A No-Go Result on Common Cause Approaches via Hardy's Paradox

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Abstract

According to a conventional view, there exists no common-cause model of quantum correlations satisfying locality requirements. In fact, Bell's inequality is derived from some locality conditions and the assumption that the common cause exists, and the violation of the inequality has been experimentally verified. On the other hand, some researchers argue that in the derivation of the inequality the existence of a common common-cause for multiple correlations is implicitly assumed and that the assumption is unreasonably strong. According to their idea, what is necessary for explaining the quantum correlation is a common cause for each correlation. However, in this paper, we will show that in *almost all entangled states* we can not construct a local model that is consistent with quantum mechanical prediction even when we require only the existence of a common cause of each correlation.

1 Introduction

A quantum correlation is a correlation between measurement results for each particle of a coupled system composed of quantum mechanical objects (e.g. electrons). As is well known, the correlation can occur in two spatially separated regions. If, as in an orthodoxy, we do not acknowledge the existence of causal connections between events in such regions, we can not think that one event causes the other. Then it is a natural idea that there exists a common cause of the two correlated events.

In usual discussions concerning the question of whether there is a common cause of a quantum correlation, as a result of the mathematical arguments, it is concluded that there exists no common-cause model consistent with quantum mechanical predictions. However, some researchers argue that a disputable requirement is tacitly assumed in such discussions. The requirement is that there exists a common common-cause of multiple quantum correlations. According to their idea what is necessary for explaining quantum correlations is a common cause for each correlation. In this paper, we discuss the possibility and impossibility of constructing a common cause model explaining quantum mechanical correlations.

There is no doubt about the importance of this theme. If we can construct a common cause model that satisfies locality requirements, we will have new possibilities to have a picture of how the quantum mechanical object behaves behind phenomena. On the other hand, if such a model does not exist, we must explore another way to understand quantum world.

2 Backgrounds

2.1 Surface Probability Space

In this subsection, we introduce a classical probability space consisting of (i) propositions representing measurement apparatus settings and measured effects and (ii) a probability measure on them. Van Fraassen (1982) called the phenomenological probabilities as surface probabilities. We use his terminology and notation.

Suppose spin measurements are performed for a combined system of spin 1/2 two particles (particle L and particle R). And suppose that by the measuring apparatus for particle L (apparatus L) one of two specific incompatible spin observables will be measured. In other words, apparatus L has two measurement settings L_i $(i = 1, 2)$. Likewise, apparatus R has two measurement settings R_j $(j = 1, 2)$. For simplicity, it is assumed that in each experiment either one of the two spin observables is necessarily measured for each of the particles L and R. In short, the apparatuses L and R have the following 4 possible measurement settings:

$$\{L_i \wedge R_j : \ i, j = 1, 2\}.$$

Here also, for simplicity, we will assume that in any measurement either measured value + or − can always be obtained without errors. Then in each measurement setting $L_i \wedge R_j$, there are the 4 possible results $L_{ia} \wedge R_{jb}$ $(a = +, -; b = +, -)$. The set of all possible measurement outcomes is as follows:

$$\{L_{ia} \wedge R_{jb} : i, j = 1, 2; \ \ a, b = +, -\}.$$

It has 16 members. We can construct Boolen algebra having these 16 measurement results as atoms. This algebra represents propositional structure concerning measurements.

We can define a probability measure on the Boolean algebra in the following way. Let $Q(L_{ia} \wedge R_{jb})$ stand for the probabilistic prediction given by quantum mechanics to a measurement result $L_{ia} \wedge R_{jb}$. $Q(L_{ia} \wedge R_{jb}) = Tr(D(P_{L_{ia}} \otimes P_{R_{jb}}))$, where D denotes the density operator of the measured system and $P_{L_{ia}}$ $(P_{R_{jb}})$ denotes the projection operator corresponding to the measured effect L_{ia} (R_{jb}). And let $\mu(L_i \wedge R_j)$ denote a rate at which the apparatuses L and R take a measurement setting $L_i \wedge R_j$. As already mentioned, we suppose that either one of the two spin observables is necessarily measured for each of the particles L and R. Then, $\sum_{i,j} \mu(L_i \wedge R_j) = 1$. For each atom of the Boolean algebra, we define the probability as follows:

$$P(L_{ia} \wedge R_{jb}) \equiv Q(L_{ia} \wedge R_{jb}) \ \mu(L_i \wedge R_j). \tag{1}$$

As can be easily seen, the sum of the probabilities assigned to the 16 atoms is 1. We can extend this definition to the whole algebra to satisfy additivity. We call the classical probability space constructed in this way *surface probability space*.

