Inflationary cosmology and the scale-invariant spectrum

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Abstract

The claim of inflationary cosmology to explain certain observable facts, which the Friedmann-Roberston-Walker models of 'Big-Bang' cosmology were forced to assume, has already been the subject of significant philosophical analysis. However, the principal empirical claim of inflationary cosmology, that it can predict the scale-invariant power spectrum of density perturbations, as detected in measurements of the cosmic microwave background radiation, has hitherto been taken at face value by philosophers.

The purpose of this paper is to expound the theory of density perturbations used by inflationary cosmology, to assess whether inflation really does predict a scale-invariant spectrum, and to identify the assumptions necessary for such a derivation.

The first section of the paper explains what a scale-invariant power-spectrum is, and the requirements placed on a cosmological theory of such density perturbations. The second section explains and analyses the concept of the Hubble horizon, and its behaviour within an inflationary space-time. The third section expounds the inflationary derivation of scale-invariance, and scrutinises the assumptions within that derivation. The fourth section analyses the explanatory role of 'horizon-crossing' within the inflationary scenario.

1 Introduction

In the past couple of decades, inflationary cosmology has been subject to trenchant criticism in some quarters. The criticism has come both from physicists (Penrose 2004, 2010, 2016; Steinhardt 2011), and philosophers of physics (Earman 1995, Chapter 5; Earman and Mosterin 1999). The primary contention is that inflation has failed to deliver on its initial promise of supplying cosmological explanations which are free from dependence on initial conditions.

Inflation was initially promoted as a theory which explained certain observable astronomical facts that the Friedmann-Roberston-Walker (FRW) models of 'Big-Bang' cosmology were forced to assume, (Guth 1981). The most prominent examples of this were dubbed the 'horizon problem' and the 'flatness problem', (McCoy 2015).

In the first case, it was pointed out that points on the cosmic microwave background sky which are separated by large angles, have temperatures which are very similar. This, despite the fact that there was insufficient time in an FRW model for these regions to have causally interacted before the time of 'recombination', when the photons in the background radiation effectively decoupled from the matter. In the second case, it was pointed out that the current value of the density parameter Ω_0 is very close to 1, despite the fact that $\Omega_0 = 1$ is an unstable fixed point of the FRW dynamics (Smeenk 2012).

At first sight, inflation was able to explain these facts as the result of evolutionary processes rather than initial conditions. Its failure to deliver on this promise is rooted in the fact that inflation was also tasked with reproducing the spectrum of density perturbations ultimately responsible for seeding galaxy formation, ('structure formation'). In order to produce the correct statistics, the scalar field responsible for the hypothetical period of exponential expansion had to be parameterised in a fashion inconsistent with any candidate field available in a Grand Unified Theory of particle physics, (Smeenk 2012).

It became clear that the predictions of inflation were extremely sensitive to the type of scalar field chosen, and to the initial conditions of that field: "The original models of inflationary cosmology...predicted an amplitude of density fluctuations that was too high by several orders of magnitude. To get the right order of magnitude, the false vacuum plateau of the inflaton field has to be very flat...For a slow roll potential, the ratio of the change in potential to the change in the scalar field must be less than 10^{-6} - 10^{-8}, and for the potential used in the 'extended' inflationary scenario the ratio must be less than 10^{-15}," (Earman and Mosterin, 1999).

Moreover, once inflation was no longer tied down to the world of particle physics, a cornucopia of different models was unleashed: "Martin, Ringeval and Vennin (2014a) have catalogued and analyzed a total of 74(!) distinct inflaton potentials that have been proposed in the literature: all of them corresponding to a minimally coupled, slowly-rolling, single scalar field driving the inflationary expansion. And a more detailed Bayesian study (Martin et al., 2014b), expressly comparing such models with the Planck satellites 2013 data about the CMB, shows that of a total of 193(!) possible models - where a 'model' now includes not just an inflaton potential but also a choice of a prior over parameters defining the potential - about 26% of the models (corresponding to 15 different underlying potentials) are favored by the Planck data. A more restrictive analysis (appealing to complexity measures in the comparison of different models) reduces the total number of favoured models to about 9% of the models, corresponding to 9 different underlying potentials (though all of the 'plateau' variety)," (Azhar and Butterfield, 2017).

Whilst inflationary cosmologists have retreated somewhat from the claim that their theory is independent of initial conditions, faith in the theory has instead been built on its empirical success. The theory, it is claimed, predicts that the spectrum of density perturbations is scale-invariant, and observations of the cosmic microwave background radiation verify this prediction.

The critics of inflation are able to point out that the class of inflationary models is so general that it could explain just about any empirical data. But the claim that inflation predicts a scale-invariant spectrum is generally accepted without reservation or further examination.

The purpose of this paper is to expound the theory of density perturbations used by inflationary cosmology, and to assess whether inflation really does predict a scale-invariant spectrum.

The first section of the paper explains what a scale-invariant power-spectrum is, and the requirements placed on a cosmological theory of such density perturbations. The second section explains and analyses the concept of the Hubble horizon, and its behaviour within an inflationary space-time. The third section expounds the inflationary derivation of scale-invariance, and scrutinises the assumptions within that derivation. The fourth section analyses the explanatory role of 'horizon-crossing' within the inflationary scenario.

2 Perturbations and the power spectrum

Given a scalar field $\rho(x)$ representing the density of matter, with a mean density $\langle\rho\rangle$, the *fluctuation field* (or 'perturbation field') $\delta_\rho(x)$ is defined by

$$\delta_\rho(x) = \rho(x) - \langle\rho\rangle \,,$$

and the *contrast field* is defined by

$$\frac{\delta_\rho(x)}{\langle\rho\rangle} = \frac{\rho(x) - \langle\rho\rangle}{\langle\rho\rangle} \,.$$

The fluctuation field can be expressed as an inverse Fourier transform:

$$\delta_\rho(\mathbf{x}) = \frac{1}{(2\pi)^3}\int A(\mathbf{k})e^{i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{k} \,.$$

This expresses the fluctuation field as the superposition of a spectrum of wave-like 'modes'. The k-th mode is $e^{\mathbf{x}\cdot\mathbf{k}}$, and the amplitude of the k-th mode is $A(\mathbf{k})$. The wavelength λ of the k-th mode is related to the wave-vector $\mathbf{k}$ by:

$$\lambda = \frac{2\pi}{|\mathbf{k}|} \,.$$

Hence, long wavelength perturbations correspond the small wave-numbers, and short wavelength perturbations correspond to large wave-numbers.

