Credence and Chance in Quantum Theory

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David Lewis' "Principal Principle" is a purported principle of rationality connecting credence and objective chance. Almost all of the discussion of the Principal Principle in the philosophical literature assumes classical probability theory, which is unfortunate since the theory of modern physics that, arguably, speaks most clearly of objective chance is the quantum theory, and quantum probabilities are not classical probabilities. This paper develops an account of how chance works in quantum theory that reveals a connection between credence and quantum chance quite unlike what is envisioned in the philosophical literature: as a theorem of quantum probability, updating a completely additive chance function on a knowledge of chance brings credence into line with chance. The account also suggests a way of construing the Humean supervenience of chance that has the virtue of dissolving some puzzles about the "undermining" of chances. A number of interpretative moves in quantum theory are needed to generate the account of quantum chance on offer here, and they can all be disputed. But engaging in these disputes is part and parcel of naturalized metaphysics, and as such it can be more productive than engaging in the battle of intuitions among analytical metaphysicians about how chance ought to work this and other possible worlds.

1 Introduction

David Lewis (1980) proposed a normative principle, which he dubbed the Principal Principle (PP), linking credence (aka degree of belief) and objective chance. Of course, it is a matter of controversy as to whether there is objective chance in the world and, if so, what it is. Lewis wanted to allow for the possibility that chance exists and, regardless of what "it" is, he thought that it should constrain rational credence in the way specified by PP. Turning this around, PP serves as a functional characterization of chance—chance is what commands rational credence. To give a concrete example of an intended application, if you know that the objective chance of heads on the next flip of a coin is 1/2 then in order for your credence to be rational you ought to assign a credence of 1/2 to said outcome, and this is so regardless of other things you might know, such as the frequency of heads and tails on past flips of the coin. Lewis intended the regardless clause to apply not just to evidence about past flips of the coin but to any "admissible" evidence, but initially no general criterion of admissibility was specified. Lewis himself came to believe that there was "bug" in his original formulation of PP, and he sought to reformulate PP in a manner that would avoid the bug while conforming to his desire allow for the Humean supervenience of chance (Lewis 1994).

A fairly sizable and ever growing literature has accreted around these topics.[1] The contributors are mainly analytical metaphysicians who produce ever more nuanced treatments which are sprinkled with interesting insights and clever, and even brilliant, moves. But since the discussion seems to be constrained only by intuitions about how chance ought to work in the actual and other possible worlds it is not surprising that fundamental disagreements have arisen both about how to capture in precise form the idea behind Lewis' PP and about how to justify PP as a principle of rationality of belief. Even more disconcerting is the fact that the literature makes little contact with quantum theory despite the fact that this is inarguably one of the most successful theories of modern physics and arguably the theory that speaks most clearly of objective chance. This lack of contact is not an oversight that is easily corrected; for the bulk of the philosophical discussions of the issues surrounding Lewis' PP assume classical probability theory whereas quantum probabilities cannot be construed as probabilities on a classical probability space. In particular, Bayesian conditionalization, used in updating credence functions in classical probability, is inappropriate for credence functions defined over a non-commutative event algebra such as the one encountered in quantum theory.

In the present paper I propose a turn away from analytical metaphysics towards naturalized metaphysics. The form of naturalization I have in mind is not a fulfillment of the logical positivist dream of replacing metaphysics by a purely empirical enquiry. It is empirically based in that it justifies a focus on quantum theory by the overwhelming evidence of the empirical success of the theory. At the same time, it leaves room for and, indeed, requires philosophical analysis of a metaphysical (or perhaps better, meta-physical) bent; but the metaphysics (or meta-physics) is constrained by the imperative

[1] For a sampling of the literature, see Arntzenius and Hall (2003); Bigelow, Collins, and Pargeter (1993); Black (1998); Haddock (2011); Hall (1994, 2004); Ismael (2008); Meacham (2010); Pettigrew (2012); Roberts (2001, 2013); Strevens (1995); Thau (1994); Vranus (2002, 2004).

to illuminate the content and applications of the quantum theory. Under this constraint I offer an account of how chance works in quantum theory and show how this account helps to resolve a number of disputed issues and puzzles that have arisen in the literature on PP. The resolutions take the form of proving theorems in quantum probability theory.

Needless to say, my account of quantum chance is open to dispute since it invokes a number of interpretative moves, and the interpretation of quantum theory is radically underdetermined by the need to account for its empirical adequacy. But reducing issues about the relation of credence and chance to issues about theory interpretation is, I claim, an advance in that it produces more fruitful discussion than the clash of philosophical intuitions about how chance ought to work. Also needless to say, my approach has nothing to say about how chance works in other possible worlds where quantum theory does not hold sway. I concede this territory to analytical metaphysicians and remain content with trying to understand how chance works in this world which is apparently governed by quantum theory.

The plan of the paper is as follows. In Section 2 I provide a brief critical review of the literature on PP and endeavor to identify some key points of contention, especially those that are ripe for resolution within an account of quantum chance. Section 3 presents a framework for quantum probability theory that lends itself to probing the relation between the rational credence in quantum events and their objective chances. Section 4 lays out the interpretative moves in quantum theory that undergird my account of how quantum chance works. This account allows issues about credence and chance to be settled by proving results in quantum probability theory. Details are presented in Section 5. Section 6 presents a new way of construing the supervenience of quantum chances that may, or may not satisfy Humean scruples; but does have the virtue of helping to dissolve puzzles about the "undermining" of chances. Conclusions are contained in Section 7.

2 Lewis' Principal Principle: A Brief Critical Review

2.1 Preliminaries

In what follows I rely on the formulations of various Principles due to Pettigrew (2012).[2] Notation and assumptions: Cr stands for a credence function defined over some set of sentences closed under truth functional connectives. The domain of Cr is assumed to contain not only sentences expressing propositions about such mundane matters as outcomes of coin flips but also sentences of the of the form C_{ch} expressing the proposition that chances are given by the chance function ch. It is further assumed that rationality requires agents to adopt credence functions that are coherent (i.e. satisfy the standard axioms of probability) and to update their credences by Bayesian conditionalization.[3] The issues to be addressed are whether (as Lewis' PP would have it) there is an additional requirement of rationality to the effect that rational credence must be heedful of objective chance and, if so, how to express the requirement in formal terms and how to justify it.

2.2 An ambush of alleged principles of rationality

A good entry point to the literature is a special case of PP sometimes called Miller's Principle:

> (MP) A rational agent ought to have a credence function Cr such that for all possible chance functions ch and all A in the domain of ch, $Cr(A/C_{ch}) = ch(A)$, provided that $Cr(C_{ch}) \neq 0$.

Here the slash denotes classical conditionalization, i.e. $Cr(X/Y) := Cr(XY)/Cr(Y)$.

No sooner is MP written down than a host of questions present themselves. To begin, can some chance functions be merely finitely additive

[2] I have modified Pettigrew's formulations in ways that are mainly insignificant but in some instances are consequential.

[3] The notion that the relation between credence and chance is best understood within the Bayesian framework for credence has been criticized by Roberts (2013). While I think there is some merit to Robert's position I adopt the Bayesian framework here since it allows the maximal contact with the literature.

while others are countably additive, and still others completely additive?[4] If so the spirit of MP cannot be satisfied: on pain of contradiction, a credence function cannot be heedful of chances of all stripes of additivity since Bayes conditionalizing a merely finitely additive (respectively, merely countably additive, completely additive) Cr results in a merely finitely additive (respectively, merely countably additive, completely additive) credence function, whereas the application of MP to chance functions that are variously finitely, countably, and completely additive would force the conditionalized chance functions to change their additivity character.[5] But the letter of MP might be satisfied, for example, by a merely finitely additive Cr such that $Cr(C_{ch}) = 0$ for all countably or completely additive ch.

This leads to a second question: What to say about a credence function Cr that satisfies MP vacuously by violating the proviso for all chance functions, i.e. $Cr(C_{ch}) = 0$ for all ch? Such a credence function would more properly be called chance denying than chance heeding. Which raises another question: How should MP be strengthened so as to rule out such vacuity? The strongest form would drop the proviso altogether, requiring $Cr(C_{ch}) \neq 0$ for all ch and, thereby, requiring complete chance heedfulness. But for reasons already noted no credence function can be completely chance heeding unless all chance functions share a single form of additivity. If all chance functions do share one form of additivity, do there then exist completely chance heeding credence functions? If being completely chance heeding is too strong a requirement, should MP be emended to require that $Cr(C_{ch}) \neq 0$ for at least one ch, or for a large set of ch, or ... ?

Leaving to the side the exact form of non-vacuity to be required, suppose that MP is non-vacuously satisfied (in some suitable form) by the credence function Cr of some agent. Then this agent learns that E is true, and she Bayes updates her credence function to $Cr'(\bullet) = Cr(\bullet/E)$. Now apply MP to Cr', and with the help of a little arithmetic conclude that the two applications of MP entail a consistency constraint: for any chance function

[4] Finite (respectively, countable, complete) additivity applies the additivity principle to finite sets (respectively, denumerable sets, sets of any cardinality) of mutually exclusive alternatives. The distinctions among different forms of additivity become important for the discussion of quantum probabilities, as will become evident in Sections 3 and 4 below.

