INERTIAL TRAJECTORIES IN DE BROGLIE-BOHM QUANTUM THEORY: AN UNEXPECTED PROBLEM[1]

PABLO ACUÑA
INSTITUTO DE FILOSOFÍA
PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO
AVENIDA EL BOSQUE 1290, 2530388, VIÑA DEL MAR, CHILE
pablo.acuna.l@pucv.cl

A salient feature of de Broglie-Bohm quantum theory is that particles have determinate positions at all times and in all physical contexts. Hence, the trajectory of a particle is a well-defined concept. One then may expect that the closely related notion of inertial trajectory is also unproblematically defined. I show that this expectation is not met. I provide a framework that deploys six different ways in which de Broglie-Bohm theory can be interpreted, and I state that only in the canonical interpretation the concept of inertial trajectory is the customary one. However, in this interpretation the description of the dynamical interaction between the pilot-wave and the particles, which is crucial to distinguish inertial from non-inertial trajectories, is affected by serious difficulties, so other readings of the theory intend to avoid them. I show that in the alternative interpretations the concept at issue gets either drastically altered, or plainly undefined. I also spell out further conceptual difficulties that are associated to the redefinitions of inertial trajectories, or to the absence of the concept.

1. INTRODUCTION

The quantum theory of motion firstly introduced by Louis de Broglie, and later and independently by David Bohm, has several attractive features. Most notably: it is not affected by the measurement problem, it provides an intuitive and visualisable explanation of quantum phenomena, it offers a clear account of the classical limit, particles are always distinguishable, and they have definite positions at all times and in all contexts. This last feature entails that the notion of *trajectory* is well-defined in the theory. In all other interpretations of quantum mechanics, the state of a system can be such that its representation in the position operator basis corresponds to a superposition. In general, the state of a particle for which we know its initial position evolves in such a way that its later states are not position eigenstates[1], so even if we measure its location at a later time, we cannot talk about the trajectory of that particle between its initial and final position. In de Broglie-Bohm (dB-B) theory, the state of a quantum system is described by the wavefunction Ψ *and* the configuration Q of the particles, where the coordinates of Q are given by the position of each particle. Besides, the particle positions deterministically evolve according to an equation of motion. Thus, in dB-B theory particles have a definite position at all times, so that the notion of trajectory is well-defined.

Ever since Newton, an exhaustive twofold classification of trajectories—inertial and accelerated—has played a central role in modern physics. An inertial trajectory is traversed by a free-body, i.e., by a body that is not affected by forces, or by a body on which the exerted forces mutually cancel. According to Newton's first law, inertial trajectories are uniform and rectilinear, and after relativistic physics we say that they correspond to (timelike) geodesics in the corresponding spacetime structure. On the other hand, bodies which are affected by impressed forces follow non-inertial trajectories that do not correspond to geodesics. Thus, in order to trace the distinction between inertial and non-inertial trajectories

[1] This work is published in *International Studies in the Philosophy of Science*, **30**:201-230 (2017). For quoting purposes, please refer to the published version.

we obviously need the concept of trajectory, and we also need a concept of force and an underlying spacetime structure.

Thus, it is obviously pointless to expect that interpretations of quantum mechanics in which the notion of trajectory is absent can trace the inertial/non-inertial distinction. The situation is very different in dB-B theory, in this case we can certainly expect that the distinction can be consistently drawn. As we said above, the notion of trajectory is fully grounded. Besides, the dynamical laws in dB-B theory are Galilean invariant, so that the spacetime structure that naturally corresponds to the theory is the same as in classical mechanics. Furthermore, Bohm's (1952) original presentation postulates an equation of motion that is structurally identical to Newton's second law—but which includes a distinctive quantum force. That is, in Bohm's formulation of the theory we have all the ingredients required to trace the distinction between inertial and non-inertial trajectories.

Bohm's is not the only way to understand the theory, though. Alternative ways to interpret the meaning of the wavefunction have been proposed. In the *nomological* interpretation, the wavefunction does not refer to an element in the ontology of the theory, rather, it represents a physical law. In the *dispositionalist* interpretation, Ψ does not represent a physical entity either, but a dispositional property possessed by the quantum system. Now, the excision of the Ψ-field from the theory's ontology implies that there is no *quantum* force, but this does not mean that very notion of force is overthrown. Thus, we may expect that the nomological and dispositionalist interpretations are also able to cogently trace the distinction between inertial and non-inertial trajectories.

Apart from the meaning of the wavefunction, there is another aspect in the theory that makes room for interpretation. The equation of motion governing the particles can be either the second-order one introduced by Bohm, or a first-order equation in which the trajectories of the particles are determined directly by their velocities. Now, the choice of a first-order equation of motion immediately suggests that the concept of force will be affected. The customary notion of force is associated to accelerations, yet, in a first-order context there remains the possibility of conceiving forces as resulting in velocities—and these first-order forces can be in turn used to reformulate the concept of inertial motion.

Thus, in spite of the differences between them, it seems, at least at face value, that the interpretive alternatives for dB-B theory can retain the conceptual machinery required to cogently define and characterize inertial trajectories. In this article I explicitly unfold the resulting definitions and characterizations in the different interpretations of dB-B theory, and I critically assess them. This analysis is clarifying and heuristically relevant for two main reasons. First, because in most of the available interpretive alternatives the issue of inertial trajectories has not been directly tackled. Second, because the analysis shows that in the alternative readings of the theory the redefined notions of inertial trajectory (or the plain absence of the concept) are connected to important difficulties. The goal of this article is thus twofold: I intend to reveal the precise characterization of inertial motion (or the lack of it) in each interpretive alternative, and to spell out the problems that are connected to those characterizations (or to the absence of the concept).

In section 2, I introduce a general interpretive framework for dB-B theory. As I mentioned, there are three ways to interpret the meaning of the wavefunction: as a physical field, as a law, or as a dispositional property. The chosen interpretation of Ψ can be further specified by selecting either the first-order or the second-order formulation of the theory, so that six different basic interpretations result. After outlining this schema, I proceed to address the subject of inertial trajectories in each of the interpretive setups.

In the first part of section 3, I deal with the physical-field/second-order interpretation. Bohm's original proposal falls under this category, and its characterization of inertial trajectories is customary and straightforward. The difficulties that affect this formulation are well known, but it is useful to reconsider them here. On the one hand, I claim that they undermine the characterization of inertial motion in Bohm's proposal. On the other hand, there are two recent (not so well-known) variations on the physical-field/second-order view that intend to avoid such problems, so it is worth to examine whether they succeed—I claim that they do not. In the second part of section 3, I address the physical-field/first-order interpretation. This stance requires a redefinition of the concept of force: forces are now associated to velocities rather than to accelerations. Consequently, the notion of inertial motion must also be redefined, and it turns out to be a state of absolute rest. I show that this is intrinsically problematic, for it requires a privileged frame of reference that dB-B theory cannot afford.

In the first half of section 4, I deal with the nomological/second-order interpretation. I show that since in this approach Ψ does not represent an entity, inertial trajectories that are not rectilinear and uniform are allowed. I claim that this result is problematic, for it constitutes a violation of a basic principle in spacetime theories. In the second part of section 4, I address the nomological/first-order interpretation. In this reading of the theory, the concept of force is altogether absent, so the distinction between inertial and non-inertial motion cannot even be traced. I argue that the lack of a concept of inertial motion illustrates a problem of incommensurability: this interpretation builds a conceptual wall between the quantum and the classical realms.

In section 5, I deal with the dispositionalist interpretation. I show that the situation is analogous to the nomological view. In this reading the wavefunction does not represent a physical object either, so in the second-order formulation there are inertial trajectories that are not rectilinear and uniform, and then the aforementioned friction with a central principle in spacetime theories comes up. If we move to the first-order formulation, the concept of inertial trajectory cannot be defined, so we face the threat of incommensurable descriptions of the quantum and the classical realm.

In section 6, I state a general conclusion. I claim that only the canonical (physical-field/second order) interpretation of dB-B theory meets the expectation of a clear conceptual definition of inertial motion. However, this virtue is undermined by several problems. The rest of the interpretations of the theory intend to avoid those problems, but then the concept of inertial motion becomes either problematically redefined, or problematically absent.

2. INTERPRETING DE BROGLIE-BOHM THEORY

De Broglie-Bohm theory was originally presented by Louis de Broglie in 1926-7, and then independently reintroduced by David Bohm in (1952).[2] There is, however, an important difference between de Broglie's and Bohm's formulation. Whereas de Broglie postulated a first-order law of motion determining the trajectories of quantum particles, so that forces and accelerations play no fundamental role in the dynamics of the theory, Bohm introduced a second-order law of motion, similar in form to Newton's second law, but including an extra-term that he took as representing a quantum potential.

More precisely, the first-order version of the theory is given by the following postulates:

(P1) The state of an n-particle system is represented by (Ψ, Q), where $\Psi(q,t)$ is the wavefunction of the system, $q \in \mathbb{R}^{3n}$ is the generic configuration of the wavefunction, $Q \equiv (Q_1, Q_2, \dots, Q_n) \in \mathbb{R}^{3n}$ is the actual configuration of the particles, and $Q_i \in \mathbb{R}^3$ is the position of the ith particle.

(P2) The temporal evolution of the wavefunction is governed by the Schrödinger equation $i\hbar\frac{\partial\Psi}{\partial t} = \left(-\sum_{i=1}^{n}\frac{\hbar^2}{2m_i}\nabla_i^2 + V(q,t)\right)\Psi$, where $V(q,t)$ is the classical potential, m_i is the mass of the ith particle, $\nabla_i^2 = \nabla_i \cdot \nabla_i$, and ∇_i is the gradient with respect to the coordinates of the ith particle.

