An Intrinsic Theory of Quantum Mechanics:
Progress in Field's Nominalistic Program, Part I

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Abstract

In this paper, I introduce an intrinsic account of the quantum state. This account contains three desirable features that the standard platonistic account lacks: (1) it does not refer to any abstract mathematical objects such as complex numbers, (2) it is independent of the usual arbitrary conventions in the wave function representation, and (3) it explains why the quantum state has its amplitude and phase degrees of freedom.

Consequently, this account extends Hartry Field's program outlined in *Science Without Numbers* (1980), responds to David Malament's long-standing impossibility conjecture (1982), and establishes an important first step towards a genuinely intrinsic and nominalistic account of quantum mechanics.

I will also compare the present account to Mark Balaguer's (1996) nominalization of quantum mechanics and discuss how it might bear on the debate about "wave function realism." In closing, I will suggest some possible ways to extend this account to accommodate spinorial degrees of freedom and a variable number of particles (e.g. for particle creation and annihilation).

Along the way, I axiomatize the quantum phase structure as what I shall call a "periodic difference structure" and prove a representation theorem as well as a uniqueness theorem. These formal results could prove fruitful for further investigation into the metaphysics of phase and theoretical structure.

Keywords: quantum mechanics, wave function, phase structure, mathematical nominalism, intrinsic physical theory, indispensability argument, mathematical platonism, metaphysics of science.

Contents

1 Introduction

No doubt quantum mechanics is empirically successful (at least in the non-relativistic domain). But what it means remains highly controversial. Since its initial formulation, there have been many debates (in physics and in philosophy) about the ontology of a quantum-mechanical world. Chief among them is a serious foundational question about how best to understand the quantum-mechanical laws and the origin of quantum randomness. That is the topic of the quantum measurement problem (which arguably is a problem in physics which we should evaluate the solutions on the basis of empirical and super-empirical virtues). At the time of writing this paper, the following are serious contenders for being the best solution: Bohmian mechanics (BM), spontaneous localization theories (GRW0, GRWf, GRWm, CSL), and Everettian quantum mechanics (EQM and Many-Worlds Interpretation (MWI)).

There are deeper questions about quantum mechanics that have a philosophical and metaphysical flavor. Opening a standard textbook on quantum mechanics, we find an abundance of mathematical objects: Hilbert spaces, operators, matrices, wave functions, and etc. But what do they represent in the physical world? Are they ontologically serious to the same degree or are some merely dispensable instruments that facilitate calculations? In recent debates in philosophy of physics, there is considerable agreement that the universal wave function, modulo some mathematical degrees of freedom, represents something genuinely physical — the quantum state. In contrast, matrices and operators are convenient summaries but in no way essential to a fundamental description of the world.

However, the meaning of the quantum state is still unclear. We know its mathematical representation very well: a wave function, which is crucially involved in the dynamics of BM, GRW, and EQM. In the position representation, a scalar-valued wave function is a square-integrable function from the configuration space $\mathbb{R}^{3N}$ to the complex plane $\mathbb{C}$. But what does the wave function really mean? There are two ways of pursuing this question:

1. What kind of "thing" does the wave function represent? Does it represent a physical field on the configuration space, something quasi-nomological, or a *sui generis* entity in its own ontological category?
2. What is the physical basis for the mathematics used for the wave function? Which mathematical degrees of freedom of the wave function are physically genuine? What is the metaphysical explanation for the merely mathematical or gauge degrees of freedom?

Much of the philosophical literature on the metaphysics of the wave function has pursued the first line of questions.[1] In this paper, I will pursue the second one.

In particular, I will introduce an intrinsic theory of the quantum state. It answers the second line of questions by making explicit the physical basis for the usefulness of the mathematics of the wave function and providing a metaphysical explanation for why certain degrees of freedom in the wave function (the scale of the amplitude and the overall phase) are merely gauge. My intrinsic theory will also have the feature that the fundamental ontology does not include abstract mathematical objects such as complex numbers, functions, vectors, or sets.

My theory is therefore nominalistic in the sense of Hartry Field (1980). Recall: in his influential monograph *Science Without Numbers: A Defense of Nominalism*, Field advances a new approach to philosophy of mathematics by explicitly constructing nominalistic counterparts of the platonistic physical theories. In particular, he nominalizes Newtonian gravitation theory.[2] In the same spirit, Frank Arntzenius and Cian Dorr (2011) develop a nominalization of differential manifolds, laying down the foundation of a nominalistic theory of general relativity. Up until now, however, there has been no successful nominalization of quantum theory. In fact, it has been an open problem–both conceptually and mathematically–how it is to be done. The non-existence of a nominalistic quantum mechanics has encouraged much skepticism about Field's program of nominalizing fundamental physics and much optimism about the Quine-Putnam Indispensability Argument for Mathematical Objects. Indeed, there is a long-standing conjecture, due to David Malament (1982), that Field's nominalism would not succeed in quantum mechanics.

Therefore, being nominalistic, my intrinsic theory of the quantum state would advance Field's nominalistic project and provide (the first step of) an answer to Malament's skepticism. Moreover, it will shed light on several related issues in the metaphysics of quantum mechanics.

Here is the roadmap: I will first explain (in §2) the two visions for a fundamental physical theory of the world: the intrinsic vision and the nominalistic vision. I will

[1] Albert (1996), Loewer (1996), Wallace and Timpson (2010), North (2012), Ney (2012), Maudlin (2013), Goldstein and Zanghì (2013), Miller (2013), Bhogal and Perry (2015), Chen (forthcoming), and Chen (ms.).

[2] It is not quite complete as it leaves out integration.

then discuss why quantum theory may seem to resist the intrinsic and nominalistic reformulation. Next (in §3), I will write down an intrinsic and nominalistic theory of the quantum state. Finally (in §4), I will discuss (1) how this account bears on the nature of phase and the debate about wave function realism, and (2) in which ways my account is superior to the account in Balaguer (1996). I will also briefly sketch how to extend the present account of the quantum state to a variable number of particles (e.g. in the presence of particle creation and annihilation), how to develop a Schrödinger dynamics in terms of the intrinsic relations, and how to carry out nominalistic integration to obtain probabilities (to be written in a sequel paper).

Along the way, I axiomatize the quantum phase structure as what I shall call a "periodic difference structure" and prove a representation theorem as well as a uniqueness theorem. These formal results could prove fruitful for further investigation into the metaphysics of quantum mechanics.

2 The Two Visions and the Quantum Obstacle

There are, broadly speaking, two grand visions for what a fundamental physical theory of the world should look like. (And of course there exist other visions and aspirations.) The first is what I shall call the intrinsicalist vision, the requirement that the fundamental theory be written in a form without any reference to arbitrary conventions such as coordinate systems and units of scale. The second is the nominalistic vision, the requirement that the fundamental theory be written without any reference to mathematical objects. The first one is familiar to mathematical physicists from the development of synthetic geometry and differential geometry. The second one is familiar to philosophers of mathematics and philosophers of physics working on the ontological commitment of physical theories. First, I will describe the two visions, explain their motivations, and illustrate with some examples. Next, I will explain why quantum mechanics seems to frustrate both programs.

2.1 The Intrinsicalist Vision

The intrinsicalist vision is best illustrated with some history of Euclidean geometry. Euclid succeeded in showing that complex geometrical facts can be demonstrated using rigorous proof on the basis of simple axioms. However, Euclid's axioms do not mention real numbers or coordinate systems, for they were not yet discovered. They are stated with only predicates of congruence and betweenness. With these concepts, Euclid was able to derive a large body of geometrical propositions stated in terms of congruence and betweenness.

