrestrictedly the transition from a concept $\Phi(\varepsilon)$ to its extension $\acute{\varepsilon}\Phi(\varepsilon)$. If one were to produce a zealous critique in Frege's style, one could say that his supposedly infallible tools of analysis were insufficient to make him analyze properly the principles on which he was relying.

intrinsically apt to express cardinalities, and the real numbers (√5, π, *e*) are intrinsically "designed", so to say, to measure magnitudes. One might perhaps talk of a form of essentialistic objectualism, but I prefer to call it Frege's "singularism".

In the case of Dedekind, we find nothing like that, but rather a form of structuralism. *Anything* could be (or better: play the role of) a natural or a real number, provided it belongs to a set which has the characteristic structure of the set of natural numbers – a "simply infinite" set – *viz*. the set of real numbers – a totally and densely ordered field with the Cut property. There is nothing singular or concrete about the objects, since any object whatsoever may belong to such a set. But the *structure* of their totality is uniquely determined, "invariant" as Dedekind said. Otherwise put, the choice of representatives is arbitrary, but the general structure (or "concept", to use the terminology of Hilbert and Dedekind) is not at all.[46]

Thus Frege's mathematical ontology is quite demanding, taking talk of mathematical objects in a very strict sense, which gives it a certain metaphysical 'thickness'. It deserves to be mentioned that he shared Kant's views on existence, denying that mere logical consistency could be a sufficient basis for existence – in sharp contrast to Hilbert and Dedekind, who on this matter were essentially on a par. The mathematical ontology of Dedekind (and Hilbert) is more flexible and tends to be non-metaphysical, since it is all a matter of potentialities (logical possibilities) rather than actualities. Hence its metaphysical 'lightness'.

In the case of Dedekind logic and pure mathematics, rather than a theory of *a priori* objects, would be *a theory of a priori structures*. (Naturally, he would not have used this clarifying word, but perhaps "concepts"; and he would consider related "objects", which however in his approach are merely possible objects of thought.) According to Dedekind, and as we saw above in Kantian spirit, two things must exist as a consequence of the pure laws of logic: one called the Self, another called the thought-world (*Gedankenwelt*). But no other existential assumption is required to prove that there is a simply infinite set (he believed in 1888), hence to prove that the set ***N*** of natural numbers exists – nor, he assumed, to show the existence of the set ***R*** of real numbers or the set ***C*** of complex numbers. The argument presented by Dedekind for such claims works purely at the conceptual level, at the level of logical possibilities, and carefully avoids concrete assumptions about the existence of particular things (other than the two just mentioned) or of anything empirical.

To summarize, these differences emphasize a half epistemic, half metaphysical point on which Frege and Russell were close, while Dedekind was not. The disparities make clear that there were different brands of logicism, and that logicism does not necessarily lead to

[46] This is related to the centrality of categoricity results in Dedekind's work. See Reck (2003), Awodey & Reck (2002).

singularism or "essentialistic objectualism". This difference was very important philosophically,[47] but it does not affect the issue whether Dedekind's basic principles qualify as logicistic or not. Taking into account also the work of other authors, I would suggest that there were two influential brands of logicism in 1900: one might be called *singular logicism* and is represented by Frege, being characterized by an intensionalist view of logic and by singularism in the above sense; the other might be called *structural logicism* and is represented by Dedekind, Hilbert, and to some extent Schröder, being characterized by an extensional conception of logic and by structuralism (which includes a "deflationary" view of mathematical ontology).

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[47] For instance, Russell's inclinations around 1900 were similar to Frege's, and this prompted him to be uncomfortable with Dedekind's viewpoint: he did not like structuralism.

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