In the surface probability space, as you can see from how to construct the probability space,

$$L_i \wedge R_j = (L_{i+} \wedge R_{j+}) \vee (L_{i+} \wedge R_{j-}) \vee (L_{i-} \wedge R_{j+}) \vee (L_{i-} \wedge R_{j-}).$$

Thus, the probabilities of both sides are equal. Then, from the definition (1) and the additivity of the probabilities, the following equation holds.

$$P(L_i \wedge R_j) = \mu(L_i \wedge R_j). \tag{2}$$

Clearly, from the structure of the propositions in the surface probability space, $L_{ia} \wedge R_{jb} \wedge L_i \wedge R_j = L_{ia} \wedge R_{jb}$. Using this fact, (1), and (2), we can see that the following equations hold for quantum mechanical probability $Q(L_{ia} \wedge R_{jb}) = Tr(D(P_{L_{ia}} \otimes P_{R_{jb}}))$, $Q(L_{ia}) = Tr(D(P_{L_{ia}} \otimes I))$ and $Q(R_{jb}) = Tr(D(I \otimes P_{R_{jb}}))$.

$$\begin{aligned} Q(L_{ia} \wedge R_{jb}) &= P(L_{ia} \wedge R_{jb} | L_i \wedge R_j), \\ Q(L_{ia}) &= P(L_{ia} | L_i \wedge R_j), \\ Q(R_{jb}) &= P(R_{jb} | L_i \wedge R_j). \end{aligned}$$

In this way, in the surface probability space, the quantum mechanical probabilities are represented as the conditional probabilities given each measurement setting.

From the above explanation, it is obvious that the following fact holds.

Fact 1. *Suppose spin measurements are performed for a combined system of spin 1/2 two particles, and that the apparatuses L and R have two measurement settings respectively. Then for any spin state, we can construct the surface probability space.*

Here, we supposed that each apparatus has two measurement settings, but even if the number of measurement settings is any finite, we can construct the surface probability space in a similar way.

2.2 Conditions Used to Derive Bell's Inequality

In the surface probability space, a quantum mechanical correlation is represented as

$$\begin{aligned} Corr(L_{ia}, R_{jb}) &\equiv Q(L_{ia} \wedge R_{jb}) - Q(L_{ia})Q(R_{jb}) \\ &= P(L_{ia} \wedge R_{jb} | L_i \wedge R_j) - P(L_{ia} | L_i \wedge R_j) P(R_{jb} | L_i \wedge R_j). \end{aligned} \tag{3}$$

If $Corr(L_{ia}, R_{jb}) \neq 0$, then we say that L_{ia} and R_{jb} are correlated. When a two-particles system is in an entangled state, a quantum mechanical correlation occurs by choosing spin components appropriately. Since a quantum mechanical correlation can occur in spatially separated regions, it is unlikely that direct causal connection exists between the correlated events. In such case, it is natural to think that there is a common cause for the two correlated events and that the correlation is caused by the occurrence of the event. However, with respect to attempts to explain quantum mechanical correlations by a common causal explanation, the following mathematical result is well known (e.g. Bub 1997, and Redhead 1987).

Fact 2. *Let $\{C_k\}_{k\in K}$ be a family of events which satisfy (i) if $i \neq j$ $(i, j \in K)$, then $C_i \wedge C_j = \emptyset$, (ii) $P(C_k) \neq 0$ for any $k \in K$, and (iii) $\sum_k P(C_k) = 1$. Assuming an event belonging to $\{C_k\}_{k\in K}$ satisfies the following conditions, Bell's inequality is derived* [1].