[5] On the account of quantum chance offered here quantum chances are always completely additive. Of course, complete additivity reduces to countable or finite additivity in appropriate circumstances.

ch and any A and E in the domain of ch, $ch(AE) = ch(A)ch(E)$ provided that $Cr(EC_{ch}) \neq 0$. If the constraint isn't satisfied and if MP is necessary for rational credence then the agent is condemned as irrational at the second stage even though she was deemed rational at the initial stage and she reached the second stage by following the prescribed rule for rational updating. This unpalatable consequence has led some commentators to propose that MP should be restricted to initial or tabula rasa credence functions (see Pettigrew 2012 and the references therein). But such a move constitutes a retreat from the idea that knowledge of chance should make rational credence line up with chance.[6]

These issues reemerge when attention turns from MP to the full PP, which is intended to cover cases where the agent learns what the chances are and other things as well. One stab at capturing PP in schematic form would read:

> (SPP) A rational agent ought to have a credence function Cr such that for all chance functions ch and all A and E in the domains of ch and Cr, if E is admissible for A then $Cr(A/C_{ch}E) = ch(A)$, provided that $Cr(C_{ch}E) \neq 0$.

As noted above, David Lewis thought that if E reports the outcomes of past flips of a coin and A asserts that the outcome of a future flip will be, say, Heads, then E is admissible for A. But to turn the schematic SPP into a definite principle requires a general specification of the conditions for admissibility of evidence.

One proposal for specficity is to relativize admissibility of E for A to chance functions and to take E to be admissible for A relative to the chance function ch just in case A and E are stochastially independent for ch, i.e. $ch(AE) = ch(A)ch(E)$. This leads to what Pettigrew (2012) dubs Levi's PP:

> (LPP) A rational agent ought to have a credence function Cr such that for all possible chance functions ch and all A and E in the domains of ch and A, if E and A are stochastically independent for ch, then $Cr(A/C_{ch}E) = ch(A)$, provided that $Cr(C_{ch}E) \neq 0$.

[6] No such retreat is needed when quantum chance is at issue; see Section 5.1.

LPP and MP are logically equivalent (Pettigrew 2012), which is not surprising in view of the discussion above. Thus, LPP is subject to all the worries that beset MP.

A rather different approach, due to Hall (1994) and Thau (1994), is to forego trying to implement the schema SPP in favor of a new PP:

> (NPP) A rational agent ought to have a credence function Cr such that for all possible chance functions ch and all A and E in the domain of ch, $Cr(A/C_{ch}E) = ch(A/C_{ch}E)$, provided that $Cr(C_{ch}E) \neq 0$ and that $ch(C_{ch}E) \neq 0$.

NPP supposes that the sentences C_{ch} are in the domains of chance functions, which some commentators take as problematic (see Haddock 2011). How are expressions such as $ch(C_{ch'}) = x$ to be read? As: the ch-chance is x that chances are given by the chance function ch' (?). Are there such chances of chances? What other reading are available, and how do we choose among them?[7]

If $ch(C_{ch}) = 1$ for all ch then NPP and MP are equivalent. But the proponents of NPP reject the condition that $ch(C_{ch}) = 1$ for all ch since they want to allow for "self-undermining" chance functions such that $ch(C_{ch}) < 1$. MP takes a dim view of such functions. By MP $Cr(C_{ch}/C_{ch}) = ch(C_{ch})$ provided that $Cr(C_{ch}) \neq 0$; but by the probability calculus $Cr(C_{ch}/C_{ch}) = 1$. Thus, either there are no self-undermining chance functions or else there are and MP requires that a rational Cr ignores them (i.e. MP requires that $Cr(C_{ch}) = 0$ for any self-undermining ch). By contrast NPP is compatible with self-undermining chance functions, and in their presence does not reduce to MP. The motivation for wanting to allow for self-undermining chance functions derives from another hobby horse of David Lewis'—the conviction that chances must be Humean supervenient—plus the conviction that such supervenience is not possible if chances cannot be self-undermining.[8]

There is disturbing dilemma for NPP that its proponents have not acknowledged. If the probability function $ch(\bullet/C_{ch}E)$ on the rhs of the NPP

[7] Given the proposal for understanding quantum chance developed below, there is a sensible reading for the quantum analogs of the expressions $ch(C_{ch'})$ in terms of transition probabilities; see Section 4.

[8] On the account of quantum chance offered here, quantum chances are never self-undermining. In Section 6 I will argue that the non-self-undermining character of quantum chances does not undermine their Humean supervenience, properly understood.

equality is not a chance function then NPP is not only not an expression of the idea that knowledge of chance makes rational credence line up with chance but rather a denial of it. So suppose that $ch(\bullet/C_{ch}E)$ and, a fortiori $ch(\bullet/C_{ch})$, are chance functions. In the case where E is a tautology NPP enjoins a rational agent to choose a credence function such that $Cr(A/C_{ch}) = ch(A/C_{ch})$, provided that $Cr(C_{ch}) \neq 0$ and $ch(C_{ch}) \neq 0$. By our supposition $ch(\bullet/C_{ch})$ is equal to a chance function $ch'(\bullet)$. If $ch' = ch$ then ch is not self-undermining, and since NPP was supposed to allow for self-undermining we can assume that $ch' \neq ch$. Since $C_{ch'}$ and C_{ch} are logically incompatible when $ch' \neq ch$ it follows $Cr(C_{ch'}/C_{ch}) = 0 = ch'(C_{ch'})$ and, therefore, $ch'(C_{ch}) > 0$ for any $ch \neq ch'$. Such a ch' seems insane because there seems to be no plausible reading that makes sense of these properties. Thus, either there is no self-undermining ch, which is contrary to the motivation of NPP; or else there is a self-undermining ch, and NPP enjoins a rational agent who has learned that C_{ch} is true to assign credence values given by an insane chance function.

On the other hand, it should be noted that no-self-undermining for chance functions also seems to have some unpleasant consequences. If chance functions are never self-undermining it follows that $ch(C_{ch'}) = 0$ if $ch' \neq ch$. Now let E be such that $ch(E) \neq 0$ so that $ch(\bullet/E)$ is well-defined. Suppose that $ch(\bullet/E)$ is a chance function $ch'(\bullet)$. If $ch(\bullet/E) \neq ch(\bullet)$ and, thus, $ch \neq ch'$ and if there is no self-undermining then a contradiction is reached since $ch'(C_{ch'}) = 1$ but $ch(C_{ch'}/E) = ch'(C_{ch'}) = 0$. To avoid contradiction it must be that $ch(A/E) = ch(A)$ for all E such that $ch(E) \neq 0$. This in turn implies that $ch(A) = 0$ or 1 for all A. In sum, non-self-undermining means either that chances are degenerate deterministic $0-1$ chances, or else that conditionalizing a chance function ch on evidence can produce a probability function $ch(\bullet/E)$ that is not a chance function. Taking the 'or else' escape hatch has (as seen above) unpleasant consequences for NPP; but it does no harm to various implementations of SPP, such as LPP. Nevertheless, one would like to know why and under what conditions $ch(\bullet/E)$ fails to be a chance function.[9]

Another approach starts from the idea that rational credence in a proposition is an expression of epistemic uncertainty about the chances that said proposition is true. One way to formalize this idea is given by what Ismael (2008) dubs the Generalized Principal Principle:

[9] The account of quantum chance given here does provide an answer; see Section 5.5.

GPP A rational agent ought to have a credence function Cr such that for all A in the domains of chance functions

$$Cr(A) = \sum_{ch} Cr(C_{ch})ch(A)$$

where the sum ranges over all possible chance functions.

It is easy to see that MP entails GPP.[10] Conversely, GPP entails MP if there are no self-undermining chance functions.[11] Thus, the problems visited upon MP will also affect GPP. On the other hand, if chances can be self-undermining, as the proponents of NPP claim, then GPP does not entail MP. Nevertheless, GPP is in tension with NPP. From NPP it follows that $Cr(A) = \sum_{ch} Cr(C_{ch})ch(A/C_{ch})$, and this expression reduces to the GPP formula if and only if there are no self-undermining chance functions such that $Cr(C_{ch}) > 0$. The proponents of NPP can now duke it out with the proponents of GPP.

Although superficial, this brief review of the literature indicates that before we can settle on a formulation of a PP-like principle there is a need to address a number of issues about the nature of chance. Are all chance functions of the same additivity stripe? Why or why not? Can chance functions be self-undermining? Why or why not? Does the conditionalization of a chance function produce a chance function? Absent an account of what chance is and how it operates it is hard to see how to answer such questions. But supposing that we do manage to settle on a version of PP, there remains to issue of its normative status. Norms require justification. What is the justification for this alleged norm?

2.3 Justification for norms of rational credence

There is a striking contrast between the justification of the widely accepted norms for rational credence of coherence and updating by Bayesian conditionalization vs. the justification of MP, LPP, NPP, and their ilk. There are a number of considerations that militate in favor of coherence as a necessary condition for rational credence: Dutch book arguments, arguments from

[10] By the principle of total probability $Cr(A) = \sum_{ch} Cr(AC_{ch})$, where without loss of generality the sum is taken over all ch such that $Cr(C_{ch}) \neq 0$. And $\sum_{ch} Cr(AC_{ch}) = \sum_{ch} Cr(C_{ch})Cr(A/C_{ch}) = \sum_{ch} Cr(C_{ch})ch(A)$ where the last equality follows by MP.