Real numbers and coordinate systems were introduced to facilitate the derivations. With the full power of real analysis, the metric function defined on pairs of tuples of coordinate numbers can greatly speed up the calculations, which usually take up many steps of logical derivation on Euclid's approach. But what are the significance of the real numbers and coordinate systems? When representing a 3-dimensional Euclidean space, a typical choice is to use $\mathbb{R}^3$. It is clear that such a representation

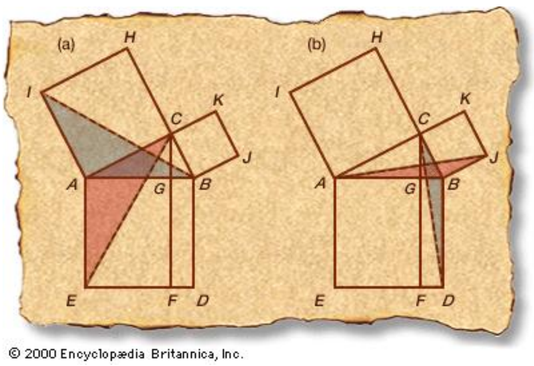

Figure 1: Euclid's Windmill proof of the Pythagorean Theorem. No coordinate systems or real numbers were used. Cartesian coordinates were invented much later to facilitate derivations.

has much surplus (or excess) structure: the origin of the coordinate system, the orientation of the axis, and the scale are all arbitrarily chosen (sometimes conveniently chosen for ease of calculation). There is "more information" or "more structure" in $\mathbb{R}^3$ than in the 3-dimensional Euclidean space. In other words, the $\mathbb{R}^3$ representation has gauge degrees of freedom.

The real, intrinsic structure in the 3-dimensional Euclidean space–the structure that is represented by $\mathbb{R}^3$ up to the Euclidean transformations–can be understood as an axiomatic structure of congruence and betweenness. In fact, Hilbert 1899 and Tarski 1959 give us ways to make this statement more precise. After offering a rigorous axiomatization of Euclidean geometry, they prove a representation theorem: any structure instantiates the betweenness and congruence axioms of 3-dimensional Euclidean geometry if and only if there is a 1-1 embedding function from the structure onto $\mathbb{R}^3$ such that if we define a metric function in the usual Pythagorean way then the metric function is homomorphic: it preserves the exact structure of betweenness and congruence. Moreover, they prove a uniqueness theorem: any other embedding function defined on the same domain satisfies the same conditions of homomorphism if and only if it is a Euclidean transformation of the original embedding function: a transformation on $\mathbb{R}^3$ that can be obtained by some combination of shift of origin, reflection, rotation, and positive scaling.

The formal results support the idea that we can think of the genuine, intrinsic features of 3-dimensional Euclidean space as consisting directly of betweenness and congruence relations on spatial points, and we can regard the coordinate system ($\mathbb{R}^3$) and the metric function as extrinsic representational features we bring to speed up calculations. (Example: Figure 1. Exercise: please try to prove the Pythagorean Theorem with and without real-numbered coordinate systems.) The merely representational artifacts are highly useful but still dispensable.

There are several advantages of having an intrinsic formulation of geometry. First, it eliminates the need for a large class of arbitrary conventions: where to place the origin, how to orient the axis, and what scale to use. Second, in the absence of these arbitrary conventions, we can look directly into the real structure of the geometrical objects without worrying that we are looking at some merely representational artifact (or gauge degrees of freedom). This gives us a more perspicuous picture of the geometrical reality.

The lessons we learn from the history of Euclidean geometry can be extended to other parts of physics. For example, people have long noticed that there are many gauge degrees of freedom in the representation of both scalar and vector valued physical quantities: temperature, mass, potential, and field values. There has been much debate in philosophy of physics about what structure is physically genuine and and what is merely gauge. It would therefore be helpful to go beyond the scope of physical geometry and extend the intrinsic approach to physical theories in general.

Hartry Field (1980), building on previous work by Krantz et al. (1971), ingeniously extends the intrinsic approach to Newtonian gravitation theory. The result is an elimination of arbitrary choices of zero field value and units of mass. His conjecture is that all physical theories can be "intrinsicalized" in one way or another.

2.2 The Nominalist Vision

As mentioned earlier, Field (1980) provides an intrinsic version of Newtonian gravitation theory. But the main motivation and the major achievement of his project is a defense of nominalism, the thesis that there are no abstract entities, and, in particular, no abstract mathematical entities such as numbers, functions, and sets.

The background for Field's nominalistic project is the classic debate between the mathematical nominalist and the mathematical platonist, the latter of whom is ontologically committed to the existence of abstract mathematical objects. Field identifies that a main problem of maintaining nominalism is the apparent indispensability of mathematical objects in formulating our best physical theories:

> Since I deny that numbers, functions, sets, etc. exist, I deny that it is legitimate to use terms that purport to refer to such entities, or variables that purport to range over such entities, in our ultimate account of what the world is really like.
>
> This appears to raise a problem: for our ultimate account of what the world is really like must surely include a physical theory; and in developing physical theories one needs to use mathematics; and mathematics is full of such references to and quantifications over numbers, functions, sets, and the like. It would appear then that nominalism is not a position that can reasonably be maintained.[3]

In other words, the main task of defending nominalism would be to respond to the Quine-Putnam Indispensability Argument:[4]

[3] Field (2016), Preliminary Remarks, p.1.

[4] The argument was originally proposed by W. V. Quine and later developed by Putnam (1971). This version is from Colyvan (2015).

P1 We ought to be ontologically committed to all and only those entities that are indispensable to our best theories of the world. [Quine's Criterion of Ontological Commitment]

P2 Mathematical entities are indispensable to our best theories of the world. [The Indispensability Thesis]

C Therefore, we ought to be ontologically committed to mathematical entities.

In particular, Field's task is to refute the second premise–the Indispensability Thesis. Field proposes to reformulate all physical theories into attractive nominalistic versions that do not quantify over mathematical objects.

Field's nominalistic versions of physical theories would have significant advantages over their platonistic counterparts. First, the nominalistic versions illuminate what exactly in the physical world provide the explanations for the usefulness of any particular mathematical representation. After all, even for a platonist, she might accept that numbers and coordinate systems do not really exist in the physical world but merely represent some concrete physical reality. She merely disputes that we can do physics without the mathematical objects. Second, as Field has argued, the nominalistic physics seems to provide better explanations than the platonistic counterparts, for the latter would involve explanation of physical phenomena by things (such as numbers) external to the physical processes themselves.

Field has partially succeeded by writing down an intrinsic theory of physical geometry and Newtonian gravitation, as it contains no explicit first-order quantification over mathematical objects, thus qualifying his theory as nominalistic. But what about other theories? Despite the initial success of his project, there has been significant skepticism about whether his project can extend beyond Newtonian gravitation theory to more advanced theories such as quantum mechanics.

2.3 Obstacles From Quantum Theory

We have looked at the motivations for the two visions for what the fundamental theory of the world should look like: the intrinsic vision and the nominalistic vision. They should not be thought of as competing against each other. They often converge on a common project. Indeed, Field's reformulation of Newtonian Gravitation Theory is both intrinsic and nominalistic.[5]

Both have had considerable success in certain segments of classical theories. But with the rich mathematical structures and abstract formalisms in quantum mechanics, both seem to run into obstacles.