Now, while the mean value of δ_ρ is zero, $\langle\delta_\rho\rangle = 0$, the mean of its square-value is non-zero, $\langle\delta_\rho^2\rangle \neq 0$. The mean of the square-value is simply the variance σ_{δ_ρ} in the fluctuation field:

$$\sigma_{\delta_\rho} = \langle\delta_\rho^2\rangle \neq 0 \,.$$

The variance can be expressed in terms of the amplitudes of the perturbational modes as follows:

$$\sigma^2_{\delta_\rho} = \frac{1}{(2\pi)^3} \int |A(\mathbf{k})|^2 \mathrm{d}\mathbf{k}$$

Assuming the perturbations to the density field are sampled from a homogeneous and isotropic random field, then the random field will be spherically symmetric about any point, and this 3-dimensional integral over the space of mode-vectors $\mathbf{k}$ can be simplified into an integral over wave-numbers $k = |\mathbf{k}|$:

$$\sigma^2_{\delta_\rho} = \frac{1}{(2\pi)^3} \int_0^\infty |A(k)|^2 4\pi k^2 \, \mathrm{d}k \,,$$

where $4\pi k^2$ is the surface area of a sphere of radius k in the space of wave-vectors.

The square of the modulus of the amplitudes is called the *power spectrum*:

$$P_\rho(k) = |A(k)|^2 \,.$$

Hence, the variance can be expressed as

$$\sigma^2_{\delta_\rho} = \frac{1}{2\pi^2} \int P_\rho(k) k^2 \mathrm{d}k \,.$$

The significance, then, of the power spectrum, is that it determines the contribution of the mode-k perturbations to the total variance.

According to inflationary cosmology, the power spectrum of the density perturbations is given by a power law:

$$P_\rho(k) = Ak^n$$

where A is some constant (not to be confused with the Fourier coefficients above), and the exponent n is called the *spectral index*. Inflationary cosmology purportedly predicts that $n \approx 1$. A power spectrum with such an exponent is said to be (approximately) *scale-invariant*.

Note that $P_\rho(k) \sim k$ entails that the amplitude of the perturbations increases with k. Greater wave-numbers correspond to shorter wavelengths, so shorter wavelength perturbations have a greater amplitude. Scale-invariance of the power-spectrum does *not* mean that the amplitude of the perturbations is the same on every scale.

Such language is commonly used in association with power laws, where the invariance is only manifest on a log-scale. In other words:

$$\log_k(P_\rho(k)) = \log_k(Ak^n) = \log_k(A) + \log_k(k^n) \sim 1 \,.$$

The power spectrum of the matter density field is not the only power spectrum of interest. From the perspective of Newtonian cosmology, (often used to simplify calculations in the study of structure formation), the matter density couples to the gravitational potential Φ via the Poisson equation:

$$\nabla^2\Phi = 4\pi G\rho\,.$$

Hence, a perturbation-pattern in the matter field must have a corresponding perturbation pattern in the gravitational potential. Accordingly, there is a power spectrum P_Φ for the perturbations in the gravitational potential.

Denote the fluctuation field in the potential as $\delta_\Phi = \Phi - \langle\Phi\rangle$, and express it as an inverse Fourier transform:

$$\delta_\Phi(\mathbf{x}) = \frac{1}{(2\pi)^3}\int B(\mathbf{k})e^{i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{k}\,,$$

where $B(\mathbf{k})$ is the amplitude of the mode-k fluctuation. The Poisson equation is linear, so

$$\nabla^2\Phi = \nabla^2(\delta_\Phi + \langle\Phi\rangle) = \nabla^2\delta_\Phi + \nabla^2\langle\Phi\rangle = 4\pi G(\delta_\rho + \langle\rho\rangle) = 4\pi G(\delta_\rho) + 4\pi G(\langle\rho\rangle)\,.$$

The perturbations therefore satisfy their own Poisson equation:

$$\nabla^2\delta_\Phi = 4\pi G\delta_\rho\,.$$

Inserting the inverse Fourier transform expression for the gravitational fluctuations into the left-hand-side of this Poisson equation yields:

$$\begin{aligned}\nabla^2\delta_\Phi &= \sum_{j=1}^{3}\frac{\partial^2\delta_\Phi}{dx_i^2} = \sum_{j=1}^{3}\frac{1}{(2\pi)^3}\int B(\mathbf{k})\frac{\partial^2 e^{i\mathbf{x}\cdot\mathbf{k}}}{dx_j^2}\mathrm{d}\mathbf{k}\\ &= \sum_{j=1}^{3}\frac{1}{(2\pi)^3}\int B(\mathbf{k})k_j^2 e^{i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{k}\\ &= \frac{1}{(2\pi)^3}\int B(\mathbf{k})\mathbf{k}^2 e^{i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{k}\,.\end{aligned}$$

It follows that the corresponding matter fluctuation field is:

$$\delta_\rho(\mathbf{x}) = \frac{1}{(2\pi)^3}\int A(\mathbf{k})e^{i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{k}\,,$$

where the amplitude $A(\mathbf{k})$ of the mode-k fluctuations is given by:

$$A(\mathbf{k}) = \frac{B(\mathbf{k})\mathbf{k}^2}{4\pi G}\,.$$

Now, given that the power spectrum is square of the modulus of the amplitudes, it follows that:

$$P_\rho(k) = |A(k)|^2 \sim |B(k)k^2|^2 = |B(k)|^2 k^4\,.$$

In other words, the power spectrum in the matter density fluctuations is obtained from the power spectrum in the gravitational potential fluctuations by multiplying the latter by k^4.

If the power spectrum in the matter field fluctuations has the form $P_\rho \sim k$, it follows that the power spectrum in the gravitational potential fluctuations has the form $P_\Phi \sim k^{-3}$.

Hence, if the matter field power spectrum is scale-invariant, then the gravitational power spectrum isn't. However, at this juncture, a common mathematical ruse is employed to preserve a sense of invariance: a log-scale is used.