[11] Using GPP, $Cr(AC_{\widehat{ch}}) = \sum_{ch} Cr(C_{ch})ch(AC_{\widehat{ch}})$. If there is no self-undermining then $ch(AC_{\widehat{ch}}) = 1$ or 0 according as $ch = \widehat{ch}$ or not. Thus, the sum on the rhs of the equality reduces to $Cr(C_{\widehat{ch}})\widehat{ch}(A)$. So $Cr(A/C_{\widehat{ch}}) = \widehat{ch}(A)$ if $Cr(C_{\widehat{ch}}) \neq 0$.

decision theory, scoring rule arguments, Cox's theorem, and others.[12] There are also arguments (perhaps less decisive) for updating by conditionalization. By contrast there is a notable lack of persuasive of justification for various versions of PP, despite the seeming initial appeal of the informal idea behind Lewis' principle.

One possible route to justification for PP-like principles, explored by Pettigrew (2012), is to ape the scoring rule argument for coherence. Roughly, this argument proceeds by defining a class scoring rules that measure how well a credence function tracks the truth over the range of possible worlds, and then it is shown that a credence function Cr is incoherent if and only if it is dominated by a coherent Cr' in the sense that Cr' scores as well as Cr in all worlds and better in some according to all the rules in said class. Pettigrew proposes scoring rules that reward closeness of credence to chance rather than closeness to truth, and he shows that a coherent Cr fails to satisfy MP and its ilk if and only if it is dominated by a Cr' satisfying MP and ilk according to all the rules in said class. As impressive as this argument is, it has little polemical force in favor of PP-like principles since those who have doubts about such principles will also doubt that rational credence should track chance (even assuming that there is such).

Ismael's GPP could be justified by the claims that rational credence *just is*, by its very meaning, an expression of epistemic uncertainty about objective chances and that GPP is the correct formalization of this idea. But the first claim would be vehemently rejected by personalists of the de Finetti school who deny the existence of objective chance (see de Finetti 1972, 1974). And even those who are willing to entertain objective chance but have initial doubts about Lewis' PP and its ilk would find question begging the claim that rational credence just is epistemic uncertainty about chance. The claim that rational credence can be represented *as if* it were an expression of epistemic uncertainty about chances—which is all that GPP requires—is another matter. Justification for this claim must come from representation theorems, which require substantive assumptions. We will see that quantum theory can, under some conditions, provide the basis for a representation theorem that bears a formal resemblance to GPP, but it is not straightforwardly interpreted as saying that credence in a quantum event is epistemic

[12] But these various arguments deliver different verdicts for additivity principles. This will become relevant to the discussion below of relation of credence and quantum chance.

uncertainty about what the chance of said event is. Quantum theory does yield an analog of MP, but in doing so it shows that the quantum MP is not an extra principle of rationality but merely an expression of principles of quantum probability. More generally, proving representation theorems is the key to answering many of the quantum probability analogs of the questions that arose above about the relation of credence and chance in classical probability theory. Or at least this is so on the account provided below of how chances work in quantum theory.

In advance note that much of the apparatus and nomenclature developed in this philosophical treatments of Lewis' PP becomes irrelevant because quantum probabilities are defined on a non-Boolean algebra of propositions. The next section reviews the quantum probability theory that will be used in an account of quantum chance.

3 Quantum probabilities

3.1 Quantum probabilities in the algebraic formulation of quantum physics

One reason for adopting the algebraic approach is that it is general and flexible enough to encompass ordinary quantum mechanics (QM), relativistic quantum field theory (QFT), and quantum statistical mechanics (QSM). In addition, the algebraic apparatus also helps to reveal relationships between credence and quantum chance that philosophers have been slow to appreciate.

In the version of the algebraic approach used here a quantum system is characterized by two objects: a von Neumann algebra $\mathfrak{N}$ of observables acting on Hilbert space $\mathcal{H}$, which may be separable or non-separable[13]; and a set of states $\mathfrak{S}$ on $\mathfrak{N}$, the members of which are regarded as physically realizable for the system at issue. This subsection reviews some basic facts about algebras

[13] A separable $\mathcal{H}$ has a countable basis. A von Neumann algebra $\mathfrak{N}$ acting on Hilbert space $\mathcal{H}$ is an algebra of bounded operators closed in the weak operator topology. By von Neumann's double commutant theorem the closure condition is equivalent to the condition that $\mathfrak{N}'' := (\mathfrak{N}')' = \mathfrak{N}$, where X' stands for the commutant of X, i.e. the set of all bounded operators that commute with every element of X.

The reader interested in the details of the relevant operator algebra theory can consult Bratteli and Robinson (1987) and Kadison and Ringrose (1997).

that serve as the basic for a formulation of quantum probability theory. The following subsection discusses quantum states.

Ordinary QM (sans superselection rules) deals with case where $\mathfrak{N}$ is the Type I factor algebra $\mathfrak{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. The account of quantum chance given below concentrates on this case; but the supporting mathematical apparatus is developed for general $\mathfrak{N}$, and in Section 4.3 I will consider how the Type III algebras encountered in QFT raise some thorny issues about the nature of quantum chance.[14] For most applications of ordinary QM a separable $\mathcal{H}$ suffices. I will mention in passing some interesting issues that arise when $\mathcal{H}$ is non-separable.

A projection $E \in \mathfrak{N}$ is a self-adjoint element such that $E^2 = E$. The range $Ran(E)$ of E is a closed subspace of $\mathcal{H}$, and for $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ the projections are in one-one correspondence with the closed subspaces of $\mathcal{H}$. The projections $\mathcal{P}(\mathfrak{N})$ have a lattice structure that derives from a partial order whereby $E_1 \leq E_2$ iff $Ran(E_1) \subseteq Ran(E_2)$.[15] That $\mathcal{P}(\mathfrak{N})$ form a lattice means that it is closed under meet $E_1 \wedge E_2$ and join $E_1 \vee E_2$ of $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$ which are defined respectively as the greatest lower bound and the least upper bound. They are respectively the projections corresponding to $Ran(E_1) \cap Ran(E_2)$ and the closure of $Ran(E_1) \cup Ran(E_2)$. Projections E_1 and E_2 are said to be orthogonal iff $Ran(E_1) \cap Ran(E_2) = \varnothing$. When E_1 and E_2 are orthogonal $E_1 \wedge E_2 = E_1 E_2 = E_2 E_1 = E_2 \wedge E_1 = 0$ and $E_1 \vee E_2 = E_1 + E_2$.

The elements of the projection lattice $\mathcal{P}(\mathfrak{N})$ are referred to as quantum propositions (also yes-no questions, or quantum events). Quantum probability theory may be thought of as the study of quantum probability functions Pr on $\mathcal{P}(\mathfrak{N})$ (see Hamhalter 2003). Pr is required to satisfy the analogs of the basic axioms of classical probability:

(a) $\Pr : \mathcal{P}(\mathfrak{N}) \to [0, 1]$

(b) $\Pr(I) = 1$, where I is the identity operator

(c) $\Pr(E \vee F) = \Pr(E + F) = \Pr(E) + \Pr(F)$ for all orthogonal pairs $E, F \in \mathcal{P}(\mathfrak{N})$.

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[14] Details about the different types of von Neumann algebras can be found in Bratelli and Robinson (1987) and Kadison and Ringrose (1997). One of the essential differences concerns the type of projections (see below) in the algebras: Type I algebras contain minimal projectors; Type II contain no minimal projectors but do contain finite dimensional projectors; and Type III contain only infinite dimensional projectors.

[15] This is equivalent to requiring that $E_1 \leq E_2$ iff $E_2 - E_1$ is a positive operator.

The condition (c) may be strengthened to require

(c*) $\Pr(\sum_{a\in\mathcal{I}} E_a) = \sum_{a\in\mathcal{I}} \Pr(E_a)$ for any family $\{E_a\} \in \mathcal{P}(\mathfrak{N})$ of mutually orthogonal projections.[16]

If $card(\mathcal{I}) \leq \chi_0$ then (c*) is the condition of countable additivity, and when $\mathcal{I}$ is allowed to have any cardinality it is known as the requirement of complete additivity. When $\mathcal{H}$ is separable the distinction between complete and countable additivity vanishes, and when $\mathcal{H}$ is finite dimensional the distinctions between finite, countable, and complete additivity vanish.

3.2 Quantum states and quantum probabilities.

Quantum states $\mathfrak{S}$ on $\mathfrak{N}$ are normed positive linear functional mapping elements of $\mathfrak{N}$ to $\mathbb{C}$. A particularly important subclass of quantum states that will play an outsized role in what follows are the normal states $\mathcal{N}$, states with a density operator representation, i.e. there is a trace class operator ϱ on $\mathcal{H}$ with $Tr(\varrho) = 1$ such that $\omega(A) = Tr(\varrho A)$ for all $A \in \mathfrak{N}$.[17] For future reference, some additional nomenclature should be noted. A pure state ω is defined by the property that there are no distinct states ω_1 and ω_2 and real numbers λ_1 and λ_2, $\lambda_1 + \lambda_2 = 1$, such that $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$. Impure states are also referred to as mixed states. A vector state is a state ω such that there is a vector $\psi \in \mathcal{H}$ with $\omega(A) = \langle\psi|A|\psi\rangle$ for all $A \in \mathfrak{N}$. Vector states are normal, and for the algebra $\mathfrak{B}(\mathcal{H})$ the vector states coincide with the pure states.[18]

Quantum states induce quantum probability functions on $\mathcal{P}(\mathfrak{N})$: it is easy to verify that for any $\omega \in \mathfrak{S}$

$$\Pr^\omega(E) := \omega(E) \text{ for } E \in \mathcal{P}(\mathfrak{N})$$

satisfies the requirements (a)-(c) for a quantum probability function. Furthermore, if $\omega \in \mathcal{N}$ then $\Pr^\omega$ is completely additive, whereas if $\omega \notin \mathcal{N}$ then $\Pr^\omega$ will be merely countably or merely finitely additive depending on the

[16] If $\mathcal{I}$ is infinite, the convergence of $\sum_{a\in\mathcal{I}} E_a$ is taken in the weak operator topology.

