Large cardinals and the iterative conception of set

Neil Barton*

4 December 2017†

Abstract

The independence phenomenon in set theory, while pervasive, can be partially addressed through the use of large cardinal axioms. One idea sometimes alluded to is that maximality considerations speak in favour of large cardinal axioms consistent with **ZFC**, since it appears to be 'possible' (in some sense) to continue the hierarchy far enough to generate the relevant transfinite number. In this paper, we argue against this idea based on a priority of subset formation under the iterative conception. In particular, we argue that there are several conceptions of maximality that justify the consistency but falsity of large cardinal axioms. We argue that the arguments we provide are illuminating for the debate concerning the justification of new axioms in iteratively-founded set theory.

Introduction

Large cardinal axioms (discussed in detail below) are widely viewed as some of the best candidates for new axioms of set theory. They are (apparently) linearly ordered by consistency strength, have substantial mathematical consequences for independence results (both as a tool for generating new models and for consequences within the models in which they reside), and often appear natural to the working set theorist, providing fine-grained information about different properties of transfinite sets. They are considered mathematically interesting and central for the study of set theory and its philosophy.

In this paper, we do not deny any of the above views. We will, however, argue that the status of large cardinal axioms as *maximality* principles is questionable. In particular, we will argue that there are conceptions of maximality in set theory on which large cardinal axioms are viewed as *minimising* principles that serve to restrict the subsets formed under the iterative conception of set.

Our strategy is as follows: We first (§1) explain how large cardinals have been seen to be related to the iterative conception of set, and how they might be viewed as maximality principles. Specifically, we will canvass the idea that large cardinal axioms assert that the stages in the iterative conception go as far as a certain ordinal, and so assuming that the iterative conception is maximal and has a stage for every

*Kurt Gödel Research Center for Mathematical Logic (KGRC), Währinger Straße, 25, 1090, Vienna, Austria. E-mail: neil.barton@univie.ac.at.

†The author wishes to thank Monroe Eskew, Sy Friedman, Luca Incurvati, Michael Potter, Chris Scambler, Matteo Viale, and an audience in Cambridge for helpful discussion. They are also very grateful for the generous support of the FWF (Austrian Science Fund) through Project P 28420 (*The Hyperuniverse Programme*).

possible ordinal, we should accept the truth of any consistent large cardinal axiom. We then (§2) argue that the iterative conception of set advocates a conceptual priority of width over height, and present three troublesome cases for the idea that consistency of a large cardinal axiom entails its truth. In particular, we show how certain conceptions of a 'rich' process of subset formation can preclude the existence of large cardinals in the universe, despite their consistency. On this picture, we argue, large cardinals serve as *minimising* principles rather than maximising principles. Next, (§3) we consider the roles played by large cardinals in contemporary set theory, and argue that even in the anti-large cardinal framework, large cardinal axioms can still play their usual foundational roles of indexing consistency strength and justifying axioms of definable determinacy. We consider (§4) the question of whether theories incorporating anti-large cardinal principles are restrictive, and argue that some of the theories we have considered perform well with respect to Maddy's notion of restrictiveness (and slight variations thereof). Finally (§5) we identify an open question concerning how to move forward with the debate and make some concluding remarks.

1 Large cardinals and the iterative conception of set

In this section, we provide some required background on large cardinals and the iterative concept of set.[1] We then explain how one might think that the iterative conception legislates in favour of large cardinals on the basis of their status as maximality principles.

1.1 Large cardinals

Given a set theory capable of axiomatising a reasonable fragment of arithmetic (i.e. able to support the coding of the relevant syntactic notions), we start our discussion with the following celebrated theorem:

Theorem 1. [Gödel, 1931] (Second Incompleteness Theorem). No consistent[2] recursive theory capable of axiomatising primitive recursive arithmetic is can prove its own consistency sentence[3] (often denoted by '$Con(\mathbf{S})$').

Given then some appropriately strong set theory $\mathbf{S}$, we can then obtain a *strictly stronger* theory by adding $Con(\mathbf{S})$ to $\mathbf{S}$.[4] So, if we accept the standard axioms of Zermelo-Fraenkel set theory with Choice (henceforth '$\mathbf{ZFC}$') then, $\mathbf{ZFC}+Con(\mathbf{ZFC})$ is a strictly stronger theory, and $\mathbf{ZFC} + Con(\mathbf{ZFC} + Con(\mathbf{ZFC}))$ is strictly stronger still. More generally:

[1]One might feel that this section covers well-known ground. We include it simply for clarity and because our main point is rather philosophical in nature: The place of large cardinals in the iterative conception requires further sharpening of how sets are formed in the hierarchy. For this reason, we hope that the philosophical claims of the paper will be readable and open to scrutiny by a relatively wide audience, even if some of the technical details are somewhat tricky in places. Time-pressed readers are invited to proceed directly to §1.3.

[2]Strictly speaking, this is Rosser's strengthening, but we suppress the usual discussion of ω-inconsistency for clarity.

[3]The consistency sentence for a theory $\mathbf{T}$ is a sentence in the language of $\mathbf{T}$ that states that there is no code of a proof of $0 = 1$ (or some other suitable contradiction) in $\mathbf{T}$.

[4]Of course, $\mathbf{S} + Con(\mathbf{S})$ might itself be inconsistent, even if $\mathbf{S}$ is consistent [for example when $\mathbf{S} = \mathbf{PA} + \neg Con(\mathbf{PA})$]. Many thanks to for pointing this out to me.

Definition 2. A theory $\mathbf{T}$ has *greater consistency strength* than $\mathbf{S}$ if $Con(\mathbf{T}) \Rightarrow Con(\mathbf{S})$, but $Con(\mathbf{S}) \not\Rightarrow Con(\mathbf{T})$. They are called *equiconsistent* iff $Con(\mathbf{T}) \Leftrightarrow Con(\mathbf{S})$.[5]

The interesting fact for current purposes is that in set theory we are not limited to increasing consistency strength solely through adding Gödel-style diagonal sentences. The axiom which asserts the existence of a transitive model of $\mathbf{ZFC}$ is stronger still (such an axiom implies the consistency of theories with transfinite iterations of the consistency sentence for $\mathbf{ZFC}$). As it turns out, by postulating the existence of certain kinds of models, embeddings, and varieties of sets, we discover theories with greater consistency strength. For example:

Definition 3. A cardinal κ is *strongly inaccessible* iff it is uncountable, regular (i.e. there is no function from a smaller cardinal unbounded in κ), and a strong limit cardinal (i.e. if $|x| < \kappa$ then $|\mathcal{P}(x)| < \kappa$).

Such an axiom provides a model for *second-order* $\mathbf{ZFC}_2$ [namely $(V_\kappa, \in, V_{\kappa+1})$]. These cardinals represent the first steps on an enormous hierarchy of logically and combinatorially characterised objects.[6] More generally, we have the following rough idea: A large cardinal axiom is a principle that serves as a natural stepping stone in the indexing of consistency strength.

In the case of inaccessibles, many of the logical properties attaching to the cardinal appear to derive from its brute size. For example, it is because of the fact that such a κ cannot be reached 'from below' by either of the axioms of Replacement or Powerset that $(V_\kappa, \in, V_{\kappa+1})$ satisfies $\mathbf{ZFC}_2$. In addition, this is often the case for other kinds of cardinal and consistency implications. A *Mahlo cardinal*, for example, is a strongly inaccessible cardinal κ beneath which there is a stationary set (i.e. an $S \subseteq \kappa$ such that S intersects every closed and unbounded subset of κ) of inaccessible cardinals. The fact that such a cardinal has higher consistency strength than that of strong inaccessibles (and mild strengthenings thereof) is simply because it contains many models of these axioms below it.

It is not the case, however, that consistency strength is inextricably tied to size. For example, the notion of a *strong*[7] cardinal has lower consistency strength than that of *superstrong*[8] cardinal, but the least strong cardinal is larger than the least superstrong cardinal.[9] The key point is that despite the fact that the least superstrong is not as *big* as the least strong cardinal, one can always *build* a model of a strong cardinal from the existence of a superstrong cardinal (but not vice versa). Thus, despite the fact that a superstrong cardinal can be 'smaller', it still validates the consistency of the existence of a strong cardinal.

Before we move on to our discussion of the iterative conception, we note two phenomena concerning large cardinals that make them especially attractive objects of study:

[5] A subtlety here is exactly what the base theory we should prove this equiconsistency claims is. Number theory will do (since consistency statements are number-theoretic facts), but we will keep discussion mostly at the level of a suitable set theory (e.g. $\mathbf{ZFC}$).

[6] Often, combinatorial and logical characterisations go hand in hand, such as in the case of measurable cardinals. However, sometimes it is not clear how to get one characterisation from another. Recently, cardinals often thought of as having only combinatorial characterisations have been found to have embedding characterisations. See [Holy et al., S] for details.

[7] A cardinal κ is *strong* iff for all ordinals λ, there is a non-trivial elementary embedding (to be discussed later) $j : V \longrightarrow \mathfrak{M}$, with critical point κ, and in which $V_\lambda \subseteq \mathfrak{M}$.

[8] A cardinal κ is *superstrong* iff it is the critical point of a non-trivial elementary embedding $j : V \longrightarrow \mathfrak{M}$ such that $V_{j(\kappa)} \subseteq \mathfrak{M}$.

[9] See [Kanamori, 2009], p. 360.

Fact 4. The 'natural' large cardinal principles appear to be linearly ordered by consistency strength.

One can gerrymander principles (via metamathematical coding) that would produce only a partial-order of consistency strengths[10], however it is an empirical fact that the large cardinal axioms that set theorists have naturally come up with and view as interesting *are* linearly ordered. This has resulted in the following:

Fact 5. Large cardinals serve as the the natural indices of consistency strength in mathematics.

In particular, if consistency concerns are raised about a new branch of mathematics, the usual way to assess our confidence in the consistency of the practice is to provide a model for the relevant theory with sets, possibly using large cardinals.[11] For example, worries of consistency were raised during the emergence of category theory, and were assuaged by providing a set-theoretic interpretation, which then freed mathematicians to use the category-theoretic language with security. For instance, Grothendieck postulated the existence of universes (equivalent to the existence of inaccessible cardinals), and Mac Lane is very careful to use universes in his expository textbook for the working mathematician.[12] These later found application in interpreting some of the cohomological notions used in the original Wiles-Taylor proof of Fermat's Last Theorem (see [McLarty, 2010]). Of course now category theory is a well-established discipline in its own right, and quite possibly stands free of set-theoretic foundations.[13] Nonetheless, set theory was useful in indexing the consistency strength of the emerging mathematical field. More recently, several category-theoretic principles (even some studied in the 1960s) have been calibrated to have substantial large cardinal strength.[14]

This observation concerning the role of large cardinals in contemporary mathematics point to a central desideratum for their use:

> **Interpretative Power.** Large cardinals are required to *maximise interpretative power*: We want our theory of sets to facilitate a unified foundational theory in which all mathematics can be developed.[15]

[10]See [Koellner, 2011] for discussion.

[11]See here, for example, Steel:

> "The central role of the theories axiomatized by large cardinal hypotheses argues for adding such hypotheses to our framework. The goal of our framework theory is to maximize interpretative power, to provide a language and theory in which all mathematics, of today, and of the future so far as we can anticipate it today, can be developed." ([Steel, 2014], p. 11)

[12]See [Mac Lane, 1971], Ch.1, §6. Also interesting here is [McLarty, 1992], Ch. 12.