A1 $P(L_{ia} \wedge R_{jb}|L_i \wedge R_j \wedge C_k) = P(L_{ia}|L_i \wedge R_j \wedge C_k)P(R_{jb}|L_i \wedge R_j \wedge C_k)$.

A2.1 $P(L_{ia}|L_i \wedge R_j \wedge C_k) = P(L_{ia}|L_i \wedge C_k)$.

A2.2 $P(R_{jb}|L_i \wedge R_j \wedge C_k) = P(R_{jb}|R_j \wedge C_k)$.

A3 $P(C_i|L_i \wedge R_j) = P(C_k)$.

The reasons for imposing the respective conditions are as follows. If $\{C_k\}_{k\in K}$ is a set of events satisfying condition A1, the quantum mechanical correlation disappears when each event belonging to the set occurs. Therefore, the events are considered as a candidate for the cause of the correlation.

Condition A2.1 is equivalent to the following condition: $P(L_{ia}|L_i \wedge R_l \wedge C_k) = P(L_{ia}|L_i \wedge R_m \wedge C_k)$ $(l \neq m)$. This condition says, once the cause of the correlation occurs, the probability of the measurement result of particle L will not change even if the setting of apparatus R is changed (just before measurement). The same is true for the condition A2.2.

A3 only requires that *each event of* $\{C_k\}_{k\in K}$ is statistically independent of the setting of the measuring device. However actually, when this condition is satisfied, any event constructed from $\{C_k\}_{k\in K}$ (e.g. $C_i \wedge C_j, C_i^{\perp}$) is statistically independent of the setting of the measuring device. Since C_k is a candidate of a common cause of a correlation between measurement results that can occur in two spatially separated regions, it is natural to think that C_k occurs in the intersection of backward light cones of each spatiotemporal region. Considering that we can change the device settings even after C_k occurred, it is reasonable to require the condition A3.

Bell's inequality has been verified experimentally and its violation has been confirmed (e.g. Aspect, Dalibard, and Roger 1982). When the inequality is violated, there exists no mathematical model that satisfies all the above requirements.

[1] In this paper, we discuss only when $\{C_k\}_{k\in K}$ is a countable set. However, if the notation is slightly changed, the same conclusion is obtained even when the parameters are continuous.

2.3 Szabó's Common Cause Model and a No-Go Result

Hofer-Szabó, Rédei, and Szabó (1999) pointed out that the derivation of Bell's inequality implicitly presupposes the existence of a **common** common-cause for multiple correlations. Originally, the principle of the common cause is merely requiring the existence of a common cause for each correlation (see Reichenbach 1956). Nevertheless, in the derivation of Bell's inequality, the same family of events $\{C_k\}_{k\in K}$ is considered as common cause events of multiple correlations. According to their idea, it is sufficient that each quantum correlation $Corr(L_{ia}, R_{jb}) \neq 0$ has respectively common cause events $\{C_k^{ij}\}_{k\in K}$.

Actually, Szabó (1998, 2000) constructed a mathematical model that contains the common cause of each correlation and also satisfies some conditions related to locality. The following fact holds.

Fact 3. *There exists an extension*[2] *of the surface probability space which includes a family of events $\{C_k^{ij}\}_{k\in K}$ $\quad (i,j=1,2)$ which satisfy the below conditions (B1-3)*[3]*. (Here, as in Fact 2, $\{C_k^{ij}\}_{k\in K}$ is a family of events which satisfy (i) if $l \neq m$, then $C_l^{ij} \wedge C_m^{ij} = \emptyset$, (ii) $P(C_k^{ij}) \neq 0$ for any $k \in K$, and (iii) $\sum_k P(C_k^{ij}) = 1$.)*

B1 $P(L_{ia} \wedge R_{jb} | L_i \wedge R_j \wedge C_k^{ij}) = P(L_{ia} | L_i \wedge R_j \wedge C_k^{ij}) P(R_{jb} | L_i \wedge R_j \wedge C_k^{ij})$

B2.1 $P(L_{ia} | L_i \wedge R_j \wedge C_k^{ij}) = P(L_{ia} | L_i \wedge C_k^{ij})$

B2.2 $P(R_{jb} | L_i \wedge R_j \wedge C_k^{ij}) = P(R_{jb} | R_j \wedge C_k^{ij})$

B3 $P(C_k^{ij} | L_i \wedge R_j) = P(C_k^{ij})$

Although Szabó's model satisfies B1-3, it is still questionable whether this model is truly local. Actually, as he himself pointed out (see "postscript" in Szabó 1998), his model does not satisfy the following stronger condition than B3.