[17] This is what physicists call the Born rule for calculating probabilities see below.

[18] When the algebra of observables is a Type III von Neumann algebra, as with the local algebras in the algebraic formulation of relativistic QFT, all normal states are vector states and all vector states are impure. This introduces a number of complications in the discussion of quantum probabilities.

dimension of $\mathcal{H}$. If $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ then unless the dimension of $\mathcal{H}$ is huge—technically, a measurable cardinal—complete additivity for quantum probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ reduces to countable additivity (see Eilers and Horst 1975 and Drish 1979). This reduction does not hold for quantum probabilities on the projection lattice of Type III algebras (see Arageorgis et al. 2017).

3.3 Downward and upward

As mentioned above, one can start with states on $\mathfrak{N}$ and move downward to the state-induced quantum probabilities on $\mathcal{P}(\mathfrak{N})$. The most fundamental theorem of quantum probability theory shows that, with a mild restriction, one can also move in the upward direction from probabilities on $\mathcal{P}(\mathfrak{N})$ to states on $\mathfrak{N}$. The original version of the theorem concerned the case of ordinary QM with $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and $\mathcal{H}$ separable:

> *Gleason's theorem.* Let $\mathcal{H}$ be a separable Hilbert space of dimension 3 or greater. Then any quantum probability function Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ has a unique extension to a state ω_{Pr} on $\mathfrak{B}(\mathcal{H})$. Further, if Pr is countably additive (respectively, merely finitely additive) then ω_{Pr} is normal (respectively, non-normal).

When $\mathcal{H}$ is non-separable it is still true that any quantum probability function Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ has a unique extension to a state ω_{Pr} on $\mathfrak{B}(\mathcal{H})$; but the the last clause of the theorem has to be emended to: if Pr is completely additive (respectively, non-completely additive) then ω_{Pr} is normal (respectively, non-normal). Gleason's theorem has also be generalized to cover quite general von Neumann algebras:

> Generalized *Gleason theorem.* Let $\mathfrak{N}$ be a von Neumann algebra acting on $\mathcal{H}$ and suppose that that $\mathfrak{N}$ does not contain any summands of Type I_2. Then any quantum probability function Pr on $\mathcal{P}(\mathfrak{N})$ has a unique extension to a state ω_{Pr} on $\mathfrak{N}$. Further, if Pr is completely additive (respectively, non-completely additive) then ω_{Pr} is normal (respectively, non-normal). [19]

[19] See Hamhalter (2003) and Maeda (1990). A Type I_2 summand has the form $E_1 + E_2 = I$ where E_1 and E_2 are pairwise orthogonal projections.

3.4 Quantum conditionalization

Consider a probability measure Pr on $\mathcal{P}(\mathfrak{N})$ and a $F \in \mathcal{P}(\mathfrak{N})$ such that $\Pr(F) \neq 0$. The Lüders conditionalized measure is defined by $\Pr(E//F) := \Pr(FEF)/\Pr(F)$, $E \in \mathcal{P}(\mathfrak{N})$. When E and F commute $\Pr(E//F) = \dfrac{\Pr(EFF)}{\Pr(F)} = \dfrac{\Pr(EF^2)}{\Pr(F)} = \dfrac{\Pr(EF)}{\Pr(F)} = \dfrac{\Pr(FE)}{\Pr(F)}$ and, thus, Lüders' conditionalization reduces to classical Bayesian conditionalization. Of course, there are many candidates for a quantum conditionalization rule that share the virtue of reducing to classical conditionalization for abelian algebras. But there is a solid reason for thinking that Lüders conditionalization is the correct generalization of classical conditionalization to quantum probability. In analogy to its classical counterpart, Lüders conditionalization is the unique quantum conditionalization rule with a desirable property; namely, for $F \in \mathcal{P}(\mathfrak{N})$ such that $\Pr(F) \neq 0$, $\Pr(\bullet//F)$ is the unique probability on $\mathcal{P}(\mathfrak{N})$ such that for any $E \in \mathcal{P}(\mathfrak{N})$, if $E \leq F$ then $\Pr(E//F) = \Pr(E)/\Pr(F)$, i.e. $\Pr(\bullet//F)$ is the renormalization of $\Pr(\bullet)$ to$\Pr'(\bullet)$ where$\Pr'(F) = 1$ (see Cassinelli and Zanghi 1983).[20] Updating by Lüders conditionalization will taken for granted here.

4 The meta-physics of quantum chance

4.1 Physically realizable quantum states

To repeat, on the approach to quantum theory pursued here a quantum system is characterized by an algebra $\mathfrak{N}$ of observables and a set of states $\mathfrak{S}$ on $\mathfrak{N}$. The threshold issue for what follows is: which of the mathematically possible expectation value functionals $\mathfrak{S}$ are physically realizable? The implicit answer of 99.9% of textbooks on quantum theory is that the subclass of physically realizable states is limited to the normal states $\mathcal{N}$, for these texts assume that probabilities are to be calculated by the "Born rule" (aka trace prescription). But assumption isn't an argument. In view of the importance of the issue, there is a surprising lack of convincing arguments for the assumption. It would take us too far afield to review the matter, and

[20] The special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ is treated in Bub (1977) and Hughes (1989, Appendix B).

the interested reader is simply referred to the articles by Ruetsche (2011) and Arageorgis et al. (2017). Here I follow the conventional wisdom and concentrate on normal states.

4.2 Down vs. up; objective vs. personal probabilities

The interpretation of quantum probabilities is one of the most contested issues in the foundations of physics. For present purposes the most important dispute is between what I will call the top-down vs. bottom-up views on the relation between quantum states and quantum probabilities. On the top-down view, states are ontologically prior to probabilities; normal quantum states codify objective, i.e. observer independent, features of a physical system; and, therefore, the probabilities they induce on the projection lattice are objective. On the bottom-up view probabilities are ontologically prior to states; appealing to Gleason's theorem, quantum states can be seen as bookkeeping devices to represent and keep track of probabilities. Further, following de Finetti's lead in rejecting the notion of objective, physical probabilities, quantum Bayesians (QBians as they call themselves) want to hew to a personalist reading of the probabilities assigned to elements of the projection lattice (see Caves et al. 2008 and Fuchs 2010). Studying this clash of views provides a quick route into some of the most fundamental foundations problem in quantum theory. But again following this route would take us far afield, and the interested reader is referred to the articles by Timpson (2008) and Earman (2017) and the references given therein. Here I adopt the top-down view. My reasons are admittedly ad hominem: if and only if the top-down view is adopted, does quantum theory provide a case study for the relation between credence and chance; rejecting the view removes the lifeline that lets us crawl out of the rabbit hole leading to analytical metaphysics Wonderland.

4.3 Which quantum states induce quantum chances?

For quantum probabilities to constitute chances we not only want them to have an objective, observer independent, basis, but we also want them to be free of epistemic components. The most stringent form of the non-epistemic requirement would require a no-hidden-variable result to the affect that, given plausible assumptions, it is not possible to interpret quantum probabilities as resulting from ignorance of the precise values of variables not supplied by quantum theory. Yet again we have hit a large topic that cannot be reviewed here. For present purposes I simply want to note that the non-epistemic requirement can be used to exclude impure or mixed states as inducing chances.

In ordinary QM a normal impure state is a mixture (i.e. a convex linear combination) of normal pure (= vector) states, and what is referred to as the ignorance interpretation of mixtures holds that such a state means that the system really is in some pure state with the mixture weights serving as measures of the epistemic uncertainty about which pure state is the actual one. Such an epistemic component would signal that the probabilities induced by the state not pure chances. The well-worn objection to the ignorance interpretation of mixtures is that, in general, impure states do not admit a unique decomposition into pure states and, thus, an indiscriminate use of the ignorance interpretation would lead to inconsistency. But sometimes the non-uniqueness issue can be resolved by attending to the actual history of the way in which the state is prepared, thereby selecting one decomposition. For example, arrange a mechanism that (a) flips a fair classical coin and (b) prepares the object system in a pure state φ_1 (respectively, φ_2) if the coin lands heads (respectively, tails). If no observer looks at the result of the coin flip then the object system is in a state ω that is a 50-50 mixture of φ_1 and φ_2. Unproblematically, in these circumstances the mixture weights represent epistemic uncertainty. The fact that for any mixed state in ordinary QM a unique decomposition can be singled out by such a story is enough for present purposes to motivate the ansatz that only normal pure states induce chances. This ansatz applies only to global states, that is, states on the algebra of observables $\mathfrak{N}_{tot}$ of the total system and not to states on the algebra of observables $\mathfrak{N}_{sub} \subset \mathfrak{N}_{tot}$ of a subsystem; for the restriction $\omega|_{\mathfrak{N}_{sub}}$ to $\mathfrak{N}_{sub}$ of a pure state ω on $\mathfrak{N}_{tot}$ can be a mixed state.