[13]We should note, however, that the interpretation of category theory through the postulation of set-theoretic universes is a widely accepted technique, whilst there is no 'canonical' accepted axiomatisation for category theory comparable to the role of **ZFC** in set-theoretic mathematics. More specifically, category-theoretic foundations usually propose interpreting mathematics in some topos or other. However, these topoi can vary wildy: Aside from being topoi, the category of sets (as given by **ETCS**), the category of smooth spaces (as given by **SDG**) the category of categories (as given by **CCAF**) are very different category-theoretically. In contrast, **ZFC**-extensions, while they can vary to a large degree, do have a similar structure in mind (namely the cumulative hierarchy). There is also the question of whether category theory, as an *algebraic* theory, requires some underlying theory of concrete objects. See [Hellman, 2006] and [Mclarty, 2004] for discussion.

[14]See [Bagaria and Brooke-Taylor, 2013] for details. The consistency strength is really quite high; many category-theoretic statements turn out to be equivalent to Vopěnka's Principle.

[15]This idea is strongly emphasised in [Steel, 2014] and has a strong affinity with Penelope Maddy's principles UNIFY and MAXIMIZE (see [Maddy, 1997] and [Maddy, 1998]). We will discuss the latter in due course.

Maximising interpretative power entails maximising consistency strength; we want a theory that is able to incorporate as much consistent mathematics as is possible, and hence (assuming the actual consistency of the relevant cardinals) require the consistency strength of our framework theory to be very high.

1.2 The iterative conception of set

It seems then that large cardinals are important foundationally, but do not follow from our usual canonical set theory (**ZFC**). We might then ask the natural question: What reason (aside from their usefulness[16]) do we have for accepting them?

When analysing whether or not we should accept an axiom, it is important to bear in mind the background concept of set against which we measure them. The contemporary conception of set is, for most philosophically-minded mathematicians, the *iterative conception*. There are other conceptions of set[17], however the iterative conception is normally the paradigm within which set-theoretically inclined mathematicians operate (especially those interested in large cardinals) and so is the conception we consider here. The discovery of the set-theoretic paradoxes at the turn of the century, necessitated (assuming a revision of logic is not on the table) a conception of set on which not every condition $\phi(x)$ determines a set. The iterative conception incorporates this though the idea that sets are formed in *stages*. In other words, we obtain the sets by iterating a process of set formation along the ordinals. At the initial stage of construction we do not have anything, and so form the set containing nothing (i.e. the empty set). At the next stage, we form all possible sets available at previous stages. We continue going in this way, and at a limit stages collect together everything we have formed at a previous stage.

The picture is informal, but is often formally construed through the repeated application of the powerset operation and union through the ordinals:

Definition 6. The *Cumulative Hierarchy* (or V) is defined by transfinite recursion over the ordinals as follows:

(i) $V_0 = \emptyset$,

(ii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, for successor ordinal $\alpha + 1$,

(iii) $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$, where λ is a limit ordinal,

(iv) $V = \bigcup_{\gamma\in On} V_\gamma$.

The iterative conception is often seen as theoretically appealing. First, it appears to block the set-theoretic paradoxes: Since the relevant problematic conditions have objects satisfying them unboundedly in the cumulative hierarchy, there is no set of all of them.[18] Secondly, it does so in a way that, one might think, is natural and seemingly well-motivated. Whether or not we would have come up with the iterative

[16] Some authors (e.g. [Maddy, 2011]) regard the usefulness of an axiom as *key* to its acceptance (say because of the relevant foundational goals of set theory). Since we are focussed on the very specific issue of how consistency might be linked to truth for large cardinals given considerations of *iterativity*, we set aside this issue here.

[17] For discussions of different conceptions of set, the seminal [Fraenkel et al., 1973] is an important early text. More specifically, [Holmes, 1998] (Ch.8) and [Forster, 1995] provide some remarks about a possible conception for **NFU**, and Incurvati and Murzi (in [Incurvati, 2014], [Incurvati, 2012], and [Incurvati and Murzi, 2017]) discuss various different conceptions of set.

[18] A slight complication here for the Burali-Forti paradox is how we interpret the notion of *ordinal* in set theory. Usually a canonical representative is chosen with the property that such representatives appear unboundedly in a cumulative hierarchy: Common choices here are von Neumann ordinals (very much the canonical option), or Scott-Potter ordinals (see [Potter, 2004]).

conception of set independently of the discovery of the paradoxes (as Boolos comes close to suggesting[19]) is a difficult question, but it is certainly the case that there is a natural 'picture' behind this resolution of the paradoxes, and one that meshes well with our canonical theory of sets.

1.3 Relating large cardinals and the iterative conception

The above serves as an introduction for the uninitiated, but will be familiar to specialists. Given these seemingly natural axioms and the usual conception of set, a natural question is the extent to which there is a relationship between the two.

Note first that what is satisfied by the cumulative hierarchy[20] depends on two main factors:

(1.) What sets get formed at each additional stage.

(2.) How far the stages extend upwards.

The former issue we shall refer to as issues of *width* and the latter as issues of *height*. The relationship between the two determines what sentences are true in the cumulative hierarchy; once we fix what sets are formed at each additional stage and how far the stages go we thereby settle on the reference of our set-theoretic concepts and definitions. Given a principle that mathematics should be as unconstrained as possible, and that mathematical existence is relatively undemanding, the thoughts that there should be *as many* sets formed as possible at each additional stage, and that the stages should go on *as far* as possible are appealing (i.e. the cumulative hierarchy should be 'maximal').

Discussing height, Incurvati writes (in a survey on maximality principles in set theory):

> "We are told that the cumulative process of construction is indexed by ordinals, but how far does this process go? An initial and frequently given answer is that the process should be iterated as far as possible:
>
> **Height Maximality.** There are as many levels of the hierarchy as possible." ([Incurvati, F], p4)

As Incurvati notes, however:

> "However, Height Maximality does not tell us much until the idea of iteration 'as far as possible' is developed to some extent." ([Incurvati, F], p.4)

Here is where large cardinals come in. In order to capture height maximality, one might think that we should appeal to large cardinals. After all, don't large cardinals simply assert that 'very large' order-types exist? Incurvati continues:

> "To answer this question, a number of principles have been invoked. The ones that are probably best known are principles telling us, effectively, that the hierarchy goes at least as far as a certain ordinal. These include the Axiom of Infinity and the standard large cardinal axioms, such as (in order of increasing consistency strength): inaccessible, Mahlo, weakly compact, ω-Erdős, measurable, strong, Woodin, and supercompact." ([Incurvati, F], p.4)

[19]See [Boolos, 1971], p. 219.

[20]If there is a single such thing—for simplicity we shall assume that there is despite the subtlety of the question for the philosophy of set theory.

Similar remarks are also found elsewhere in the literature, for example in the work of Maddy:

> "As with any large cardinal, positing a supercompact can be viewed as a way of assuring that the stages go on and on; for example, below any supercompact cardinal κ there are κ measurable cardinals, and below any measurable cardinal λ, there are λ inaccessible cardinals." ([Maddy, 2011] pp. 125–126)

and Hauser, who refers to large cardinals as "global existence postulates motivated in part by a priori considerations about the inexhaustibility of the universe of all sets" ([Hauser, 2001], p. 257), and in set theory textbooks, such as Frank Drake's pleasantly written volume on large cardinals:

> "But probably the main reason to study them [measurable cardinals] is the more open-minded interest in the properties which follow from assuming that very large cardinals exist; we want to consider the universe of set theory as being the cumulative type structure, continued through all possible ordinals, so that if it is possible to go so far that we get to a cardinal that is measurable, then we should do so." ([Drake, 1974], p. 186)

The thought then might be the following. Since large cardinals assert that the stages go as far as a particular large ordinal (i.e. they are good characterisations of height maximality), and since the iterative conception incorporates the idea that the stages should go as far as possible, then any large cardinal whose existence is consistent with **ZF** (or possibly **ZFC**) should exist.[21]

[21] This has been seen by some as an attractive principle. For example Koellner writes (in an endnote to his PhD thesis):

> "Dodd and Jensen showed that this [a certain embedding principle] is equivalent to the statement that there is an inner model with a measurable cardinal. So we have a justification of such a model. Note, however, that this is quite different from a justification of the existence of a measurable cardinal. A further argument would be required to move from the consistency to the existence of a measurable cardinal. I suspect that such an argument can be supplied—large cardinals (in contrast, say, to an ω_2-well-ordering of the reals) seem to be the type of things which require for their existence only their consistency. But I will not pursue this thought here." ([Koellner, 2003], p100)

Similar ideas might be extracted from the work of Cantor. For example, the following is a famous quotation:

> "If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into "one thing", I call it a consistent multiplicity or a "set"." ([Cantor, 1899]: p.114)

Hallett, develops this Cantorian idea concerning 'consistent' multiplicities:

> "Let us grant that the Absolute is not counted in the scale of transfinite numbers. But why should numerability mean just numerability in the transfinite scale? Why does the Absolute not give rise to a further domain of mathematical activity, to super-transfinite numbers, Absolute numbers, or whatever? Why is it as Cantor says an Absolute maximum? One answer that Cantor would give is that to try to mathematize the Absolute would be simply a category mistake: everything mathematizable (or numerable) is already in the realm of the finite and transfinite and the Absolute is simply that which embraces all these. There are no numbers beyond all transfinite numbers waiting to enumerate the Absolute. This is not to say that we may not discover new types of number, perhaps with surprising properties. For example, Hausdorff later discovered numbers ω_α such that $\alpha = \omega_\alpha$, and since then much larger ordinals have been defined or isolated. But if—to take one example—'the smallest uncountable measurable cardinal' is a genuine number (i.e. if this concept is self-consistent

As we shall argue later, this argument does not hold water (without supplementation). The main issue, as we shall see, concerns how the stages are formed, and hence what theory we measure 'consistency' against.

2 The priority of width over height

We will put pressure on two aspects of this argument. First: (a) assuming that sets are formed through iteration of the power set operation, we dispute the claim that that consistency of a large cardinal axiom with **ZFC** is enough to guarantee the existence of such a cardinal, and (b) we will raise a more foundational worry concerning exactly how stages are formed.

When we look at the iterative conception, we note that part of the idea is to form 'as many sets as possible' at each additional stage, and continue this process for 'as long as possible'. Our core point is the following: It might be that the formation of certain subsets at each additional stage *precludes* the formation of a certain stage with a large cardinal property attached. There is certainly nothing a priori about the iterative conception that requires that particular formation of subsets *must* permit all large cardinal axioms consistent with $\mathbf{ZF}(\mathbf{C})$.

Indeed, it is interesting that there are several principles that have anti-large cardinal properties. Aside from an axiom asserting the brute non-existence (or inconsistency) of a large cardinal, good examples here are axioms of constructibility (such as $V = L$) and consequences thereof (such as $\square$-principles). However, it is not clear why any of these conceptions of subset formation should be *true* of V. $V = L$ for example seems to represent a *minimality* condition on what sets are formed at additional stages, and so does not seem to cohere well with our concept of forming *all possible* sets at each additional stage.[22]

Importantly for our current discussion, there are principles that seek to *maximise* the *width* of the hierarchy (i.e. the sets formed at each additional stage) that have anti-large cardinal properties. This raises serious challenges for the view that the consistency of a large cardinal axiom with **ZFC** is enough to guarantee the existence of a cardinal with the property; it might simply be that forming *all possible* sets at successor stages precludes the existence of a stage indexed by an ordinal with the relevant property. In other words, there are interpretations of maximality which validate the consistency but not the truth of large cardinal axioms. We now provide some examples, before explaining their philosophical consequences.