[5]Moreover, the intrinsicalist and nominalistic visions can also come apart. For example, we can, in the case of mass quantities, adopt an intrinsic but platonistic theory of mass ratios. We can also adopt an extrinsic but nominalistic theory of mass relations by using some arbitrary object (say, my water bottle) as standing for unit mass and assigning comparative relations between that arbitrary object and every other object.

David Malament was one of the earliest critics of the nominalistic vision. He voiced his skepticism in his influential review of Field's book. Malament states his general worry as follows:

> Suppose Field wants to give some physical theory a nominalistic reformulation. Further suppose the theory determines a class of mathematical models, each of which consists of a set of "points" together with certain mathematical structures defined on them. Field's nominalization strategy cannot be successful unless the objects represented by the points are appropriately physical (or non-abstract)...But in lots of cases the represented objects *are* abstract. (Malament (1982), pp. 533, emphasis original.)

Given his *general* worry that, often in physical theories, it is abstracta that are represented in the state spaces, Malament conjectures that, in the specific case of quantum mechanics, Field's strategy of nominalization would not "have a chance":

> Here [in the context of quantum mechanics] I do not really see how Field can get started at all. I suppose one can think of the theory as determining a set of models—each a Hilbert space. But what form would the recovery (i.e., representation) theorem take? The only possibility that comes to mind is a theorem of the sort sought by Jauch, Piron, *et al*. They start with "propositions" (or "eventualities") and lattice-theoretic relations as primitive, and then seek to prove that the lattice of propositions is necessarily isomorphic to the lattice of subspaces of some Hilbert space. But of course no theorem of this sort would be of any use to Field. What could be worse than *propositions* (or *eventualities*)? (Malament (1982), pp. 533-34.)[6]

As I understand it, Malament suggests that there are no good places to start nominalizing non-relativistic quantum mechanics. This is because the obvious starting point, according to Malament and other commentators, is the Hilbert space. However, *prima facie*, this starting point seems strange for at least two reasons. First, it is hard to see what concrete physical structure can provide the basis for a representation theorem of the kind sought after by the nominalist. But more importantly, as we have learned from the debates in quantum foundations, although the Hilbert space is a convenient device for the mathematical formulation, it is dispensable for a sound ontological interpretation of quantum theories, platonistic or otherwise. To connect quantum formalism to the real-world laboratory results, we represent the quantum state as a wave function on a high-dimensional configuration space, which is constructed from particle configurations in the physical space. The Hilbert space is merely a space of possible wave functions; but each wave function is

[6] Malament also gives the example of classical Hamiltonian mechanics as another specific instance of the general worry. But this is not the place to get into classical mechanics. Suffice to say that there are several ways to nominalize classical mechanics. Field's nominalistic Newtonian Gravitation Theory is one way. Arntzenius and Dorr (2011) provides another way.

defined as a function from the configuration space to complex values. Thus, being much more closely related to the physical space(time), the configuration space seems to be a more natural place to start for the nominalization project. However, the problem that Malament raises remains, because (*prima facie*) it is not clear how to think about the configuration space in the nominalistic framework. Therefore, at least *prima facie*, quantum mechanics seems to frustrate the nominalistic vision.

Moreover, the mathematics of quantum mechanics comes with much conventional structure that is hard to get rid of. For example, we know that the exact value of the amplitude of the wave function is not important. For that matter, we can scale it with any arbitrary positive constant. It is true that we usually choose the scale such that we get unity when integrating the amplitude over the entire configuration space. But that is merely conventional. We can, for example, write down the Born rule with a proportionality constant to get unity in the probability function:

$$P(x \in X) = Z \int_X |\Psi(x)|^2 dx,$$

where Z is a normalization constant.

Another example is the overall phase of the wave function. As we learn from modular arithmetic, the exact value of the phase of the wave function is not physically significant, as we can add a constant phase factor to every point in configuration space and the wave function will remain physically the same: producing exactly the same predictions in terms of probabilities.

All these gauge degrees of freedom are frustrating from the point of view of the intrinsicalist vision. What exactly is going on in the real world that allows for these gauge degrees of freedom but not others? What is the most metaphysically perspicuous picture of the quantum state, represented by the wave function? Many people would respond that the quantum state is projective, meaning that the state space for the quantum state is not the Hilbert space, but its quotient space: the projective Hilbert space. It can be obtained by quotienting the usual Hilbert space with the equivalence relation $\psi \sim Re^{i\theta}\psi$. But this does little to relieve frustrations. These people face a similar question: what exactly is going on in the real world that allows for quotienting with this equivalence relation but not others?[7] No one, as far as I know, has offered an intrinsic picture of the quantum state, even in the non-relativistic domain.

In short, at least *prima facie*, both the intrinsicalist vision and the nominalist vision are challenged by quantum mechanics.

[7]These questions, I believe, are in the same spirit as Ted Sider's 2016 Locke Lecture (ms.), and especially his final lecture on theoretical equivalence and against what he calls"quotienting by hand."

3 An Intrinsic and Nominalistic Account of the Quantum State

In this section, I will propose a new theory of the quantum state based on some crucial lessons we learned from the debates about wave function realism.[8] As we shall see, it does not take much to overcome the "quantum obstacle." For simplicity, I will focus on the case of a quantum state for a constant number of identical particles without spin.

3.1 The Mathematics of the Quantum State

First, let me explain my strategy for nominalizing non-relativistic quantum mechanics.

1. I will start with a Newtonian space-time, whose nominalization is readily available.[9]
2. I will use symmetries as a guide to fundamentality and identify the intrinsic structure of the quantum state on the Newtonian space-time. This will be the goal for the remaining part of the paper. (Here we focus only on the quantum state, because it is novel and it seems to resist nominalization. But the theory leaves room for additional ontologies of particles, fields, mass densities supplied by specific interpretations of QM; these additional ontologies are readily nominalizable.)
3. In future work, I will develop nominalistic translations of the dynamical equations and generalize this account to accommodate more complicated quantum theories.

Before we get into the intrinsic structure of the quantum state, we need to say a bit more about its mathematical structure. For the quantum state of a spinless system at a time t (see Figure 2), we can represent it with a scalar-valued wave function:

$$\Psi_t : \mathbb{R}^{3N} \to \mathbb{C},$$

where N is the number of particles in the system, $\mathbb{R}^{3N}$ is the configuration space of N particles, and $\mathbb{C}$ is the complex plane. (For the quantum state of a system with spin, we can use a vector-valued wave function whose range is the spinor space—$\mathbb{C}^{2^N}$.)

My strategy is to start with a Newtonian space-time (which is usually represented by a Cartesian product of a 3-dimensional Euclidean space and a 1-dimensional

[8]Here I'm taking the "Hard Road" to nominalism. As such, my goal is to reformulate quantum mechanics such that *within the theory* it no longer refers (under first-order quantifiers) to mathematical objects such as numbers, functions, or sets. To arrive at my theory, and to state and prove the representation theorems, I use and refer to a lot of mathematics. But these are parts of the meta-theory to explicate the relation between my account and the platonistic counterpart and to argue (by *reductio*) against the indispensability thesis. Thanks to Andrea Oldofredi for suggesting that I make this clear.

[9]It is an interesting question what role Galilean relativity plays in non-relativistic quantum mechanics. In future work, I'd like to say more about its significance and how it relates to the nominalistic theory.