The situation is more complicated when the algebra of observables is not the simple Type I $\mathfrak{B}(\mathcal{H})$ but a Type III algebra, for such an algebra does not admit any normal pure state. The fact that a mixed state on a Type III algebra does not admit any decomposition into normal pure states might lead one to conclude that these mixed states induce probabilities that count as chances. But, as will be seen shortly, there is no suitable analog of the classical C_{ch} for mixed states. This quandary can, perhaps, be resolved by a more detailed look at how Type III algebras arise in relativistic QFT. Von Neumann algebras of observables $\mathfrak{N}(\mathcal{O})$ are associated with open bounded regions $\mathcal{O}$ of Minkowski spacetime $\mathcal{M}$ (see Haag 1992 and Horuhzy 1990). In most applications these local algebras are Type III while the global algebra $\mathfrak{N}(\mathcal{M})$ generated by the $\mathfrak{N}(\mathcal{O})$ as $\mathcal{O}$ ranges over all regions is Type I. As an illustration, the Minkowski vacuum state ω_0 for the Klein-Gordon field is a normal pure state on $\mathfrak{N}(\mathcal{M})$. Its restriction $\omega_0|_{\mathfrak{N}(\mathcal{O})}$ to a local region $\mathcal{O}$ is a mixed state; indeed, if $\mathcal{O}$ has non-null causal complement then $\omega_0|_{\mathfrak{N}(\mathcal{O})}$ is a mixed state par excellance—it is thermal state (technically, a KMS state at finite temperature). This suggests a way to take the sting out of the consequence of our ansatz that, because it is a mixed state, $\omega_0|_{\mathfrak{N}(\mathcal{O})}$ does not induce a chance function on $\mathcal{P}(\mathfrak{N}(\mathcal{O}))$; namely, since ω_0 is a normal pure state on $\mathfrak{N}(\mathcal{M})$ it does induce a chance function on the global projection lattice $\mathcal{P}(\mathfrak{N}(\mathcal{M}))$, which includes the $\mathcal{P}(\mathfrak{N}(\mathcal{O}))$ as sublattices. More discussion of these issues is needed, but in what follows I will concentrate on ordinary QM.

It is worth noting that in the case of a classical (i.e. abelian) algebra $\mathfrak{N}$ a pure state φ (normal or not) on $\mathfrak{N}$ is dispersion free, i.e. $\varphi(A^2) = \varphi(A)^2$ for all $A \in \mathfrak{N}$ (see Bratteli and Robinson, 1987, Cor. 2.3.21). For a projection operator $E \in \mathfrak{N}$ this means that $\varphi(E) = \varphi(E^2) = \varphi(E)^2$, and since $\varphi(E) \in [0, 1]$, $\varphi(E)$ must be 0 or 1. Consequently, for abelian $\mathfrak{N}$ if the probabilities the pure state φ induces are regarded as chances then those chances are deterministic $0-1$ chances. Thus, it is the non-abelian nature of a quantum $\mathfrak{N}$ that allows quantum theory to assign non-deterministic chances.

4.4 What is the quantum analog of the classical C_{ch}?

In order to formulate quantum analogs of Lewis' PP and its variants for classical probability we need to identify elements of $\mathcal{P}(\mathfrak{N})$ that serve as the counterparts of the classical C_{ch}. For a normal state on $\mathfrak{N}$ and $E \in \mathcal{P}(\mathfrak{N})$ I propose the reading of $\varphi(E) = x$ as: if the system is in state φ then the probability is x that a measurement of E will return a positive result; and, per the above discussion, when φ is a normal pure state this probability can be interpreted as the objective chance that a measurement of E will return a positive result. With this reading in mind, our goal then is to find an element $?_\varphi \in \mathcal{P}(\mathfrak{N})$ such that a positive result of a measurement of $?_\varphi$ ensures that the system is in state φ. The goal can be reached with the help of the von Neumann projection postulate which, in the present setting comes to this: if a system is initially in state ω and a measurement of $E \in \mathcal{P}(\mathfrak{N})$, $\omega(E) \neq 0$, gives a positive result then the post-measurement state of the system is $\omega_E(\bullet) := \omega(E \bullet E)/\omega(E)$. Thus, we want $?_\varphi$ to serve as a *filter* for φ within the class of all normal states $\mathcal{N}$ on $\mathcal{P}(\mathfrak{N})$ (pure or not), anyone of which could serve as a physically possible initial state of the system; that is, we want $?_\varphi$ to have the property that $\omega_{?_\varphi}(\bullet) = \dfrac{\omega(?_\varphi \bullet ?_\varphi)}{\omega(?_\varphi)} = \varphi(\bullet)$ for any normal state ω (pure or not) such that $\omega(?_\varphi) \neq 0$.

At this juncture we can use some basic facts about states on von Neumann algebras:

> *Fact 1.* Impure states do not have filters within the class of all normal states.
>
> *Fact 2.* Normal pure states do have filters within the class of all normal states. A filter for a normal pure state φ is its support projection S_φ. [21]

The support projection S_φ for a normal state is the smallest projection in $\mathcal{P}(\mathfrak{N})$ to which φ gives probability 1. Thus, for a normal pure state I propose to identify the sought after $?_\varphi$ with S_φ.[22]

For $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ the class of normal pure states is identical with the class of vector states.[23] The support projection S_φ for such a state φ is the projection $E_{|\varphi\rangle}$ onto the ray spanned by a vector $|\varphi\rangle$ corresponding to φ. It is easy to verify that S_φ is the unique filter for φ. For normal pure states φ and φ' the expression $\varphi'(S_\varphi)$ agrees with the standard expression for transition probability from φ' to φ since $\varphi'(S_\varphi) = \varphi'(E_{|\varphi\rangle}) = \langle\varphi'|E_{|\varphi\rangle}|\varphi'\rangle = \langle\varphi'|E_{|\varphi\rangle}|\varphi'\rangle = ||E_{|\varphi\rangle}|\varphi'\rangle||^2 = |\langle\varphi'|\varphi\rangle|^2$. It remains to justify the interpretation of $\omega(S_\varphi)$ as the transition probability from ω to φ when ω is a normal impure state and φ is a normal pure state. I leave this task to the reader.

Fact 1 lends ad hominem support to the ansatz that, for the total system algebra, only pure states induce chances; for if an impure state φ induced chances there would be no way to evaluate the relation between such chances

[21] See Ruetsche and Earman (2011) for proofs of these facts.

[22] Returning to the case of relativistic QFT mentioned in Section 4.3, the support projection for the Minkowski vacuum state ω_0 (a normal pure state) does not belong to any local algebra and, consequently, no local measurement that any finite observer could hope to make can establish (or refute) the proposition expressed by this projection.

[23] If $\mathfrak{N}$ is Type III the class of all normal states, pure and mixed, is identical with the class of vector states. Thus, in contrast to the Type I case, some vector states for Type III algebras are mixed states. And, as mentioned above, there are no normal pure states for a Type III algebra.

and rational credence since there are no suitable candidates for the role of $?_{\varphi}$.

Since for any normal pure state $\varphi(S_{\varphi}) = 1$, the chances induced by such a state are never self-undermining. This is one feature of quantum chance that must figure into the discussion of the relation of credence and quantum chance, on which I embark in the following section.

4.5 How chance changes

On the account of quantum chance on offer, quantum chances are induced by quantum states, and when the state changes the chances change accordingly. There are two processes of change: Schrödinger evolution between measurements and the change given by the von Neumann projection postulate as the upshot of a measurement outcome, the latter of which is referred to as state vector or collapse when the pre-measurement state is a pure vector state and the measurement concerns a observable represented by a self-adjoint operator with a pure point spectrum (in which case the measurement outcome is an eigenvalue of said observable). Thus, the account on offer here is inextricably entangled with notorious controversies over the nature of the quantum measurement process. In self-defense it can be said it is hard to see how any account of quantum chance can manage to avoid these controversies. It does no good to rehearse them here, and a few brief remarks about the implications for quantum chance will have to suffice.

There are many ways to classify treatments of quantum measurement, but for present purposes the most relevant division is between collapse and non-collapse interpretations. Non-collapse interpretations include modal interpretations (see Lombardi and Dieks 2014), many worlds interpretations (see Vaidman 2015); Bohmian mechanics (see Goldstein 2013); and consistent histories (see Griffiths 2014). Collapse interpretations include GRW interpretations (see Ghirardi 2011) which propose physical mechanisms for producing state vector collapse.[24] Here I wish to note that non-collapse interpretations must at least pay lip service to collapse as codified in the von Neumann projection postulate. The reason is simple: Lüders updating in-

[24] The QBian treatment of measurement can also be classified as a collapse interpretation. But QBians reject an objectvist take on quantum states, and for them state vector collapse is nothing but a change in the representation of the credence function of a QBian agent when she updates on a measurement outcome. See Earman (2017) for some critical remarks on this treatment of measurement.

disputably gives the empirically correct post-measurement probabilities; so insofar as quantum probabilities are induced by quantum states, there must be a concomitant change in the post-measurement quantum state, and that change is precisely that given by the von Neumann projection postulate. Thus, the non-collapse interpretations have to acknowledge that everything works (mirable dictu) *as if* there is collapse. Of course, if there is no collapse then post-measurement probabilities are not those induced by the post-measurement quantum state, and they must be calculated by other means. The implications of those other means for the existence and nature of quantum chance needs to be examined for each of the non-collapse interpretations. I will not attempt this large project here, and will continue to assume for present purposes that the von Neumann projection postulate is to be taken literally rather than being given an *as if* reading.

5 The relation between rational credence and quantum chance

5.1 A MP-like principle for quantum probability

We are now in a position to obtain a quantum analog of MP, not as an additional rationality constraint on credence functions on the lattice of quantum propositions but as a theorem of quantum probability.