> or coherent) then it is not a new Absolute number, but a normal increasable transfinite number. We have discovered it within the realm of the transfinite. The same would hold of all numbers we might define or hope to introduce." ([Hallett, 1984], p43)

The idea then, for Hallett's Cantor, is that in the case of cardinals and ordinals, if you can isolate a *coherent* or *consistent* concept, then there is such an ordinal with the relevant property. That is just what it is for the universe to be Absolute; it contains all numbers we could coherently talk about.

[22] As Drake puts it:

> "Note that this is a case where the word axiom is used simply to indicate that we shall look at models of this sentence; there seems to be no very good argument to say that it should hold of the cumulative type structure. Most set theorists regard it as a restriction which may prevent one from taking every subset at each stage, and so reject it (this includes Gödel, who named it)." ([Drake, 1974], p131)

2.1 The Axiom of Choice

In this subsection, we'll argue that the Axiom of Choice could provide us with an interpretation of maximality on which certain large cardinal axioms are consistent but false.

We can formulate the Axiom of Choice as follows:

Axiom 7. (Axiom of Choice). Let $\mathcal{F}$ be a non-empty family of pairwise disjoint non-empty sets. Then there is a set C that contains exactly one element of every member of $\mathcal{F}$.

While Choice is often regarded as receiving justification from a wide range of sources (especially notable here is its equivalence with diverse natural statements across mathematics) we might think that it follows naturally from the iterative conception. Suppose we have some family $\mathcal{F}$ of pairwise-disjoint non-empty sets first formed in some $V_{\alpha+1}$ (nothing new is formed at limit ordinals, the previous sets are simply collected together). Then we know that every element of $\mathcal{F}$ is first formed at latest at stage V_α, and hence all members of elements of sets are first formed at latest by stage V_α. But then, assuming that *all* subsets are formed at additional stages, a choice set must be formed at latest at stage $V_{\alpha+1}$ (what could possibly prevent it from existing?). Indeed, Kreisel went so far as to say:

> "For the fat (or "full") hierarchy, the axiom of choice is quite evident." ([Kreisel, 1980], p. 192)[23]

[Potter, 2004] (p. 257) explains how to recast the discussion in terms of second-order logic. The details need not detain us, but salient is that through using a *logical* choice function in second-order logic (rather than through coding a set of ordered-pairs), one can derive the second-order choice principle from the second-order separation principle (in conjunction with some other unobjectionable assumptions). We thus arrive at a position where, on the basis of the iterative conception, we hold that Choice receives a natural motivation on the basis of the view that we form all possible sets at an additional stage, and this motivation can be recast in terms of an argument in second-order logic.[24]

The issue finds relevance when we consider the definition of cardinals through elementary embeddings. We have already seen mention of measurable cardinals earlier, now the time has come to define them:

Definition 8. A cardinal κ is *measurable* iff it is the critical point of a non-trivial $j : V \longrightarrow M$, for some transitive inner model $\mathfrak{M} = (M, \in)$.[25]

[23]As [Potter, 2004] notes, similar remarks are to be found in [Ramsey, 1926]. However, by 1926 the iterative conception had not yet been fully isolated, and so it is questionable whether Ramsey's views flowed from a conception that was iterative *as well as* combinatorial, rather than a straight-up combinatorialism.

[24]Similar remarks can be found in [Paseau, 2007] concerning Boolos' views on the Axiom of Choice:

> "There is also an alternative to Boolos's suggestion that the Axiom of Choice should be derived from a stage version of Choice. One could instead see Choice as flowing from a combinatorial understanding of the set-formation process. If one thinks that any arbitrary combination of sets below some given stage constitutes a property, then a generalisation of Spec [i.e. Separation] to cover all possible properties whatsoever—as opposed to those expressible in some formal language, as in Boolos's initial presentation—expresses the intuitive thought that at any given stage all the possible sets available for formation are indeed formed. As it is usually conceived, and as Boolos himself conceives it, the iterative conception includes this combinatorial idea. And combinatorialism straightforwardly implies a Choice axiom." ([Paseau, 2007], pp. 35–36)

[25]See [Drake, 1974] for a relatively friendly introduction to measurable cardinals, and [Kanamori, 2009]

One route to providing stronger definitions of these cardinals is to increase the similarity between V and $\mathfrak{M}$. For example, the following provides a definition of a cardinal far stronger than measurable:

Definition 9. Let λ be an ordinal. A cardinal κ is *λ-supercompact* iff κ is the critical point of a non-trivial elementary embedding $j : V \longrightarrow M$ for some transitive inner model $\mathfrak{M} = (M, \in)$, $j(\kappa) > \lambda$, and ${}^{\lambda}M \subseteq M$.

Here, for sufficiently large λ, specifying that M is closed under λ-sequences substantially strengthens the kind of cardinal defined. (An additional subtlety here is that part of what strengthens the cardinal is that j sends κ above λ, but we set this aside for now.) More generally, many such strengthenings of the notion of measurability make use of this strategy. Carrying this idea to its natural endpoint suggests the following principle:

Definition 10. A cardinal κ is *Reinhardt* iff κ is the critical point of a non-trivial elementary embedding $j : V \longrightarrow V$.

However, we can now state the following:

Theorem 11. [Kunen, 1971] There are no Reinhardt cardinals.

Importantly, Kunen's proof makes essential use of the Axiom of Choice[26], as do more recent proofs[27]. However, it is unknown whether or not the existence of a Reinhardt cardinal is inconsistent in **ZF**. In investigating this question, Koellner and Woodin in unpublished work[28] have developed strengthenings of these axioms within **ZF**. For example:

Definition 12. A cardinal κ is *Super-Reinhardt* iff for every ordinal λ there is a $j : V \longrightarrow V$ with critical point κ and such that $j(\kappa) > \lambda$.

Interestingly, it turns out that these 'choiceless cardinals' in turn form an elegant hierarchy of consistency strengths[29]. What should our reaction to this situation be? In his PhD thesis, Koellner remarked:

> "There is an entire hierarchy of "choiceless cardinals" and it may be the case that the hierarchy of consistency strength outstrips that which assumes choice. In the end it may turn out to be reasonable to view AC as a limitative axiom on a par with $V = L$." ([Koellner, 2003], p. 100)

Assuming (highly non-trivially) that the existence of a Reinhardt cardinal is in fact consistent with **ZF**, one might think that we should drop Choice. After all, then

and [Jech, 2002] for detailed technical discussion. These cardinals admit of a wide variety of characterisations, many notably first-order in character.

[26]This is because Kunen uses the theorem in [Erdős and Hajnal, 1966] that for any infinite ordinal λ, there is a function ${}^{\omega}\lambda \longrightarrow \lambda$ such that whenever $A \subseteq \lambda$ and $|A| = |\lambda|$ then F"$({}^{\omega}A) = \lambda$.

[27]For example those that use the Solovay Splitting Lemma that if κ is a regular uncountable cardinal then any stationary subset of κ can be partitioned into κ many disjoint stationary sets (such as the proof due to Woodin presented in [Schindler, 2014]).

[28]See [Koellner, 2014] for a summary of some of these ideas.

[29]There is a question of exactly what the consistency strength of a Reinhardt cardinal is within **ZF**, given that Choice has to be given up. [Woodin, 2011] (p. 456) mentions a result that the theory **ZF**+"There exists a weak Reinhardt cardinal" implies the consistency of **ZFC**+"There exists a proper class of ω-huge cardinals". In conversation, Woodin has stated that the consistency of **ZF**+"There is a super-Reinhardt cardinal" implies the consistency of $\mathbf{ZFC} + I_0$. Discussion of this issue is available on MathOverflow at [(https://mathoverflow.net/users/2362/tim campion),].

there is a natural theory of sets (**ZF**), one which can be given an iterative story, and under which it is consistent to go on and form a Reinhardt cardinal.

Insofar as we accept the earlier iterative story concerning the justification of the Axiom of Choice, however, we should not be moved by the thought that AC might be limitative in a similar way to $V = L$. Simply put, we would already be confident that the formation of powersets at each additional stage guarantees that there is no such cardinal. Continuing the hierarchy 'as far as possible' does not go so far as to include choiceless cardinals, since AC is true when we form 'all possible' sets at each additional stage.

Nonetheless, the consistency of a Reinhardt cardinal could still be witnessed. We could perfectly well have a Reinhardt cardinal in countable models of **ZF** or even an inner model of V satisfying **ZF**. It is just that such a model has to miss out some subsets, specifically those that guarantee the existence of the relevant choice functions.[30] Indeed, this has long been appreciated; for some time Jensen was a staunch adherent of $V = L$, yet held that we might have countable transitive models containing large cardinals. Drake is clear about the situation:

> "Perhaps it is worth indicating the sort of reason why the mere fact that a definition makes a cardinal look very large is not sufficient to convince us that such cardinals must exist in the cumulative type structure, if only we continue it far enough. Suppose there is, in some transitive model of **ZFC**, a strongly compact cardinal. Then there must be a countable, transitive model of **ZFC**, M say, in which there is a strongly compact cardinal (according to M); suppose α is an ordinal which is strongly compact in M. Then the various α-additive measures which must exist in M will only be measures in M because a great many possible subsets are missing in M, so that the purported measure does not have to consider them...This sort of consideration highlights the fact that even if we are convinced that strongly compact cardinals are consistent with **ZFC**, we have not answered the question whether they exist in the cumulative type structure." ([Drake, 1974], pp. 315–318)

Aside from countable models, we might have a model containing all ordinals, but leaving out some choice sets. Thus, it is at least epistemically possible that we have a κ that is Reinhardt in some inner model, but no Reinhardt in V. Given that we hold (for the purposes at hand) that Choice should be true, the only sense in which one 'could' continue the hierarchy to include a Reinhardt κ is to leave subsets out when iterating the powerset operation. Thus, in this possibly maximal context the large cardinal axiom "There exists a Reinhardt cardinal" is actually a *minimising* principle, necessitating the omission of subsets. It is in this sense, given a prior justified width maximality operation, that width is prior to height.

2.2 The Inner Model Hypothesis

We will see a similar feature with respect to a variety of principles known as *inner model hypotheses*. Again, we will see that this class of principles provides a conception of maximality and forming as many sets as possible at each additional stage on

[30] An additional subtlety here is that the consistency of a Reinhardt cardinal may be witnessed by an *outer* model of **ZF**, and not every outer model of **ZF** can be widened to a model of **ZFC**. In that case though, the witnessing model is (from the point of view of V) not a bona fide two-valued set-theoretic structure, but rather a Boolean-valued structure (which may be captured through the use of a forcing relation), and so we omit its consideration here.

which large cardinal axioms serve to minimise rather than maximise. We begin by explaining these ideas and then later will discuss exactly how they might be thought to follow from maximality in width. We begin with the following:

Definition 13. Let ϕ be a parameter-free first-order sentence. The *Inner Model Hypothesis* (or IMH) states that if ϕ is true in an inner model of an outer model of V, then ϕ is already true in an inner model of V.