$\mathcal{C}$-independence Let $\mathcal{C}$ be the Boolean subalgebra generated by $\{C_k^{11}\}_{k\in K} \cup \{C_l^{12}\}_{l\in L} \cup \{C_m^{21}\}_{m\in M} \cup \{C_n^{22}\}_{n\in N}$. Then, for any atom Z of $\mathcal{C}$,

$$P(Z | L_i \wedge R_j) = P(Z) \qquad (i,j=1,2).$$

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[2]The classical probability space $(\mathcal{B}', P')$ (where $\mathcal{B}'$ a Boolean algebra and P' is a probability measure on it) is called an extension of $(\mathcal{B}, P)$ (where $\mathcal{B}$ a Boolean algebra and P is a probability measure on it), if there exists a Boolean homomorphism $h : \mathcal{B} \to \mathcal{B}'$ such that $P'(h(X)) = P(X)$ for all $X \in \mathcal{B}$.

[3]B1-3 are different from the conditions Szabó (1998, 2000) explicitly stated in his paper. However, as can be seen from (11) and the definition of probability (25) in his paper, his model satisfies the following four conditions: (a) $P(L_i \wedge R_j) = P(L_i)P(R_j)$, (b) $P(C_k^{ij} \wedge L_i \wedge R_j) = P(C_k^{ij})P(L_i \wedge R_j)$, (c) $P(L_{ia} \wedge R_j \wedge C_k^{ij}) = P(L_{ia} \wedge C_k^{ij})P(R_j)$; $P(L_i \wedge R_{ib} \wedge C_k^{ij}) = P(R_{jb} \wedge C_k^{ij})P(L_i)$, (d) $P(L_{ia} \wedge R_{jb} | C_k^{ij}) = P(L_{ia} | C_k^{ij}) P(R_{jb} | C_k^{ij})$. Then, we can easily show that his model satisfies B1-3.

The reason for imposing this condition is as follows. As mentioned above, it is natural to think that a common-cause event occurs in the intersectional region of the past light cones of the two spatially separated regions where measurements are made. Also, for each correlation $Corr(L_{ia}, R_{jb})$, an event belonging to $\{C_k^{ij}\}_{k \in K}$ must occur because $\sum_k P(C_k^{ij}) = 1$. Note that is true for any correlations. Thus, any one of the atoms of $\mathcal{C}$ occurs in the intersectional region of the past light cones. However, even after such an event occurred, we can change the setting of the measuring apparatus. Therefore, $\mathcal{C}$-independence is a requirement to be satisfied.

Although Szabó's model does not satisfy $\mathcal{C}$-independence, there may exist other models that satisfy all the conditions. However, it is already known that in the situation where three pairs of the same spin components are measurable for a two-particles system in the singlet state, there exists no mathematical model which satisfies all the conditions (Graßhoff, Portmann, and Wüthrich 2005).

Fact 4. *Suppose the followings:*

- *two spin-1/2 particles system is in the singlet spin state,*
- *three incompatible spin components are measurable for a particle on each side and those measurable spin components are the same for the particle L and the particle R.*

Then, assuming B1, B2, $\mathcal{C}$-independence, Bell's inequality is derived.

As with common common-causes, when Bell's inequality is violated, there exists no mathematical model which satisfies all of the above demands and is consistent with quantum mechanical predictions.

3 A No-Go Result for Almost All Entangled State

In this section, we will show that for almost all entangled states, there exists no mathematical model which satisfies B1, B2, and $\mathcal{C}$-independence and is consistent with quantum mechanical predictions. We will use Hardy's famous argument (Hardy 1993).

3.1 Hardy's Argument

First, let us confirm the famous mathematical fact (Schmidt decomposition[4]) which holds for any pure state on the tensor product space of two finite dimensional Hilbert spaces. However, since the following discussion uses only the tensor product space of 2-dimensional Hilbert space, we describe the mathematical fact only for 2-dimensional case.