> *Proposition 1.* Let $\mathfrak{N}$ be a von Neumann algebra that does not contain any summands of Type I_2, and let Pr be a completely additive probability function on the projection lattice $\mathcal{P}(\mathfrak{N})$. Then for any normal pure state φ on $\mathfrak{N}$, $\Pr(F//S_\varphi) = \varphi(F)$ for all $F \in \mathcal{P}(\mathfrak{N})$, provided that $\Pr(S_\varphi) \neq 0$.

Proof: Provided that $\Pr(S_\varphi) \neq 0$, $\Pr(\bullet//S_\varphi) = \Pr(S_\varphi \bullet S_\varphi)/\Pr(S_\varphi)$. The hypothesis of the proposition enables the generalized Gleason theorem to be invoked to conclude that a completely additive Pr corresponds to a unique normal state ω. Thus, $\Pr(\bullet//S_\varphi) = \omega(S_\varphi \bullet S_\varphi)/\omega(S_\varphi)$. By Fact 2 S_φ is a filter for φ within the class of all normal states and, thus, $\Pr(\bullet//S_\varphi) = \omega(S_\varphi \bullet S_\varphi)/\omega(S_\varphi) = \varphi(\bullet)$.[25]

[25] Since for an impure φ there is no filter (recall Fact 1) Prop. 1 does not hold for S_φ or any alternative to S_φ in $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. So if, contrary to my ansatz, impure states induce

Nothing prevents the probability function in Prop. 1 from being interpreted as the credence function Cr of some agent. If per the discussion of Section 4, chances for propositions in $\mathcal{P}(\mathfrak{N})$ are induced by normal pure states on $\mathfrak{N}$ and only by such states, a completely additive Cr on $\mathcal{P}(\mathfrak{N})$ has—as a truth of quantum probability theory—the property that, when Lüders conditionalized on the proposition S_φ that chances are those given by φ, the conditional credence assigned to any proposition $F \in \mathcal{P}(\mathfrak{N})$ is equal to the φ-chance of F.

There is some faux generality to Prop. 1 since, as noted above, normal pure states do not exist for Type II or III algebras, so from here on focus on the case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$. When $\dim(\mathcal{H}) \geq 3$ the first supposition for of Prop. 1 is satisfied. As for the second supposition there are four subcases to consider.

(i) $\dim(\mathcal{H}) < \infty$. Here complete additivity on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ reduces to finite additivity.

(ii) $\dim(\mathcal{H}) = \infty$ but $\mathcal{H}$ is separable. Here complete additivity on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ reduces to countable additivity.

(iii) $\mathcal{H}$ is non-separable and $\dim(\mathcal{H})$ is not a measurable cardinal. Here again complete additivity on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ reduces to countable additivity.

(iv) $\mathcal{H}$ is non-separable and $\dim(\mathcal{H})$ is a measurable cardinal. Here complete additivity on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ does not reduce to countable additivity.

Since it is being assumed that updating of credence takes place by Lüders conditionalization, the question of whether or not learning what the chances are, without the help of an additional principle of rationality, forces coherent credence to line up with chance values reduces to the question of which form of additivity is essential to rationality of credence.

Case (i) gives no controversy since it is universally agreed that finite additivity is essential to rationality of credence and, thus, a quantum MP is a theorem of quantum probability when $\dim(\mathcal{H}) \geq 3$. Cases (ii)-(iv) do generate controversy; in particular, de Finetti (1972, 1974) and his followers do not

quantum chance probabilities then the discussion of credence and quantum chance has to be altered.

accept countable additivity, much less complete additivity, as necessary for rationality. In favor of mere finite additivity it can be noted that the scoring rule argument cannot be used to justify countable additivity (see Arageorgis et al. 2017). But against mere finite additivity it can be pointed out that the Dutch book arguments, originally introduced by de Finetti to motivate finite additivity for personal probabilities, can be generalized to motivate countable additivity (see Howson 2008). However, the Dutch book construction does not suffice to motivate complete additivity if the an agent is only required to accept bets he judges to be strictly favorable as opposed to merely fair (see Skyrms 1992). Additional motivation for countable and complete additivity comes from the facts that merely finitely additive and merely countably additive probabilities fail to be conglomerable (Hill and Lane 1985) and that this failure makes them prey to money-pump constructions (Seidenfeld and Schervich 1983). However, statisticians do not find these considerations to be dispositive, and the status of countable and complete additivity for personal probabilities remains in dispute (see Kadane et al. 1986 and Seidenfeld 2001). Thus, at present only a conditional conclusions can be drawn from Prop. 1 for cases (ii)-(iii) and case (iv) respectively: if $\dim(\mathcal{H}) \geq 3$ and countable (respectively, complete) additivity is necessary for rational credence, then a quantum MP is not an additional principle of rationality but a theorem of quantum probability. Those who insist on thinking that PP-like principles are valid norms of rationality can step into the gap here and claim that for quantum chance and credence these principles boil down to the prescription that rational credence requires complete additivity. Whatever the merits of this proposal it certainly represents a comedown for what were initially conceived to be lofty principles connecting credence and chance.

A state ω on $\mathfrak{N}$ is deemed faithful just in case $\omega(A) = 0$ implies $A = 0$ for all $A \in \mathfrak{N}$. For a probability function Pr corresponding to a faithful normal state the proviso in Prop. 1 is satisfied for all normal states φ. Faithful normal states do exist for $\mathfrak{B}(\mathcal{H})$ when $\mathcal{H}$ is separable. However, this desirable property of being chance tracking for all chance functions is an ephemeral property since the Lüders conditionalization of a Pr corresponding to a faithful state can result in a probability function that itself does not correspond to a faithful state. When $\mathcal{H}$ is non-separable there are no faithful normal states on $\mathfrak{B}(\mathcal{H})$ (see Bratelli and Robinson 1987, p. 85) and credence functions are incapable of being universally chance tracking even ephemerally.

Now to the case of $\dim(\mathcal{H}) = 2$. If we lived in a world accurately described using a Hilbert space of dimension 2 then complete additivity would reduce to finite additivity, all states on $\mathfrak{B}(\mathcal{H})$ would be normal; but Gleason's theorem would not apply. In such a world there would be completely additive credence functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that do not correspond to *any* quantum state on $\mathfrak{B}(\mathcal{H})$, and such a credence function does not have the property that it is brought into alignment with chance by learning that the chances are. I see no persuasive reason to think that such a credence function is irrational.

Two final remarks are appropriate. The first concerns the need perceived by some commentators to limit the application of MP to initial or tabula rasa credence functions. In the quantum case there is no corresponding need: Prop. 1 is a valid proposition of quantum probability regardless of whether Pr is a tabula rasa or a much learned credence function. Second, the results of this subsection hold for the case of a classical, i.e. abelian, algebra $\mathfrak{N}$ where Lüders conditionalization coincides with classical Bayesian conditionalization. But insofar as chance is encoded in pure states the resulting relation between credence and chance is uninteresting since (recall Section 4.3) for abelian $\mathfrak{N}$ the chances are degenerate $0-1$ probabilities. In this sense an interesting quantum counterpart of MP relies on the non-commutativity characteristic of quantum observables.

5.2 A GPP-like principle for quantum probabilities

> *Proposition 2.* Let Pr be a completely additive probability function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\dim(\mathcal{H}) \geq 3$. Then
>
> $\Pr(F) = \sum_i \Pr(S_{\varphi_i})\varphi_i(F)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$
>
> where the sum is taken over a countable set $\{\varphi_i\}$ of mutually orthogonal normal pure states on $\mathfrak{B}(\mathcal{H})$ with $\sum_i \Pr(S_{\varphi_i}) = 1$.

Proof: Since by Gleason's theorem a completely additive Pr corresponds to a normal state on $\mathfrak{B}(\mathcal{H})$ and since every normal state on $\mathfrak{B}(\mathcal{H})$ lies in the norm closure of the hull of convex linear combinations of normal pure states, Pr can be written in the form $\Pr(F) = \sum_i \lambda_i\varphi_i(F)$, $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, where the φ_i are mutually orthogonal normal pure states, and the mixture weights satisfy $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$. Moreover, even when $\mathcal{H}$ is non-separable the set $\{\varphi_i\}$ is countable (see Kadison and Ringrose 1997, Vol. 2, Theorem 7.1.12). Since $\Pr(S_{\varphi_j}) = \sum_i \lambda_i\varphi_i(S_{\varphi_j})$, and since $\varphi_i(S_{\varphi_j}) = 1$ or 0 according as $i = j$ or $i \neq j$, $\Pr(S_{\varphi_i}) = \lambda_i$ as required.

With the Pr in Prop. 2 interpreted as the credence function Cr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ of some agent, it is tempting to read Prop. 2 as saying that a completely additive Cr can be represented as if it is an expression of epistemic uncertainty (given by the coefficients $Cr(S_{\varphi_i})$) about what the objective chances (given by the φ_i) are. However, unlike representation theorems in classical probability, such as de Finetti's celebrated representation theorem for exchangeable credence functions (de Finetti 1937), the representation given in Prop. 2 is not unique; for if Cr corresponds to a mixed state ω^{Cr} on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ then ω^{Cr} may be decomposable in many different ways into distinct families of mutually orthogonal pure states, $\{\varphi_i\}, \{\varphi'_j\}, \ldots$. As indicated in Section 4.3, information about how the state ω^{Cr} was prepared may indicate that the system is actually in one of the pure states of a particular decomposition. If not, the coefficients $Cr(S_{\varphi_i})$ do not have straightforward ignorance interpretation as epistemic uncertainty about objective chances, and despite the formal resemblance between the formulae of the classical GPP and Prop. 2, the idea behind GPP does not find a straightforward expression in quantum probability.