(P3) With the wavefunction in polar form $\Psi = R(q,t)e^{iS(q,t)/\hbar}$, the trajectory of the particles is determined by

$$m_i \frac{dQ_i}{dt} = \hbar \mathrm{Im} \frac{\nabla_i \Psi}{\Psi} = \nabla_i S \tag{1}$$

(P4) The distribution of the particles in the system associated to Ψ is given by $P = R^2 = |\Psi|^2$.[3]

On the other hand, Bohm's second-order approach was the following. Plugging the wavefunction in polar form into the Schrödinger equation, and separating the imaginary and the real parts, he obtained the formulas

$$\frac{\partial S}{\partial t} + \sum_{i=1}^{n} \frac{(\nabla_i S)^2}{2m_i} + V - \sum_{i=1}^{n} \left(\frac{\hbar^2}{2m_i}\right) \frac{{\nabla_i}^2 R}{R} = 0 \tag{2}$$

$$\frac{\partial R^2}{\partial t} + \sum_{i=1}^{n} \nabla_i \cdot \left(R^2 \frac{\nabla_i S}{m_i}\right) = 0 \tag{3}$$

Equation (2) is identical in form to the Hamilton-Jacobi equation in classical mechanics except for the term $U(q,t) = -\sum_{i=1}^{n} \left(\frac{\hbar^2}{2m_i}\right) \frac{{\nabla_i}^2 R}{R}$, that Bohm interpreted as a quantum potential. He concluded, also in analogy with Hamilton-Jacobi formulation of classical mechanics, that Eq. (3) says that the (assumed) statistical distribution $P = R^2 = |\Psi|^2$ of the particles is conserved over time. Now, since Hamilton-Jacobi mechanics is fully compatible with Newton's second law of motion $\frac{d\mathbf{p}}{dt} = -\nabla V$, the analogy can be further exploited in order to introduce a second-order law of motion[4]

$$\frac{d\mathbf{p}_i}{dt} = -\nabla_i (V + U) \tag{4}$$

Bohm pointed out that empirical consistency with standard quantum theory requires to restrict the initial value of the momenta of the particles according to Eq. (1)—a restriction that also guarantees the empirical consistency between the first-order and the second-order approaches. Thus, Bohm's theory can be formulated by replacing (P3) with the postulate

(P3′) In an n-particle system, the trajectories of the particles are governed by Eq. (4), where $U(q,t) = -\sum_{i=1}^{n} \left(\frac{\hbar^2}{2m_i}\right) \frac{{\nabla_i}^2 R}{R}$ is a quantum potential, with the initial momenta of the particles given by Eq. (1)

A quick inspection of (P3′) shows that the first-order formulation of the theory is more economical than the second-order one. Eq. (1) can directly determine the trajectories of the particles, so the introduction of Eq. (4) as the law of motion seems superfluous. The usual justification of (P3′) is explanatory. Though the second-order law is unnecessary for the empirical predictions of the theory, it allows that the explanatory framework of Newtonian mechanics can be imported into the quantum realm. That is, the motion of quantum particles can be understood as determined by forces proportional to accelerations, forces which are in turn determined by the classical potential V and the quantum potential U. That is, the second-order approach results in a quasi-Newtonian quantum theory.[5]

However, the introduction of a *physical* field Ψ—the so-called pilot-wave—associated to the quantum potential, comes at a cost. First, unlike ordinary physical fields, Ψ does not have a source. Second, although Ψ affects the particles through the quantum force determining their trajectories, the particle trajectories do not affect the Ψ-field back. Third, Ψ is defined and evolves in $3n$-dimensional configuration space, not in physical 3-space. These properties of the wavefunction cast doubts on its interpretation as a physical field. Consequently, two alternative ways to understand the meaning of Ψ have been proposed.

The *nomological* approach (Dürr, Goldstein, and Zanghì 1997, Goldstein and Zanghì 2013) rejects the view that Ψ represents a physical entity. Instead, this term is understood as denoting a component of physical law:

> The wavefunction of the universe is not an element of physical reality. We propose that the wavefunction belongs to an altogether different category of existence than that of substantive physical entities, and that its existence is nomological rather than material. We propose, in other words, that the wavefunction is a component of physical law rather than of the reality described by the law. (Dürr, Goldstein, and Zanghì 1997, 35)

As a way to clarify, proponents of this approach introduce an analogy with the Hamiltonian in classical mechanics. The Hamiltonian function H determines the trajectories of classical systems, but it is not affected back by the motion of those systems. Furthermore, it is defined in phase space, which is even more abstract and greater in dimensions than configuration space. However, these properties of H are not problematic, for H does not refer to a physical entity, rather, it is a mathematical representation of nomological structure. The main idea in the nomological interpretation is to understand the wavefunction Ψ just as we understand the Hamiltonian function H.

A second way to avoid the physical field consists in taking the wavefunction as representing a *dispositional property* of the quantum system.[6] Following Belot (2012), let us consider the phase space of a classical system. Each trajectory in phase space represents a possible history of dynamic states of the system, and the possible trajectories are in turn determined by Hamilton's equations. Now, each point of the system's phase space can be taken as specifying a value of a position-momentum property that the system can adopt. Thus, we can understand phase space as a whole as determining a family of possible position-momentum property-values. The basic idea in the dispositional interpretation of the wavefunction is analogue to this understanding of phase space. We can consider the space of quantum states as determining a family of property-values, so that by specifying the wavefunction of a system we single out the value of the property that is effectively possessed by the system at a certain time. A possible history of the property-values of the system is a trajectory through the space of quantum states, where the possible trajectories are determined by the Schrödinger equation. Now, the Φ-property at issue, determined by Ψ *via* Eq. (1), is a disposition that determines the velocity that each particle in the system can adopt:

> The Bohmian law of motion [Eq. (1)] is in effect a recipe that takes as input a system's wavefunction [Ψ] and gives as output a function that assigns to each possible particle configuration [$Q = (Q_1, Q_2, \ldots, Q_n) \in \mathbb{R}^{3n}$] the velocities that the particles would have were the system in that configuration and were the quantum state of the system given by [Ψ]. So we can think of the Bohmian law of motion as a rule via which the wavefunction [Ψ] of a system determines a complicated dispositional property Φ of the system—the dispositional property that determines how fast (and in what direction) each of the particles would move

> for each possible configuration of the system of particles. Correspondingly we can think of the Bohmian law of motion as giving us a rule via which the solution $[\Psi(t)]$ of the system's Schrödinger equation determines a one-parameter family Φ_t of such properties, one for each instant of time. (Belot 2012, 78)

The ontology of dB-B theory under the dispositional interpretation is given by systems of particles that possess a dispositional property Φ. Each particle has a definite position at each time, and the system of particles as a whole has a dispositional property that determines the velocity that each particle would take, were the system in a certain state and configuration.

In sum, we get a six-fold framework for interpreting dB-B theory that results from crossing the three available readings of the wavefunction with the two possible choices for the equation of motion. Let us now consider the issue of inertial motion in each of these six interpretations.

3. PHYSICAL FIELD(S)

3.1. SECOND-ORDER

We begin with interpretations that postulate a quantum field in the theory's ontology, in second-order formulation. In Bohm's original proposal, which is also advocated by Holland (1993) and Cushing (1994), we have a physical field (the pilot-wave) Ψ with potential U, that exerts a quantum force $-\nabla U$ that contributes to determine the particle trajectories. Accordingly, the condition for inertial quantum trajectories in this interpretation is customary and straightforward: classical and quantum forces in Eq. (4) must both vanish or cancel each other.[7] However, this reading of the theory lacks a convincing description of how the pilot-wave exerts the quantum force on the particles. Since the quantum force—a concept that is crucial to trace the distinction between inertial and non-inertial motion—is affected by two important problems, the foundations of the distinction in Bohm's formulation are rather shaky.

The see the first problem, recall that the wavefunction is defined and evolves in configuration $3n$-dimensional space. If, accordingly, the quantum field is assumed to exist in $3n$-space, a form of configuration space realism must be assumed, which is already a radical step to take (cf. Holland 1993, 277-278). Anyhow, even if we grant configuration space realism, it is difficult to make up an intelligible view of how the quantum force affects the motion of particles. The picture is that the pilot-wave Ψ exerts a force $-\nabla U$ on the particles affecting their trajectories. But, considering that Ψ inhabits and evolves in configuration space, whereas the particle lives and evolves in physical space, one wonders how this causal-mechanical interaction can occur. We encounter here a sort of Cartesian dualism involving physical and configuration space. Solé refers to this issue as 'the problem of communication': "as the wavefunction lives in a different space from that of the particles, it becomes rather complicated to tell a coherent causal story about how the former influences the movement of the latter" (Solé 2013, 367).[8]

The second problem consists in that, in this interpretation, the theory violates Anandan and Brown's action-reaction principle: "we shall say that two physical entities satisfy the action-reaction (AR) principle, if they interact in such a manner that each entity both acts on and is acted on by the other entity. [...] A physical theory is *dynamically complete* if all the entities postulated in the theory pairwise satisfy the AR principle" (Anandan and Brown 1995, 351). In the case of the pilot-wave and the particles, the AR principle is not pairwise satisfied. The evolution of the quantum field is fully specified by the Schrödinger equation, so that Ψ is dynamically indifferent to the particle trajectories (cf. Holland 1993, 79). Thus, although the pilot-wave acts on the particles by guiding them through the physical exertion of a force, it is not affected back by the particle trajectories in any way—the theory is dynamically incomplete. This is problematic because, as a general principle, one may expect and require that mechanical interactions, such as the one between the pilot-wave and the particle, respect the AR principle.

A recent proposal (Norsen, 2010; Norsen, Marian, and Oriols 2015) provides an outline to formulate dB-B theory in such a way that its ontology is composed only by quantum particles and fields in 3-space. In this way, the root of the aformentioned problems may be removed, so that the quantum force—which is crucial to define inertial motion—may stand on more solid ground. Norsen explains his proposal by means of a toy-model. He invites us to consider a two-particle system such that the corpuscles move in a single spatial dimension. The time evolution of the system's wavefunction $\Psi(x,t)$ is given by the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\Psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\Psi}{\partial x_2^2} + V(x,t)\Psi\bigg|_{x_i=X_i} \tag{5}$$

Along with the quantum particle, a field in physical space—represented by the *conditional wavefunction*—is included in the ontology. The conditional wavefunction of a particle (in the context of our two-particle system) is "simply the full, configuration-space wavefunction evaluated at the actual location of the other particle" (Norsen 2010, 1866). Thus, "each conditional wavefunction, because it depends only on the spatial coordinate associated with the single-particle in question, can be regarded as a wave that propagates in physical space" (Norsen, Marian, and Oriols 2015, 9). In our two-particle system, the conditional wavefunctions for particles 1 and 2 are, respectively

$$\psi_1(x,t) = \Psi(x, X_2, t) \tag{6}$$

$$\psi_2(x,t) = \Psi(X_1, x, t) \tag{7}$$

Now, an ontology constituted only by particles and their pilot-waves in physical space (the conditional fields) is not enough in order to obtain predictive adequacy and empirical equivalence with standard quantum mechanics, for conditional wavefunctions do not contain the information about correlations when the subsytems are entangled. Norsen then derives a law for the time evolution of the conditional wavefunction contaning terms that carry the missing entanglement information. By taking the time-derivative of Eq. (6) and using Eq. (5), he gets the following equation for the evolution of ψ_1 (Norsen 2010, 1869-1870):

$$i\hbar\frac{\partial\psi_1}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi_1}{\partial x^2} + V(x, X_2, t)\psi_1 + A + B \tag{8}$$

where

$$A = i\hbar\frac{dX_2}{dt}\frac{\partial\Psi(x,x_2,t)}{\partial x_2}\bigg|_{x_2=X_2} \qquad B = -\frac{\hbar^2}{2m_2}\frac{\partial^2\Psi(x,x_2,t)}{\partial x_2^2}\bigg|_{x_2=X_2},$$

and an analogous procedure can be applied on Eq. (7) to obtain the equation for the evolution of ψ_2. The terms A and B, that carry the entanglement information, cannot be represented in terms of the entities in physical space introduced so far (the particles and the conditional fields). Norsen thus introduces new elements in the ontology, namely, the 'entanglement fields'

$$\psi_1' = \frac{\partial\Psi(x,x_2,t)}{\partial x_2}\bigg|_{x_2=X_2} \tag{9}$$

$$\psi_1'' = \left.\frac{\partial^2 \Psi(x, x_2, t)}{\partial x_2^2}\right|_{x_2 = X_2}, \tag{10}$$

so that Eq. (8) for the time evolution of ψ_1 becomes

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2 \psi_1}{\partial x^2} + V(x, X_2, t)\psi_1 + i\hbar \frac{dX_2}{dt}\psi_1' - \frac{\hbar^2}{2m_2}\psi_1'' \tag{11}$$

The role that the fields ψ_1' and ψ_1'' play is, in simple words, to tell particle 1 how to entangle with particle 2. According to Eq. (11), ψ_1' and ψ_1'' contribute to determine the time evolution of ψ_1, which in turn determines the trajectory of particle 1; and according to Eqs. (9) and (10), the position of particle 2 determines both ψ_1' and ψ_1''—so the theory is non-local (as expected).