[4]For proof, see Nielsen and Chuang (2000).

Schmidt decomposition Let $\mathcal{H}_L^2$ and $\mathcal{H}_R^2$ be the spin state (2-dimensional) spaces of the particle L and the particle R, respectively. Suppose $|\Phi\rangle$ is a pure state of the composite system (particles L and R). Then, there exists an orthonormal basis $\{|+\rangle_L, |-\rangle_L\}$ for $\mathcal{H}_L^2$, and an orthonormal basis $\{|+\rangle_R, |-\rangle_R\}$ for $\mathcal{H}_R^2$ such that

$$|\Phi\rangle = \alpha\,|+\rangle_L \otimes |+\rangle_R + \beta\,|-\rangle_L \otimes |-\rangle_R\,,$$

where α, β are non-negative real numbers satisfying $\alpha^2 + \beta^2 = 1$ known as *Schmidt coefficients*.

Next, we define the state vector $|a_L\rangle, |b\rangle_L$ on $\mathcal{H}_L^2$ by using Schmidt coefficients as follows:

$$|a\rangle_L \equiv \frac{1}{\sqrt{(1-\alpha\beta)(\alpha+\beta)}}(\beta\sqrt{\beta}\,|+\rangle_L - \alpha\sqrt{\alpha}i\,|-\rangle_L),$$

$$|b\rangle_L \equiv \sqrt{\frac{\beta}{\alpha+\beta}}\,|+\rangle_L + \sqrt{\frac{\alpha}{\alpha+\beta}}i\,|-\rangle_L\,.$$

And, using $|a_L\rangle$ and $|b\rangle_L$, we define the projection operators A_L and B_L on $\mathcal{H}_L^2$ as follows:

$$A_L \equiv |a\rangle_L \langle a|_L\,,\ B_L \equiv |b\rangle_L \langle b|_L\,.$$

We define the projection operators A_R and B_R on $\mathcal{H}_R^2$ in the same way.

Suppose a two-particles system is in a pure state $|\Psi\rangle$ such that for Schmidt coefficients α, β, (I) $\alpha, \beta \neq 0$ (i.e. not a product state) and (II) $\alpha \neq \beta$ (i.e. not the singlet state). Then, The quantum mechanical probability for A_L, B_L, A_R, B_R satisfies the following four relations:

C1 $Q^{|\Psi\rangle}(A_L \wedge A_R) = \frac{\alpha^2\beta^2(\alpha-\beta)^2}{(1-\alpha\beta)^2} > 0$,

C2 $Q^{|\Psi\rangle}(B_R|A_L) = 1$,

C3 $Q^{|\Psi\rangle}(B_L|A_R) = 1$,

C4 $Q^{|\Psi\rangle}(B_L \wedge B_R) = 0$.

Hardy's argument goes as follows. Suppose $A_L \wedge A_R$ is true (this is possible by C1). Then, using C2 and C3, $B_L \wedge B_R$ is true. However, this contradicts with C4. Here, note that if the state of a two-particles system is a product state or the singlet state, then the value of C1 is 0, thus Hardy's argument does not hold.

3.2 Preparation for Derivation

If the events corresponding to the four projections A_L, B_L, A_R, B_R exist in the Boolean algebra of a classical probability space, and the quantum mechanical probabilities $Q^{|\Psi\rangle}(\,\cdot\,)$ are assigned to those events respectively, then, as explained in the previous section, Hardy's argument holds.

However, in the surface probability space, quantum mechanical probabilities are represented as conditional probabilities. As mentioned in Fact 1, any measurement results on A_L, B_L, A_R, B_R are representable in the surface probability space. Indeed, as we will see, we can represent the Hardy relations C1-4 in the surface probability space.

We use the following notation:

L_1 :	the apparatus L is set to measure A_L,
L_{1+} :	the result of A_L-measurement is +1,
L_2 :	the apparatus L is set to measure B_L,
L_{2+} :	the result of B_L-measurement is +1,
R_1 :	the apparatus L is set to measure A_R,
R_{1+} :	the result of A_R-measurement is +1,
R_2 :	the apparatus L is set to measure B_R,
R_{2+} :	the result of B_R-measurement is +1.