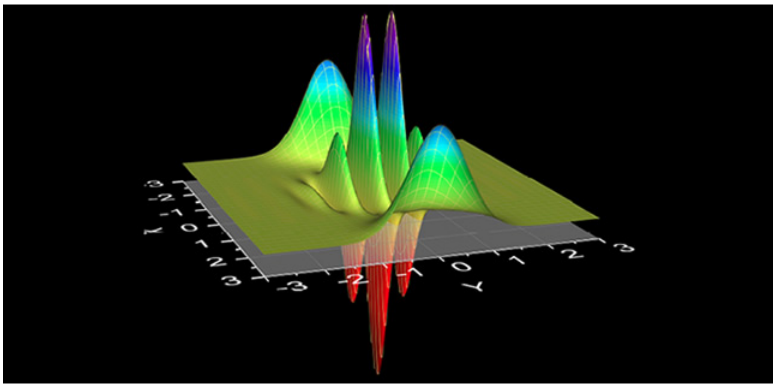

Figure 2: A wave function for a two-particle system in 1-dim space. Source: www.physics.auckland.ac.nz

time). If we want to nominalize the quantum state, what should we do with the configuration space $\mathbb{R}^{3N}$? As is now familiar from the debate about wave function realism, there are two ways of interpreting the fundamental physical space for a quantum world:

1. $\mathbb{R}^{3N}$ represents the fundamental physical space; the space represented by $\mathbb{R}^3$ only *appears* to be real; the quantum state assigns a complex number to each point in $\mathbb{R}^{3N}$. (See Figure 2. Analogy: classical field.)

2. $\mathbb{R}^3$ represents the fundamental physical space; the space represented by $\mathbb{R}^{3N}$ is a mathematical construction—the configuration space; the quantum state assigns a complex number to each region in $\mathbb{R}^3$ that contains N points (i.e. irregular and disconnected regions are allowed). (Analogy: multi-field)

Chen (forthcoming) argues that given our current total evidence, option (2) is a much better interpretation of non-relativistic quantum mechanics. I will not rehearse the arguments here. But one of the key ideas is that we can think of the complex-valued function as really "living on" the 3-dimensional physical space, in the sense that it assigns a complex number not to each *point* but each N-element *region* in physical space. We call that a "multi-field."[10]

Taking the wave function into a framework friendly for further nominalization, we can perform the familiar technique of decomposing the complex number $Re^{i\theta}$ into two real numbers: the amplitude R and the phase θ. That is, we can think of the compex-valued multi-field in the physical space as two real-valued multi-fields:

$$R(x_1, x_2, x_3, ..., x_N), \theta(x_1, x_2, x_3, ..., x_N).$$

Here, since we are discussing Newtonian space-time, the $x_1.....x_N$ are simultaneous space-time points. We can think of them as: $(x_{\alpha_1}, x_{\beta_1}, x_{\gamma_1}, x_t)$, $(x_{\alpha_2}, x_{\beta_2}, x_{\gamma_2}, x_t)$, ..., $(x_{\alpha_N}, x_{\beta_N}, x_{\gamma_N}, x_t)$.

[10]Unfortuantely, this is a misleading label for this object. I am following Belot (2012) here. See Arntzenius and Dorr (2011) for a completely different object called the "multi-field."

Now the task before us is just to nominalize and intrinsicalize the two multi-fields. In §3.2 and §3.3, we will find two physical structures (Quantum State Amplitude and Quantum State Phase), which, via the appropriate representation theorems and uniqueness theorems, justify the use of complex numbers and explain the gauge degrees of freedom in the quantum wave function.[11]

3.2 Quantum State Amplitude

The amplitude part of the quantum state is (like mass density) on the ratio scale, i.e. the physical structure should be invariant under ratio transformations

$$R \to \alpha R.$$

We will start with the Newtonian space-time and help ourselves to the structure of **N-Regions**: collection of all regions that contain exactly N simultaneous space-time points (which are irregular and disconnected regions). We start here because we would like to have a physical realization of the platonistic configuration space. The solution is to identify configuration points with certain special regions of the physical space-time.[12]

In addition to **N-Regions**, the quantum state amplitude structure will contain two primitive relations:

- A two-place relation Amplitude–Geq ($\succeq_A$).
- A three-place relation Amplitude–Sum (S).

[11]In the case of a vector-valued wave function, since the wave function value consists in 2^N complex numbers, where N is the number of particles, we would need to nominalize 2^{N+1} real-valued functions:

$$R_1(x_1, x_2, x_3, ..., x_N), \theta_1(x_1, x_2, x_3, ..., x_N), R_2(x_1, x_2, x_3, ..., x_N), \theta_2(x_1, x_2, x_3, ..., x_N), ...$$

But there is an additional wrinkle of how to best handle the gauge degrees of freedom in the orientation in the spin-space. I shall leave that to future work.

[12]Notes on mereology: As I am taking for granted that quantum mechanics for indistinguishable particles (sometimes called identical particles) works just as well as quantum mechanics for distinguishable particles, I do not require anything more than Atomistic General Extensional Mereology (AGEM). That is, the mereological system that validate the following principles: Partial Ordering of Parthood, Strong Supplementation, Unrestricted Fusion, and Atomicity. See Varzi (2016) for a detailed discussion.

However, I leave open the possibility for adding structures in **N-Regions** to distinguish among different ways of forming regions from the same collection of points, corresponding to permuted configurations of distinguishable particles. We might need to introduce additional structure for mereological composition to distinguish between mereological sums formed from the same atoms but in different orders. This might also be required when we have entangled quantum states of different species of particles. To achieve this, perhaps we can borrow some ideas from Kit Fine's "rigid embodiment" and add primitive ordering relations to enrich the structure of mereological sums.

For helpful explanations for how to understand the Symmetrization and Anti-Symmetrization Postulates in the context of constructing dynamics for indistinguishable particles, see Dürr et al. (2006) and Dürr et al. (2007). For philosophical discussions about how it relates to the debate about wave function realism, see Chen (forthcoming).

Interpretation: $a \geq_A b$ iff the amplitude of N-Region a is greater than or equal to that of N-Region b; $S(a,b,c)$ iff the amplitude of N-Region c is the sum of those of N-Regions a and b.

Define the following short-hand (all quantifiers below range over only N-Regions):

1. $a =_A b := a \geq_A b$ and $b \geq_A a$.

2. $a >_A b := a \geq_A b$ and not $b \geq_A a$.

Next, we can write down some axioms for Amplitude–Geq and Amplitude–Sum.[13] Again, all quantifiers below range over only N-Regions. $\forall a,b,c:$

G1 (Connectedness) Either $a \geq_A b$ or $b \geq_A a$.

G2 (Transitivity) If $a \geq_A b$ and $b \geq_A c$, then $a \geq_A c$.

S1 (Associativity*) If $\exists x$ s.t. $S(a,b,x)$ and $\forall x'$ [if $S(a,b,x')$) then $\exists y$ s.t. $S(x',c,y)$], then $\exists z$ s.t. $S(b,c,z)$ and $\forall z'$ [if $S(b,c,z')$) then $\exists w$ s.t. $S(a,z',w)$] and $\forall f,f',g,g'$ [if $S(a,b,f) \wedge S(f,c,f') \wedge S(b,c,g) \wedge S(a,g,g')$, then $f' \geq_A g'$].

S2 (Monotonicity*) If $\exists x$ s.t. $S(a,c,x)$ and $a \geq_A b$, then $\exists y$ s.t. $S(c,b,y)$ and $\forall f,f'$ [if $S(a,c,f) \wedge S(c,b,f')$ then $f \geq_A f'$].

S3 (Density) If $a >_A b$, then $\exists d,x$ [$S(b,d,x)$ and $\forall f$, if $S(b,x,f)$, then $a \geq_A f$].