The last ingredient is a law for the time evolution of the entanglement fields ψ_1' and ψ_1''. To obtain it, a procedure similar to the reasoning leading to Eq. (8) can be applied. It turns out that the resulting Schrödinger-like equation for the evolution of ψ_1' includes a term

$$\left.\frac{\partial^3 \Psi(x, x_2, t)}{\partial x_2^3}\right|_{x_2 = X_2}$$

that Norsen defines as a further entanglement field

$$\psi_1''' = \left.\frac{\partial^3 \Psi(x, x_2, t)}{\partial x_2^3}\right|_{x_2 = X_2} \tag{12}$$

But now an expression for the evolution of ψ_1''' is needed. Such an expression includes a fourth-order entanglement field ψ_1'''', a field that also requires a time evolution equation including a fifth-order entanglement field, and so on, *ad infinitum.* Thus, in order to accomplish the goal of formulating dB-B theory solely in terms of particles and fields defined in physical space, an infinite amount of entanglement fields must be introduced in the ontology.

After this overview of Norsen's proposal we can make explicit and discuss the underlying characterization of how quantum forces determine particle trajectories, from the second-order dynamics standpoint.[9] In Norsen's formulation, the ontology of the theory is given exclusively by entities in physical space, so the problem of communication should not even come up. More precisely, in Norsen's proposal Eq. (1) for the velocity of the ith particle in a many-particle system becomes

$$m_i \frac{dQ_i}{dt} = \hbar \mathrm{Im} \frac{\nabla \psi_i}{\psi_i} \tag{13}$$

Eq. (13) contains only terms defined in physical space—ψ_i is here the conditional wavefunction. We can now apply Bohm's strategy to obtain the corresponding quasi-Newtonian second-order law of motion in Norsen's draft-theory. That is, we introduce the conditional wavefunction ψ_i in polar form into the equation governing its evolution. In our two-particle system example, if we plug $\psi_1 = Re^{iS/\hbar}$ into Eq. (11) and separate imaginary and real parts, a quantum potential $U^{\psi_1} = \frac{\hbar^2}{2m_1}\frac{\nabla^2 R}{R} + \frac{\hbar^2}{2m_2}\frac{\psi_1''}{R}$ can be read off (which differs from Bohm's potential by the second term in the right hand side). In the general case of systems of n-particles moving in three dimensions, a second-order law of motion of the form

$$\frac{d\mathbf{p}_i}{dt} = -\nabla_i\left(V + U^{\psi_i}\right) \quad (14)$$

can be derived (see footnote 4). Eq. (14) is analogous to Eq. (4), but now the quantum force $-\nabla U^{\psi_i}$, that contributes to determine the trajectory of the ith particle, is exerted by the conditional field ψ_i, a field that is defined and evolves in physical space. Hence, no problem of communication.

Norsen also argues that the problem of the violation of the AR principle does not hold in his proposal, for the particle trajectories do affect the evolution of the pilot-wave. As we saw in our two-particle system example, the evolution of the pilot-wave ψ_1 is indeed influenced by the trajectory of particle 2, so Norsen states that

> Each particle's motion is dictated just by its own associated pilot-wave field, but the evolution of each pilot-wave field is influenced by all the other particles. Not only, then, do the particles influence the pilot-wave fields, but the particles can quite reasonably be understood as (indirectly) affecting each other (through the various fields). Perhaps those who dislike the causality posited by the usual pilot-wave theory, then, will find the theory sketched here more tolerable. (Norsen 2010, 1879)

Thus, Norsen's draft-version of dB-B theory seems to be an amenable environment for an understanding of quantum forces and (inertial) particle trajectories in quasi-Newtonian terms, and it seems to avoid both the problem of communication and the dynamical incompleteness that affect Bohm's interpretation.

However, and despite Norsen's statements to the contrary, the problem of the violation of the AR principle does come up in his proposal. It is true that the evolution of the pilot-wave ψ_i is not dynamically indifferent to the trajectories of particles, for Eq. (11) says that the evolution of ψ_i determines the trajectory of the ith particle, and that ψ_i is affected by the trajectories of all the other-than-the-ith particles in the system. However, inspecting Eq. (11) more carefully we see that the evolution of ψ_i is not affected *back* by the trajectory of the ith particle. As Norsen himself states in the quotation above, "each particle's motion is dictated *just* by its own associated pilot-wave field, but the evolution of each pilot-wave field is influenced by all *the other* particles". Furthermore, although the trajectories of the other-than-the-ith particles non-locally affect the trajectory of the ith-one by determining the evolution of ψ_i, the pilot-wave ψ_i does not affect the trajectory of any of those particles. Thus, in Norsen's proposal the AR principle is actually violated in a twofold way: ψ_i acts on the ith particle determining its trajectory, but ψ_i is not affected *back* by the ith particle; and the positions of the rest of the particles in the system affect the evolution of ψ_i, but ψ_i does not affect *back* the trajectories of those particles.

The problem of the violation of the AR principle, in the context of Bohm's theory, results from the fact that the evolution of the wavefunction is fully determined by the Schrodinger equation, and according to this law the position of the particle has no effect on the temporal development of Ψ. In Norsen's proposal something similar is the case. If we inspect Eq. (11), it is clear that the position of the ith particle in the system is not relevant for the time evolution of the conditional field ψ_i—so that the AR principle cannot be obeyed. In sum, Norsen's proposal does avoid the problem of communication, but it is dynamically incomplete.

Another recent variation on the physical-field/second-order approach, that intends to avoid dynamical incompleteness and the problem of communication, was introduced by Belousek (2003). This proposal does not take $\Psi = Re^{iS/\hbar}$ as representing a pilot-wave. However, I classify it among the interpretations that include physical fields because Belousek states that S, R and U do have an ontological meaning. In his proposal, the function S describes a 'velocity field' *via* Eq. (1): "Because all *actual* velocities, and hence trajectories, obtained by evaluating [Eq. (1)] at any given configuration point at any time will be three-dimensional quantities, S would have its primary physical significance in reference to actual single-particle trajectories in 3-dimensional space" (Belousek 2003, 161). Something similar holds for the function R: it describes a 'potential field' in configuration space by means of the quantum potential equation $U(q,t) = -\sum_{i=1}^{n} \frac{\hbar^2}{2m_i} \frac{\nabla_i{}^2 R}{R}$. Now, the potential field has its physical significance in 3-space, for it determines, *via* Eq. (4), actual forces acting on particles that follow trajectories in 3-space. That is, in Belousek's interpretation, the fields S and R acquire a physical meaning inasmuch as they determine the trajectories of the particles by means of the velocity field, the potential field U, and the quantum forces $-\nabla U$.

In Belousek's proposal there is no problem of communication, for the ontology is completely defined in 3-space. The problem of violation of the AR principle does not come up either, for there is no pilot-wave to be affected back by the particles. However, the resulting description of the way in which trajectories are determined by the quantum forces is very difficult to accept. The denial of entities in configuration space has a very awkward consequence: the quantum forces are ontologically primitive, in the sense that they are not exerted by any physical entity—they simply exist 'out there on their own'.[10] Although this proposal attempts to retain the explanatory power of the quantum force included in Eq. (4), the maneuver of excising the quantum field, while retaining the quantum potential and its corresponding forces, conveys a dubious ontological element.

Summing up, since the assessed interpretations postulate a second-order law of motion, in which $-\nabla U$ (or $-\nabla U^{\psi}$) represents a quantum force, the condition for inertial motion is natural and the same in all three cases: quantum and classical forces must be zero or cancel each other. However, we have that, despite the initial promise, the physical-field(s)/second-order formulations of dB-B theory fail to provide a fully intelligible quasi-Newtonian picture of the way in which the field(s) determine the trajectories of the particles. Taking Ψ as a pilot-wave in configuration space leads to the problem of communication and to dynamical incompleteness. Replacing Ψ with the conditional and the entanglement fields avoids the former problem, but deepens the latter. By excising the quantum field while retaining the quantum potential both problems are avoided, but on the price of introducing mysterious quantum primary forces. Thus, although in these three interpretations we find a simple characterization of inertial motion, it gets undermined by the problems associated with the quantum force.

We can now move on to assess a reading of the theory in which Ψ also represents a physical field, but this time choosing the first-order equation of motion.

3.2. First-order

As we saw in section 2, Eq. (1) directly determines the empirical content and predictions of the theory. Thus, the introduction of Eq. (4) as the law of motion, with Eq. (1) as a condition for the initial momenta of the particles, seems to be redundant and unnecessary. Taking the more economical choice precludes that we can make use of the quasi-Newtonian explanatory framework, but that does not constitute, *per se*, a drawback. Besides, if first-order dB-B theory is empirically adequate and explains satisfactorily, this may be the right approach:

> David Bohm [...] and many others [...] present the theory as being Newtonian in appearance. [...] But this takes us off target. Differentiation of [Eq. (1)] is redundant. Bohmian mechanics is a first order theory. [...] Casting Bohmian mechanics into a Newtonian mould is not helpful for understanding the behavior of the trajectories in quantum mechanical situations, because understanding means first explaining things on the basis of the equations that define the theory. Redundancies are more disturbing than helpful. Analogies may of course help, but they are secondary. (Dürr and Teufel 2009, 150)

Now, in order to spell out the precise way in which the equations that define the theory explain physical phenomena, we must specify the meaning attributed to the wavefunction. Valentini (1992, 1996, 1997) assumes the first-order approach and interprets the term Ψ as denoting a pilot-wave. In this proposal, since Eq. (1) constitutes the law of motion, the agency of the pilot-wave Ψ on the particles is given by an "Aristotelian force" ∇S that induces a velocity, not an acceleration:

> The pilot-wave Ψ should be interpreted as a new causal agent, more abstract than forces or ordinary fields. This causal agent is grounded in configuration space—which is not surprising in a fundamental "holistic", nonlocal theory. Heuristically, however, its action in three-space may be visualized in terms of "Aristotelian forces". The "Aristotelian force" $f_i = \nabla_i S$ on the right-hand side of [Eq. (1)] is analogous to the Newtonian force $F_i = -\nabla_i V$. [...] The ratio of Newtonian force to mass gives the acceleration. While according to [Eq. (1)], the ratio of "Aristotelian force" to mass gives the *velocity*. (Valentini 1997, 216)

Valentini's interpretation gets affected by the problem of communication, and it is also dynamically incomplete. The pilot-wave Ψ is defined and evolves in configuration space, whereas the particle motions it determines, *via* the Aristotelian force, occur in physical space. The modification in the definition of the concept of force leaves this problem untouched: how does the pilot-wave in configuration space exerts an Aristotelian force on particles in physical space? As to dynamical incompleteness, the situation is here the same as in Bohm's second-order approach: Ψ is governed by the Schrödinger equation, so the trajectory of the guided particle is irrelevant for the evolution of the wavefunction. The pilot-wave determines the trajectory of the particle, but the particle's trajectory does not affect back the pilot-wave.