There is a question of how exactly to formalise the Inner Model Hypothesis. Since proper outer models of V do not exist on many ontological frameworks, if we take a literal interpretation of the meaning of 'outer model' then the principle is utterly trivial (since either the outer model is V itself or does not exist). However, if we accept some coding of the notion of outer model (or at least *satisfaction* in outer models), we can formulate the principle as having significant consequences.[31]

How might we figure the Inner Model Hypothesis into an iterative account? The story is slightly more speculative than that of Choice, as by its very nature the IMH is a global principle that can not be given a local first-order formulation. Nonetheless, we might think of the Inner Model Hypothesis as formulating the idea that all *possible* sets are formed at each additional stage. Essentially the Inner Model Hypothesis states that any statement consistent with the structure of V is already realised in V. In this way, it can be viewed as a form of *absoluteness principle*. Indeed, it is equivalent to the following absoluteness principle:

Definition 14. A formula is *persistent*-Σ^1_1 iff it is of the following form:

$$(\exists M)M \models \psi$$

where ψ is first-order.

Definition 15. *Parameter-free persistent* Σ^1_1*-absoluteness* is then the claim that if ϕ is persistent-Σ^1_1 and true in an outer model of V, then ϕ is true in V.

Theorem 16. [Friedman, 2006] The Inner Model Hypothesis is equivalent to parameter-free persistent Σ^1_1-absoluteness.

This in turn can be viewed as a generalisation of the following theorem of **ZFC**:

Theorem 17. (Lévy-Shoenfield Absoluteness) Let ϕ be a Σ_1 sentence. If ϕ is true in an outer model of V, then ϕ is true in V.

We wish to take the following points from the above observations. First, the Inner Model Hypothesis can be thought of as a principle that asserts that anything (of a particular kind) that 'could' have been realised by the formation of subsets already has a witness. In this way, it makes sense of the claim that we form all 'possible' subsets at each additional stage. Second, it does so by generalising an idea already present in **ZFC**. In this respect, it resembles a reflection principle for height: A standard principle of absoluteness true in **ZFC** is generalised to a language of

[31] All we need for the results of the present paper is that satisfaction for tame class forcings can be formalised, and since these all have a definable forcing relation there is no obstacle here. In fact much of the strength of the IMH is captured by Lévy-absoluteness for Σ_1 formulas with parameter ω_1 for ω-preserving outer models which are tame, Δ_2-definable class forcing extensions. Thus, for many of the results stemming from the IMH, one does not need the full force of arbitrary outer models; the formula to which absoluteness is to be applied can just be first-order (Σ_1) with parameter ω_1. If satisfaction in *arbitrary* well-founded outer models is desired, $\mathbf{NBG} + \Sigma^1_1$-Comprehension suffices (since atisfaction in arbitrary outer models can be coded as long as it is possible to produce a code for $Hyp(V)$—the least admissible set containing V).

higher-order. While there are also substantial disanalogies between the two kinds of principle, we merely wish to motivate the idea that the Inner Model Hypothesis might be taken as a sharpening of the notion of 'as many subsets as possible' in a *similar* way to set-theoretic reflection principles making sense of the hierarchy going 'as far as possible'.

Let us then, as before, suppose that we take this motivation for the Inner Model Hypothesis to be sound, and hold that the power set operation should support parameter-free persistent Σ^1_1-absoluteness (equivalently the Inner Model Hypothesis). We immediately have the following result:

Theorem 18. [Friedman, 2006] If the Inner Model Hypothesis holds, there are no inaccessible cardinals in V.

On our current understanding, this would mean that there could be no (significant) large cardinals in V; a conception of the formation of powersets on which there is a high degree of absoluteness to outer models (in making sense of the notion of possible sets) precludes their existence. Here though, an interesting contrast is highlighted with the example of choiceless cardinals. There we were only able to conjecture that it might be possible to leave out subsets to obtain large cardinals. In the current context, however, the existence of large cardinals in inner models is positively *implied*:

Theorem 19. [Friedman et al., 2008] The Inner Model Hypothesis implies that there are measurable cardinals of arbitrarily large Mitchell order in inner models.[32]

Thus, while the Inner Model Hypothesis does not permit the existence of large cardinals in V, it *does* vindicate their existence in inner models and thus their use in consistency proofs. Large cardinals, whilst not *true*, are acceptable for determining what combinations of sets are possible in satisfying particular formal theories, even if they are strictly incompatible with the *full* powerset operation.[33] However, on the current conception of maximality they act as *minimising* principles; whilst they are witnessed as consistent we must omit subsets in order for them to hold.

2.3 Forcing axioms

We have seen thus far that there are conceptions of maximality on which large cardinals are minimising rather than maximising principles. We will now consider a very extreme version of maximality which calls into question large-cardinal-like axioms of **ZFC**.

[32] The Mitchell ordering is a way of ordering the normal measures on a measurable cardinal. Roughly, it corresponds to the strength of the measure, where a measure U_1 is below another U_2 in the Mitchell order if U_1 belongs to the ultrapower obtained through U_2. See [Jech, 2002] Ch. 19.

[33] Indeed, a worry we might have about the Inner Model Hypothesis is whether or not it is consistent. This is somewhat assuaged by the following:

Theorem 20. [Friedman et al., 2008] Assuming the consistency of the existence of a Woodin cardinal with an inaccessible above, the Inner Model Hypothesis is consistent.

Thus, we have a rough guide as to the consistency strength of the Inner Model Hypothesis (somewhere between many measurables and a Woodin with an inaccessible above). Should the believer in the Inner Model Hypothesis be (significantly) perturbed by the non-existence of Woodin cardinals or inaccessibles in V in getting this consistency proof? It is at least plausible that they should not; they hold that the subset operation supports the Inner Model Hypothesis, and thus supports many inner models with large cardinals. The hypothesis of the consistency of an inner model of a Woodin cardinal with an inaccessible above is thus substantially *less* worrisome than it would be otherwise.

We proceed through considering *forcing axioms*. To facilitate understanding of the ideas later in this section, we first provide a very coarse and intuitive sketch of the forcing technique.

Forcing, loosely speaking, is a way of adding subsets of sets to certain kinds of model. For some model $\mathfrak{M}$ and atomless partial order $\mathbb{P} \in \mathfrak{M}$, we (via an ingenious definition of ways of 'naming' possible sets and 'evaluating' these names) add a set G that intersects every dense set of $\mathbb{P}$ in $\mathfrak{M}$. The resulting model (often denoted by '$\mathfrak{M}[G]$'), can be thought of as the smallest object one gets when one adds G to $\mathfrak{M}$ and closes under the operations definable in $\mathfrak{M}$.

A *forcing axiom* expresses the claim that the universe has been saturated under forcing of a certain kind. For example we have the following axiom:

Definition 21. Let κ be an infinite cardinal. $\mathsf{MA}(\kappa)$ is the statement that for any forcing poset $\mathbb{P}$ in which all antichains are countable (i.e. $\mathbb{P}$ has the countable chain condition), and any family of dense sets $\mathcal{D}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter G on $\mathbb{P}$ such that if $D \in \mathcal{D}$ is a dense subset of $\mathbb{P}$, then $G \cap D \neq \emptyset$.

Definition 22. *Martin's Axiom* (or just MA) is the statement that for every κ smaller than the cardinality of the continuum, $\mathsf{MA}(\kappa)$ holds.

One can think of Martin's axiom in the following way: The universe has been saturated under forcing for all posets with a certain chain condition and less-than-continuum-sized families of dense sets.

There are several kinds of forcing axiom, each corresponding to different permissions on the kind of forcing poset allowed (the countable chain condition is quite a restrictive requirement). Many of these have interesting consequences for the study of independence, notably many imply that CH is false and that in fact $2^{\aleph_0} = \aleph_2$. If we think of forcing as a way of generating subsets we might think that saturation under forcing represents a good approximation to having all possible subsets at successor stages.[34]

This idea is supported by the following equivalent characterisation of MA:

Definition 23. *Absolute*-MA. We say that V satisfies *Absolute*-MA iff whenever $V[G]$ is a generic extension of V by a partial order $\mathbb{P}$ with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $V[G] \models \exists x \phi(x)$ then there is a y in V such that $\phi(y)$.

This formulation makes it particularly perspicuous the sense in which a forcing axiom (in this case MA) can be thought of as maximising the universe under 'possible' sets; if we could force there to be a set of kind ϕ (for a particular kind of ϕ and $\mathbb{P}$), one already exists in V.[35]

[34]Some set theorists are sympathetic to this idea. For example, Magidor writes:

> "Forcing axioms like Martin's Axiom (MA), the Proper Forcing Axiom (PFA), Martin's Maximum (MM) and other variations were very successful in settling many independent problems. The intuitive motivation for all of them is that the universe of sets is as rich as possible, or at the slogan level: A set [whose] existence is possible and there is no clear obstruction to its existence [exists]...
>
> ...What do we mean by "possible"? I think that a good approximation is "can be forced to [exist]"... I consider forcing axioms as an attempt to try and get a consistent approximation to the above intuitive principle by restricting the properties we talk about and the the forcing extensions we use. " ([Magidor, U], pp. 15–16)

[35]For some discussion of the coding of Absolute-MA (and similar principles) for the philosopher inclined towards a 'definite' picture of the set-theoretic universe, see [Barton and Friedman, F].

Feeding this into our iterative story, we might say that, given that the universe is iteratively formed, any subsets that can be generated by forcing at a given stage should exist. The intuition is somewhat witnessed by the interaction between forcing axioms and choice-like principles. To see this, we first require some definitions:[36]

Definition 24. Let κ be a cardinal and $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a partial order. $\mathsf{FA}_\kappa(\mathbb{P})$ is the statement that for all families $\mathcal{D} = \{D_\alpha | \alpha < \kappa\}$ of predense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ meeting all these predense sets.

Definition 25. Given a class Γ of partial orders $\mathsf{FA}_\kappa(\Gamma)$ holds iff $\mathsf{FA}_\kappa(\mathbb{P})$ holds for all $\mathbb{P} \in \Gamma$.

Definition 26. Let λ be a cardinal. A partial order $\mathbb{P}$ is $(< \lambda)$-closed iff every decreasing chain $\{p_\alpha | \alpha < \gamma\}$ indexed by some $\gamma < \lambda$ has a lower bound in $\mathbb{P}$.

Definition 27. Γ_λ denotes the class of $(< \lambda)$-closed posets. Ω_λ denotes the class of posets for which $\mathsf{FA}_\lambda(\mathbb{P})$ holds.

We can now point out the following:

Theorem 28. $\mathsf{FA}_\kappa(\Gamma_\kappa)$ is equivalent (modulo **ZF**) to the Axiom of Choice.[37]

What should we take from this? If we accept the earlier argument that AC makes precise the claim that as many sets as possible are formed at successor stages, then there is a very clear sense in which this can be viewed as the existence of certain generics for forcing notions.[38] This supports the idea that forcing is a way of 'generating' new subsets, and perhaps we should view saturation under generics as part of taking 'all possible' sets at each additional stage.