S4 (Non-Negativity) If $S(a,b,c)$, then $c \geq_A a$.

[13] Compare with the axioms in Krantz et al. (1971) Defn.3.3: Let A be a nonempty set, $\geq$ a binary relation on A, B a nonempty subset of $A \times A$, and $\circ$ a binary function from B into A. The quadruple $< A, \geq, B, \circ >$ is an extensive structure with no essential maximum if the following axioms are satisfied for all $a,b,c \in A$:

1. $< A, \geq >$ is a weak order. [This is translated as G1 and G2.]

2. If $(a,b) \in B$ and $(a \circ b, c) \in B$, then $(b,c) \in B, (a, b \circ c) \in B$, and $(a \circ b) \circ c \geq_A a \circ (b \circ c)$. [This is translated as S1.]

3. If $(a,c) \in B$ and $a \geq b$, then $(c,b) \in B$, and $a \circ c \geq c \circ b$. [This is translated as S2.]

4. If $a > b$, then there exists $d \in A$ s.t. $(b,d) \in B$ and $a \geq b \circ d$. [This is translated as S3.]

5. If $a \circ b = c$, then $c > a$. [This is translated as S4, but allowing N-Regions to have null amplitudes. The representation function will also be zero-valued at those regions.]

6. Every strictly bounded standard sequence is finite, where $a_1, ..., a_n, ...$ is a standard sequence if for $n = 2, .., a_n = a_{n-1} \circ a_1$, and it is strictly bounded if for some $b \in A$ and for all a_n in the sequence, $b > a_n$. [This is translated as S5. The translation uses the fact that Axiom 6 is equivalent to another formulation of the Archimedean axiom: $\{n | na$ is defined and $b > na\}$ is finite.]

The complications in the nominalistic axioms come from the fact that there can be more than one N-Regions that are the Amplitude-Sum of two N-Regions: $\exists a,b,c,d$ s.t. $S(a,b,c) \wedge S(a,b,d) \wedge c \neq d$. However, in the proof for the representation and uniqueness theorems, we can easily overcome these complications by taking equivalence classes of equal amplitude and recover the amplitude addition function from the Amplitude-Sum relation.

S5 (Archimedean Property) $\forall a_1, b$, if $\neg S(a_1, a_1, a_1)$ and $\neg S(b, b, b)$, then $\exists a_1, a_2, ..., a_n$ s.t. $b >_A a_n$ and $\forall a_i$ [if $b >_A a_i$, then $a_n \geq_A a_i$], where a_i's, if they exist, have the following properties: $S(a_1, a_1, a_2)$, $S(a_1, a_2, a_3)$, $S(a_1, a_3, a_4)$, ..., $S(a_1, a_{n-1}, a_n)$.[14]

Since these axioms are the nominalistic translations of a platonistic structure in Krantz et al. (Defn. 3.3), we can formulate the representation and uniqueness theorems for the amplitude structure:

Theorem 3.1 (Amplitude Representation Theorem) *<N-Regions, Amplitude–Geq, Amplitude–Sum> satisfies axioms (G1)—(G2) and (S1)—(S5), only if there is a function* R *: N-Regions* $\rightarrow \{0\} \cup \mathbb{R}^+$ *such that* $\forall a, b \in$ *N-Regions:*

1. $a \geq_A b \Leftrightarrow R(a) \geq R(b)$;
2. *If* $\exists x$ *s.t.* $S(a, b, x)$, *then* $\forall c$ *[if* $S(a, b, c)$ *then* $R(c) = R(a) + R(b)$*].*

Theorem 3.2 (Amplitude Uniqueness Theorem) *If another function* R' *satisfies the conditions on the RHS of the Amplitude Representation Theorem, then there exists a real number* $\alpha > 0$ *such that for all nonmaximal element* $a \in$ *N-Regions,* $R'(a) = \alpha R(a)$.

Proofs: See Krantz et al. (1971), Sections 3.4.3, 3.5, pp. 84-87. Our primitives are slightly different from the ones used by Krantz et al., because we assume that there may be several N-Regions whose amplitude is the sum of that of two N-Regions. To modify their proof, we can construct equivalence classes N-Regions / $=_A$, where $a =_A b$ if $a \geq_A b \wedge b \geq_A a$, on which we can define an addition function from the Amplitude-Sum relation.

[14] S5 is an infinitary sentence, as the quantifiers in the consequent should be understood as infinite disjunctions of quantified sentences. However, S5 can also be formulated with a stronger axiom called Dedekind Completeness, whose platonistic version says:

Dedekind Completeness. $\forall M, N \subset A$, if $\forall x \in M, \forall y \in N, y > x$, then there exists $z \in A$ s.t. $\forall x \in M, z > x$ and $\forall y \in N, y > z$.

The nominalistic translation can be done in two ways. We can introduce two levels of mereology so as to distinguish regions of points and regions of regions of points. Alternatively, as Tom Donaldson, Jennifer Wang, and Gabriel Uzquiano suggest to me, perhaps I can make do with plural quantification in the following way. For example (with $\propto$ for the logical predicate "is one of"), here is one way to state the Dedekind Completeness with plural quantification:

Dedekind Completeness Nom Pl. $\forall mm, nn \in$ N-Regions, if $\forall x \propto mm, \forall y \propto nn, y > x$, then there exists $z \in A$ s.t. $\forall x \propto mm, z > x$ and $\forall y \propto nn, y > z$.

We only need the Archimedean property in the proof. Since Dedekind Completeness is stronger, the proof in Krantz et al. (1971), pp. 84-87 can still go through if we assume Dedekind Completeness Nom Pl.

This way of writing down the last axiom has the virtue of avoiding the infinitary sentences included in S5. Note: this is the point where we have to trade off certain nice features of first-order logic and standard mereology with the desiderata of the intrinsic and nominalistic account. (I have no problem with infinitary sentences in S5. But one is free to choose instead to use plural quantification to formulate the last axiom as Dedekind Completeness Nom Pl.) This is related to Field's worry in *Science Without Numbers*, Ch. 9, "Logic and Ontology."

The representation theorem suggests that the intrinsic structure of Amplitude-Geq and Amplitude-Sum guarantees the existence of a faithful representation function. But the intrinsic structure makes no essential quantification over numbers, functions, sets, or matrices. The uniqueness theorem explains why the gauge degrees of freedom are the positive multiplication transformations and no further, i.e. why the amplitude function is unique up to a positive normalization constant.

3.3 Quantum State Phase

The phase part of the quantum state is (like angles on a plane) of the periodic scale, i.e. the intrinsic physical structure should be invariant under overall phase transformations

$$\theta \to \theta + \phi \bmod 2\pi.$$

We would like something of the form of a "difference structure." But we know that according to standard formalism, just the absolute values of the differences would not be enough, for time reversal on the quantum state is implemented by taking the complex conjugation of the wave function, which is an operation that leaves the absolute values of the differences unchanged. So we will try to construct a signed difference structure such that standard operations on the wave function are faithfully preserved.[15]

We will once again start with **N-Regions**, the collection of all regions that contain exactly N simultaneous space-time points.

The intrinsic structure of phase consists in two primitive relations:

- A three-place relation Phase–Clockwise–Betweenness (C_P),
- A four-place relation Phase–Congruence ($\sim_P$).

Interpretation: $C_P(a,b,c)$ iff the phase of N-Region b is clock-wise between those of N-Regions a and c (this relation realizes the intuitive idea that 3 o'clock is clock-wise between 1 o'clock and 6 o'clock, but 3 o'clock is not clock-wise between 6 o'clock and 1 o'clock); $ab \sim_P cd$ iff the signed phase difference between N-Regions a and b is the same as that between N-Regions c and d.