Suppose we have an expression for the total variance in the gravitational perturbations:

$$\sigma_\Phi^2 = \frac{1}{2\pi} \int_0^\infty P_\Phi(k) k^2 \, \mathrm{d}k \, .$$

Given that $\mathrm{d}(\ln k)/\mathrm{d}k = 1/k$, a change of variables can be implemented using the substitution $\mathrm{d}k = \mathrm{d}(\ln k)\, k$:

$$\sigma_\Phi^2 = \frac{1}{2\pi} \int_{-\infty}^\infty P_\Phi(k) k^3 \, \mathrm{d} \ln k \, .$$

Defining an expression $\triangle(k) = \frac{1}{2\pi} P_\Phi(k) k^3$, this represents the contribution to the total variance, per unit logarithmic interval in k, (Coles and Lucchin p266). This can also be placed in the form of a power law:

$$\triangle(k) \sim k^n \, .$$

If $P_\Phi(k) \sim k^{-3}$, it follows that $\triangle(k) \sim k$. Hence, whilst $P_\Phi(k)$ might not be scale-invariant, its logarithmic sibling $\triangle(k)$ is.

So, we've defined what it means for perturbations in the matter density and gravitational potential to be scale-invariant. But does inflation genuinely predict that the spectrum will be scale-invariant, and if so, how? The explanation hinges upon the concept of the Hubble horizon, so it is to that which we turn next.

3 Inflation and the Hubble Horizon

The Hubble horizon around a point in space is defined to be the set of points which are receding from that point at the speed of light c. Let's review what this means in a broad relativistic context, before considering the nature of the Hubble horizon within inflationary space-times.

Prior to the invention of inflation, general relativistic cosmology represented the universe as a Friedmann-Roberston-Walker (FRW) spacetime. Geometrically, an FRW model is a 4-dimensional Lorentzian manifold $\mathcal{M}$ which can be expressed as a 'warped product' (McCabe 2004):

$$I \times_a \Sigma \, .$$

I is an open interval of the 1-dimensional pseudo-Euclidean manifold $\mathbb{R}^{1,1}$, and Σ is a complete and connected 3-dimensional Riemannian manifold. The warping function a is a smooth, real-valued, non-negative function upon the open interval I, otherwise known as the 'scale factor'.

If we denote by t the natural coordinate function upon I, and if we denote the metric tensor on Σ as γ, then the Lorentzian metric g on $\mathcal{M}$ can be written as

$$g = -dt \otimes dt + a(t)^2 \gamma \, .$$

One can consider the open interval I to be the time axis of the warped product cosmology. The 3-dimensional manifold Σ represents the spatial universe, and the scale factor $a(t)$ determines the time evolution of the spatial geometry.

Now, a Riemannian manifold (Σ, γ) is equipped with a natural metric space structure (Σ, d). In other words, there exists a non-negative real-valued function $d : \Sigma \times \Sigma \to \mathbb{R}$ which is such that

$$\begin{aligned} d(p,q) &= d(q,p) \\ d(p,q) + d(q,r) &\geq d(p,r) \\ d(p,q) &= 0 \text{ iff } p = q \end{aligned}$$

The metric tensor γ determines the Riemannian distance $d(p,q)$ between any pair of points $p, q \in \Sigma$. It defines the length of all curves in the manifold, and the Riemannian distance $d(p,q)$ is defined as the infimum of the length of all the piecewise smooth curves between p and q.

When cosmologists refer to 'comoving' spatial coordinates and distances, they are referring to the Riemannian distance $d(p,q)$.

In the warped product space-time $I \times_a \Sigma$, the spatial distance between (t,p) and (t,q) is $a(t)d(p,q)$. Hence, if one projects onto Σ, one has a time-dependent distance function on the points of space,

$$d_t(p,q) = a(t)d(p,q) \, .$$

Each hypersurface Σ_t is a Riemannian manifold $(\Sigma_t, a(t)^2\gamma)$, and $a(t)d(p,q)$ is the *physical* distance between (t,p) and (t,q) due to the metric space structure (Σ_t, d_t). Whilst the comoving distance has no physical dimensions, the scale factor supplies the dimensions (i.e., empirical units) for the physical distance.

The rate of change of the distance between a pair of points in space, otherwise known as the 'recession velocity' v, is given by

$$\begin{aligned} v = \frac{d}{dt}(d_t(p,q)) &= \frac{d}{dt}(a(t)d(p,q)) \\ &= \dot{a}(t)d(p,q) \\ &= \frac{\dot{a}(t)}{a(t)} a(t)d(p,q) \\ &= H(t)a(t)d(p,q) \\ &= H(t)d_t(p,q) \, , \end{aligned}$$

where $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter. Hence, the rate of change of distance between a pair of points is proportional to the spatial separation of those points, and the constant of proportionality is the Hubble parameter.

The set of points receding at the speed of light is defined by setting $v = c$, so that

$$c = H(t)d_t(p, q) = \dot{a}(t)d(p, q)\,.$$

Hence, the physical radius $d_H(t)$ of the Hubble sphere (i.e., the Hubble horizon) is:

$$d_H(t) = \frac{c}{H(t)} = c\frac{a(t)}{\dot{a}(t)}\,.$$

If a system of units is chosen in which $c = 1$, it follows that the physical Hubble radius is:

$$d_H(t) = H^{-1}(t)\,.$$

Points which are separated by a distance smaller than d_H will be receding slower than the speed of light, and points which are separated by a distance greater than d_H will be receding faster than the speed of light.

In terms of comoving distances, the Hubble radius is:

$$\frac{d_H(t)}{a(t)} = \frac{1}{a(t)H(t)} = \frac{1}{\dot{a}(t)}\,.$$

In a conventional FRW cosmology, the expansion is deceleratory, $\ddot{a}(t) < 0$. Hence, to find pairs of points (p, q) which satisfy the equation $c = \dot{a}(t)d(p, q)$ it is necessary to look at points separated by ever-greater comoving coordinate distances: $\dot{a}(t)$ is getting smaller, so $d(p, q)$ has to increase. The comoving coordinate radius of the Hubble sphere therefore expands with the passage of time in a conventional FRW model. The consequence of this is that points inside the Hubble sphere at one time remain inside it for all future time, and points initially outside the Hubble sphere eventually fall inside it.

Things are significantly different during inflation. The scale factor $a(t)$ is an exponential function of time:

$$a(t) = e^{\chi t}\,.$$

From the definition of the exponential function is follows that the time derivative is:

$$\dot{a}(t) = \chi e^{\chi t}\,.$$

Now, the Hubble parameter is defined as $H(t) \equiv \dot{a}(t)/a(t)$. Hence, in the case of inflation,

$$H(t) = \frac{\chi e^{\chi t}}{e^{\chi t}} = \chi\,.$$

In other words, the Hubble parameter is constant during inflation, $H(t) = H$, and is equal to the coefficient in the exponent of the scale factor:

$$a(t) = e^{Ht}.$$

Note, of course, that whilst $H(t)$ is constant during inflation, the scale factor is increasing at an exponential rate, hence $\dot{a}(t) > 0$.