5.3 Admissibility of evidence

In the quantum context the question of the admissibility of evidence is a matter of proving results in quantum probability theory about whether learning S_φ forces credence to line up with the the φ-chances when additional evidence E is acquired along side of S_φ, where as usual S_φ is the support projection for a normal pure state φ. Because of the non-commutative nature of quantum propositions, the "along side" has to be treated with care. Three cases need to be considered.

First, "along side" could mean that E and S_φ are learned simultaneously. In this case standard quantum doctrine on simultaneously measurability requires that E and S_φ commute. In addition, the proviso (which needs to be attached to a PP-like principle) that $\Pr(S_\varphi E) = \Pr(ES_\varphi) \neq 0$ requires that $S_\varphi F = FS_\varphi \neq 0$. Since S_φ is a minimal projection for $\mathfrak{B}(\mathcal{H})$ the upshot is that $S_\varphi E = ES_\varphi = S_\varphi$ and E is entailed by S_φ. So, trivially, $\Pr(F//ES_\varphi) = \Pr(F//S_\varphi E) = \Pr(F//\ S_\varphi) = \varphi(F)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, and in this sense E is admissible. This quite unlike what philosophers imagine can happen in the classical case where it seems that the simultaneous learning of E and C_{ch} does not require the latter to entail the former.

Now suppose that E is learned before S_φ. For any E such that $\Pr(E) \neq 0$

the updated$\Pr'(F) := \Pr(F//E)$ is subject to Prop. 1. For the second updating on S_φ to $\Pr"(\bullet) := \Pr'(\bullet//S_\varphi)$ the proviso that $\Pr'(S_\varphi) = \dfrac{\Pr(ES_\varphi E)}{\Pr(E)} \neq 0$ holds iff $\Pr(S_\varphi) \neq 0$. Thus, as long as $\Pr(E) \neq 0$ and $\Pr(S_\varphi) \neq 0$ Prop. 1 ensures that $\Pr'(F//S_\varphi) = \Pr(F//S_\varphi) = \varphi(F)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. The upshot is that learning that S_φ brings credence into line with φ-chances if E is learned first.

Finally, consider the case where S_φ is learned before E. Then by Prop. 1, for any Pr such that $\Pr(S_\varphi) \neq 0$ the updating on S_φ gives $\Pr'(F) := \Pr(F//S_\varphi) = \varphi(F)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Then for any $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\Pr'(E) = \varphi(E) \neq 0$, the second updating on E gives $\Pr"(F) := \Pr'(F//E) = \varphi(F)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ just in case $\varphi(EFE) = \varphi(E)\varphi(F)$. In the classical (abelian) case when E and F commute this last condition reduces to $\varphi(EF) = \varphi(FE) = \varphi(E)\varphi(F)$, which is the condition of stochastic independence relative to φ (recall LPP from Section 2.2).

This ends the main discussion of the relation between credence and quantum chance. The next two subsections take up some loose ends left over from the critical review of the literature on Lewis's PP.

5.4 No self-undermining for quantum chances

As noted above $\varphi(S_\varphi) = 1$ for any normal pure state and, as a consequence, the chance functions induced by such states are never self-undermining. In the classical probability case no-self-undermining for chance functions ($ch(C_{ch}) = 1$ for all ch) implies that $ch(C_{ch'}) = 0$ when $ch \neq ch'$ since then C_{ch} and $C_{ch'}$ are logically incompatible. But in the quantum case $\varphi(S_{\varphi'})$—which as we have seen can be interrupted as the transition probability from state φ to state φ'—need not be zero when $\varphi \neq \varphi'$. The transition probabilities $\varphi(S_{\varphi'})$ and $\varphi'(S_\varphi)$ are zero just in case the support projections S_φ and $S_{\varphi'}$ are incompatible in the sense of orthogonality, i.e. $S_{\varphi'}S_\varphi = 0 = S_\varphi S_{\varphi'}$, which for $S_\varphi, S_{\varphi'} \in \mathfrak{B}(\mathcal{H})$ means that the vectors corresponding to φ and φ' respectively are orthogonal. These differences between the classical and quantum cases will become important in Section 6 in connection with the discussion of the Humean supervenience of chance.

5.5 When conditionalized chances are not chances

We also have the means to illuminate some of the issues left hanging in Section 2.2. Recall that one of the issues for classical probability was whether conditionalizing a chance function ch on E, where $ch(E) > 0$, results in another chance function $ch'(\bullet) := ch(\bullet/E)$ when $ch(\bullet/E) \neq ch(\bullet)$. If the answer is affirmative then classical probability calculus and the fact that C_{ch} and $C_{ch'}$ are incompatible when $ch \neq ch'$ entail that ch' is an insane chance function ($ch'(C_{ch'}) = 0$).

For quantum chance the issue can be posed as follows: If Pr^{φ} is a chance function induced by the normal pure state φ, under what conditions on $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ (apart from $\mathrm{Pr}(E) \neq 0$) is $\widehat{\mathrm{Pr}^{\varphi}}(\bullet) := \mathrm{Pr}^{\varphi}(\bullet//E)$ a chance function? This is equivalent to: If φ is a normal pure state and $\varphi(E) \neq 0$, is the state $\widehat{\varphi}(\bullet) := \varphi(E \bullet E)/\varphi(E)$ to which $\widehat{\mathrm{Pr}^{\varphi}}(\bullet)$ corresponds a normal pure state? $\widehat{\varphi}$ is always normal, but to be pure its support projector $S_{\widehat{\varphi}}$ must be a minimal (one-dimensional) projection. This will be the case if E is the projection onto a ray not orthogonal to the ray corresponding to φ; then the new chance function $\widehat{\mathrm{Pr}^{\varphi}}(\bullet) = \mathrm{Pr}^{\widehat{\varphi}}(\bullet)$ is not insane and, indeed, is not self-undermining. For other E's $\widehat{\varphi}$ may fail to be a pure state, in which case the conditionalized chance function is not a chance function.

6 Humean supervenience of chance and paradoxes of undermining

6.1 The prospects for Humean supervenience

Some philosophers are leery of talk of objective chance because they worry that it carries a commitment to non-Humean powers. One way to assuage such worries is to argue that chance supervenes a base of Humean facts. For present purposes it is enough to work with the rough characterization that X supervenes on Y means that facts about Y are the truth makers for facts about X, i.e. all possible worlds that agree on facts about Y also agree on facts about X. Different strengths of supervenience arise depending upon whether the possible worlds are construed as physically possible, metaphysically possible, or logically possible; and different forms of Humean supervenience devolve from different views on what to allow in the Humean supervenience basis. It seems to be widely assumed that if chances are to be Humean supervenient then, for want of a better option, they must be supervenient on relative frequencies. Presumably actual finite relative frequencies would be welcomed into the approved Humean supervenience basis, but perhaps not hypothetical or limiting relative frequencies. In any case, I think that the prospects for supervenience of chances on relative frequencies is dim, but I will not argue the case here.[26] Rather I want to suggest a different way of viewing the supervenience of quantum chances as characterized by the account developed above.

The most obvious conclusion to draw from the view of quantum chances that emerges from Sections 4 and 5 is that the truth makers for the chances induced by a normal pure quantum state φ are not relative frequencies, actual or hypothetical, but rather the facts that make true the proposition expressed by the support projection S_φ of φ. Since these support projections belong to the projection lattice of quantum propositions, the basic interpretational structure of quantum theory assumes that the facts making S_φ true can, in principle, be established by the outcomes of experiments in exactly the same sense in which the outcomes of experiments establish facts about the value of basic physical quantities such a position and energy.[27] If this way of conceiving the supervenience of quantum chance fails to meet the scruples of some Humeans—say, because they want a more circumscribed supervenience basis—then my advice to them is to adjust their scruples to accommodate quantum physics. If they refuse on the grounds that to allow the truth makers of the S_φ into the supervenience base is to countenance non-Humean powers, then I conclude that the quantum world is redolent of non-Humean powers. But I would add that these powers are not very scary since their presence or absence can be confirmed and disconfirmed by observation and experiment.

In the following subsections I argue that the account of the supervenience of quantum chances on offer here helps to cut though the travail unleashed

[26] Ismael (1996) argues against a logical connection between chances conceived on Teller's (1995) intrinsic property model and relative frequencies, whether actual or hypothetical. There certainly are problems in trying to identify chance with relative frequencies. If chances can have irrational values they cannot be identified with finite frequencies; and if chances are completely additive—as they are in quantum physics—then they cannot be identified with limiting relative frequencies which can fail to be even countably additive. While rejecting the identification of chance with frequency, Hájek (1997, 2009) holds that it must be possible for chance to match frequency, from which he concludes that chance must be finitely additive. Since on my account quantum chances are completely countably additive I reject his conclusion.

[27] Just as Teller's (1995) analysis of the nature of quantum probabilities suggests.

by Lewis' worries about undermining.

6.2 The paradoxes of undermining

These paradoxes refer to a family of problems that supposedly arise from attempting to combine four elements: taking objective chance seriously; insisting on Humean supervenience of chances; accepting Lewis' PP (or some variant) as a key link between credence and chance; and trying to juggle all of these elements within classical probability theory. I accept the first and second elements; but I reject the fourth. This rejection has, as seen above, major implications for the third element, and it leads to a dissolution of the paradoxes as applied to quantum chances.