The intrinsic structures of Phase–Clockwise–Betweenness and Phase–Congruence satisfy the following axioms for what I shall call a "periodic difference structure":

All quantifiers below range over only N-Regions. $\forall a,b,c,d,e,f$:

C1 At least one of $C_P(a,b,c)$ and $C_P(a,c,b)$ holds; if a,b,c are pair-wise distinct, then exactly one of $C_P(a,b,c)$ and $C_P(a,c,b)$ holds.

[15] Thanks to Sheldon Goldstein for helpful discussions about this point. David Wallace points out (p.c.) that it might be a virtue of the nominalistic theory to display the following choice-point: one can imagine an axiomatization of quantum state phase that involves only absolute phase differences. This would require thinking more deeply about the relationship between quantum phases and temporal structure, as well as a new mathematical axiomatization of the absolute difference structure for phase.

C2 If $C_P(a,b,c)$ and $C_P(a,c,d)$, then $C_P(a,b,d)$; if $C_P(a,b,c)$, then $C_P(b,c,a)$.

K1 $ab \sim_P ab$.

K2 $ab \sim_P cd \Leftrightarrow cd \sim_P ab \Leftrightarrow ba \sim_P dc \Leftrightarrow ac \sim_P bd$.

K3 If $ab \sim_P cd$ and $cd \sim_P ef$, then $ab \sim_P ef$.

K4 $\exists h, cb \sim_P ah$; if $C_P(a,b,c)$, then $\exists d', d''$ s.t. $ba \sim_P d'c, ca \sim_P d''b$; $\exists p,q, C_P(a,q,b), C_P(a,b,p)$, $ap \sim_P pb, bq \sim_P qa$.

K5 $ab \sim_P cd \Leftrightarrow [\forall e, fd \sim_P ae \Leftrightarrow fc \sim_P be]$.

K6 $\forall e,f,g,h$, if $fc \sim_P be$ and $gb \sim_P ae$, then $[hf \sim_P ae \Leftrightarrow hc \sim_P ge]$.

K7 If $C_P(a,b,c)$, then $\forall e,d,a',b',c'$ [if $a'd \sim_P ae, b'd \sim_P be, c'd \sim_P ce$, then $C(a',b',c')$].

K8 (Archimedean Property) $\forall a, a_1, b_1$, if $C_P(a,a_1,b_1)$, then $\exists a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$, such that $C_P(a,a_1,a_n)$ and $C_P(a,b_n,b)$, where $a_n a_{n-1} \sim_P a_{n-1}a_{n-2} \sim_P ... \sim_P a_1 a_2$ and $b_n b_{n-1} \sim_P b_{n-1} b_{n-2} \sim_P ... \sim_P b_1 b_2$.[16]

Axiom (K4) contains several existence assumptions. But such assumptions are justified for a nominalistic quantum theory. We can see this from the structure of the platonistic quantum theory. Thanks to the Schrödinger dynamics, the wave function spreads out continuously over space and time, which will ensure the richness in the phase structure.

With some work, we can prove the following representation and uniqueness theorems:

Theorem 3.3 (Phase Representation Theorem) *If < N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence> is a periodic difference structure, i.e. satisfies axioms (C1)—(C2) and (K1)—(K8), then for any real number $k > 0$, there is a function ψ : N-Regions $\times$ N-Regions $\rightarrow [0,k)$ and there is a function f : N-Regions $\rightarrow [0,k)$ such that $\forall a,b,c,d \in$ N-Regions:*

1. $C_P(c,b,a) \Leftrightarrow f(a) \geq f(b) \geq f(c)$ *or* $f(c) \geq f(a) \geq f(b)$ *or* $f(b) \geq f(c) \geq f(a)$;
2. $ab \sim_P cd \Leftrightarrow f(a) - f(b) = f(c) - f(d)$ *(mod k).*
3. $\psi(a,b) = f(a) - f(b)$ *(mod k).*

Theorem 3.4 (Phase Uniqueness Theorem) *If another function f' satisfies the conditions on the RHS of the Phase Representation Theorem, then there exists a real number β such that for all element $a \in$ N-Regions , $f'(a) = f(a) + \beta$ (mod k).*

[16] Here it might again be desirable to avoid the infinitary sentences / axiom schema by using plural quantification. See Fn. 11.

Proofs: see Appendix.

Again, the representation theorem suggests that the intrinsic structure of Phase–Clockwise–Betweenness and Phase–Congruence guarantees the existence of a faithful representation function of phase. But the intrinsic structure makes no essential quantification over numbers, functions, sets, or matrices. The uniqueness theorem explains why the gauge degrees of freedom are the overall phase transformations and no further, i.e. why the phase function is unique up to an additive constant.

Therefore, we have written down an intrinsic and nominalistic theory of the quantum state, consisting in merely four relations on the regions of physical space-time: Amplitude-Sum, Amplitude-Geq, Phase-Congruence, and Phase-Clockwise-Betweenness. As mentioned earlier but evident now, the present account of the quantum state has several desirable features: (1) it does not refer to any abstract mathematical objects such as complex numbers, (2) it is independent of the usual arbitrary conventions in the wave function representation, and (3) it explains why the quantum state has its amplitude and phase degrees of freedom.[17]

3.4 Comparisons with Balaguer's Account

Before discussing related topics, let me briefly compare my account with Mark Balaguer′s account (1996) of the nominalization of quantum mechanics.

Balaguer′s account follows Malament's suggestion of nominalizing quantum mechanics by taking seriously the Hilbert space structure and the representation of "quantum events" with closed subspaces of Hilbert spaces. Following orthodox textbook presentation of quantum mechanics, he suggests that we take as primitives the *propensities* of quantum systems as analogous to probabilities of quantum experimental outcomes.

> I begin by recalling that each quantum state can be thought of as a function from events (A, Δ) to probabilities, i.e., to $[0, 1]$. Thus, each quantum state specifies a set of ordered pairs $< (A, \Delta), r >$. The next thing to notice is that each such ordered pair determines a propensity property of quantum systems, namely, an r–strengthed propensity to yield a value in Δ for

[17] Seeing my use of mereology and quantification over space-time points, some readers might wonder whether the present account of the quantum state is sufficiently nominalistic. I think it is often hard to give clear answers to such questions, given the diverse use of the term "nominalism." Hartry Field addresses such questions in *Science Without Numbers* (1980), but he still agonizes over some of the choices he makes. Interestingly, W. V. Quine, after reading Field's book, sent a letter to Field, which is now included in the second edition of *Science Without Numbers* (2016). The second paragraph of Quine's letter is most salient here:

> "What to count as nominalism is a question of no great importance, though I shall get back to it. More important is the ontological economy and relative homogeneity that you achieve, whatever one's views of the objects that you keep. More important still, perhaps, is the economy of theory that you gain by what you call intrinsic formulation; namely, the resolving out of conventional units and measures in favor of the objective invariances that underlie the quantitative laws." (P-55)

a measurement of A. We can denote this propensity with "(A, Δ, r)". (Balaguer, 1996, p.218.)

After giving several informal arguments that we can prove representation theorems for propensities,[18] he defends the idea that they are "nominalistically kosher." By interpreting the Hilbert space structures as propensities instead of propositions, Balaguer makes some progress in the direction of making quantum mechanics "more nominalistic."