Consider the Hubble sphere about a point p. For any point q initially inside that Hubble sphere, if inflation lasts sufficiently long, that point will eventually lie outside the Hubble sphere. This can be understood in terms of either physical distance, or in terms of comoving distance.

The *physical* radius of the Hubble sphere remains approximately constant during inflation. Precisely because it is a phase of rapid expansion, the physical distance $d_t(p,q) = a(t)d(p,q)$ between points is rapidly increasing, hence points q initially inside the Hubble sphere of p will pass through it once they satisfy $c = Hd_t(p,q)$. In other words, any pair of points initially receding slower than the speed of light will eventually be receding faster than the speed of light.

Whilst the physical radius of the Hubble sphere remains fixed during inflation, its comoving radius $1/\dot{a}(t)$ is rapidly shrinking because $\dot{a}(t)$ is rapidly increasing. Hence, for any point q separated from p by a comoving distance $d(p,q)$ smaller than the Hubble radius at the onset of inflation, if inflation lasts sufficiently long the comoving radius of the Hubble sphere will eventually shrink inside $d(p,q)$.

Inflation applies this logic to perturbational modes. If one thinks in terms of physical distance, the perturbational modes pass outwards through the Hubble sphere, whilst if one thinks in terms of comoving distance, the Hubble sphere shrinks inside the modes.

In this context, inflationary cosmologists refer to comoving modes k, (i.e., modes with comoving wavenumbers k) as being subhorizon modes if $k \gg aH$, and superhorizon modes if $k \ll aH$.

To best understand this, take the condition $k \ll aH$ as an example, and re-write it as $k/a \ll H$, or $a/k = a\lambda/2\pi \gg H^{-1}$, where λ denotes the comoving wavelength, and $a\lambda$ denotes the physical wavelength. Hence, when $a\lambda/2\pi \gg H^{-1}$, the physical wavelength of the mode is greater than the physical Hubble radius.

Immediately, however, we might question some of the language used. A perturbational mode is an extended disturbance. Whilst the extremities of the extended entity might become separated by a distance greater than the Hubble radius, there will always be other parts of the mode which are separated by a distance smaller than the Hubble radius. A mode will therefore straddle the Hubble radius, rather than passing outside it.

Numerous authors have pointed out that the Hubble horizon, as a 3-dimensional hypersurface in space-time, is *not* a general relativistic event horizon. After inflation ceases, the points which passed through the Hubble horizon will not only be able to send signals to points inside the Hubble horizon, but will begin to fall back inside it themselves.

However, if inflation *were* to continue indefinitely, then the Hubble horizon would indeed become an event horizon. Hence, the Hubble horizon during an inflationary phase could be deemed a counterfactual event horizon; i.e., if the part of space-time representing the finite period of inflation is isometrically mapped into a space-time where inflation lasts forever, then the Hubble horizon from the inflationary period is mapped onto part of an event horizon from the permanently inflating space-time.

4 The inflationary origin of structure formation

According to inflation, galactic structure formation in the universe is ultimately seeded by the quantum fluctuations in the inflaton field $\widehat{\phi}$. As a quantum field, $\widehat{\phi}$ is an operator-valued distribution acting on a Hilbert space $\mathscr{H}$. A state $v \in \mathscr{H}$ defines the space-time history of the quantum field in the sense that it defines an expectation value $\langle\widehat{\phi}(x)\rangle = \langle v|\widehat{\phi}v\rangle$ at each point x in space-time.[1]

In general, the value of a quantum field is indefinite at each point in space, yielding a distribution of values when subjected to measurement-like interactions. At each point this distribution has a mean value, (dubbed the expectation value). This entails that there is a spatial field of expectation values. Even if the field of expectation values is spatially homogeneous, the measured realisation of the field will not be exactly homogeneous.

However, quantum 'fluctuations' are not comparable to the perturbations in classical fields. Unless one adopts a hidden-variables or many-worlds interpretation of quantum theory, quantum fluctuations in a quantity indicate that the values of that quantity are objectively indefinite. Classical fluctuations, by contrast, simply indicate that the field does not have the same definite value throughout space.

The fundamental claim of inflation is that quantum fluctuations provide the seed for the subsequent classical field fluctuations, transforming one type of fluctuation into the other. For this to be possible, there would need to be some form of measurement-like interaction process which operates throughout space, transforming a field of expectation values into a classical field.

The explanations proffered by advocates of inflationary cosmology rarely touch on these issues. This is the summary offered by Kolb and Turner:

> "Causal microphysics operates only on distance scales less than order [of] the Hubble radius, as the Hubble radius represents the distance a light signal can travel in an expansion time. As each mode k crosses outside the horizon, it decouples from microphysics and 'freezes in' as a classical fluctuation," (1990, p286)

One can raise a number of obvious objections to this account. First, as explained in the preceding section, the Hubble radius does *not* represent "the distance a light signal can travel in an expansion time." The distance from which

[1] Neglecting the difficulties defining a field operator at a point of space-time rather than a small open subset.

light signals can be received corresponds to the particle horizon, not the Hubble horizon.[2]

Second, modes straddle the Hubble horizon rather than exiting the horizon, hence what sense does it make to claim that they "decouple from microphysics"? Whilst the extremities of a mode may be receding faster than the speed of light once it straddles the horizon, the interior parts of the mode are not, and can continue to engage in local causal interactions.

Third, if straddling the Hubble horizon is sufficient to trigger a quantum-to-classical transition, this is tantamount to proposing that it acts as a trigger for wave-function collapse, or some form of 'decoherence'. This would be a radical and controversial proposal, with deep consequences for the interpretation of quantum theory, (for a wider discussion, see Perez *et al*, 2006).

So inflation can only provide a conceptually adequate explanation for the seeds of galactic structure formation if it also provides a resolution of the fundamental interpretational issues in quantum theory.

This is clearly an explanatory deficiency, but it doesn't constitute a fatal blow against the inflationary account of structure formation. Other branches of modern science, such as solid-state physics, also provide explanations which ultimately require a resolution of quantum theory's interpretational issues. The absence of such a resolution does not destroy the viability of these explanations, it just entails that the explanations are incomplete.

So let's follow the inflationary account of structure formation all the way through to the point where it generates a scale-invariant spectrum, to determine if there are other, more fatal vulnerabilities.