As will be well know to specialists, there are some limitations in this regard. For instance, consider the following theorems:

Theorem 29. Letting $\mathfrak{c}$ denote the cardinality of the continuum, $\mathsf{MA}(\mathfrak{c})$ is inconsistent with **ZFC**.[39]

Theorem 30. In **ZFC** there is a non-countable-chain-condition $\mathbb{P}$ such that for a $(\leq \aleph_1)$-sized family of dense subsets $\mathcal{D}$ of $\mathbb{P}$, there is no filter G on $\mathbb{P}$ intersecting every member of $\mathcal{D}$ (i.e. $\mathsf{MA}_{\mathbb{P}}(\aleph_1)$ is false).[40]

It seems then that there are some limitations on what generics one can have. Given **ZFC**, we cannot just assert the existence of generic sets willy-nilly. However, the above two proofs depend on notions of uncountability; the first on the existence of $\mathfrak{c}$, and the second on the existence of $\aleph_1$.

Here then is a controversial suggestion: We might regard axioms asserting the existence of uncountable sets (e.g. the Powerset Axiom, or the claim that ω_1 exists) as certain kinds of large cardinal axiom, whilst using *forcing* (along with some *definable* powerset operation) as our way of generating all possible subsets.

These claims are certainly plausible. For powerset, we have (trivially) that both $Con(\mathbf{ZFC}-\text{Powerset})$ and $Con(\mathbf{ZC})$ follow from $Con(\mathbf{ZFC})$, whilst neither $Con(\mathbf{ZC})$

[36] Here we follow the exposition of [Viale, 2016].

[37] See [Viale, 2016], for discussion. A full proof is available in [Parente, 2012].

[38] In fact, it turns out that a wide variety of statements (including choice principles, Łos-style Theorems, and certain large cardinal axioms) can be unified in the guise of forcing axioms (again, see [Viale, 2016]). While the philosophical ramifications of these facts bear exploring, we lack the space to do so here.

[39] See [Kunen, 2013], p. 175, Lemma III.3.13.

[40] See [Kunen, 2013], pp. 175–176, Lemma III.3.15.

nor $Con(\mathbf{ZFC} - \text{Powerset})$ imply $Con(\mathbf{ZFC})$. Similarly, the assertions that "ω_1 exists", "ω_2 exists" etc. have ever increasing interpretative strength over the theory $\mathbf{ZFC}-$Powerset. Moreover, iterations of Powerset and uncountable sets behave something like a large cardinal axioms with respect to determinacy axioms; Borel determinacy requires ω_1-many iterations of the Powerset Axiom,[41] in a similar way to other determinacy axioms reversing to inner models with large cardinals (we shall see discussion of this fact regarding determinacy axioms later).[42]

With this in mind, we might view the limitative theorems concerning forcing axioms as indicative of a fundamental tension between forming every possible set given a *particular point* in the set-theoretic construction, and the formation of *all* subsets of an infinite set at a successor stage. Rather, we might think, in order to form *all possible* subsets in the hierarchy, they have to be formed in a piecemeal process; we get new subsets of certain sets appearing unboundedly in V. We can motivate such a position by generalising an idea of Cohen's:

> "A point of view which the author feels may eventually come to be accepted is that CH [the continuum hypothesis] is obviously false...$\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$ etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently." ([Cohen, 1966], p. 151)

Cohen's thought is based on the following idea: Given the immense richness of the powerset operation, and the flexibility afforded by the forcing technique, we can make the continuum have almost any value (consistent with it lacking certain large cardinal properties or contradicting König's Theorem). So perhaps we should say that it resists having a *specifiable* $\aleph$-number, instead being outside those we can define.[43]

But if the continuum is really generated by such a principle, why insist (aside from a prior adherence to the Powerset Axiom) that $\mathfrak{c}$ has an aleph value *at all*? Scott (in a forward to Bell's textbook on Boolean-valued models[44]) expresses the following thought:

> "I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible to have strong axioms, which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is

[41] See [Friedman, 1971].

[42] See [Koellner, 2014] for discussion of the links between large cardinals and determinacy.

[43] One interesting axiom that might capture this thought is the *Strong Inner Model Hypothesis*. Since our focus lies elsewhere for the moment, we do not consider it here, but see [Friedman, 2006] for discussion.

[44] See [Bell, 2011] for the third edition.

> not even a set) when at last all cardinals are absolutely destroyed. But really pleasant axioms have not been produced by me or anyone else, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models." ([Scott, 1977], p. xv)

The idea then is that perhaps that since we can force the continuum to be larger than any particular ordinal, maybe we should accept that it is, in fact, a proper class in V. Hallett, after appreciatively quoting the above two passages, sums the point up nicely:

> "Thus, the continuum evades all our attempts to characterize it by size (Cohen), so maybe we should start with this transcendence as a datum (Scott)." ([Hallett, 1984],

Building on Scott, our "new point of view" will be to regard the universe as generated not through the powerset axiom, but through saturation under *forcing* (combined with a definable power set operation).[45] We consider the following axiomatisation:

Definition 31. The theory of *Forcing Saturated Set Theory* or $\mathbf{FSST}$ comprises the following axioms:

1. All axioms of $\mathbf{ZFC}$ $-$ Powerset.
2. (Definable Powerset Axiom) $(\forall x)(\exists y)y = Def(x)$ (where $Def(x)$ is the definable powerset of x).[46]
3. (Forcing Saturation Axiom) If $\mathbb{P}$ is a forcing poset, and $\mathcal{D}$ is a family of dense sets, then there is a filter $G \subseteq \mathbb{P}$ intersecting every member of $\mathcal{D}$.

Thus, under $\mathbf{FSST}$, we view the 'richness' of available subsets as given by saturation under forcing for any set-sized family of dense sets.

Below, we explain how one might think of $\mathbf{FSST}$ as arising from an iterative process. For now, we pause briefly to note some of the theory's properties. Interestingly, $\mathbf{FSST}$ is shown to capture the intuition of Meadows by the following:

Theorem 32. $\mathbf{FSST}$ is equivalent to the theory $\mathbf{ZFC}-$Powerset+"Definable powersets exist"+"Every set is countable".[47]

[45] A salient alternative approach to ours, one which *expands* the notion of *continuum* to an 'absolute' continuum, uses Conway's notion of 'surreal number'. An explanation of this idea is available in [Ehrlich, 2012].

[46] This is in fact redundant, since for any set x, one can construct $L(x)$ in the theory $\mathbf{ZFC}-$Powerset. We include it simply to emphasise the iterative picture.

[47]

Proof. (1.) $\mathbf{FSST} \Rightarrow \mathbf{ZFC}-$Powerset+"Definable powersets exist"+"Every set is countable".

The only thing to show for this direction is to show that $\mathbf{FSST}$ implies that every set is countable. To see this, let α be the order-type of a well-ordering of an arbitrary set x (α is our putative 'uncountable' cardinal). Then, the poset $Col(\alpha, \omega)$ is obtainable by taking definable powersets. Letting $\mathcal{D}$ be an α-sized family of dense sets on $Col(\alpha, \omega)$ (again, obtained by the Definable Powerset Axiom) and using the Forcing Saturation Axiom, there is a generic G for this family, and so there is a collapsing function from α to ω.

(2.) $\mathbf{ZFC}-$Powerset+"Definable powersets exist"+"Every set is countable" $\Rightarrow \mathbf{FSST}$.

Again, we just have to show that we can obtain the Forcing Saturation Axiom from the axiom that every

By the above theorem, we have the immediate:

Corollary 33. **FSST** is consistent relative to the theory **ZFC**−Powerset+"ω_1-exists".

Proof. This is a quick consequence of the fact that **FSST** is equivalent to **ZFC**−Powerset+"Every set is countable", and the latter has a model in $H(\omega_1)$ (i.e. the sets with hereditary cardinality ω_1). □

With these properties of the theory in play, we might ask how **FSST** could arise from the consideration of an iterative process. What we want here is some recursive process of forming subsets along the ordinals, such that the resulting structure models **FSST**. One naive suggestion that occurred to the author was to take definable power sets and saturate under all generics at each successor stage:

Definition 34. *The Naive Forcing Saturated Hierarchy* is defined as follows (within **FSST**):

(i) $N_0 = \emptyset$

(ii) $N_{\alpha+1} = Def(N_\alpha) \cup \{G | \exists \mathbb{P} \in N_\alpha \exists \mathcal{D} \in N_\alpha$"$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D}$"$\}$

(iii) $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$

(iv) $N = \bigcup_{\alpha \in On} F_\alpha$.

Such a hierarchy looks like it would satisfy **FSST** by design (since we have thrown in all generics at successor stages). Unfortunately the idea does not work. This is because once the Cohen poset has been formed, one immediately puts in all reals which are Cohen-generic for arithmetically-definable families of dense sets. But then we would immediately get all reals at the following stage, and so the hierarchy breaks down.

We thus need a more subtle perspective. Since the generics need to be fed in slowly and unboundedly, we consider a well-order R on the universe, and modify the definition of the hierarchy as follows:

Definition 35. *The Forcing Saturated Hierarchy* is defined as follows (within **FSST**):

(i) $F_0 = \emptyset$

(ii) $F_{\alpha+1} = Def(F_\alpha) \cup \{G | \exists \mathbb{P} \in F_\alpha \exists \mathcal{D} \in F_\alpha$"$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D} \wedge G$ is the R-least generic for $\mathbb{P}$ and $\mathcal{D}$"$\}$

---

set is countable. So, let $\mathbb{P}$ be a forcing poset and $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$. Since every set is countable, we can enumerate $\mathcal{D}$ in order-type ω. So, without loss of generality, $\mathcal{D} = \langle D_n | n \in \omega \rangle$. Since every set is countable, $\mathbb{P}$ can also be enumerated in order-type ω, let 'f' denote the relevant enumerating function. We can then define via recursion (and using the parameter f) the following function ϕ from $\mathcal{D}$ to $\mathbb{P}$:

$$\phi(D_0) = \text{"The } f\text{-least } p \in D_0\text{"}$$

$$\phi(D_{n+1}) = \text{"The } f\text{-least } p \in D_{n+1} \text{ such that } p \leq_{\mathbb{P}} \phi(D_n)\text{"}$$

Effectively ϕ successively picks elements of each member of $\mathcal{D}$, ensuring that we always go below our previous pick in the forcing order (this is allowed because each $D \in \mathcal{D}$ is dense in $\mathbb{P}$, and so such a p always exists). By Replacement, $ran(\phi)$ exists, and the object obtained is a generic for $\mathbb{P}$, and so the Forcing Saturation Axiom holds. □

(iii) $F_\lambda = \bigcup_{\beta<\lambda} F_\beta$

(iv) $F = \bigcup_{\alpha\in On} F_\alpha$.

This hierarchy (defined within $\mathbf{FSST}$) will satisfy $\mathbf{FSST}$ (since the necessary objects are formed by design) as long as there is such a well-ordering of the universe R (equivalent, in this context, to CH).[48] On this perspective, we think of the hierarchy as formed by taking definable power sets, and adding a single generic for each pair of forcing poset and family of dense sets. Thus under this perspective, a set is 'possible' if it is obtainable by definable power set or the 'next' (codified in the sense of R) generic for some $\mathbb{P}$ and $\mathcal{D}$.