Using this notation, C1-4 are represented as follows:

D1 $P(L_{1+} \wedge R_{1+}|L_1 \wedge R_1) = \frac{\alpha^2\beta^2(\alpha-\beta)^2}{(1-\alpha\beta)^2} > 0$,

D2 $P(R_{2+}|L_{1+} \wedge L_1 \wedge R_2) = 1$,

D3 $P(L_{2+}|R_{1+} \wedge L_2 \wedge R_1) = 1$,

D4 $P(L_{2+} \wedge R_{2+}|L_2 \wedge R_2) = 0$.

From D2 alone, we can not derive that there exists a positive correlation between L_{1+} and R_{2+}, i.e. $Corr(L_{1+}, R_{2+}) > 0$ (see (3)). However, calculating by using the already introduced state $|\Psi\rangle$ and the projection operators A_L, B_R, we have $P(R_{2+}|L_1 \wedge R_2) = \alpha\beta < 1$. From this and D2, we can derive $Corr(L_{1+}, R_{2+}) > 0$ [5]. In a similar way, (although $Corr(L_{2+}, R_{1+}) > 0$ is not derived from D3 alone,) we can derive $Corr(L_{2+}, R_{1+}) > 0$ from D3 and $P(L_{2+}|L_1 \wedge R_2) < 1$. In the followings, we will discuss the impossibility of the common-causal explanation of the positive correlations $Corr(L_{1+}, R_{2+}) > 0$ and $Corr(L_{2+}, R_{1+}) > 0$.

In the next subsection, we will show that there exists no common cause of the positive correlations ($Corr(L_{1+}, R_{2+}) > 0$ and $Corr(L_{2+}, R_{1+}) > 0$) which satisfies B1, B2, and C-independence. In order not to worry about trivial things in its derivation, we assume three things.

Assumption 1 $P(L_i \wedge R_j) > 0 \;\; (i, j = 1, 2)$

[5] In general, from $P(Y|X) = 1$ and $P(Y) < 1$, we can derive $P(X \wedge Y) - P(X)P(Y) > 0$. Define the new probability measure $\tilde{P}(\,\cdot\,) \equiv P(\,\cdot\,|L_1 \wedge R_2)$. From D2, $\tilde{P}(R_{2+}|L_{1+}) = 1$. Also, from $P(R_{2+}|L_1 \wedge R_2) < 1$, $\tilde{P}(R_{2+}) < 1$. Then, $\tilde{P}(L_{1+} \wedge R_{1+}) - \tilde{P}(L_1)\tilde{P}(R_{1+}) > 0$. Therefore, $Corr(L_{1+}, R_{2+}) > 0$.

Assumption 2 $\sum_{i,j} P(L_i \wedge R_j) = 1 \ (i, j = 1, 2)$

Furthermore, for the family of common-cause events $\{C_k^{ij}\}_{k \in K}$ of the correlation $Corr(L_{i+}, R_{j+}) > 0$, we assume the followings as in (i) - (iii) of Fact 3.

Assumption 3 **(i)** if $l \neq m \ (l, m \in K)$, then $C_l^{ij} \wedge C_m^{ij} = \emptyset$,

(ii) $P(C_k^{ij}) \neq 0$ for any $k \in K$,

(iii) $\sum_k P(C_k^{ij}) = 1$.

Also, in the derivation, we will use the following two lemmas. Since the both can be shown easily, those proofs are omitted.

Lemma 1. *Suppose* $P(Y|X) = 1$*, then for any event* Z

$$\textit{either} \quad P(Y|X \wedge Z) = 1 \quad \textit{or} \quad P(X \wedge Z) = 0.$$

Lemma 2. *Suppose* $P(Y \wedge Z) \neq 0$*, then the following two expressions are equivalent.*

- $P(X \wedge Y|Z) = P(X|Z)P(Y|Z)$.
- $P(X|Y \wedge Z) = P(X|Z)$.

3.3 Derivation of a New No-Go Result

In this subsection, we will show the following fact.