For notational simplicity, the amplitudes of the inflaton perturbation modes are often denoted by ϕ_k. With this notation, the power spectrum is claimed to be (Coles and Lucchin p277):

$$P_\phi = |\phi_k|^2 = \frac{H^2}{2k^3} \,,$$

where H is the Hubble constant of the inflationary period. Let's see how this expression can be derived.

The representation of cosmological perturbations depends upon how a space-time is foliated into a one-parameter family of spacelike hypersurfaces. The choice of such a foliation is referred to by inflationary cosmologists as a choice of 'gauge' to indicate that the choice is of no physical significance. There are two popular choices: a flat gauge, in which the hypersurfaces have constant scalar curvature, but non-zero fluctuations of the inflaton, $\delta\phi$; and a 'comoving' gauge, in which the hypersurfaces have constant ϕ, but non-zero curvature perturbations.

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[2] The particle horizon at time t_0 is defined by

$$a(t_0)\int_0^{t_0} \frac{c}{a(t)}\, dt \,.$$

The exposition which follows uses a comoving 'gauge', in which $\delta\phi = 0$, as popularised by Maldacena (2003). The derivation closely tracks that provided by Baumann (2012).

Given a comoving curvature perturbation $\mathcal{R}$, a composite variable v is defined:

$$v = z\mathcal{R}\ , \text{ where } z = a\frac{\dot{\phi}}{H} = a^2\frac{\dot{\phi}}{\dot{a}}\,.$$

Sometimes called the *Mukhanov variable*, v can also be related to the 'slow-roll' parameter ϵ which characterises the dynamics of the inflaton field:

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}\,.$$

It follows that

$$z^2 = a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\epsilon\,.$$

It is the composite variable $v(\mathbf{x}, t)$ which is quantized. First, the conventional time variable t is replaced with conformal-time τ:

$$\tau = \int \frac{1}{a(t)}dt\,.$$

Then $v(\mathbf{x}, \tau)$ is expressed as an inverse Fourier transform:

$$v(\mathbf{x}, \tau) = \frac{1}{(2\pi)^3}\int v_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3\mathbf{k}\,.$$

The Fourier coefficients $v_{\mathbf{k}}(\tau)$, which specify the amplitude of each mode, are time-dependent. These coefficients must satisfy the Mukhanov-Sasaki equation:

$$v''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right)v_{\mathbf{k}} = 0\,,$$

where $k = ||\mathbf{k}||$, and the primes indicate derivatives with respect to conformal time. This equation indicates that each mode behaves in conformal time like a simple harmonic oscillator, albeit one with a time-dependent angular frequency $\omega_k(\tau)$. In other words, the equation has the form of the governing equation for a simple harmonic oscillator:

$$v''_{\mathbf{k}} + \omega_k^2 v_{\mathbf{k}} = 0\,,$$

with

$$\omega_k(\tau) = \sqrt{\left(k^2(\tau) - \frac{z''(\tau)}{z(\tau)}\right)}\,.$$

Thus, the time-dependence of the modes has been isolated in the form of the functions $v_{\mathbf{k}}(\tau)$, and the evolution of the different modes are independent from each other.

The most general solution of the Mukhanov-Sasaki equation can be written as a linear combination of a time-dependent function $v_k(\tau)$ and its complex conjugate $v_k^*(\tau)$:

$$v_{\mathbf{k}}(\tau) = a^-_{\mathbf{k}} v_k(\tau) + a^+_{-\mathbf{k}} v_k^*(\tau)\,,$$

where the function $v_k(\tau)$ depends only on the magnitude of the mode k, but the constants are functions of the mode-vector $\mathbf{k}$. The constants satisfy the condition $a^+_{\mathbf{k}} = (a^-_{\mathbf{k}})^*$.

The inverse Fourier transform is now expanded as the following:

$$\begin{aligned} v(\mathbf{x},\tau) &= \frac{1}{(2\pi)^3}\int v_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{k} \\ &= \frac{1}{(2\pi)^3}\int [a^-_{\mathbf{k}} v_k(\tau) + a^+_{-\mathbf{k}} v_k^*(\tau)] e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{k} \\ &= \frac{1}{(2\pi)^3}\int [a^-_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a^+_{\mathbf{k}} v_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}] \mathrm{d}^3\mathbf{k} \end{aligned}$$

In the canonical quantization of a massive scalar field in Minkowski space-time, the two terms correspond to the fact that the the energy-momentum vector p has to satisfy $\| p \| = m$. This equation is satisfied by a pair of disconnected 3-dimensional hypersurfaces in energy-momentum space: the forward mass hyperboloid and the backward mass hyperboloid. Hence, the Fourier transform decomposes into the sum of two integrals, one taken over the forward mass hyperboloid, and one taken over the backward mass hyperboloid. In the expression above, $a^-_{\mathbf{k}}$ has support on the forward mass hyperboloid, and $a^+_{\mathbf{k}}$ has support on the backward mass hyperboloid.

If $v_{\mathbf{k}}$ is fixed, the time-independent constants are determined by the choice of the time-dependent function $v_k(\tau)$. For any τ,

$$a^-_{\mathbf{k}} = \frac{v_k^{*\prime}(\tau) v_{\mathbf{k}}(\tau) - v_k^*(\tau) v'_{\mathbf{k}}(\tau)}{v_k^{*\prime}(\tau) v_k(\tau) - v_k^*(\tau) v'_k(\tau)}\,.$$

Now, under the canonical quantization of this system, annihilation operators $\hat{a}^-_{\mathbf{k}}$ are substituted in place of $a^-_{\mathbf{k}}$, and creation operators $\hat{a}^+_{\mathbf{k}}$ are substituted in place of $a^+_{\mathbf{k}}$:

$$\hat{v}(\mathbf{x},\tau) = \frac{1}{(2\pi)^3}\int [\hat{a}^-_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^+_{\mathbf{k}} v_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}] \mathrm{d}^3\mathbf{k}$$

Thus, for the quantized field $\hat{v}(\mathbf{x},\tau)$, the only part of the Fourier coefficients remaining are the time-dependent functions $v_k(\tau)$.

Inflationary cosmology postulates that the quantized field $\hat{v}(\mathbf{x},\tau)$ was initially in its ground state, or 'vacuum state'. However, herein lies a problem: each different choice for the form of the time-dependence $v_k(\tau)$ defines a different vacuum state.