Adopting the Aristotelian first-order formulation in the context of Norsen's approach would solve the problem of communication—we can simply take Eq. (13) as the first-order equation of motion of the theory. By doing so, Valentini's Aristotelian forces are not exerted by a field Ψ, but by the conditional fields ψ_i, which are defined and evolve in physical space. However, Norsen's proposal in first-order formulation is as dynamically incomplete as in its second-order version. This problem remains the same after replacing the quantum force $-\nabla U$ with Valentini's Aristotelian force ∇S.

We could also refer to Belousek's interpretation and bring it to a first-order context. That is, we jettison U and $-\nabla U$, and we replace them with the Aristotelian forces (in physical space) $\nabla_i S$ (see Solé 2013, 377). Belousek's interpretation avoids the problem of communication because the constituents of its ontology are all entities in physical space. It also respects the AR principle, for the forces determining the particle trajectories are not exerted by a pilot-wave. However, just as in the second-order version, we get the problem of the primitive—this time Aristotelian—forces existing out there on their own.

Furthermore, when we consider force-free motion, a distinctive problem rises up in the Aristotelian forces approach. We saw that the definition of inertial motion in second-order versions of the theory that include a quantum force is analogous to its definition in classical mechanics: a free particle does not accelerate. In first-order formulations of the theory that include a quantum Aristotelian force, the description of inertial motion is very different. An inertial particle is now a particle that is not affected by Aristotelian forces. This implies that force-free state of motion is rest, for according to Eq. (1), if the Aristotelian force ∇S is zero, the velocity is zero.

Now, if the force-free state of motion is rest, the natural question that comes up is rest *in what frame?* Since the Schrödinger equation and Eq. (1) are Galilean invariant, it seems impossible to single out a privileged reference frame in which we can define true, absolute rest (cf. Brown, Elby, and Weingard 1996, section 6). But since in this interpretation rest corresponds to the inertial state of motion, it cannot be frame-relative. Valentini (1997) faces this difficulty by arguing that Galilean invariance is a fictitious symmetry of the theory. To see why, he asks us to consider, in the context of classical mechanics, a frame of reference Γ and a frame Γ' with acceleration $\mathbf{a}$ with respect to Γ. The coordinate transformations between these frames are

$$x_i' = x_i - \frac{1}{2}\mathbf{a}t^2 \qquad t = t' \tag{15}$$

The corresponding transformations for the classical potential V and the classical force $-\nabla_i V$ are

$$V' = V - \frac{1}{2}\sum_i m_i \mathbf{a}^2 t^2 + \sum_i m_i \mathbf{a} \cdot x_i \tag{16}$$

$$-\nabla_i' V' = -\nabla_i V + m_i \mathbf{a} \tag{17}$$

Under transformations (16) and (17), Newton's law of motion $m_i \frac{d^2 x_i}{dt^2} = -\nabla_i V$ is invariant. Now, the usual view is that these transformations include fictitious inertial forces $m_i\mathbf{a}$ that equally affect all bodies in the non-inertial frame, and that are thus proportional to the mass of each body. Given the fictitious character of these 'forces', transformations (15)-(17) represent non-fundamental, fictitious symmetries that do not reflect the spatiotemporal structure that corresponds to the theory.

Let us now consider, in the context of dB-B theory, a frame Λ and a frame Λ' that moves with velocity $\mathbf{v}$ with respect to Λ. The corresponding coordinate transformations are the Galilean transformations

$$x_i' = x_i - \mathbf{v}t \qquad t' = t \tag{18}$$

The Schrödinger equation and Eq. (1) are invariant under these transformations. The corresponding transformation for the wavefunction is $\Psi' = \Psi \exp\left[i\left(\frac{1}{2}\sum_i m_i \mathbf{v}^2 t - \sum_i m_i \mathbf{v} \cdot x_i\right)\right]$, so that

$$S' = S + \frac{1}{2}\sum_i m_i \mathbf{v}^2 t - \sum_i m_i \mathbf{v} \cdot x_i \tag{19}$$

$$\nabla_i' S' = \nabla_i S - m_i \mathbf{v} \tag{20}$$

Valentini's argument relies on the similarity between Eqs. (16) and (19), and between Eqs. (17) and (20). He states that just as the fictitious forces $m_i\mathbf{a}$ in Eqs. (16) and (17) are proportional to the masses of the bodies in the accelerated frame, the Aristotelian forces $m_i\mathbf{v}$ in Eqs. (19) and (20) are proportional to the masses of the bodies in the uniformly moving frame. Therefore, the Aristotelian forces $m_i\mathbf{v}$ in Eqs. (19) and (20) should also be regarded as fictitious. Now, we saw that since the $m_i\mathbf{a}$ are fictitious forces, transformations (15)-(17) are non-fundamental, fictitious symmetries of Newtonian dynamics. One of the frames Γ and Γ' is dynamically privileged: the frame in which fictitious forces vanish is inertial. Accordingly, Valentini states, if the $m_i\mathbf{v}$ are fictitious Aristotelian forces, transformations (18)-(20) are non-fundamental, fictitious symmetries of dB-B dynamics. One of the frames Λ and Λ' is dynamically privileged: the one in which the fictitious Aristotelian forces vanish is truly at rest—so an inertial particle is at rest in that frame. Thus, Valentini concludes, despite the apparent Galilean invariance of the Schrödinger equation and of Eq. (1), true rest in dB-B theory can be well defined:

> The supposed "Galilean invariance" of the pilot-wave theory is, in our view, a first-order analogue of the above *fictitious* invariance (of second-order) classical mechanics. Just as the true, physical invariance group of classical mechanics leaves acceleration and (Newtonian) force invariant, so the true, physical invariance group of pilot-wave dynamics leaves velocity and (Aristotelian) force invariant. (Valentini 1997, 219)

Valentini readily notes that the preferred frame and the state of true motion are empirically undetectable. He responds by stating that the situation is similar in classical mechanics. Consider again the frames Γ and Γ′. The effects of absolute acceleration in the non-inertial frame can be cancelled by transforming the corresponding forces according to Eq. (17). However, Valentini claims that, after all, the fictitious forces may be real. They are regarded unreal because they appear to have no source, but they may be generated, in a Machian-like way, by accelerations with respect to distant matter. Thus, the privileged status of one of the frames Γ and Γ′ (the one in which fictitious forces vanish) relies on the *assumption* that real forces have their origin in nearby bodies.

I think that Valentini's approach carries important problems connected to the notion of force-free motion that it postulates.[11] First, the undetectability of the truly force-free state of motion and of the associated preferred frame is a very troublesome result. Even if Valentini were convincing in that classical mechanics is problematic in a similar way, a *tu quoque* argument does not solve the difficulty—if the situation is analogous in classical mechanics, then shame on it as well.

Besides, Valentini's reasoning is unconvincing. If the difference between real and fictitious forces were just an assumption or a matter of conventional consensus, then there would be no grounds to trace a fundamental dynamical distinction between inertial and accelerated frames in classical physics. Without the distinction, the symmetries of transformations (15)-(17) would not be fictitious, but fundamental. But then Valentini's argument could not even get off the ground, for how could he argue, based on the analogy, that the symmetries of transformations (18)-(20) are fictitious in dB-B theory? Valentini's proposal thus relies on the fundamental dynamical difference between real and fictitious forces in Newtonian mechanics—otherwise the analogical judgment that the transformations (15)-(17) and (18)-(20) are fictitious symmetries in Newtonian mechanics and in dB-B theory, respectively, would make no sense. Now, the difference between fictitious and real forces certainly allows to discern, in empirical terms, between inertial and accelerated frames in classical physics, but the alleged difference between real and fictitious Aristotelian forces cannot play an analogous role in Valentini's interpretation of dB-B theory—his *tu quoque* argument certainly fails.[12]

Since Valentini's reasoning relies on a fundamental distinction between real and fictitious second-order forces, the analogy supporting his main claim—namely, that transformations (18)-(20) are fictitious symmetries in dB-B theory—is rather weak. In classical mechanics, the distinction between real and fictitious forces is not only conceptual, it is empirically grounded. However, the distinction between the real and fictitious Aristotelian forces is merely formal. Valentini's analogy rests on the mathematical similarity between (16)-(17) and (19)-(20), but the dynamical difference between second-order real and fictitious forces is absent in the case of real and fictitious Aristotelian forces. His suggestion that the distinction between the classical forces relies on an assumption intends to ease this difficulty, but if the suggestion holds, the main argument falls—for Valentini's main claim essentially relies on a fundamental distinction between real and fictitious Newtonian forces.

Another related difficulty is that the preferred frame in Valentini's proposal is not only empirically undetectable, but also explanatorily superfluous. Although the motivation to introduce a preferred frame is the conceptual economy that is gained by formulating the theory in first-order terms, there are no *dynamical-explanatory* motivations for postulating such a frame. This point gets clearer when we compare Valentini's interpretation with other theories that postulate a preferred frame. Newton's original formulation of classical mechanics included a privileged frame at rest in absolute space. Although this frame and absolute velocities are undetectable, inertial effects of absolute accelerations are empirically detectable—just recall the rotating bucket—and Newton interpreted those effects as manifestations of absolute motion. In Lorentz's ether theory, the preferred ether-rest frame is also undetectable, but it certainly plays a dynamical-explanatory role. The physical underpinning of the Lorentz-contraction, for example, was given by the motion of bodies through the ether (see Janssen 1995, Acuña 2014). In Aristotle's physics, the privileged frame has its origin in the center of a spherical universe because that point determines the natural motions of terrestrial and celestial bodies: depending on what element they are made of, sub-lunar objects move towards or away from it, whereas ethereal bodies move in circles centered in it. Unlike these theories, in Valentini's proposal the postulated privileged frame plays no dynamical-explanatory role. In Newton's, Lorentz's, and Aristotle's theories, the preferred frame (although empirically undetectable in the first two cases) is relevant in the dynamical explanation of observable effects. In Valentini's rendition of the dB-B theory, though, the only motivation for its introduction is a formal-conceptual demand: the first-order law of motion and the Aristotelian forces presuppose a preferred frame to make sense of the force-free state of motion.[13]

In sum, physical-field/second-order interpretations mirror their second-order counterparts with respect to the problem of communication, dynamical incompleteness, and primitive forces. On the other hand, whereas the physical-field/second-order view describes inertial motion in traditional terms, the Aristotelian forces approach entails a radical change in the notion of force-free motion: since Aristotelian forces are proportional to velocities, the force-free state of motion is absolute rest. Now, this concept requires an undetectable and explanatorily superfluous privileged frame that is hardly acceptable.

4. NOMOLOGICAL WAVEFUNCTION

4.1. SECOND-ORDER

We can now turn to interpretations of dB-B theory that assign a nomological meaning to the wavefunction. Before we directly explore the notion on inertial motion in the second-order and first-order formulations of the nomological approach, some preliminary considerations are necessary. A subtlety in this reading, that is very relevant here, consists in that the term that is assigned a nomological meaning is the wavefunction *of the universe* Ψ, which we may distinguish from the *effective* wavefunction ψ of a subsystem of the universe (Dürr et al. 1992, section 5). The primary domain of applicability of dB-B theory, the system that (Ψ, Q) describes, is the universe as a whole. The physical description of a specific subsystem of the universe arises from the description of the whole, and the theoretical term that allows such a description is the *effective* wavefunction ψ.