The situation is not quite as clean as with $\mathbf{ZFC}$ and the V_α, since there is no theorem that *every* set of $\mathbf{FSST}$ must belong to the F-hierarchy.[49] Moreover, one might feel that the use of a notion of 'next' generic to be added is somewhat ad hoc. Certainly, if the $\mathbf{FSST}$ perspective is to be developed in any detail, these issues would have to be addressed. For the moment, however, we simply note that the above description nonetheless shows how we might think of $\mathbf{FSST}$ as corresponding to a coherent iterative process.

The existence of an iterative story not only gives us confidence that $\mathbf{FSST}$ is cogent, but also shows how we might have models of $\mathbf{ZFC}$ by missing out subsets. By looking at $\mathbf{FSST}$ through the lens of $H(\omega_1)$ in any $\mathbf{ZFC}$ model, we can see that on the one hand, we may have bounded countable transitive models of $\mathbf{ZFC}$ (and its extensions by large cardinal axioms). If a particular large cardinal axiom is true in a set-sized transitive $\mathbf{ZFC}$-model, then there is (by the usual Löwenheim-Skolem and Mostowski collapse argument) a countable transitive model of $\mathbf{ZFC}$ with the large cardinal axiom. Since these will reside in $H(\omega_1)$, there is a model of $\mathbf{FSST}$ containing a countable transitive model satisfying the large cardinal axiom and $\mathbf{ZFC}$. Moreover, we may also have inner models of $\mathbf{ZFC}$. To see this, work within $\mathbf{ZFC}$, and suppose that $0^\sharp$ exists. Then $L_{\omega_1} \models \mathbf{ZFC}$ (since ω_1 is inaccessible in L under the existence of $0^\sharp$). Further $L_{\omega_1} \subseteq H(\omega_1)$ with $L_{\omega_1} \notin H(\omega_1)$, and so it is possible to have an unbounded inner model of $\mathbf{ZFC}$ in a universe satisfying $\mathbf{FSST}$. However, the only way we can have this (given the F-hierarchy), is to leave out the relevant collapsing generics at successor stages.

Thus, by viewing axioms asserting the existence of uncountable sets as species of *large cardinal* axioms, we have another theory where 'maximality in width' is incompatible with large cardinal existence, *despite* the consistency of the axioms. Indeed it is possible to have models of $\mathbf{ZFC}$ on this picture, but only by *leaving out* certain subsets (i.e. generics). The study of $\mathbf{ZFC}$ becomes like the study of measurable cardinals on the IMH-like picture; ultimately useful for studying logical properties and

[48]In fact, the presentation of the hierarchy can be streamlined somewhat: One does not need to 'manually' add the generics in the presence of the well-order. As it turns out, the generics go in automatically as soon as the poset and the collection of associated dense sets become countable. As long as the reals have a definable well-order with countable proper initial segments, one can can simply form the hierarchy where F_α is the least $\mathbf{ZFC}-$Powerset model containing the first α-many reals in the sense of the well-order. Since these F_αs are countable and their union is a model of $\mathbf{ZFC}-$Powerset + "Every set countable", such a version of the F-hierarchy would model $\mathbf{FSST}$. In virtue of this relationship with countability, one could also substitute collapsing functions witnessing the countability of sets for generics with the same result.

[49]Looking at $\mathbf{FSST}$ from the perspective of $H(\omega_1)$ in $\mathbf{ZFC}$, if $0^\sharp$ exists, it is a countable object [and hence a member of $H(\omega_1)$], but might not be a member of the F-hierarchy (since one cannot obtain $0^\sharp$ by set forcing). Thus (assuming large cardinals) there is a model of $\mathbf{FSST}$ that contains sets not in the F-hierarchy. Of course, one could just remedy the situation by adopting $V = F$ as an axiom. Since we would like to leave it open that there might be subset addition techniques other than set forcing that could be fed into the current picture, we do not pursue that strategy.

building models, but not actually *true*. Regarding a similar state of affairs, Meadows writes:

> "Observing this situation and given our claim there are not any really uncountable infinities, we might imagine ourselves as, so to speak, navigating an endless collection of these countable models in something like the generic multiverse we have described. While the illusion of multiple infinite cardinalities is witnessed inside each of the universes, we are free to move between them...I would like to make the provocative suggestion that forcing is a kind of natural revenge or dual to Cantor's theorem: where Cantor gives us the transfinite, forcing tears it down." ([Meadows, 2015], p205–206)

In this way, **FSST** codifies Meadow's intuition[50], and the picture we have described represents a peculiar fusion between so called 'actualist' and 'potentialist' frameworks. The universe of **FSST** exists absolutely and tells us what sets exist. The **ZFC**-worlds however, are all ultimately countable transitive models or inner models, and can be extended in many and varied ways. Again, importantly, we have a picture on which the existence of certain cardinals is incompatible with a notion of taking 'all sets possible' at each additional stage, and the 'large cardinal' axioms "ω_1 exists", "ω_2 exists", Powerset, and the usual large cardinals only serve to minimise subsets rather than maximise, despite the fact that they can perfectly well be consistent. A theorist who holds that **FSST** is the right theory for capturing the iterative process of subset formation should not be moved by the consistency of the Powerset Axiom (or any other large cardinal axiom) to its truth; ironically, for the friend of **FSST**, you can only have the Powerset Axiom by *missing out* subsets.

3 The proper place for large cardinals

Do we actually repudiate **ZFC**? Should we start using **FSST** as our set theory? On the one hand we think this is an interesting proposal, worthy of philosophical and technical scrutiny.[51] On the other, we regard **ZFC** (or possibly some extensions thereof) as our current best-justified theory of sets, and do not wish to demand that the entire practice of mathematics be overhauled. We only wish to point out that the status of large cardinals as maximality principles (and the related claim that consistency with **ZFC** is sufficient for large cardinal existence) is unconvincing without substantial further argumentation.

However, a substantive question remains: What happens to the study of large cardinals if we *do* adopt one of these anti-large-cardinal perspectives? Large cardinal theory would still remain an important mathematical subject matter and the axioms would still need to play a deep foundational role in our mathematical and set-theoretic reasoning. In this section, we'll examine some uses of large cardinals in foundational discussions and argue that these uses are not necessarily threatened by the current framework of maximality principles resulting in anti-large cardinal features.

[50] At least insofar as sets are concerned. We actually have two infinite 'sizes'; countably infinite, and proper-class-sized (equivalently continuum-sized). Only the former, however, corresponds to sizes of *sets*.

[51] One interesting philosophical and technical question is how to handle the continuum which becomes a proper class in **FSST**. Clearly some sort of second-order class theory is required, but we leave it open what form this might take.

What then are the uses of large cardinals in foundational discussions? We have already seen two in §1: (1.) To index the consistency strength of theories in a linearly ordered fashion, and (2.) To provide a framework theory that maximises interpretative power. However, we also see the following uses.

First, large cardinals are used in the case for so called "axioms of definable determinacy". The full details will be familiar to specialists and obscure to non-specialists, so we omit them here.[52] Roughly these axioms assert (schematic) claims about second-order arithmetic, postulating the existence of winning strategies for games played with natural numbers. Importantly, some authors have argued that these axioms have various pleasant consequences we would like to capture.[53] Moreover, whilst it is a theorem of $\mathbf{ZFC}$ that not all games are determined, certain restricted forms can be proved from large cardinals. For example:

Theorem 36. Borel Determinacy is provable in $\mathbf{ZFC}$, but any proof requires ω_1-many applications of the Powerset Axiom.

Theorem 37. Analytic Determinacy is provable in $\mathbf{ZFC}+$"There exists a measurable cardinal", but is independent from $\mathbf{ZFC}$.

Theorem 38. Projective Determinacy is provable in $\mathbf{ZFC}+$ "For every $n \in \mathbb{N}$, there are n-many Woodin cardinals", but is independent from $\mathbf{ZFC}+$ "There exists a measurable cardinal".

Theorem 39. The Axiom of Determinacy for $L(\mathbb{R})$ is provable in $\mathbf{ZFC}+$ "There are ω-many Woodin cardinals with a measurable above them all", but is not provable in $\mathbf{ZFC}+$ "For every $n \in \mathbb{N}$, there are n-many Woodin cardinals"

Again, we will not go through the definitions of Borel, Analytic, Projective, or $L(\mathbb{R})$ here. Suffice to say, each admits progressively more sets of reals with a more permissive notion of definability, and each is resolved by strictly stronger large cardinal axioms. So, assuming that out 'best' theory of sets should contain axioms of definable determinacy, it remains to explain how we might obtain them in the absence of large cardinals.

A second use of large cardinals is in the building and studying of different models. In particular, we want to construct various 'L-like' inner models from large cardinals. For many large cardinal axioms we can (using large cardinals) build a model containing the cardinal, but also with a good deal of information (in particular, these L-like models satisfy various so called 'fine-structural' properties). Again, the details are rather technical, so we omit them.[54] The point is the following: Often in set theory we have very little information about the properties of certain sets, as exhibited by the independence phenomenon. This is not so for large cardinals with L-like inner models, where (whilst there are open questions) there is a large amount of highly tractable information concerning the objects. The construction of inner models from large cardinals thus represents an important and technically sophisticated area of study.

[52]The interested reader is directed to [Schindler, 2014] for a recent presentation of the technical details, and [Koellner, 2006], [Maddy, 2011], and [Koellner, 2014] for a philosophical discussion.

[53]See, for example, [Maddy, 2011] and [Welch, 2017]. One salient fact is that Projective Determinacy yields high degree of completeness for the hereditarily countable sets [i.e. there are no known statements apart from Gödel style diagonal sentences independent from the theory $\mathbf{ZFC} + \mathsf{PD} + V = H(\omega_1)$]. [Koellner, 2014] provides a detailed survey of the literature. Koellner is quick to point out that axioms of definable determinacy seem to be the consequence of any strong 'natural' theory extending $\mathbf{ZFC}$ (e.g. $\mathbf{ZFC} + \mathsf{PFA}$), but we shall concern ourselves here with only the argument from large cardinals.

[54]For the state of the art concerning inner model theory and the challenges faced, see [Sargsyan, 2013] and [Woodin, 2017].

Extracting the philosophical upshots from the technical details, we note that some of the main uses for large cardinals are the following:[55]

(1.) To index the consistency strength of mathematical theories in a linearly-ordered fashion.

(2.) To provide a framework theory that maximises interpretive power.

(3.) To be used in the justifying axioms of definable determinacy.

(4.) To build various kinds of models (for their own intrinsic interest, not just issues of consistency), as in the Inner Model Programme.

Can the friend of anti-large cardinal maximality principles use large cardinal axioms for these purposes? Since the case of AC and Reinhardt cardinals does not go against any currently well-regarded large cardinals [and thus trivially will be able to incorporate each of (1.)–(4.)] we set it aside. The interesting cases are where we either have an inner-model-hypothesis-style principle or a situation in which every set is countable.

Point (1.) can be dealt with very quickly on both counts. In order to study the consistency strengths of mathematical theories, we only require that the theories be true in *some* model or other, not necessarily in V. More generally, note that there are the following 'levels' to where an axiom A can be true:

(i) A could be true in V.

(ii) A could be true in an inner model.

(iii) A could be true in a transitive model.

(iv) A could be true in a countable transitive model.

(v) A could be true in some model (whatever it may be).

For consistency statements, any model will do, and so any of (i)–(v) are acceptable places for considering A. As we have seen, there is no obstacle to having any of (ii)–(v) for the friend of anti-large-cardinal principles.