Fact 5. *For any state such that Schmidt coefficients* α, β *satisfy* $\alpha, \beta \neq 0$ *and* $\alpha \neq \beta$*, there exists no extension from the surface probability space to a classical probability space which includes common-cause events of the positive correlations (*$Corr(L_{1+}, R_{2+}) > 0$ *and* $Corr(L_{2+}, R_{1+}) > 0$*) satisfying B1, B2, and* $\mathcal{C}$*-independence.*

Let $\{C_k^{12}\}_{k \in K}$ and $\{C_l^{21}\}_{l \in L}$ be families of events that satisfy B1, B2, $\mathcal{C}$-independence, and the conditions in Assumption 3 for the positive correlations $Corr(L_{1+}, R_{2+}) > 0$ and $Corr(L_{2+}, R_{1+}) > 0$, respectively. We define the two events $C_{R_{2+}}$, $C_{L_{1-}}$ as follows:

$$C_{R_{2+}} \equiv \vee \{C_k^{12} (k \in K) : P(L_{1+} \wedge C_k^{12} \wedge L_1 \wedge R_2) \neq 0\},$$
$$C_{L_{1-}} \equiv \vee \{C_k^{12} (k \in K) : P(L_{1+} \wedge C_k^{12} \wedge L_1 \wedge R_2) = 0\}.$$

Then, from these definitions and Assumption 3,

$$C_{R_{2+}} \wedge C_{L_{1-}} = \emptyset,$$

and

$$P(C_{R_{2+}} \vee C_{L_{1-}}) = 1.$$

Therefore,

$$P(C_{R_{2+}} \wedge X) = P(\neg C_{L_{1-}} \wedge X) \quad \text{for any event } X. \tag{4}$$

Suppose that $C_k^{12} \leq C_{R_{2+}}$. Then, applying Lemma 1 to Hardy relation D2, we have

$$P(R_{2+}|L_{1+} \wedge C_k^{12} \wedge L_1 \wedge R_2) = 1.$$

By using Lemma 2 for this,

$$P(R_{2+}|C_k^{12} \wedge L_1 \wedge R_2) = 1$$

holds. Furthermore, using B2.2, we have

$$P(R_{2+}|C_k^{12} \wedge R_2) = 1.$$

From the above argument, the following relation holds.

$$\text{If } C_k^{12} \leq C_{R_{2+}}, \text{ then } P(R_{2+}|C_k^{12} \wedge R_2) = 1.$$

Then, using the definition of $C_{R_{2+}}$, we have

$$P(R_{2+}|C_{R_{2+}} \wedge R_2) = 1. \tag{5}$$

Suppose that $C_k^{12} \leq C_{L_{1-}}$. Then $P(L_{1+}|C_k^{12} \wedge L_1 \wedge R_2) = 0$ [6]. Applying B2.1 to this equation, we have

$$P(L_{1+}|C_k^{12} \wedge L_1) = 0.$$

Consequently, the following relation holds.

$$\text{If } C_k^{12} \leq C_{L_{1-}}, \text{ then } P(L_{1+}|C_k^{12} \wedge L_1) = 0.$$

Then, using the definition of $C_{L_{1-}}$, we have

$$P(L_{1+}|C_{L_{1-}} \wedge L_1) = 0. \tag{6}$$

Because $P(L_{1+} \wedge C_{L_{1-}} \wedge L_1) = 0$,

$$P(C_{L_{1-}}|L_{1+} \wedge L_1) = 0.$$

From the propositional structure in the surface probability space $L_{1+} \wedge L_1 = L_{1+}$ holds. Therefore, we have

$$P(C_{L_{1-}}|L_{1+}) = 0.$$

Rewrite this,

$$P(\neg C_{L_{1-}}|L_{1+}) = 1.$$

[6] This conditional probability is definable because $P(C_k^{12} \wedge L_1 \wedge R_2) \neq 0$ from Assumption 1, Assumption 3 (ii), and $\mathcal{C}$-independence.