In this context, a vacuum state $|\mathbf{0}\rangle$ is merely required to be one which is mapped to the zero vector by all the annihilation operators:

$$\hat{a}^-_{\mathbf{k}}|\mathbf{0}\rangle = 0\,, \text{ for all } \mathbf{k} \in \mathbb{R}^3\,.$$

Assuming that the field operator $\hat{v}(\mathbf{x},\tau)$ is fixed as the quantization of the classical field $v = z\mathcal{R}$, changing the form of the time-dependence $v_k(\tau)$ entails that the annihilation and creation operators have to change as well, according to the following expression:

$$\hat{a}^-_{\mathbf{k}} = \frac{v_k^{*\prime}(\tau)\hat{v}_{\mathbf{k}}(\tau) - v_k^*(\tau)\hat{v}'_{\mathbf{k}}(\tau)}{v_k^{*\prime}(\tau)v_k(\tau) - v_k^*(\tau)v'_k(\tau)}\,.$$

If the annihilation operators change, then so too does the vacuum state.

Now, this is also true for such a quantum field in Minkowski space-time. However, in the case of Minkowski space-time the angular frequency of the modes is $\omega_k = k$, and there is a $v_k(\tau)$ such that the corresponding vacuum state provides the minimum energy eigenstate of the Hamiltonian *at all times*. This is

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\,.$$

In contrast, in a non-stationary space-time (i.e., one without a global time-like Killing vector field), the angular frequency of the modes is time-dependent $\omega_k(\tau)$, and the vacuum state $|\mathbf{0}\rangle_{\tau_0}$ which minimises the eigenvalue of the Hamiltonian at one moment τ_0, will generally be different from the vacuum state $|\mathbf{0}\rangle_{\tau_1}$ which minimises the eigenvalue of the Hamiltonian at another moment τ_1.

Specifically, at a particular time τ_0, the mode-function $v_k(\tau)$ defined by the initial conditions:

$$v_k(\tau_0) = \frac{e^{-i\omega_k(\tau_0)\tau_0}}{\sqrt{2\omega_k(\tau_0)}}\,,$$

and

$$v_k(\tau_0) = -i\omega_k(\tau_0)v_k(\tau_0)\,,$$

determines a vacuum state $|\mathbf{0}\rangle_{\tau_0}$ which provides the minimum energy eigenstate of the Hamiltonian at time τ_0, but isn't guaranteed to do so at all other times.

The solution proposed to this problem by inflationary cosmologists starts with the claim that an inflationary space-time is a quasi-de Sitter space-time. The 'quasi' is merely an acknowledgement that in 'slow-roll' inflation, the effective value of the Hubble parameter H will be slowly changing, whereas 'true' de Sitter space-time has a fixed Hubble parameter.

The second step is to note that de Sitter space-time has a conformal time-parameter running from $-\infty$ in the far past to 0 in the future. (Inflation is considered to terminate at a small negative value of τ). Given that $a(t) = \exp(Ht)$, as $\tau \to -\infty$, the rate-of-change of the scale factor diminishes to zero, and de Sitter space-time purportedly approaches Minkowski space-time. The vacuum state in de Sitter space-time which tends towards the vacuum state of Minkowski space-time in this limit is termed the *Bunch-Davies* vacuum. This vacuum state corresponds to the following initial condition on the time-dependent $v_k(\tau)$:

$$\lim_{|k\tau|\to\infty} v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \, .$$

Now the Mukhanov-Sasaki equation has the following general solution:

$$v_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \, .$$

With $\beta = 0$ and $\alpha = 1$, this becomes:

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \, .$$

In the limit $\tau \to -\infty$, it follows that for any k, $|k\tau| \to \infty$, and $i/k\tau \to 0$, hence it follows that

$$\lim_{|k\tau|\to\infty} v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \, ,$$

which thereby satisfies the definition of the Bunch-Davies vacuum.

One immediate objection to this line of argument is that inflationary cosmologists use only a patch of de Sitter space-time. In fact, the spatially flat space-time with a scale factor $a(t) = \exp(Ht)$ corresponds to just half of de Sitter space-time, and the negatively curved inflationary space-time with a scale factor $\sinh(Ht)$, corresponds to the interior of a light-cone in de Sitter space-time. Whilst the Friedmann-Roberston-Walker models of 'Big Bang' cosmology are incomplete and inextendible, the inflationary space-times are incomplete, but extendible, (see Figure 1).

The past boundary defined by the condition $\tau \to -\infty$ corresponds to a null hypersurface in the full de Sitter space-time. Timelike geodesics which are not orthogonal to the flat spacelike hypersurfaces used to foliate this half-space, can pass straight through the $\tau = -\infty$ boundary, (Aguirre, 2008). This is a somewhat troubling property for a cosmological spacetime to possess.

The complete de Sitter space-time can be foliated by a one-parameter family of compact, positively curved, spacelike Cauchy hypersurfaces, with a scale factor of the form $a(t) = \cosh(Ht)$. (In fact, the complete de Sitter space-time only admits compact Cauchy hypersurfaces). This entails that the complete de Sitter space-time describes a universe which is contracting from the infinite past to a finite radius, before expanding into the infinite future. This hardly resembles an inflationary cosmology.

Moreover, it is also true that the 4-dimensional de Sitter geometry is space-time homogeneous; any point is equivalent to every other point. So points inside the half-space chosen to represent a spatially flat inflating cosmology are equivalent to points outside the half-space. This begs the question, 'Why should one accept anything defined in terms of the coordinate boundary $\tau = -\infty$ as true initial conditions for a cosmological model?'

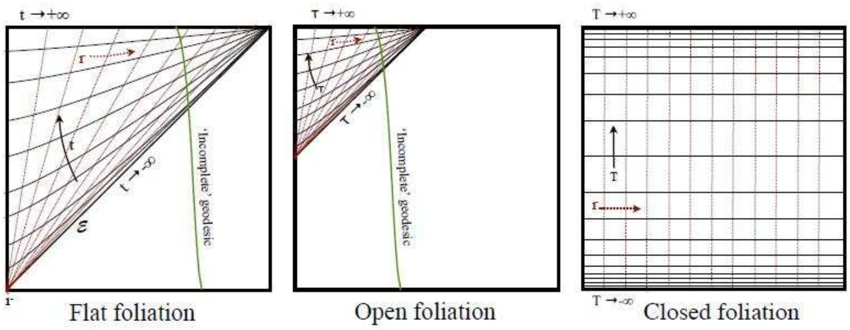

Figure 1: From left to right: Conformal diagrams for (i) the half-space of de Sitter space-time, used in spatially flat inflationary models; (ii) the interior of a light-cone in de Sitter space-time, used for inflationary models with negative spatial curvature; and (iii) the complete de Sitter space-time, with positive spatial curvature. (From Aguirre 2008)

One answer is that the Bunch-Davies vacuum *can* be defined as a vacuum state for the whole of de Sitter space-time; in fact, it is invariant under $O(4,1)$, the symmetry group of de Sitter space-time (Hollands and Wald, 2014). For a massive scalar field, there is a one-complex parameter family of vacuum states in de Sitter space-time, all of which are invariant under $O(4,1)$, and which include the Bunch-Davies vacuum as a special case (Allen, 1985). The Bunch-Davies vacuum, however, is the only such state which satisfies the Hadamard condition (Hollands and Wald, 2014). The stress-energy tensor for a quantized scalar field only makes sense for a Hadamard state, hence the Hadamard condition is arguably a physical requirement. This singles out the Bunch-Davies vacuum as an appropriate vacuum state for a massive quantum field in de Sitter space-time.