Here is a reconstruction of how trouble supposedly arises from combining the four elements. (The discussion that follows reverts initially to the terminology and notation of the philosophical literature.) If there are Humean facts on which a chance function ch supervenes then there are also Humean facts which undermine ch; for there must also be Humean facts on which a chance function ch' different from ch supervenes, so if these latter facts were to obtain ch would not be the true chance function. Suppose that the chance function ch allows that there is a non-zero chance for undermining facts for ch, i.e. there is a sentence U_{ch} in the domain of ch asserting that undermining facts obtain and $ch(U_{ch}) > 0$. Then it seems that MP imposes contradictory demands on credence functions unless $Cr(C_{ch}) = 0$. From MP it follows that a rational agent ought to adopt a credence function giving $Cr(U_{ch}/C_{ch}) = ch(U_{ch}) > 0$, provided that $Cr(C_{ch}) \neq 0$. But if U_{ch} undermines ch then U_{ch} and C_{ch} are incompatible (so it is said) so that $Cr(U_{ch}/C_{ch}) = 0$. To escape contradiction (while maintaining MP) a rational agent must set $Cr(C_{ch}) = 0$ when $ch(U_{ch}) > 0$.

Why should this be a problem for Lewis? He thought that Humean supervenience required that some (or all) chance functions ch give a non-zero chance to some undermining U_{ch}. It would seem that this will be so, for example, if chance supervenes on relative frequencies in a large but finite samples; for then there will be a non-zero chance for observing frequencies that undermine the actual chances. What the argument of the preceding paragraph shows is that MP makes it impossible for rational agents to assign a non-zero credence to the possibility of the type of Humean supervenience Lewis desired.

In diagnosing the tension notice that $ch(U_{ch}) > 0$ implies that ch is self-undermining in the sense studied above. (Since U_{ch} and C_{ch} are supposed to be incompatible U_{ch} entails $\neg C_{ch}$ and, thus, $ch(U_{ch}) \leq ch(\neg C_{ch}) = 1 - ch(C_{ch})$. So if $ch(U_{ch}) > 0$ it follows that $ch(C_{ch}) < 1$.[28]) But MP entails that a rational agent ought to assign a flatly zero initial credence to any C_{ch} where ch is self-undermining. Thus, the obvious diagnosis and treatment of the paradox seems to be that in order for rational agents to allow for the kind of undermining thought necessary for Humean supervenience, self-undermining chance functions must be admitted. This means that MP must be abandoned or modified; and for those Humeans who do not choose abandonment, the modification would have to consist of moving to a principle like NPP that allows for self-undermining chance functions (recall Section 2.2). Or at least this is one popular line of analysis; there are also competing lines that vie for attention in the philosophical literature.[29]

The situation looks considerably different when classical probability is exchanged for quantum probability and when the supervenience of quantum chances is properly understood.

6.3 Supervenience and undermining for quantum chances

On the account of quantum chance developed here quantum chance functions are never self-undermining; MP and various versions of PP are automatically satisfied for completely additive credence functions[30]; on a sufficiently liberal construal of the Humean supervenience basis, quantum chances are Humean supervenient; and quantum chances behave quite differently than the chances imagined in the philosophical literature on PP. In short, there are any number of reasons to think that the assumptions behind and the logic of the undermining paradoxes reviewed above simply do not apply to quantum chance. This said, it remains to try to understand more fully the issues of supervenience and undermining for quantum chance.

To begin, it helps to distinguish between ontic vs. epistemic truth-makers/underminers. As argued above, the ontic truth makers for the quantum chance function induced by a normal pure state φ are the facts corresponding to the truth of the proposition expressed by the support projection

[28]If U_{ch} and C_{ch} are logically incompatible and the domain of ch is closed under logical implication and negation, then since U_{ch} is supposed to be in the domain of ch so is C_{ch}.

[29]See, for example, Ismael (2008), Roberts (2001), Strevens (1995), Vranus (2004).

[30]Apart from the special case a Hilbert space of dimension 2.

S_φ of φ, and ontic underminers are facts corresponding to the truth of the propositions expressed by elements U_φ of the projection lattice that are incompatible with S_φ in the sense of orthogonality, i.e. $U_\varphi S_\varphi = S_\varphi U_\varphi = 0$. (The role of such a U_φ might, for example, be played by the support projector S_ξ for a state ξ orthogonal to φ.) Recall that the undermining paradox for classical probabilities started with the assumption that there are chance functions ch such that $ch(U_{ch}) > 0$. In the quantum case there is no chance function arising from a normal pure state φ that has the analogous property since $\varphi(U_\varphi) = 0$ when U_φ is incompatible with S_φ in the sense of orthogonality. But this does not pose any problem for the supervenience of quantum chance; indeed, $\varphi(U_\varphi) = 0$ is simply a consistency condition on quantum chance.

While frequency data do not serve as prospective ontic underminers of quantum chance, they can serve as epistemic underminers. Frequency data from IID trials that are strongly at variance with the probabilities induced by a state φ common to the IID systems would provide reason to doubt that φ is the correct state and, thus, reason to doubt that the alleged truthmakers for S_φ are in fact true. Supposing that such data can be expressed as a proposition D_φ belonging to the projection lattice, it can be the case that $\varphi(D_\varphi) > 0$. But this does not lead to antinomy since, although D_φ is epistemically incompatible with S_φ, it not incompatible with S_φ in the sense required for an ontic underminer that would generate a formal paradox in quantum probability, i.e. $D_\varphi S_\varphi = S_\varphi D_\varphi = 0$.

In response it might be objected that this story of the dissolution of the undermining paradoxes takes an internal perspective in assuming that the quantum theory is true. Switching to a external perspective which does not take quantum theory for granted, the thoroughgoing Humean must hold that the truth of the theory and the status of its postulates as laws of nature must supervene on Humean facts; and here the distinction between epistemic and ontic underminers blurs. On Lewis' "best systems" analysis of laws, the law status of the postulates of quantum theory rides on their providing the best compromise between strength and simplicity in accounting for non-probabilistic facts (e.g. the energy levels of atoms) and also on the fit between the probabilities provided by the theory and frequency data both from the past and the future. The quantum theory provides a simple and comprehensive account of the known non-probabilitistic facts about the quantum world, and quantum probabilities fit magnificently with frequency data obtained in past experiments. But the theory also assigns positive probability to possible frequencies of outcomes of future experiments that would imply on the best systems analysis of laws that quantum theory does not provide the probability laws of our world. Thus, it might seem that from the external perspective paradox is regained.

This appearance is superficial. There is no formal paradox here but only an equivocation on two senses of "incompatibility." To repeat, in order to get a formal paradox requires that there is a chance inducing quantum state φ and a $D_\varphi \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ expressing frequency data for an experiment governed by φ such that the support projection S_φ for the state φ is orthogonal to D_φ; but there is no such D_φ. What allegiance to Lewis' best systems analysis of laws gives is a different sense of incompatibility between S_φ and D_φ; namely, that assuming chances are what the best system probabilistic laws supply, if the frequency data encoded in D_φ were to occur when S_φ is true, then the postulates of the quantum theory would not be laws of nature and, consequently, the chances would not be given by φ. As Ismael puts it, "There is a confusion, very easy to make, between reasoning within the scope of a theory and reasoning *in propria persona* about which theory is correct" (2008, p. 296).

7 Conclusion

Lewis' PP takes on a different character in the account of quantum chance on offer here: PP-like principles are theorems of quantum probability for Bayesian agents whose credence functions are defined over the lattice of quantum propositions and satisfy the same form of additivity as do quantum chances. In a sense this is a vindication of Lewis' intuition that chance is what commands credence; but here the 'commands' is not what is required by an additional principle of rationality designed to force rational credence to follow chance but rather what is provable as a theorem of the account of quantum chance. It is not implausible to suggest that proving such a result serves as a criterion of adequacy on an account of chance. If there is a valid residual normative component to PP-like principles in the quantum setting it lies in the demand that rational credence should have the same form of additivity as chance. An agent who disobeys this demand can learn what the chances are but at the same time know that his credences will never be brought into line with chance values via Bayesian learning. Thus, he will know that if he uses his credence function in Nature's Casino he will be placing bets out of kilter with Casino odds and, consequently, that in the long run he faces almost sure ruin as judged by the objective chances, although his subjective odds carry no such dire warning. This is an odd if not irrational situation. The account on offer also shows how chance can supervene on non-chancy facts while not being subject to undermining paradoxes. If the Humeans find this account unacceptable because the supervenience basis does not meet their scruples then I say so much the worse for Humean supervenience.

The account of quantum chance on offer is based on several interpretational moves in quantum theory, all of which can be disputed. The basic existential challenge comes from QBism which maintains that the relation of credence and chance is a pseudo-topic since all probabilities in quantum theory are to be given a personalist/subjectivist reading. Less immediately existential but still serious is the challenge that comes from non-collapse interpretations of quantum measurement. If post-measurement probabilities are to be calculated by Lüders conditionalization—as the empirical evidence indicates—but the von Neumann projection postulate is rejected, then post-measurement probabilities are not those induced by the post-measurement quantum state, and the account on offer is undermined. What this means for the existence of quantum chance and its relation to rational credence depends on the details of the various non-collapse interpretations. Evaluating these challenges is a large and demanding project. But it can be undertaken in the confidence that it is more apt to produce understanding of credence and chance than the arm chair speculations of analytical metphysicians.

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