Using (4), we have

$$P(C_{R_{2+}}|L_{1+}) = 1. \tag{7}$$

Similarly, we define the two events $C_{L_{2+}}$ and $C_{R_{1-}}$ as follows:

$$C_{L_{2+}} \equiv \vee\{C_l^{21}(l \in L) : P(R_{1+} \wedge C_l^{21} \wedge L_2 \wedge R_1) \neq 0\},$$

$$C_{R_{1-}} \equiv \vee\{C_l^{21}(l \in L) : P(R_{1+} \wedge C_l^{21} \wedge L_2 \wedge R_1) = 0\}.$$

In a similar way as the two previous paragraph, we have

$$P(L_{2+}|C_{L_{2+}} \wedge L_2) = 1, \tag{8}$$

$$P(C_{L_{2+}}|R_{1+}) = 1. \tag{9}$$

Finally, a contradiction is derived as follows. From (7) and (9),

$$P(C_{L_{2+}} \wedge C_{R_{2+}}|L_{1+} \wedge R_{1+}) = 1. \tag{10}$$

By Hardy relation D1, we have $P(L_{1+} \wedge R_{1+}) > 0$. Therefore, using (10),

$$P(C_{L_{2+}} \wedge C_{R_{2+}}) > 0. \tag{11}$$

Applying Lemma 1 to (5),

$$\text{either } P(R_{2+}|C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) = 1 \text{ or } P(C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) = 0. \tag{12}$$

Also, from (11), Assumption 1, and $\mathcal{C}$-independence,

$$P(C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) \neq 0. \tag{13}$$

Then, by (12) and (13), we have

$$P(R_{2+}|C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) = 1. \tag{14}$$

In the same way, from (8), we have

$$P(L_{2+}|C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) = 1. \tag{15}$$

Then, from (14) and (15), we have

$$P(L_{2+} \wedge R_{2+}|C_{L_{2+}} \wedge C_{R_{2+}} \wedge L_2 \wedge R_2) = 1.$$

From this equation and Hardy relation D4, a contradiction is derived.

From the above argument, for any entangled states which Schmidt coefficients α, β satisfy $\alpha, \beta \neq 0$ and $\alpha \neq \beta$, we can not extend the surface probability space to a classical probability space such that the common cause of each correlation satisfies B1, B2, and $\mathcal{C}$-independence.

4 Discussion

In this paper, It was shown that, in almost all entangled states, there exists no common-cause model satisfying B1, B2, and $\mathcal{C}$-independence consistent with quantum mechanical predictions. Unfortunately, at least the author thinks that B1, B2, and $\mathcal{C}$-independence are all essential conditions for satisfying locality.

The significance of the mathematical result in this paper is as follows. As is well known, there are various no-go theorems about hidden variable theories. The no-go theorem presented in this paper is a theorem applicable to almost all entangled states except specific states (e.g. the singlet state), as suggested by the title of Hardy's paper.

The no-go theorem presented in this paper was derived under weak constraints.

- Not a deterministic hidden-variable but a probabilistic hidden-variable (as van Fraassen [1982] pointed out correctly, the common cause of correlation can be understood as a probabilistic hidden-variable).
- Not common common-cause, but common cause of each correlation.
- Not a situation where 6 observables are measurable, but a situation where 4 observables are measurable (see Fact 4).

On the other hand, if we weaken $\mathcal{C}$-independence to B3, there is a common cause model including common cause of each correlation (see Fact 3). In this way, the author thinks that this theorem is a limit point to derive inconsistency with quantum mechanical predictions.

Also, Hardy's argument has been considered to indicate the impossibility of deterministic hidden-variable theories so far (e.g. Shimony (2009) and Higashi (2009)). However, by the derivation of the no-go theorem in this paper, it was revealed that Hardy's argument can also be applied to probabilistic hidden variable theories.

Nonetheless, what is shown in this paper is merely a limitation of common cause approaches in classical probability spaces. We do not discuss any common cause approach in quantum probability spaces. When considering the common cause of each correlation in a classical probability space, common-cause events are commutative. Therefore, we have to think about the truth value of the propositions such as conjunction, disjunction, and etc of them. On the other hand, in a quantum probability space, common-cause events does not need to be commutative, thus we have not to think about their truth value simultaneously. However, when considering the common cause approach in quantum probability spaces, we will face another problem of how to interpret the propositions of non-commutative observables. We will consider this problem on another occasion.

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