For a massless field, there are no vacuum states invariant under the action of the de Sitter symmetry group. However, as long as the inflaton is assumed to be a massive scalar field, a viable candidate exists for a physically unique vacuum state in de Sitter space-time.

Hence, to summarise: in the de Sitter half-space corresponding to a spatially flat inflationary cosmology, the Bunch-Davies vacuum is special because it resembles the Minkowski vacuum in the past limit in which the space-time becomes Minkowski-like; and in the complete de Sitter space-time, the Bunch-Davies vacuum is the only state which is both invariant under the full symmetry group, and which satisfies the Hadamard condition.

Nevertheless, the ability of inflationary cosmology to provide explanations free from initial conditions and assumptions is undermined again, this time by the fact that de Sitter space-time does not resemble Minkowski space-time beyond the limits of the coordinate patch defined by $\tau = -\infty$.

To return to our derivation: Given the time-dependent $v_k(\tau)$ corresponding to the Bunch-Davies vacuum, the time-behaviour of the modes in the 'superhorizon' limit is considered. Recall that the ratio of k, the comoving wave-number, to aH, the reciprocal of the comoving Hubble horizon $(aH)^{-1}$, defines whether a mode is a subhorizon mode, or a superhorizon mode, at a particular point in time. Moreover, the conformal time can be approximated by $\tau = -a^{-1}H^{-1}$. Hence,

$$\left|\frac{k}{aH}\right| = |k\tau|\,.$$

The superhorizon limit is defined by $|k\tau| \ll 1$. Hence, the time-dependence of the modes is considered in the limit $k\tau \to 0$:

$$\begin{aligned}
\lim_{|k\tau|\to 0} v_k(\tau) &= \lim_{|k\tau|\to 0} \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \\
&= \lim_{|k\tau|\to 0} \left(\frac{e^{-ik\tau}}{\sqrt{2}\cdot\sqrt{k}} - \frac{e^{-ik\tau}i}{\sqrt{2}\cdot k^{3/2}\tau}\right) \\
&= \lim_{|k\tau|\to 0} \left(\frac{e^{-ik\tau}(k\tau - i)}{\sqrt{2}\cdot k^{3/2}\tau}\right) \\
&= \frac{-i}{\sqrt{2}\cdot k^{3/2}\tau} \\
&= \frac{1}{i\sqrt{2}\cdot k^{3/2}\tau}
\end{aligned} \quad .$$

Now, the so-called 'zero-point' fluctuations of the quantized field $\hat{v}$, (i.e., the fluctuations in the vacuum state) are given by:

$$\langle \mathbf{0}|\hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}}|\mathbf{0}\rangle = |v_k|^2\,.$$

Recall that for the quantized field, the classical Fourier coefficients $v_{\mathbf{k}}$ have been quantized, and in the process split into a linear combination of creation and annihilation operators and time-dependent terms, $\hat{v}_{\mathbf{k}} = \hat{a}^-_{\mathbf{k}} v_k(\tau) + \hat{a}^+_{\mathbf{k}} v^*_k(\tau)$. Hence, the power spectrum $P_{\hat{v}}$ of a quantum field is not given by the square modulus of the Fourier coefficients, but by the square modulus of the remaining time-dependent components $P_{\hat{v}} = |v_k|^2$.

Hence, on the scales which inflation transforms into superhorizon scales, the scales which subsequently become relevant to structure formation, we have:

$$P_{\hat{v}} = |v_k|^2 = \left|\frac{1}{i\sqrt{2}\cdot k^{3/2}\tau}\right|^2 = \frac{1}{2k^3\tau^2} = \frac{(aH)^2}{2k^3}\,,$$

which is the k^{-3} form of an invariant power spectrum.

At least, this is the power spectrum of the Mukhanov variable $v = z\mathcal{R}$. The power spectrum of the curvature fluctuations themselves is simply:

$$P_{\mathcal{R}} = \frac{1}{z^2}P_{\hat{v}} = \frac{1}{2a^2\epsilon}P_{\hat{v}} = \frac{1}{4k^3}\frac{H^2}{\epsilon} = \frac{1}{2k^3}\frac{H^4}{\dot{\phi}^2}\,.$$

The curvature fluctuations are related to the fluctuations in the inflaton field by the following expression:

$$\mathcal{R} = H\frac{\delta\phi}{\dot{\phi}} .$$

Hence,

$$P_{\mathcal{R}} = \left(\frac{H}{\dot{\phi}}\right)^2 P_\phi .$$

It follows that:

$$P_\phi = \frac{\frac{1}{2k^3}\frac{H^4}{\dot{\phi}^2}}{(H/\dot{\phi})^2} = \frac{H^2}{2k^3} ,$$

which is exactly the expression we set out to derive.

5 Horizon crossing

In the previous section we tracked the derivation of the perturbations in the inflaton field, obtaining the following expression:

$$P_\phi = \frac{H^2}{2k^3} .$$

As we saw, the perturbation spectrum in the inflaton (a quantum matter field) had the same k^{-3} form as the perturbation spectrum in the spatial curvature. Effectively, this is a proxy for the perturbation spectrum in the gravitational metric, which is thence approximated by a Newtonian gravitational potential. Given that the perturbation spectrum in the inflaton is $P_\phi \sim k^{-3}$, if the gravitational potential inherits this spectrum, $P_\Phi(k) \sim k^{-3}$, this will correspond to a scale-invariant spectrum $P_\rho \sim k$ in the matter density perturbations.