For any such subsystem, given the generic configuration q of the universal system, a splitting $q = (x, y)$ obtains, where x is the generic configuration of the subsystem, and y is the generic configuration of the complemenent-subsystem (formed by all the particles not in the subsystem), so that $\Psi = \Psi(x, y)$. Accordingly, given the actual configuration Q of the particles in the universe, there is a splitting $Q = Q(X, Y)$, where X is the actual configuration of the particles in the subsystem, and Y is the actual configuration of the rest of the particles in the universe. The wavefunction describing the x-subsystem at a time t is the *conditional wavefunction* $\psi_t(x) = \Psi_t(x, Y_t)$—which we already met in Norsen's proposal. In general, the conditional wavefunction does not evolve according to the Schrödinger equation, due to entanglement correlations between subsystems. However, let us assume that

$$\Psi_t(x, y) = \psi_t(x)\Phi_t(y) + \Psi_t^{\perp}(x, y) \tag{21}$$

where Φ_t and $\Psi_t^{\perp}$ have (macroscopically) disjoint supports in configuration space. If $Y_t \in \text{supp}\,\Phi_t$, then $\psi_t(x)$ is the *effective wavefunction* of the x-subsystem at time t.[14]

In the nomological interpretation, the wavefunction of the universe expresses a physical law. But since the subject matter of this article is inertial motion of quantum corpuscles, we are interested in how the effective wavefunction determines the motion of its corresponding particles. Thus, the natural question now concerns the meaning that the nomological interpretation assigns to ψ. The answer provided by Goldstein and Zanghì (2013, 275-276) is rather vague. They state that this is a secondary issue open to philosophical prejudices. What really matters, they claim, is that the nomological meaning of Ψ is clear. Anyhow, the authors openly state that the meaning attributed to ψ must be *quasi*-nomological:

> We would like to regard ψ as quasi-nomological. We mean by this that while there are serious obstacles to regarding the wavefunction of a subsystem as fully nomological, ψ does have a nomological aspect in that *it seems more like an entity that is relevant to the behavior of concrete physical reality* (the primitive ontology) *and not so much like a concrete physical reality itself.* (Goldstein and Zanghì 2013, 276, my emphasis)

The remark that I emphasize in the quotation is most relevant in our context. What Goldstein and Zanghì seem to have in mind is that, whatever its ultimate ontological status may be, ψ, just like Ψ, does not refer to a physical entity. This stance is quite consistent with the basic motivation of the nomological interpretation. Since the conditional wavefunction is defined in configuration space, if it were assigned an entity-meaning, the problems we identified in section 3 would come up once again. Now, although ψ cannot be understood in *fully* nomological terms, it is quasi-nomological because it determines the dynamical behavior of physical reality, without being itself an element of physical reality.[15]

If ψ does not represent an element of physical reality, then the problem of communication and the problem of dynamical incompleteness immediately vanish. Only the particle is a physical entity, so the fact that ψ is defined in configuration space is completely harmless. For the same reason, that ψ is dynamically indifferent to the particle trajectories does not result in dynamical incompleteness. Thus, the nomological approach seems a better way to describe the way in which the wavefunction determines quantum particles trajectories. However, as we will now see, the resulting characterizations of inertial motion carry important complications.

We can now address the second-order formulation of the nomological interpretation. If Ψ and ψ are (quasi-)nomological terms that do not refer to a physical entity, the same holds for U. In turn, if the term U is nomological, it does not represent a potential in a field. Consequently, the term $-\nabla U$ in Eq. (4) is a (quasi-)nomological term as well, so it cannot refer to a quantum force, for there is no physical entity to exert it on the particles. On the other hand, the excision of the pilot-wave from the ontology of the theory does not imply that the very notion of force is jettisoned. The absence of a pilot-wave means that there is no quantum force, but its removal does not affect the dynamical role of the classical potential V and the classical forces $-\nabla V$. The nomological/second-order approach can be understood as an attempt to avoid the problems associated to the nature of the pilot-wave, while retaining the essence of the Newtonian explanatory framework.

Now, since inertial motion is force-free, its condition under this interpretation is simply that $-\nabla V$ vanishes, regardless of the value of $-\nabla U$. But then we get a striking result: adopting the nomological, second-order reading of the theory, inertial motion is not always uniform and rectilinear. More concretely, whenever $-\nabla V = 0$ and $-\nabla U \neq 0$, free particles do not follow uniform and rectilinear trajectories. The supporter of this approach could bluntly reply by saying 'so what?!, at the quantum level inertial motion is not uniform and rectilinear'. Fair enough, but I want to point out that assuming this view generates important friction with a fundamental aspect of the spatiotemporal structure that corresponds to a Galilean-invariant theory, such as dB-B.

In Newtonian spacetime (also known as Galilean spacetime or neo-Newtonian spacetime), the affine structure is essentially associated to (or even identified with) the inertial structure. That is, motion that is free of forces corresponds to straight world-lines, whereas accelerated trajectories correspond to curved world-lines. Furthermore, the principle that inertial motion corresponds to (timelike) spacetime geodesics is a postulate not only in the four-dimensional reconstruction of Newtonian space plus time, but also in relativistic theory. Actually, the principle can be proven as a theorem both in general relativity and in geometrized Newtonian gravitation (see Weatherall 2011a, 2011b, and the references therein). That is, under the interpretation of dB-B theory that we are considering, the background spacetime structure assumed is (neo-)Newtonian, but a very important aspect in the dynamics associated to that structure—and a general dynamical principle in spacetime theories—is violated.

These remarks about the connection between dynamics and spacetime structure foreclose a possible way out for the nomological approach. To illustrate, let us assume for a moment a Humean conception of laws. Since according to this view laws belong to the best deductive system that accounts for the phenomena, both U and V are simply variables in such a system. The supporter of the nomological approach could then argue that U and V are considered on a par, so that it could be *stipulated* that the condition for inertial motion is that both $-\nabla V$ and $-\nabla U$ vanish (or mutually cancel)[16]. This would recover the definition of inertial motion as uniform and rectilinear, but the dynamical foundation and relevance of inertial motion as corresponding to force-free trajectories would be lost. That is, inertial motion would have a merely kinematical meaning.

Let us elaborate. Assuming the nomological approach, the fact that $-\nabla V$ and $-\nabla U$ are on a par in the *epistemological* sense that they are variables in the best Humean system does not mean that they are also on a par in terms of their *dynamical* significance. $-\nabla V$ is a variable in the best system that codifies certain kinds of *interactions* resulting in changes in states of motion, and that is why we interpret it as denoting a force.[17] On the other hand, in the nomological interpretation, $-\nabla U$ is a variable in the best system that cannot denote an interaction: since there is no pilot-wave, it cannot be interpreted as a force in the usual sense of the term. In the nomological view, the change of state of motion associated to $-\nabla U$ is naturally understood as a brute fact that is not caused by an interaction between physical objects. In this interpretive framework, the only way to understand the term $-\nabla U$ as a force is by directly reifying it, that is, by including the force itself as a beable. But by doing so we would be committed to something quite similar to Belousek's primitive sourceless forces. As we saw, the motivation of the nomological approach is to get rid of an uncomfortable physical entity, but if we include primitive forces as a part of the ontology of the theory, we would be accepting a physical entity that is at least as problematic as the pilot-wave in configuration space.

A more promising maneuver to avoid the friction with the geodesic principle would be to interpret dB-B theory in such a way that the term U is an expression of spacetime curvature. That is, in cases where $-\nabla V = 0$ and $-\nabla U \neq 0$, we may consider that the corresponding trajectories are indeed geodesics. Michael Dickson has suggested this possibility:

> This idea suggests that we consider a space-time in which non-classical yet free motions are geodesics, so that, in much the same way that we no longer invoke the 'force of gravity' to explain deviations from Euclidean geodesics, so also we would not invoke the 'quantum potential' to explain deviations from the Newtonian trajectories, […] the central idea being to 'geometrize away' the so-called quantum potential. The sole role for the wavefunction, then, is to determine the structure of space-time. It does not describe any other 'real features' of the world, and there are no 'forces', 'potentials', or anything of the sort accounting for the non-classicality of the motions of particles. (Dickson 2000, 707)

This is of course rather speculative. Itamar Pitowski (1991) formulated a sketch of a theory based on dB-B second-order formalism along these lines.[18] However, this draft-theory involves some features that go against the initial appeal of Bohm's, and that are troubling regarding the issue of inertial motion of quantum particles. For example, the dynamical equations in the theory have solutions in which the Bohmian particle is not localized, but 'spread-out' in a region of the spacetime manifold (see Pitowski 1991, section 4). This result seems to entail that the notion of trajectory is not (always) well defined, so the question of inertial motion becomes a non-starter. Notwithstanding these worries, Dickson's suggestion and Pitowski's approach constitute a very interesting avenue for development of dB-B theory. Unfortunately, it has not been explored in depth—at least I do not know of any other works elaborating on this basic idea.

4.2. First-order

Let us now consider what happens with the notion of inertial motion in a first-order formulation of the theory, under the nomological interpretation of the wavefunction. The law of motion is this time given by Eq. (1). Since Ψ and ψ are (quasi-)nomological terms that do not denote physical entities, so are S and R. Therefore, the equation of motion $m_i \frac{dQ_i}{dt} = \nabla_i S$ does not involve reference to a physical quantum field that determines the trajectories of particles. Thus, since there is no pilot-wave, Eq. (1) is not a law about (Aristotelian) force interactions. On the other hand, given that the law of motion is a first-order equation, accelerations and Newtonian forces play no role in the theory. Therefore, we can conclude that in this approach the notion of force is plainly absent. All the dynamics of the theory is contained in postulates (P1)-(P4), where, again, Eq. (1) is not a law about force-interactions. Thus, the theory is highly non-classical not only because of the peculiarities of the quantum world, but also because concepts that are central in Newtonian physics (namely, force, acceleration, energy, work) play no role:

> The mechanics here is Bohmian, not Newtonian, and one has to see what this new mechanics is like. Force and acceleration are not elements of the new theory, so any arguments based on that line of thought are off target […]
> The following analogy may be helpful. The wavefunction generates a velocity field (on configuration space) which defines the Bohmian trajectories. This is the analogue of the idea that the Hamiltonian generates a vector field (on phase space) which defines classical trajectories. (Dürr and Teufel 2009, 151)

The analogy with the Hamiltonian is clear in that the alluded velocity field is not a physical entity, and it underscores that the (effective) wavefunction plays a (quasi-)nomological role. Now, if the notion of force plays no role in the theory, it follows that the very concept of inertial motion makes no sense in this interpretive framework. Inertial and non-inertial trajectories are discerned in terms of the concept of force, so if we assume a nomological/first-order presentation of dB-B theory, there is just no distinction to be traced between trajectories of quantum particles. In a nomological first-order view, we have trajectories, but we have neither inertial nor non-inertial trajectories.

Considering the identified problems that affect the other interpretive frameworks, this is actually a good result. As we already saw, in the nomological reading of the wavefunction we do not have to worry about the problems of communication and of dynamical incompleteness. On the other hand, the friction with the principle that inertial motion corresponds to spacetime geodesics is rooted in the fact that the law of motion is a second-order equation. The geodesic principle makes essential reference to the concepts of force, acceleration and inertial motion. Hence, the absence of these concepts in the nomological/first-order interpretation implies that the principle does not apply—in this formulation of the theory, the principle is neither violated nor obeyed. Furthermore, that in dB-B theory the classical limit is formally clear makes the nomological first-order approach even more auspicious. Since the condition for classical behavior is well-defined, namely, $U \to 0$ (see Holland 1993, 218-224; Allori et al. 2002), the theory specifies under what circumstances the geodesic principle becomes meaningful and obeyed.