However, Balaguer's account faces two serious problems. First, Balaguer's account seems to suffer from the same foundational problem as platonistic versions of orthodox quantum mechanics. If quantum states are thought of as functions from events to probabilities, and if we go on to nominalize the experimental probabilities, then what should we make of the actual events and quantum experiments themselves? No good answer has been offered by defenders of the orthodox quantum mechanics. Moreover, as J. S. Bell argues persuasively, words such as "measurement," "observation," and "observables" should have no place in the fundamental ontology or dynamics of a physical theory; they are not only unprofessionally vague but also conceptually ambiguous.[19]

Balaguer (p.c.) suggests that his proposal is not meant to be a proposal for a complete ontology; we are free to add particles or other ontologies. But it is not clear how Balaguer's account relates to any mainstream realist interpretation of quantum mechanics. This is because all three main interpretations—Bohmian Mechanics, GRW spontaneous collapse theories, and Everettian Quantum Mechanics—crucially involve the quantum state represented by a wave function, not a function from events to probabilities. And once we add the wave function (perhaps in the nominalistic form introduced in this paper), the probabilities can be calculated (by the Born rule) from the wave function itself, which makes Balaguer's fundamental propensities redundant. Hence, it seems to me that Balaguer's nominalistic propensities are either ontologically incomplete or ontologically redundant.

An even more serious worry is about how to extend his account to the dynamics. Towards the end of his paper, he admits that his theory is not complete without the dynamics:

> [W]hat is left unnominalized is the dynamics of the theory—in particular, the Schrödinger Equation. But I don't see any reason why this can't be nominalized in the same general way that Field nominalizes the differential equations of Newtonian Gravitation Theory. It is not *trivial* that this can be done, but I do not foresee any impediments. (Balaguer, 1996, p.223.)

But this is puzzling; for it is not clear what the dynamics could be on his theory. The Schrödinger equation is a wave equation: it relates the time derivative of the wave function to the the spatial gradient of the wave function with the interaction potential.

[18] It is not clear to me which theorems these should be.

[19] Bell (1989), "Against 'Measurement,' " pp. 215-16.

As we have emphasized, the wave function comes with two pieces of information: the amplitude and the phase. Probabilities (or propensities) are given by the Born rule to be the (normalized) squared amplitude. Balaguer's nominalistic ontology, containing only propensities, would leave out some important phase information, and would be dynamically incomplete, from the point of view of the Schrödinger equation. Even if we include the probability information of all possible experiments (position measurements and any other measurements), there might not be any simple dynamical equations relating them to other probabilities at other times. In other words, Balaguer's theory is likely to have incomplete dynamics or complicated dynamics. Therefore, it seems nomologically inadequate.

In contrast, the present account of the quantum state contains information of both the amplitude and the phase parts the wave function, which would be sufficient for feeding into a nominalized version of the Schrödinger Equation, which would be simple to write down. Moreover, the primitives and the representation theorems in my account are much more perspicuous than Balaguer's account and more continuous with the main interpretations of quantum mechanics.

4 Relations to Other Issues

In this section, I will explain how my intrinsic and nominalistic account of the quantum state relates to other issues in metaphysics of quantum mechanics.

4.1 "Wave Function Realism"

It may have occurred to some readers that the present account of the quantum state provides a natural response to some of the standard objections to "wave function realism."[20] According to David Albert (1996), to be a realist about the wave function naturally commits one to accept that the wave function is a physical field defined on a fundamentally 3N-dimensional wave function space. Tim Maudlin (2013) criticizes Albert's view partly on the ground that such kind of "naive" realism would commit one to take as fundamental the gauge degrees of freedom such as the absolute values of the amplitude and the phase and recognize empirically equivalent formulations as metaphysically distinct. This "naive" realism stands in contrast with the physicists' attitude of considering the Hilbert space projectively and thinking of the quantum state as an equivalence class of wave functions ($\psi \sim Re^{i\theta}\psi$). Albert and other defenders have responded by biting the bullet and accepting the costs. If a defender of wave function realism were to take the physicists' attitude, says the opponent, it would be much less natural to think that the wave function is really a physical *field*, in the sense of something that assigns physical properties to each point

[20]Wave function realists, such as David Albert, Barry Loewer, Alyssa Ney, and Jill North, are what I call "3N-Fundamentalists." They believe that the fundamental physical space for a quantum world is 3N-dimensional. In contrast, primitive ontologists, such as Valia Allori, Detlef Dürr, Sheldon Goldstein, Tim Maudlin, Roderich Tumulka, and Nino Zanghi, are what I call "3D-Fundamentalists." They believe that the fundamental physical space is 3-dimensional.

in the 3N-dimensional space.

But the situation would be much different given our present account of the quantum state. On the intrinsic and nominalistic versions of field theories, field values at points or regions can be thought of as mathematical representation of comparative relations obtaining among space-time regions. On the intrinsic theory of the quantum state, it can be similarly thought of as two fields (amplitude and phase) on the configuration space or two multi-fields on the physical space. Regardless whether one believes in a fundamentally high-dimensional space or a fundamentally low-dimensional space, the intrinsic and nominalistic account will recover the mathematical representation unique up to certain transformations. In the case of the quantum state, we recover exactly the right equivalence class of wave functions ($\psi \sim Re^{i\theta}\psi$).

This provides some defensive resources for the "wave function realists." They can use the intrinsic account of the quantum state to identify two field-like entities on the configuration space (by thinking of the N-Regions as points in the 3N-dimensional space) without committing to the excess structure of absolute amplitude and overall phase.[21]

4.2 Future Work

Before concluding, let us briefly anticipate four lines of future research.

First, the intrinsic and nominalistic account of the quantum state described above is the first step towards an intrinsic and nominalistic theory of quantum mechanics. In future work, I will describe nomological constraints on the quantum state: the Schrödinger dynamics, the Born rule, and square-integrability.[22] One idea of nominalizing the Schrödinger equation is to decompose it into two equations, in terms of amplitude and gradient of phase of the wave function. The key would be to codify the differential operations (which Field has done for Newtonian Gravitation Theory) in such a way to be compatible with our phase and amplitude relations. To nominalize integration theory, I plan to borrow some ideas from Zee Perry's work on the theory of scalar quantities and space-time. The Born rule would present new conceptual challenges, as it is controversial what place probability can occupy in a nominalistic ontology and what the bearers of comparative probability should be. But this is a general conceptual problem for nominalistic physics, not just for nominalistic quantum mechanics.

[21] Not surprisingly, the present account may also provide some new arsenal for the defenders of the fundamental 3-dimensional space. The axiomatic structure of the quantum state fills in the concrete details to the multi-field proposal in the recent literature.

[22] At this point, even without a nominalistic theory of integration yet, we can say something about the requirement that the wave function is square-integrable. There are many equivalent conditions to square-integrability. We can, for example, require that the quantum state amplitude structure is continuous and has compact support at some point in time, say, the initial time. This, by the conservation of squared-amplitude, will guarantee that the quantum state at all times is square-integrable. Moreover, "continuity" and "compact support" are readily nominalizable with Field's resources.

Second, we have described how to think of the quantum state for a system with constant number of particles, but how should we think about particle creation and annihilation? I think the best way to get a grip on that question would be to think carefully about the ontology of a quantum field theory. One option (which may or may not be the best option) would be to think of the quantum state for such a system as being represented by a complex valued function whose domain is $\bigcup_{N=0}^{\infty} \mathbb{R}^{3N}$—the union of all configuration spaces (of different number of particles). In that case, the extension of our theory would be easy—just keep the axioms fixed but let the elements be mereological sums of any tupled physical points, not just N-tuples.