The mathematical derivation of this scale-invariance is typically accompanied by claims about the significance of the horizon-crossing process during inflation. Here's a typical example from Kolb and Turner:

"The classical evolution of a given mode k is governed by:

$$\delta\ddot{\phi}_k = 3H\delta\dot{\phi}_k + k^2\delta\phi_k/a^2 = 0$$

whose solution for super-horizon-sized modes ($k \ll aH$) is very simple: $\delta\phi_k =$ const...During inflation H and $\dot{\phi}$ vary rather slowly; moreover, the modes of cosmological interest, say 1 Mpc to 3000 Mpc, crossed outside the horizon during a period spanning only a small fraction of the total inflationary epoch - about 8 e-folds out of the total of 60 or so. As a result, the spectrum of perturbations predicted is very nearly scale invariant...As each mode crosses outside the horizon, it has the same physical size ($\sim H^{-1}$), and the Universe has the same expansion rate; thus each scale has impressed upon it a wrinkle of the same amplitude," (1990, p286-287).

The equation above is that of a harmonic oscillator with a damping term $3H\delta\dot{\phi}_k$. The Hubble horizon is significant here because a mode exceeds the scale of the Hubble horizon when it satisfies the condition $k/a \ll H$. The ratio k/a is also the coefficient in the term which specifies the 'restoring force' of the oscillator:

$$\frac{k^2}{a^2}\delta\phi_k \, .$$

Hence, when $k/a \ll H$, the coefficient in the damping term, $3H$, dominates the coefficient in the restoring-force term, and the system becomes an over-damped oscillator, which should return to its equilibrium value without further oscillations. For the inflaton field, this means that the amplitude of the perturbational modes becomes constant in time once their scale exceeds that of the Hubble horizon. But does each scale possess "a wrinkle of the same amplitude"?

Consider any pair of comoving wave-numbers (k_1, k_2). They will pass through the Hubble radius at a pair of times (t_1, t_2), where

$$a(t_i)\frac{2\pi}{k_i} = H^{-1} \, .$$

It we interpret Kolb and Turner's statement at face value, it amounts to the following pair of assertions:

$$\delta\phi_{k_1}(t_1) = \delta\phi_{k_2}(t_2) \, .$$

$$\delta\dot{\phi}_{k_1}(t_1) = \delta\dot{\phi}_{k_2}(t_2) = 0 \, .$$

If each mode had the same amplitude as it straddled the Hubble horizon, and if the amplitude of each mode ceased to grow thereafter, each mode would have the same amplitude when it fell back inside the horizon during the post-inflationary era. But this isn't true: the smaller wavelength modes have larger amplitudes.

It is certainly true that if H is constant then each mode will have the same physical size as it crosses the Hubble horizon. However, the universe does not have the same expansion rate as each mode crosses the Hubble horizon because $\ddot{a}(t) > 0$.

Whilst the Hubble parameter H is constant during inflation, the Hubble parameter is not equivalent to the expansion rate $\dot{a}(t)$. The expansion is accelerating, so the expansion rate $\dot{a}(t)$ is *increasing*. Moreover, because each mode has been growing for a different period of time before it crosses the horizon, there is no guarantee that the amplitude of each mode will be the same as it crosses the horizon. Whilst $\dot{\phi}$ might be nearly constant, this does not entail that each mode has the same amplitude as it crosses the horizon. This is a misinterpretation of scale-invariance.

After the inflationary era ends, and the universe enters deceleratory FRW expansion, the Hubble radius begins to increase. The modes which correspond to the galaxies and galaxy clusters in the presently observable universe re-enter the Hubble horizon when the universe is still radiation-dominated. At this point, they are classical perturbations. These perturbations trigger acoustic oscillations within the primordial plasma, which leave their imprint on the cosmic microwave background radiation at the time of 'recombination'.

It is often claimed that not only are the perturbational amplitudes scale-invariant when they're 'exiting' the horizon during inflation, but they are also scale-invariant when they re-enter the horizon during the subsequent post-inflationary period. A particularly clear example of this is enunciated by Brandenberger:

"In position space, we compute the root mean square mass fluctuation $\delta M/M(k,t)$ in a sphere of radius $l = 2\pi/k$ at time t. A scale-invariant spectrum of fluctuations is defined by the relation

$$\frac{\delta M}{M}(k, t_H(k)) = \text{const}\,,$$

[where $t_H(k)$ is the time at which a mode crosses back inside the Hubble radius during the post-inflationary era]. Such a spectrum was first suggested by Harrison and Zeldovich as a reasonable choice for the spectrum of cosmological fluctuations. We can introduce the 'spectral index' n of cosmological fluctuations by the relation

$$\left(\frac{\delta M}{M}\right)^2 (k, t_H(k)) \sim k^{n-1}\,,$$

and thus a scale-invariant spectrum corresponds to $n = 1$," (2004, p135-136).

This is *not* the result derived in the preceding section. Instead, an additional line of argument is required to establish the claim that the spectrum of perturbations which re-enter the horizon is also scale-invariant, (ibid., p145-146). Moreover, failure to satisfy this condition might be a necessary component in an explanation of structure formation:

"The amplitude fluctuations have when they come back inside the causal horizon is the same as it was when they left - *except* for a small factor, which depends on the ratio of pressure to energy density obtained in the universe when they say goodbye and hello again. This factor is larger for particles saying hello again in the early Big Bang phase, which is dominated by radiation, than it is later, when the universe is dominated by matter. This means that the amplitude of fluctuations coming within the horizon during the radiation-dominated phase is larger than the amplitude of fluctuations entering the horizon during the matter-dominated epoch. This effect is actually helpful for the formation of galaxies and clusters of galaxies...The scales on which [they] are forming (smaller than 400 million light years today) came within the causal horizon and said hello again when the universe was still radiation dominated. So they get the benefit of this larger factor in amplitude starting off. This boosts the formation of galaxies and clusters of galaxies, while keeping the fluctuations in the cosmic background as small as what we observe today. The Harrison-Zeldovich constant-amplitude hypothesis would not give us this extra saving grace," (Gott 2016, p91).

6 Conclusions

One of the primary questions posed at the start of this paper was whether inflation genuinely provides an explanation for the scale-invariant spectrum of density perturbations. The answer to this question is in the affirmative.

However, the inflationary explanation of the seeds of structure formation suffers from two primary deficiencies:

- It tacitly assumes there is an unspecified process for transforming quantum fluctuations into classical fluctuations.
- It lacks a unique vacuum state for a quantum field, and can only plausibly fix this problem within the extendible space-time structure assumed by the theory. Whilst the space-time patch used by inflation can be extended, doing so further dissolves the notion that inflationary cosmology is a theory of initial conditions.

The empirical credentials of inflationary cosmology ultimately rest upon a resolution of these problems.

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