This sounds rather attractive, but given the paramount relevance of the concept of inertial motion at the classical level, its absence at the quantum level illustrates a problem of incommensurability. Although the conditions for classicality are formally clear in all the interpretive frameworks of dB-B theory, in the case of the nomological-first-order interpretation there seems to be an important conceptual difficulty. As we saw, in this view the concepts of force and acceleration play no dynamical role, and neither does the term U:

> We wish to stress that since the dynamics for Bohmian mechanics is completely defined by Schrödinger's equation together with the guiding equation, there is neither need nor room for any further *axioms* involving the quantum potential [U]! Thus the quantum potential should not be regarded as fundamental, and we should not allow it to obscure, as it all too easily tends to do, the most basic structure defining Bohmian mechanics. […]
> Bohmian mechanics should be regarded as a first-order theory, in which it is the velocity, the rate of change of position, that is fundamental in that it is this quantity that is specified by the theory, directly and simply with the second-order (Newtonian) concepts of acceleration and force, work and energy playing no fundamental role. From our perspective the artificiality suggested by the quantum potential is the price one pays if one insists on casting a highly nonclassical theory into a classical mold. (Dürr, Goldstein, and Zanghì 1996, 25-26)[19]

From this passage one may infer that U plays neither a dynamical nor a formal role in the theory. However, when it comes to the issue of the classical limit, as we just commented, U does play a role, for it defines the conditions for classical behavior:

> This is not to say that these second-order concepts play no role in Bohmian mechanics; they are *emergent* notions, fundamental to the theory to which Bohmian mechanics converges in the "classical limit", namely, Newtonian mechanics. Moreover, in order most simply to see that Newtonian mechanics should be expected *to emerge* in this limit, it is convenient to transform the defining equations [Schrödinger equation and Eq. (1)] of Bohmian mechanics into Bohm's Hamilton-Jacobi form. One then sees that (the size of the) quantum potential provides a rough measure of the deviation of Bohmian mechanics from its classical approximation. (Dürr et al. 1996, 26, my emphasis)

The worry I would like to address is that, in the nomological/first-order setup, the proposed account of the connection and continuity between the quantum level and the classical level is rather vague, if present at all. I think that much more needs to be said about the way and sense in which Newtonian concepts, such as force, allegedly *emerge* in the limit $U \to 0$ (taking for granted that there is a well-defined notion of emergence, which is already a controversial issue). Does it mean that the dynamical relevance of forces vanishes when U does not vanish, in analogy with the relativistic effects that become negligible when $v \ll c$? Does it mean that in the quantum world there are just no force-interactions at all and that they somehow appear in the limit $U \to 0$? Anyhow, it seems that we are facing a conceptual (even ontological) wall between the quantum and the classical level. A celebrated feature of dB-B theory is that the clear formal account of the classical limit it offers establishes a strong conceptual and ontological continuity between the quantum and the classical realms. This is clearly so in the case of Bohm's own understanding of the theory, for example, but such a continuity becomes fuzzy, if recognizable at all, in the case of the nomological/first-order approach.

In simple words, I think that this version of the theory involves an instance of Kuhnian incommensurability (both in a semantical and an ontological sense) between dB-B theory and Newtonian mechanics, involving the concepts of force and acceleration (and probably other Newtonian concepts). In short, nomological first-order dB-B theory reduces to Newtonian mechanics in the limit $U \to 0$, but the latter theory describes the world employing concepts that make no sense from the point of view of the former. One of the manifestations of this incommensurability is that at the quantum level there is no distinction between inertial and accelerated motion, but at the classical level the distinction becomes dynamically paramount (Cf. Belousek 2003, 152-154).

5. DISPOSITIONAL PROPERTY

5.1. SECOND-ORDER

As we saw in section 2, the basic idea in the interpretation of the wavefunction as the expression of a dispositional property is that Eq. (1) is a recipe in which we plug ψ as input and we obtain as output a function that assigns to each possible configuration Q the specific velocities that the particles would take, were the particles in that configuration and were the system in the state ψ. That is, the (effective) wavefunction determines a dispositional property Φ that determines in what direction and how fast the particles in the system would move for each possible configuration. Belot (2012) proposes that a precise mathematical expression of Φ is given by a vector field X on Q, determined by ψ_t as the input in Eq. (1):

> The natural way to encode the history Φ_t of such dispositional properties is via a time-dependent vector field X on Q (i.e., a function that assigns a vector field on Q to each time t). We can think of [Eq. (1)] above as defining a map $\mathcal{B} : \psi_t \mapsto X_t$ that associates with each (suitably smooth) solution of the system's Schrödinger equation a time-dependent vector field X_t that encodes the history Φ_t. (Belot 2012, 78-79)

Since the property Φ is a disposition to adopt a certain velocity, one may think that running this interpretation in a second-order formulation of the theory would be pointless. However, Suárez (2015) states that we can do so by defining a second-order dispositional property, let us dub it Ξ, that determines how Φ evolves in time, so that the defined second-order property dynamically grounds and explains the first-order one:

> The critical difference, from a dispositional point of view, between a first order equation such as [Eq. (1)] and a second-order one such as [Eq. (4)] is that if [Eq. (1)] describes first-order dispositions then [Eq. (4)] necessarily describes second-order ones. […] [Eq. (4)] establishes the evolution of this dispositional velocity field [X] as a result of what it refers to as the 'quantum potential'. In a dispositional interpretation the quantum potential is nothing but a second-order disposition, and [Eq. (4)] then effectively describes the disposition of (first-order) velocity dispositions to evolve. (Suárez 2015, section 8)[20]

Let us then investigate what comes of inertial motion in a dispositionalist, second-order interpretation of dB-B theory. The first point we must underscore is that, following the basic motivation of the dispositional approach, the wavefunction does not refer to an element of physical reality. Just as in the nomological interpretation, the main goal is to avoid the inclusion of a physical field that is defined and evolves in configuration space rather than in 3-space. In the dispositional interpretive framework,

> the wavefunction is not a law, and has no nomological force. It has merely a descriptive, or representational, function concerning the state of the physical particles in 3-d space. Yet, it does not represent any distinct object per se in 3-d space—neither a field nor a wave nor even the particle itself. […] Its function is rather to represent […] the properties of the 3-d particles, including crucially a series of dispositional properties over and above the particles' positions. (Suárez 2015, section 6)

Accordingly, in the second-order version of the dispositional approach, the terms U and $-\nabla U$ do not refer to physical interactions of the particles with another physical object. $-\nabla U$ does not express a force, rather, it is the expression for the second-order dispositional property Ξ of the particles. On the other hand, the term V still represents a classical potential, so that its gradient $-\nabla V$ is naturally understood as the expression of a force. That is, just as in the nomological/second-order approach, the excision of the pilot-wave means that there is no quantum force, but it does not follow that the concept of (classical) force is also excised.

The upshot is that $-\nabla V$ represents a force in Eq. (4), but $-\nabla U$ does not. Consequently, just as in the second-order version of the nomological approach, the condition for inertial motion is that $-\nabla V$ vanishes, regardless of the value of $-\nabla U$. In turn, we have that, in general, inertial motion is not rectilinear and uniform, and, *a fortiori,* the friction with the principle connecting inertial trajectories and geodesics in spacetime theories comes up once again.[21]

In sum, the situation concerning the dispositional interpretation of the wavefunction within a second-order formulation of the theory resembles the case of the nomological/second-order view. There is a way to define inertial motion for quantum particles, but the resulting definition states that, in general, free particles do not follow geodesic trajectories in the spacetime structure that corresponds to the symmetries of the theory.

5.2. First-order

Let us finally consider what comes of inertial motion in the nomological, first-order interpretation. As mentioned above, if we understand Eq. (1) as a recipe that given the input ψ conveys the output Φ—a dispositional property expressed in a vector field X defined on Q—we obtain a natural interpretation of the wavefunction as the determination of a dispositional property for the particles in the quantum state to adopt a certain velocity for a specific configuration. This is already enough to extract the empirical predictions of the theory, so it is possible to drop Eq. (4) and formulate the theory in more economic terms. In this setup, we avoid the problem of communication, of dynamical incompleteness, and the primitive forces.

Now—just as in the case of the first-order nomological view—since the wavefunction does not represent an entity, and since the law of motion is given by Eq. (1), the concepts of force and acceleration are plainly absent, and without them, the notion of inertial motion cannot be defined. In this reading of the theory we have particle trajectories, but we have neither inertial nor non-inertial trajectories. This result may look attractive in this interpretive setup as well, for the problem of absolute rest does not come up, and there is no friction with the geodesic principle in spacetime theories—the principle just does not apply in this approach.

However, the difficulty connected to the classical limit of the theory that we found in the nomological first-order approach comes up here too. We still need the term U in order to define the conditions under which quantum effects vanish and classical behavior emerges. We may take U as a term with a mere formal meaning in the quantum theory, but the problem is that forces get indeed well-defined in the classical limit $U \to 0$, and, *a fortiori*, the notion of inertial motion gets also well-defined in that limit. Thus, we level the same worry as in the previous subsection: the dispositionalist/first-order interpretation involves a case of semantic and ontological incommensurability between the quantum and the classical realms.

However, I would like to suggest that the metaphysical baggage of the dispositionalist account may be of some help with respect to this difficulty. Let us recall that ψ is taken as a term that determines a dispositional property to adopt a certain velocity. Now, the ontological framework of this interpretation may allow us to conceive that when the conditions for classicality are met, the fundamental property Φ determines an acceleration rather than a velocity. That is, if $U \rightarrow 0$, the mathematical expression of Φ becomes a map $\mathcal{C} : \psi_t \rightarrow Y_t$ determined by the Newtonian equation $F = -\nabla V$, where Y_t is a vector-field on Q representing acceleration (and Eq. (1) becomes a restrictive initial condition for the particles momenta). In a word, if we define the dispositional property Φ in broader terms, that is, as a disposition of the particles in the state ψ to *move* in a certain way given a configuration Q—sometimes adopting velocity values, sometimes adopting acceleration values—this framework may be able to make room for a picture of how forces and accelerations emerge from a first-order quantum dynamics. That is, the role of the term U would be to define when the property Φ determines the adoption of a certain velocity under a certain configuration, and when it determines the adoption of a certain acceleration under a certain configuration.

6. CONCLUDING SUMMARY

In spite of the natural expectation that inertial trajectory is a well-defined concept in a quantum theory in which particles have determinate positions at all times, we have found that this expectation is met only in the case of a physical field reading of the wavefunction, in second-order formulation. However, in all three versions of this interpretation, the conceptual distinction between inertial and non-inertial trajectories is undermined by the problems associated to the foundations of the quantum force, for the latter concept is crucial to trace the distinction. None of the proposals addressed in section 3.1 can provide a fully satisfactory description of how the quantum force is exerted on the particles. Since in Bohm's proposal the wavefunction in configuration space represents the pilot-wave, the problems of communication and dynamical incompleteness come up. Norsen's formulation of the theory avoids the first problem by defining the ontology of the theory exclusively in terms of entities in physical 3-space. However, this proposal is not able to elude the problem of dynamical incompleteness, for the evolution of the conditional wavefunction is dynamically indifferent to the trajectory of the corresponding particle. Belousek's approach avoids both problems by denying Ψ the status of a physical field. Yet, since in this interpretation there is no pilot-wave, while U still represents a potential field that determines quantum forces in 3-space, the ontological status of such forces is very mysterious.