Third, we have only considered quantum states for spinless systems in this paper. The straightforward way to extend the present account to accommodate spinorial degrees of freedom would be to use two comparative relations for each complex number assigned by the wave function. This is certainly possible and conceptually similar to the situation in the present account. But there are two worries. First, it does not appear to be the most simple or most elegant extension. Moreover, this method would reify absolute orientation in the spin space, a degree of freedom that we regard as gauge. These two considerations together seem to suggest that the best way to proceed might be to reify the value space, in the same way as Arntzenius and Dorr (2011) have done in the context of differential geometry.

Fourth, as we have learned from the relational theories of motion and the comparative theories of quantities, there is always the possibility of a theory becoming indeterministic when drawing from only comparative predicates without fixing an absolute scale.[23] It would be interesting to investigate whether similar problems of indeterminism arise in our comparative theory of the quantum state.

5 Conclusion

There are many *prima facie* reasons for doubting that we can ever find an intrinsic and nominalistic theory of quantum mechanics. However, in this paper, we have made some significant progress towards constructing it. In particular, we have offered an intrinsic and nominalistic theory of the quantum state, consisting in just four relations on the regions of physical space: Amplitude-Sum, Amplitude-Geq, Phase-Congruence, and Phase-Clockwise-Betweenness. Not only does it have many desirable features, qualifying it to be a better fundamental physical theory than the platonistic version, it also provides an answer to Malament's long-standing conjecture about the possibility of nominalizing quantum mechanics. We have also discussed possible ways of extending this account to a complete theory of quantum mechanics.

Here we have focused on the quantum state, because it has no classical counterpart and it seems to resist an intrinsic and nominalistic treatment. But the nominalistic theory leaves room for including additional ontologies of particles, fields, mass

[23]See Dasgupta (2013), Baker (2014), Martens (2016), and Field (2016), preface to second edition, pp. 41-44.

densities supplied by specific solutions to the quantum measurement problem such as BM, GRWm, and GRWf; these additional ontologies are readily nominalizable.

Moreover, this study lends itself to several future lines of research on the metaphysics of quantum mechanics and quantum field theory. Finally, the formal results obtained for the periodic difference structure might prove fruitful for further investigation.

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Appendix: Proofs of Theorems 4.3 and 4.4.

Step 1. We begin by enriching < N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence> with some additional structures.

First, to simplify the notations, let us think of N-Regions as a set of regions, and let us now only consider $\Omega :=$ N-Regions / $=_P$, the set of "equal phase" equivalence classes by quotienting out $=_P$. ($a =_P b$ if they form phase intervals the same way: $\forall c \in S, ac \sim_P bc$.)

Second, we fix an arbitrary $A_0 \in \Omega$ to be the "zero phase equivalence class."

Third, we define a non-inclusive relation C on Ω according to C_P on N-Regions. ($\forall A, B, C \in \Omega$, $C(A, B, C)$ iff A, B, C are pairwise distinct and $\forall a \in A, \forall b \in B, \forall c \in C$, $C(a, b, c)$.)

Fourth, we define an addition function $\circ : \Omega \times \Omega \to \Omega$. $\forall A, B \in \Omega$, $C = A \circ B$ is the unique element in Ω such that $CB \sim AA_0$, which is guaranteed to exist by (K4) and provably unique as elements in Ω form a partition over N-Regions.

Step 2. We show that the enriched structure $< \Omega, \circ, C >$ with identity element A_0 satisfies the axioms for a periodic extensive structure defined in Luce (1971).

Axiom 0. $< \Omega, \circ >$ is an Abelian semigroup.

First, we show that $\circ$ is closed: $\forall A, B \in \Omega$, $A \circ B \in \Omega$.

This follows from (K4).

Second, we show that $\circ$ is associative: $\forall A, B, C \in \Omega$, $A \circ (B \circ C) = (A \circ B) \circ C$.

This follows from (K6).

Third, we show that $\circ$ is commutative: $\forall A, B \in \Omega$, $A \circ B = B \circ A$.

This follows from (K2).

$\forall A, B, C, D \in \Omega$:

Axiom 1. Exactly one of $C(A, B, C)$ or $C(A, C, B)$ holds.

This follows from C1.

Axiom 2. $C(A, B, C)$ implies $C(B, C, A)$.

This follows from C2.

Axiom 3. $C(A, B, C)$ and $C(A, C, D)$ implies $C(A, B, D)$.

This follows from C2.

Axiom 4. $C(A, B, C)$ implies $C(A \circ D, B \circ D, C \circ D)$ and $C(D \circ A, D \circ B, D \circ C)$.

This follows from (K7).

Axiom 5. If $C(A_0, A, B)$, then there exists a positive integer n such that $C(A_0, A, nA)$ and $C(A_0, nB, B)$.

This follows from (K8).

Therefore, the enriched structure $< \Omega, \circ, C >$ with identity element A_0 satisfies the axioms for a periodic extensive structure defined in Luce (1971).

Step 3. We use the homomorphisms in Luce (1971) to find the homomorphisms for < N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence>.

Since $< \Omega, \circ, C >$ satisfy the axioms for a periodic structure, Corollary in Luce (1971) says that for any real $K > 0$, there is a unique function ϕ from Ω into $[0, K)$ s.t. $\forall A, B, C \in \Omega$:

1. $C(C, B, A) \Leftrightarrow \phi(A) > \phi(B) > \phi(C)$ or $\phi(C) > \phi(A) > \phi(B)$ or $\phi(B) > \phi(C) > \phi(A)$;
2. $\phi(A \circ B) = \phi(A) + \phi(B) \pmod K$;
3. $\phi(A_0) = 0$.

Now, we define f : N-Regions $\to [0, K)$ as follows: $f(a) = \phi(A)$, where $a \in A$. So we have $C_P(c, b, a) \Leftrightarrow f(a) \geq f(b) \geq f(c)$ or $f(c) \geq f(a) \geq f(b)$ or $f(b) \geq f(c) \geq f(a)$.

We can also define ψ : N-Regions $\times$ N-Regions $\to [0, K)$ as follows: $\psi(a, b) = \phi(A) - \phi(B) \pmod K$, where $a \in A$ and $b \in B$. Hence, $\forall a, b \in$ N-Regions, $\psi(a, b) = f(a) - f(b) \pmod K$.

Moreover, given (K5), $\forall a \in A, b \in B, c \in C, d \in D$, $ab \sim_P cd$

$\Leftrightarrow AB \sim CD$

$\Leftrightarrow A \circ D = B \circ C$

$\Leftrightarrow \phi(A \circ D) = \phi(B \circ C)$

$\Leftrightarrow \phi(A) + \phi(D) = \phi(B) + \phi(C) \pmod K$

$\Leftrightarrow \forall a \in A, b \in B, c \in C, d \in D$, $f(a) + f(d) = f(b) + f(c) \pmod K$

$\Leftrightarrow \forall a \in A, b \in B, c \in C, d \in D$, $f(a) - f(b) = f(c) - f(d)$ (mod K)

Therefore, we have demonstrated the existence of homomorphisms.

Step 4. We prove the uniqueness theorem.

If another function f' : N-Regions $\to [0, K)$ with the same properties exists, then

$$f'(a) - f'(a_0) \bmod K = \psi(a, a_0) = f(a) - f(a_0) \bmod K = f(a),$$

which entails that

$$f'(a) = f(a) + \beta \bmod K,$$

with the constant $\beta = f'(a_0)$. QED.

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