Point (2.) can also be dealt with reasonably easily. In order to maximise interpretive power we just need *some* place where the relevant mathematics can be developed. Perhaps, however, in addition to (1.) there are additional philosophical requirements on what is acceptable. For example, maybe the models must be transitive, or if the relevant mathematics to be interpreted refers to uncountable objects, maybe the relevant models must actually be uncountable. Nonetheless the appropriate place may well be an inner model. If one is not wedded to the uncountability of the objects, even a countable transitive model may well do. As we have seen there is no special objection to existence of these kinds of structure within any of the anti-large cardinal set theories we have considered. To illustrate the point, consider the following two set theories:

(i) **ZFC**+"There is a supercompact cardinal".

(ii) **ZFC**+"There is a countable transitive model of **ZFC** + "There is a supercompact cardinal"" (**FSST** could also be substituted for the first occurrence **ZFC** here).

[55]See also [Arrigoni and Friedman, 2013] for discussion of some of these uses.

(ii) has greater consistency strength (by Gödel's Second) than (i), and can interpret more mathematics. But (ii) does not commit us to the existence of any large cardinals, they can perfectly well live in the countable transitive model and do their interpretive work there. It is a simple point, but it is simply not true that maximising interpretive strength *necessitates* the addition of large cardinal axioms (other than as true in *some* model). While this is a point that proponents of large cardinal axioms are well aware of, given the worries we have presented about iterativity, it shows that the argument from interpretive strength (via maximality) to truth requires supplementation. It is, at this stage, unclear why the relevant context *must* be V, rather than some other appropriately 'nice' context.

Point (3.) is somewhat more subtle. Some of the principles we have considered (e.g. IMH) imply that PD is false outright.[56] However, it is open whether there could be an IMH-like principle with anti-large cardinal features that is nonetheless consistent with axioms of definable determinacy.[57] Moreover, it is *not* the case that a principle having anti-large cardinal features *immediately* disqualifies the justificatory case for PD found in the literature. This is because axioms of definable determinacy do not require the *literal truth* of large cardinal axioms, but rather only the truth of the large cardinals axioms in inner models. Generally speaking this is where there are equivalences (rather than strict implications from the large cardinals to axioms of definable determinacy). For example[58]:

Theorem 41. (Woodin) The following are equivalent:

(a) Projective Determinacy (schematically rendered).

(b) For every $n < \omega$, there is a fine-structural, countably iterable inner model $\mathfrak{M}$ such that $\mathfrak{M} \models$ "There are n Woodin cardinals".

Thus it may very well be the case that PD holds, there are plenty of Woodin cardinals in inner models, but no actual Woodin cardinals in V. More must be done to argue why the existence of such models must be *explained* by truth of the large cardinals, rather than the apparent consistency of the practice.[59] The friend of anti-large cardinal principles may acknowledge that the existence of an inner model theory is

[56] This is because the IMH implies that it is not the case that for every real x, $x^\sharp$ exists.

[57] For example, suppose that one is moved by justifications for Woodin cardinals and adopts **ZFC**+"There is a proper class of Woodin cardinals" as one's canonical theory of sets. Suppose further that one holds that some IMH-like principle should hold on the basis of absoluteness considerations. We might then formulate the following principle:

Definition 40. *The Inner Model Hypothesis for Woodins* states that if ϕ is true in an inner model of an outer model of V containing a proper class of Woodin cardinals, then ϕ is true in a inner model of V.

Assuming that the existence of a proper class of Woodin cardinals can be given an inner model theory, the results of [Friedman, 2006] could then be generalised to show that over the base theory **ZFC**+"There is a proper class of Woodin cardinals", the Inner Model Hypothesis for Woodins implies that there is no inaccessible limit of Woodin cardinals in V in the presence of PD.

[58] For a comprehensive list see [Koellner, 2011].

[59] This is perhaps what lies behind the following idea of Woodin:

> "**A Set Theorist's Cosmological Principle:** The large cardinal axioms for which there is an inner model theory are consistent; the corresponding predictions of unsolvability are true because the axioms are true." ([Woodin, 2011], p. 458)

Woodin's idea is that on the basis of consistency statements, we can make predictions. For example, "There will be no discovery of an inconsistency in the theory **ZFC**+"There is a Woodin cardinal" in the next 10'000 years" is a prediction ratified by the truth of the theory **ZFC**+"There is a proper class of Woodin cardinals".

good evidence that the axiom is consistent (perhaps even in an inner model), agreeing that the diverse theoretical relationships between models of large cardinals and axioms of definable determinacy are evidence for the consistency of the practice. For them, however, this consistency is to be explained by the existence of some model (possibly of a particular kind) rather than the strict truth of the axiom. Perhaps a supplementary argument can be provided. However, for the moment, any such claim stands in need of support and clarification.

Point (4.) is also subtle, and closely linked to (3.). As we have noted there are no obvious obstacles to having various kinds of model within an anti-large cardinal framework. A technicality here is that for many such model building purposes (such as under the Inner Model Programme), a substantial degree of interest lies in the production of a 'canonical' inner model. One assumes the truth of the large cardinal axiom, and then *uses* it to construct an inner model that is (in a certain technical sense) 'unique'.

Since axioms of definable determinacy are equivalent to the production of canonical models, any anti-large cardinal framework that also has the relevant axiom of definable determinacy can also have the required 'unique' models. However, in certain anti-large cardinal contexts (e.g. under the IMH) we lose the iterability of the models, and hence the production of a 'canonical' model for a certain large cardinal axiom. However, this does not prevent us from having particular *contexts* in which a unique model can be built (in particular the constructions will be possible in any model in which the relevant large cardinal axiom is true).

Again this highlights a difference (aside from considerations of truth) from the way the friend of anti-large-cardinal principles views the set-theoretic landscape. The point is just the following: A detailed examination of structural connections between diverse fields (as occurs with axioms of definable determinacy) may well provide her with evidence for their consistency. After all, it seems likely that such connections do not just arise by chance without also displaying any inherent contradiction. But, for her, this should only convince us that there is *some* structure exemplifying these relationships (in the set-theoretic case, some appropriately 'nice' model), not that these are *true*. That requires us to link the structural properties identified to those of V, a task to which the 'maximality implies consistency, and consistency implies existence' argument was meant to address. We have, however, shown that this line of thought requires more argument.

4 Is the anti-large cardinal perspective restrictive?

Thus far we've argued that there are natural interpretations of maximality based on iterative ideas that have anti-large cardinal features whilst allowing for their consistency, and that such frameworks can incorporate most (if not all) the uses to which large cardinal axioms have been put. Nonetheless one might try and raise the charge that supposed maximality axioms with anti-large cardinal features are somehow restrictive in that they do not allow us to have certain kinds of mathematical structure.

We already argued in §2 that there should be a priority of subset formation over attaching large cardinal properties to certain ordinals in virtue of the iterative conception of set. As it turns out, we can provide arguments to the effect that these anti-large cardinal frameworks are not restrictive, and indeed from their perspective large cardinal principles seem restrictive in *more* than just the sense of requiring the omission of subsets.

Consider the notion of restrictiveness proposed by Penelope Maddy[60] (with subsequent development by Incurvati and Löwe)[61], in which theories using large cardinals maximise over others that forbid their truth (in particular $V = L$). We will not delve into the technical details, but (roughly speaking) a theory $\mathbf{T}_1$ is restrictive in Maddy's sense iff it has a consistent extension to some theory $\mathbf{T}_2$, inconsistent with $\mathbf{T}_1$ and such that you can always find a 'nice' model of $\mathbf{T}_1$ in any model of $\mathbf{T}_2$, but not vice versa. More specifically, any model of $\mathbf{T}_2$ can be restricted to a smaller model in which $\mathbf{T}_1$ holds, either through moving to an inner model, truncation at an inaccessible, or moving to an inner model of a truncation at an inaccessible, but going the other way is not possible. In this way, so Maddy argues, $\mathbf{T}_2$ "proves there are more 'isomorphism types' than $\mathbf{T}_1$". Combining this with an account of the foundational goals of set theory, Maddy then argues that this justifies the truth of large cardinals.

We will not delve into the details of Maddy's proposal (or its developments[62]), however, the following points are in order: First, what counts as an appropriately 'nice' picture may well depend on one's outlook concerning the iterative conception. For example, it is unclear why an arbitrary transitive model is unacceptable interpretation, a modification which would change the whole notion of restrictiveness. In particular if one is moved on the basis of the iterative conception to adopt $\mathbf{FSST}$, the notion of arbitrary transitive model seems appropriate (and indeed, $\mathbf{FSST}$ and $\mathbf{ZFC}$ are incomparable in the restrictiveness ordering, since $\mathbf{FSST}$ is not a consistent extension of $\mathbf{ZFC}$).

Second, it turns out that in any case, some of the theories we have considered perform quite well with respect to Maddy-style restrictiveness (or close analogues thereof). For example, if we ditch the requirement that the theories must extend $\mathbf{ZFC}$ (let us say they must extend only $\mathbf{Z}$) the theory $\mathbf{FSST}$+"There exists an inner model for $\mathbf{ZFC}+\mathsf{A}$" (where A is some large cardinal axiom) properly maximises over $\mathbf{ZFC} + \mathsf{A}$. This is because $\mathbf{FSST}$ can only be true in a model containing only countable sets, and these are all inappropriate interpretations (for restrictiveness) within $\mathbf{ZFC}$. However, there is no (obvious) obstacle to having full inner models of $\mathbf{ZFC}+\mathsf{A}$ in $\mathbf{FSST}$. Moreover, we noted earlier that if V satisfies the Inner Model Hypothesis, then V contains inner models with measurable cardinals of arbitrarily large Mitchell order. This provides the resources to say that the IMH is not hugely restrictive in Maddy's terms since it shows the existence of many measurable cardinals in inner models. Interestingly, it is once again the case IMH and larger large cardinals (with assumptions on iterability) are incompatible in Maddy's restrictiveness ordering.[63] Moreover, again, simply adding in the assumption that A is true in some inner model

[60] See [Maddy, 1998].

[61] See [Incurvati and Löwe, 2016].

[62] As an example of a relevant detail and subsequent development: Maddy is admirably explicit about problems of false positives and false negatives. A modification in [Incurvati and Löwe, 2016] (by changing the notion of 'nice' interpretation) attempts to address this issue.

[63] This is because neither $\mathbf{NBG}$+IMH nor $\mathbf{ZFC}$+"There exist infinitely many Woodin cardinals" (with assumptions on the iterability of the relevant sharps) can have the other in appropriately nice models. It can't be the case that under infinitely many Woodin cardinals, the IMH is true in (i) an inner model, (ii) a truncation at an inaccessible, or (iii) an inner model of a truncation at an inaccessible. For (i), note that if the IMH is true in an inner model $\mathfrak{M}$ of V, then it is true in V, since any outer model of V is also an outer model of $\mathfrak{M}$, and hence $\mathfrak{M}$ already contains the required inner models to witness the IMH in V. For (ii), note that since the IMH implies the negation of PD, then if the IMH is true in some V_κ for κ inaccessible, then $\neg$PD holds in V_κ, and hence $\neg$PD holds in V (contradicting the large cardinals). Case (iii) follows immediately by (ii) and (i). In the other direction (assuming the IMH), note that in case (i) since the reals are not closed under $\sharp$, there can't be countably-iterable, fine-structural inner models for "There exist n-many Woodin cardinals" for every n. For cases (ii) and (iii), note that truncation at an inaccessible is trivial in the IMH-context.

on top of the IMH, will result in a theory that properly maximises over $\mathbf{ZFC} + \mathsf{A}$.[64]

Considerations of space prevent us from a thorough analysis of how anti-large cardinal principles based on maximality behave with respect to notions of restrictiveness. For now we simply note the following philosophical upshot: Many such principles can be fed into theories that perform quite well with respect to a natural interpretation of restrictiveness (i.e. Maddy's). Given the difficulty of providing a workable technical theory of restrictiveness, we will say no more about it, but simply note that it is not *obvious* that denying large cardinals on the basis of a width-maximising principle need be restrictive.