Valentini's approach is an attempt to retain the explanatory power of the pilot-wave, while formulating the dynamics of the theory in more economic, first-order terms. However, the very idea of the quantum Aristotelian force is problematic, for the notion of a force inducing a velocity rather than an acceleration implies a drastic change in the customary notion of force-free state of motion: in Valentini's proposal, inertia is associated to absolute rest. Now, since the symmetries of the theory are given by the Galilean transformations, the privileged frame that a state of absolute rest requires cannot be cogently introduced. Valentini's attempt to define a suitable kinematics is rather unconvincing, and it conveys a preferred frame of reference that is empirically undetectable and explanatorily superfluous.

By excising the pilot-wave from the theory's ontology, the nomological interpretation of the wavefunction certainly avoids both the problem of communication and the problem of dynamical incompleteness. In the second-order formulation of the dynamics, however, the criterion for inertial motion is simply that the classical force $-\nabla V$ vanishes. Now, when this requirement is observed and $-\nabla U \neq 0$, we have instances of inertial motion that are not rectilinear and uniform. This result is problematic insofar as it implies a violation of the principle (theorem) that inertial trajectories are essentially associated to geodesics, which holds both in classical and relativistic spacetime theories.

The first-order version of the nomological view is not affected by this problem. Since the Newtonian concepts of force and acceleration play no dynamical role in this interpretation, the distinction between inertial and non-inertial trajectories cannot be traced. Although the concept of trajectory is well defined, it does not make sense to discern inertial and non-inertial trajectories. Therefore, the aforementioned principle of spacetime theories simply does not apply. Furthermore, since in dB-B theory the classical limit is formally clear, this interpretation has the resources to determine the condition under which the geodesic principle becomes meaningful and respected—namely, when $U \to 0$. However, the absence of a concept of inertial motion illustrates that in this interpretation there is an important loose end. That is, we end up with incommensurable descriptions of the quantum and the classical levels. In the former, forces and accelerations do not play any role, whereas in the latter they become dynamically essential, and so does the distinction between inertial and non-inertial motion.

We obtained a similar result in the case of the dispositional reading of the wavefunction. Since there is no pilot-wave in the ontology, the problems of communication and dynamical incompleteness are avoided from the outset. However, given that the wavefunction does not represent a physical entity, the criterion for inertial motion in the second-order formulation is that $-\nabla V$ is 0, regardless of the value of $-\nabla U$. This criterion, as we just said, permits non-geodesic inertial trajectories, so the friction with the geodesic principle in spacetime theories comes up.

Finally, since in the first-order version of the dispositional interpretation forces and accelerations play no dynamical role, the distinction between inertial and non-inertial trajectories cannot be traced, so the tension with the geodesic principle does not arise. Now, although the threat of incommensurable descriptions of the quantum and the classical realms comes up, the metaphysical baggage of the dispositionalist outlook can be of some help. Defining the property as a disposition of the quantum particles to move in a certain way, opens the possibility of understanding U as a term that distinguishes in what cases that property manifests as a disposition to adopt a certain velocity, and in what cases the property manifests as a disposition to adopt a certain acceleration.

The main results hereby obtained can be schematically summarized in the following table:

dB-B Theory and Inertial Trajectories	Ψ Physical Field	Ψ Nomological	Ψ Dispositional
Second-order	Inertial Motion: $-\nabla(V+U)=0$	Inertial Motion: $-\nabla V = 0$	Inertial Motion: $-\nabla V = 0$
	Problem of communication (Bohm) Dynamical incompleteness (Bohm, Norsen) Sourceless quantum forces (Belousek)	Non-geodesic inertial motion	Non-geodesic inertial motion
First-order	Inertial motion: $\nabla S = 0$	Inertial motion: undefined	Inertial motion: undefined
	Absolute rest and privileged frame	Incommensurability	Incommensurability

Table 1. Conditions for inertial motion, and related problems, in dB-B theory

Given this diagnosis of the situation in dB-B theory with respect to the concept of inertial trajectories, I think that the interpretive proposals that score higher are the nomological and the dispositional, with a first-order equation of motion. The problem of communication, dynamical incompleteness, mysterious quantum primitive forces, absolute rest in a Galilean invariant theory, and friction with the geodesic principle in spacetime theories, are all deep difficulties that do not look easy to solve—if solvable at all. On the other hand, the threat of incommensurability, although important, seems to be a problem that, at least in principle, can be faced—and the reply that the dispositionalist can offer is a suggestion for a starting point. That is, more than an unsurmountable problem, the incommensurability menace seems to identify a blank that needs to be filled: we certainly need a more detailed story about how the second-order dynamics of classical mechanics emerges from a quantum world governed by a first-order dynamics.

This evaluative conclusion is somewhat unexpected, for, at first sight at least, dB-B theory seemed to be a theory able to draw a cogent distinction between inertial and non-inertial motion. However, the ontological and dynamical problems associated to the pilot-wave and the quantum force, the problem of absolute rest, and the friction with the geodesic principle can be jointly avoided only by readings of the theory in which the conceptual tools to trace the distinction are no longer available. In short, by critically addressing interpretations of dB-B theory from the point of view of the concept of inertial motion, we find important difficulties in all of them, and it turns out that interpretations in which the concept is absent are the least problematic.

Acknowledgements

I thank two anonymous referees for their helpful criticisms on an earlier version of this work.

---

[1] For simplicity, I gloss over the issue that position eigenstates are not normalizable, so that, strictly speaking, a quantum state is never an eigenstate of the position operator.
Modal interpretations do not accept the eigenstate ↔ eigenvalue link, so that a quantum system can possess a definite value for a certain observable even if its state is a superposition in the basis of the corresponding operator. However, in none of the different versions of the modal interpretation the value-state necessarily corresponds to a definite value for the position observable.

[2] For a historical treatment of de Broglie's work, see Bacciagaluppi and Valentini (2009). De Broglie's original papers are collected in de Broglie and Brillouin (1928). For a conceptual analysis of Bohm's work and the historical and sociological conditions surrounding its reception, see Cushing (1994).

[3] Skow (2010) identifies an interesting worry with respect to (P3). That the theory is not affected by the measurement problem strongly relies on the distribution postulate (P4), whose conservation over time is guaranteed by Eq. (3). Now, Eq. (1) is not the only equation of motion that complies with distribution conservation, any formula of the form $\tilde{\mathbf{v}}(q,t) = \mathbf{v}(q,t) + \frac{w(q,t)}{|\Psi|^2}$, where $\mathbf{v}(q,t)$ is a solution of Eq. (1) and $w(q,t)$ is divergence free, satisfies the conservation condition Eq.(3). Dürr, Goldstein, and Zanghì (1992, 1996) offer an argument to justify the choice of Eq. (1) in that, assuming certain symmetry conditions that implement a Galilean invariance constraint, Eq. (1) is the simplest law of motion that can be derived. Skow claims that the derivation is flawed because the symmetry assumptions it relies on cannot be justified. Thus, he claims, the symmetry argument cannot count as a reason to pick Eq. (1), among all possible choices, as the first-order equation of motion of the theory.

[4] Eq. (4) can be derived in the following way (see Holland 1993, 74). We take Eq. (2) (for simplicity, in the single-particle system case), with $U = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$. Rearranging terms and applying the gradient operator ∇ we get

$$\left[\frac{\partial}{\partial t} + \left(\frac{1}{m}\right)\nabla S \cdot \nabla\right]\nabla S = -\nabla(U+V)$$

Now, since $\nabla S = m\mathbf{v} = \mathbf{p}$ (Eq. (1)), and given the operator $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, we get $\frac{d\mathbf{p}}{dt} = -\nabla(U+V)$.

[5] I use 'quasi-Newtonian' instead of 'Newtonian' given the peculiar quantum features of dB-B theory that are not present in classical physics (entanglement, non-locality, contextuality, etc.).

[6] This tripartite classification of the interpretations of the wavefunction in dB-B theory is advocated by (Belot 2012). Esfeld et al. (2014) qualify the dispositionalist account of Ψ as a type of nomological interpretation, in which the nomological wavefunction is traced back to the expression of a dispositional property: "According to this view, it is essential for a

property to induce a certain behavior in the objects that instantiate the property in question; the law then expresses that behavior" (Esfeld et al. 2014, 784). I think that the divergence between Belot and Esfeld and co-authors is mainly a matter of terminology: Belot assumes a conception of scientific laws that does not make room for dispositionalism, whereas Esfeld and co-authors understand scientific laws in broader terms, with Humeanism and dispositionalism as the two main possibilities. For the purpose of this work, Belot's classification is more schematic, and it allows to trace an important difference between the nomological view and the dispositionalist interpretation that is relevant for our subject.

[7] One must be careful here, though. Due to entanglement and the associated non-local correlations, in dB-B theory a particle is never strictly free. Anyhow, even if there were no free quantum particles in the universe, the notion of a particle following an inertial trajectory is conceptually well defined. Furthermore, there are (rather idealized) physical contexts in dB-B in which particle motion is indeed inertial, see (Allori et al. 2002). The situation is actually not substantially different than in classical mechanics. Even if there were no free Newtonian particles in the universe, the concept of a particle following an inertial trajectory is conceptually well defined, and it plays an important role in the foundations of the theory.

[8] The problem may be alleviated by arguing that 3-space somehow supervenes on $3n$-space, but Monton (2002) argues that this strategy does not work. The reason is that, in general, there are different ways in which the n objects in a system can evolve in 3-space that are compatible with a particular evolution of the corresponding objects in $3n$-space: there is no way to specify which dimensions in configuration space correspond to which particle. In the case of the dB-B theory, the evolution of the particles in 3-space is underdetermined by the evolution of the universal particle in $3n$-space.

[9] Norsen presents his toy-theory with a first-order equation of motion, but he does not comment on the possibility of a second-order dynamics. We explore this possibility here.

[10] To be fair, Belousek is aware of this problem, and because of it, he evaluates his interpretation as a provisional approach: "On this view quantum forces would not even have their origin in the quantum state itself, for it is just the interpretation of the quantum state as representing an entity subsisting in its own right that is being denied here. Instead, forces would simply exist on their own in addition to particles, and actual entities of both sorts would exist only in 3-dimensional space. One would have, then, a genuine dualistic ontology—equiprimordial particles and forces. Of course, one is left here without an account of the origin of such forces [...]. So, because it is not completely satisfactorily 'intuitive', one might well regard the causal view proposed here as provisional [...], awaiting a better physical interpretation of the quantum potential" (Belousek 2003, 163).

[11] I refer to 'Valentini's approach' to respect the authorship of the Aristotelian forces proposal. However, the following criticisms hold also for the first-order versions of Norsen's and Belousek's interpretations. That is, the problems I point out hold regardless of whether the Aristotelian forces are exerted by a quantum field in configuration space (Valentini), by a conditional field in 3-space (Norsen), or by nothing at all (Belousek).

[12] Moreover, the difference between real and inertial forces is *not* a mere assumption or a consensual convention. The reason why fictitious forces are taken as unreal is dynamical: real forces are essentially connected to *interactions* resulting in motion, whereas in the case of fictitious forces there is no such interaction. The Machian strategy that Valentini mentions to conceive fictitious forces as real is not convincing. Just like in Mach's principle, we would demand for a complete description of how the fictitious forces are generated by acceleration with respect to distant matter.

[13] To be fair, in the relativistic version of dB-B theory this problem gets softened. In a Minkowskian setup, the non-locality of the theory introduces causal correlations between spacelike separated events in the case of entangled subsystems. Furthermore, the guidance equations of the field-version of the theory are not Lorentz invariant. Thus, it seems that a privileged hyperplane of simultaneity is dynamically suggested, a hyperplane that in turn picks a privileged frame. Anyhow, the dynamic grounds to introduce a preferred frame in dB-B theory are connected to non-locality and to the failure of Lorentz invariance, not to Aristotelian forces and a state of absolute rest.

[14] Dürr, Goldstein, and Zanghì (1992) explain the meaning of $\Psi^{\perp}$ in the following way. In standard quantum mechanics, given a system that has been measured by an apparatus, the composite wavefunction is of the form $\sum_{\alpha} \psi_{\alpha} \otimes \Phi_{\alpha}$, where the different Φ_{α} are the possible experiment outcomes, as given by apparatus pointer positions, for example. In dB-B theory, only one of those α, say, α_0, is selected—depending deterministically on the initial configuration of the particle(s). To emphasize this feature of the theory, we can write the post-measurement composite state as $\psi \otimes \Phi + \Psi^{\perp}$, where $\psi = \psi_{\alpha_0}$, $\Phi = \Phi_{\alpha_0}$, and $\Psi^{\perp} = \sum_{\alpha \neq \alpha_0} \psi_{\alpha} \otimes \Phi_{\alpha}$. In simple words, $\Psi^{\perp}$ represents the 'empty' zones of the wavefunction of the composite system.

The effective wavefunction is equivalent to Holland's notion of effective factorization (Holland 1993, 287-289). Let us say that a wavefunction $\Psi = \Psi(x, y)$ is *strictly* factorizable if and only if $\Psi = \psi(x)\Phi(y)$—so that the subsystem $\psi(x)$ strictly factorizes (it gets dynamically isolated) from the rest of the universe. Strict factorizability is a highly unrealistic assumption, for interaction between particles in the subsystems typically result in entanglement correlations. However, if $\psi(x)$ observes the condition expressed in Eq. (21) and if $Y_t \in \operatorname{supp} \Phi_t$, the x-subsystem *effectively* factorizes from the rest of the universe. Now, decoherence processes determining the interactions between a subsystem and its environment typically result in effective factorization. That is, for all practical purposes at least, the entanglement correlations between the x-subsystem and its environment are dynamically idle—and thus, whenever it exists, the effective wavefunction of a subsystem evolves according to the Schrödinger equation.

[15] The difficulties with a fully nomological reading that Goldstein & Zanghì refer to are that ψ evolves over time, and that it is experimentally controllable. The first issue is troubling because laws are not supposed to change over time according to another dynamical law—in this case, according to the Schrödinger equation. This problem affects the universal wavefunction as well, and Goldstein and Zanghì (2013, 268-270) propose a tentative argument to face it—see also Solé (2013,

372). That ψ is experimentally controllable is worrisome because the form of a physical law is not supposed to be under our control, but we can prepare a quantum system with a specific ψ. For the sake of the argument, we can dodge these problems and accept the view that the effective wavefunction is a quasi-nomological term.
The nomological interpretation of the wavefunction in dB-B theory can adopt different forms depending on the particular conception of the laws of nature that is assumed (Humeanism, universalism, primitivism, etc.). However, those differences are not relevant for our subject. The important point here is that none of the different accounts of laws of nature states that a law is an entity in the physical world that dynamically interacts with other physical systems. This view that laws themselves are not elements of concrete physical reality holds also for the effective wavefunction, regardless of the metaphysics and epistemology of laws of nature one may assume.

[16] I thank Michael Esfeld (private communication) for this remark.

[17] Notice that this view does not consider the force itself as a beable, the beables are the object that exerts the force and the body on which it is exerted. A force is the measure of the interaction between these beables resulting in a change of state of motion.

[18] Pitowski did not present his proposal as a reformulation of the theory, but as a new generally covariant theory written in the spirit of Bohm's approach: "The idea is as following: for each quantum state ψ, we absorb the effects of the "quantum potential" associated with ψ into the metric g, while, at the same time, we demand that ψ satisfies a covariant equation with respect to that same metric. In that way ψ and g are coupled in (essentially) 11 partial differential equations in 11 unknowns" (Pitowski 1991, 343-344). Pitowski does not address the question of whether the proposed theory is predictively equivalent or not to standard quantum mechanics and to dB-B theory, neither its formal and conceptual connection with general relativity.

[19] Notice that in the nomological first-order interpretation of dB-B theory the concept of energy plays no role either. But then one wonders about the meaning of the term $V(q,t)$ in the Schrödinger equation—it can hardly represent a potential in this interpretation. Perhaps it has to be understood in quasi-nomological terms as well, but Dürr, Goldstein and Zanghì do not address this worry.

[20] A worry concerning Suárez's maneuver is that we could define a third-order dispositional property by differentiating Eq. (4) with respect to time, a property that would come to dynamically explain Ξ and its evolution—and we could then define a fourth-order property, and so on, (perhaps until the derivative is 0). True that such a maneuver would be against formal economy, but so is to include Eq. (4) as a part of the formalism. Besides, Suárez states that the justification of including a second-order dispositional property is explanatory, for the property expressed by U in Eq. (4) explains Φ, but the putative higher-order properties would in turn explain the lower-order ones.

[21] And just as in the case of the second-order nomological interpretation, the maneuver of *stipulating* that both terms must vanish would not do the trick. Even in a dispositional reading of the wavefunction *and* of the classical potential, that V and U are epistemically on a par (dispositional terms) does not mean that they have the same dynamical significance ($-\nabla V$ involves an interaction, $-\nabla U$ does not). Furthermore, in the nomological approach we saw that it was possible to directly reify $-\nabla U$, interpreting it as the expression of a primitive quantum force *à la* Belousek (on the pain of sacrificing the basic motivation of the approach); but such a maneuver does not make sense in the dispositional framework. If the dispositional property already determines the motion (no interaction present), why would we add a primitive quantum force?

References

Acuña, P. (2014). On the Empirical Equivalence between Special Relativity and Lorentz's Ether Theory. *Studies in History and Philosophy of Modern Physics, 46*, 283-302.

Allori, V., Dürr, D., Goldstein, S, & Zanghì, N. (2002). Seven Steps towards the Classical World. *Journal of Optics B, 4*, S482-S488.

Anandan, J., & Brown, H. (1995). On the Reality of Space-Time Geometry and the Wavefunction. *Foundations of Physics, 25*, 349-360.

Bacciagaluppi, G., & Valentini, A. (2009). *Quantum Theory at the Crossroads: reconsidering the 1927 Solvay Conference.* Cambridge University Press.

Belot, G. (2012). Quantum States for Primitive Ontologists. *European Journal for Philosophy of Science, 2*, 67-83.

Belousek, D. (2003). Formalism, Ontology and Methodology in Bohmian Mechanics. *Foundations of Science, 8*, 109-172.

Bohm, D. (1952). A suggested interpretation of the quantum theory in terms of “hidden” variables I-II. *Physical Review, 85*, 166-193.

Brown, H., Elby, A., & Weingard, R. (1996). Cause and Effect in the Pilot-Wave Interpretation of Quantum Mechanics. In J. Cushing, A. Fine, & S. Goldstein, *Bohmian Mechanics and Quantum Theory: an appraisal* (pp. 309-320). Springer.

Cushing, J. (1994). *Quantum Mechanics: historical contingency and the Copehagen hegemony.* Chicago: University of Chicago Press.

de Broglie, L., & Brillouin, L. (1928). *Selected Papers on Wave Mechanics.* Blackie & Sons.

Dickson, M. (2000). Are There Material Objects in Bohm's Theory? *Philosophy of Science, 67*, 704-710.

Dürr, D., & Teufel, S. (2009). *Bohmian Mechanics: the physics and mathematics of quantum theory.* Springer.

Dürr, D., Goldstein, S., & Zanghì, N. (1992). Quantum Equilibrium and the Origin of Absolute Uncertainty. *Journal of Statistical Physics, 62*, 843-907.

Dürr, D., Goldstein, S., & Zanghi, N. (1996). Bohmnian Mechanics as the Foundations of Quantum Mechanics. In J. Cushing, A. Fine, & S. Goldstein, *Bohmian Mechanics and Quantum Theory: an appraisal* (pp. 21-44). Springer.

Dürr, D., Goldstein, S., & Zanghi, N. (1997). Bohmian Mechanics and the Meaning of the Wave Function. In R. Cohen, M. Horne, & J. Stachel, *Experimental Metaphysics: quantum mechanical studies for Abner Shimony* (pp. 25-38). Kluwer Academic Publisher.

Esfeld, M., Hubert, M., Lazarovici, M., & Dürr, D. (2014). The Ontology of Bohmian Mechanics. *The British Journal for the Philosophy of Science, 65*, 773-796.

Goldstein, S., & Zanghì, N. (2013). Reality and the Role of the Wave Function in Quantum Theory. In D. Dürr, S. Goldstein, & N. Zanghi, *Quantum Physics without Quantum Philosophy* (pp. 263-278). Heidelberg: Springer.

Holland, P. (1993). *The Quantum Theory of Motion.* Cambridge University Press.

Janssen, M. (1995). *A Comparison between Lorentz's Ether Theory and Special relativity in the Light of the Experiments of Trouton and Noble.* Dissertation.

Monton, B. (2002). Wave Function Ontology. *Synthese, 130*, 265-277.

Norsen, T. (2010). The Theory of (Exclusively) Local Beables. *Foundations of Physics, 40*, 1858-1884.

Norsen, T., Marian, D., & Oriols, X. (2015). Can the Wave Function in Configuration Space Be Replaced by Single-Particle Wave Functions in Physical Space? *Synthese, 192*, doi:10.1007/s11229-014-0577-0.

Pitowski, I. (1991). Bohm's Quantum Potential and Quantum Gravity. *Foundations of Physics, 21*, 343-352.

Skow, B. (2010). On a Symmetry Argument for the Guidance Equation in Bohmian Mechanics. *International Studies in the Philosophy of Science, 24*, 393-410.

Solé, A. (2013). Bohmian Mechanics without Wave Function Ontology. *Studies in History and Philosophy of Modern Physics, 44*, 365-378.

Suárez, M. (2015). Bohmian Dispositions. *Synthese, 192*, doi:10.1007/s11229-015-0741-1.

Valentini, A. (1992). *On the Pilot-Wave Theory of Classical, Quantum, and Subquantum Physics.* Dissertation.

Valentini, A. (1996). Pilot-Wave Theory of Fields, Gravitation and Cosmology. En J. Cushing, A. Fine, & S. Goldstein, *Bohmian Mechanics and Quantum Theory: an appraisal* (págs. 45-66). Springer.

Valentini, A. (1997). On Galilean and Lorentz Invariance in Pilot-Wave Dynamics. *Physics Letters A, 228*, 215-222.

Weatherall, J. O. (2011). On the Status of the Geodesic Principle in Newtonian and Relativistic Physics. *Studies in History and Philosophy of Modern Physics, 42*, 276-281.

Weatherall, J. O. (2011). The Motion of a Body in Newtonian Theories. *Journal of Mathematical Physics, 52*, 032502.