5 An important open question and concluding remarks

We have identified several problems with a view about the relationship between maximality, consistency, and truth for large cardinals, and examined several theories that merit further attention in order to help us understand the role of these axioms. Before we conclude, we propose an open question for further study.

While we have stated our claims in a somewhat strong tone, we do not wish to claim that there is *definitively* no relationship between **ZFC**-consistency and the truth of large cardinal axioms. We have provided some arguments to show that large cardinal existence should not be supported by a simplistic argument that the relevant axioms somehow 'maximise height' or state that the stages go 'on and on': It might be that such an apparent 'height' cannot be realised without leaving out subsets. This raises the following question in studying the link between large cardinals and consistency:

Question 42. Is there a good criterion for 'height maximality' that can operate more independently of background theory?

One conjecture is to say that only those height principles that are downward absolute should definitely count as maximising height, other principles are too dependent on the (presently unclear) notion of arbitrary subset to be accepted as definitely true on the basis of their consistency.[65] Thus, any large cardinal absolute between V and L [or perhaps even between V and the structure $(Ord, \in)$] should count as definitely height-maximising. This idea, however, does not get us very high in the large cardinal hierarchy (still within L). It is an interesting question whether there are other criteria that would allow us to draw a clean distinction between 'height' and 'width', and whether it is these principles for which these principles which respond to **ZFC**-consistency.

In sum, it seems that there are reasons to worry about the justification of large cardinal principles based on the idea that **ZFC**-consistency tells us how far it is 'possible' to continue the hierarchy. In particular, it the manner in which new sets are formed at successor stages under the iterative conception is critical in assessing whether or not the hierarchy can tolerate a particular large cardinal axiom. Focussing on developing a more precise account of subset formation, or *why* **ZFC**-consistency and large cardinal truth should be intertwined, is thus paramount for understanding the precise role of large cardinal axioms in the development of contemporary iterative set theory.

[64] This is because, by the arguments in the footnote above, the only (transitive) place the IMH can live in the presence of large cardinals is a countable model. Then, exactly as in the **FSST** case, we have that the theory $\mathbf{NBG} + \mathsf{IMH} +$"There is an inner model for A" shows A in an inner model, but $\mathbf{ZFC} + \mathsf{A}$ cannot.

[65] One idea is that the axiom of $\sharp$-generation might be such a principle, see [Friedman, F] for discussion.

References

[Arrigoni and Friedman, 2013] Arrigoni, T. and Friedman, S.-D. (2013). The Hyperuniverse Program. *Bulletin of Symbolic Logic*, 19:77–96.

[Bagaria and Brooke-Taylor, 2013] Bagaria, J. and Brooke-Taylor, A. (2013). On colimits and elementary embeddings. *The Journal of Symbolic Logic*, 78(2):562–578.

[Barton and Friedman, F] Barton, N. and Friedman, S.-D. (F). Maximality and ontology: How axiom content varies across philosophical frameworks. Forthcoming in *Synthese*.

[Bell, 2011] Bell, J. (2011). *Set Theory: Boolean-Valued Models and Independence Proofs*. Oxford University Press.

[Boolos, 1971] Boolos, G. (1971). The Iterative Conception of Set. *The Journal of Philosophy*, 68(8):215–231.

[Cantor, 1899] Cantor, G. (1899). Letter to Dedekind. In *[van Heijenoort, 1967]*, pages 113–117. Harvard University Press.

[Cohen, 1966] Cohen, P. (1966). *Set Theory and The Continuum Hypothesis*. W.A. Benjamin, Inc.

[Drake, 1974] Drake, F. R. (1974). *Set Theory: An Introduction to Large Cardinals*. North Holland Publishing Co.

[Ehrlich, 2012] Ehrlich, P. (2012). The absolute arithmetic continuum and the unification of all numbers great and small. *Bulletin of Symbolic Logic*, 18(1):1–45.

[Erdős and Hajnal, 1966] Erdős, P. and Hajnal, A. (1966). On a problem of B. Jónsson. *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. 14, 19-23 (1966).*

[Forster, 1995] Forster, T. (1995). *Set Theory With a Universal Set: Exploring an Untyped Universe*. Clarendon Press.

[Fraenkel et al., 1973] Fraenkel, A., Bar-Hillel, Y., and Levy, A. (1973). *Foundations of Set Theory*. Elsevier.

[Friedman, 1971] Friedman, H. M. (1971). Higher set theory and mathematical practice. *Annals of Mathematical Logic*, 2(3):325 – 357.

[Friedman, 2006] Friedman, S.-D. (2006). Internal consistency and the inner model hypothesis. *Bulletin of Symbolic Logic*, 12(4):591–600.

[Friedman, F] Friedman, S.-D. (F). Evidence for set-theoretic truth and the hyperuniverse programme. *IFCoLog Journal of Logic and its Applications*. Forthcoming.

[Friedman et al., 2008] Friedman, S.-D., Welch, P., and Woodin, W. H. (2008). On the consistency strength of the inner model hypothesis. *The Journal of Symbolic Logic*, 73(2):391–400.

[Gödel, 1931] Gödel, K. (1931). On formally undecidable propositions of *Principia mathematica* and related systems I. In *[Gödel, 1986]*, pages 144–195. Oxford University Press.

[Gödel, 1986] Gödel, K. (1986). *Collected Works, Volume I: Publications 1929-1936*. Oxford University Press. Edited by: Solomon Feferman (Editor-in-chief), John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, Jean van Heijenoort.

[Hallett, 1984] Hallett, M. (1984). *Cantorian Set Theory and Limitation of Size*. Oxford University Press.

[Hauser, 2001] Hauser, K. (2001). Objectivity over Objects: a Case Study in Theory Formation. *Synthèse*, 128(3).

[Hellman, 2006] Hellman, G. (2006). Against 'absolutely everything'! In Rayo, A. and Uzquiano, G., editors, *Absolute Generality*. Clarendon Press.

[Holmes, 1998] Holmes, R. (1998). *Elementary Set Theory with a Universal Set*. Bruylant-Academia.

[Holy et al., S] Holy, P., Lücke, P., and Njegomir, A. (S). Small embedding characterizations for large cardinals. Submitted, arXiv:1708.06103 [math.LO].

[(https://mathoverflow.net/users/2362/tim campion),] (https://mathoverflow.net/users/2362/tim campion), T. C. Does con(zf + reinhardt) really imply con(zfc + i0)? MathOverflow. URL:https://mathoverflow.net/q/237031 (version: 2016-05-15).

[Incurvati, 2012] Incurvati, L. (2012). How to be a minimalist about sets. *Philosophical Studies*, 159(1):69–87.

[Incurvati, 2014] Incurvati, L. (2014). The graph conception of set. *Journal of Philosophical Logic*, 43(1):181–208.

[Incurvati, F] Incurvati, L. (F). Maximality principles in set theory. Forthcoming in *Philosophia Mathematica*.

[Incurvati and Löwe, 2016] Incurvati, L. and Löwe, B. (2016). Restrictiveness relative to notions of interpretation. *The Review of Symbolic Logic*, 9(2):238–250.

[Incurvati and Murzi, 2017] Incurvati, L. and Murzi, J. (2017). Maximally consistent sets of instances of naive comprehension. *Mind*, 126(502).

[Jech, 2002] Jech, T. (2002). *Set Theory*. Springer.

[Kanamori, 2009] Kanamori, A. (2009). *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer, 2nd edition.

[Koellner, 2003] Koellner, P. (2003). *The Search for New Axioms*. PhD thesis, Massachusetts Institute of Technology.

[Koellner, 2006] Koellner, P. (2006). On the question of absolute undecidability. *Philosophia Mathematica*, 14:153–188.

[Koellner, 2011] Koellner, P. (2011). Independence and large cardinals. In *The Stanford Encyclopedia of Philosophy*. `http://plato.stanford.edu/archives/sum2011/entries/independence-large-cardinals/`, Summer 2011 edition.

[Koellner, 2014] Koellner, P. (2014). Large cardinals and determinacy. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2014 edition.

[Kreisel, 1980] Kreisel, G. (1980). Kurt gödel: 1906-1978. *Biographical Memoirs of Fellows of the Royal Society*, 26:149–224.

[Kunen, 1971] Kunen, K. (1971). Elementary embeddings and infinitary combinatorics. *The Journal of Symbolic Logic*, 36:407–413.

[Kunen, 2013] Kunen, K. (2013). *Set Theory*. College Publications.

[Mac Lane, 1971] Mac Lane, S. (1971). *Categories for the Working Mathematician*. Springer.

[Maddy, 1997] Maddy, P. (1997). *Naturalism in Mathematics*. Oxford University Press.

[Maddy, 1998] Maddy, P. (1998). $v = l$ and maximize. In Makowsky, J. A. and Ravve, E. V., editors, *Proceedings of the Annual European Summer Meeting of the Association of Symbolic Logic*, pages 134–152. Springer.

[Maddy, 2011] Maddy, P. (2011). *Defending the Axioms*. Oxford University Press.

[Magidor, U] Magidor, M. (U). Some set theories are more equal. Unpublished.

[McLarty, 1992] McLarty, C. (1992). *Elementary Categories, Elementary Toposes*. Clarendon Press.

[Mclarty, 2004] Mclarty, C. (2004). Exploring categorical structuralism. *Philosophia Mathematica*, 12(1):37–53.

[McLarty, 2010] McLarty, C. (2010). What does it take to prove Fermat's Last Theorem? Grothendieck and the logic of number theory. *The Bulletin of Symbolic Logic*, 16(3):359–377.

[Meadows, 2015] Meadows, T. (2015). Naive infinitism. *Notre Dame Journal of Formal Logic*, 56(1):191–212.

[Parente, 2012] Parente, F. (2012). Boolean valued models, saturation, forcing axioms. Master's thesis, Università di Pisa. Unpublished Master's Thesis.

[Paseau, 2007] Paseau, A. (2007). Boolos on the justification of set theory†. *Philosophia Mathematica*, 15(1):30–53.

[Potter, 2004] Potter, M. (2004). *Set Theory and its Philosophy: A Critical Introduction*. Oxford University Press.

[Ramsey, 1926] Ramsey, F. P. (1926). The foundations of mathematics. *Proceedings of the London Mathematical Society*, s2-25(1):338–384.

[Sargsyan, 2013] Sargsyan, G. (2013). Descriptive inner model theory. *Bulletin of Symbolic Logic*, 19(1):1–55.

[Schindler, 2014] Schindler, R. (2014). *Set Theory: Exploring Independence and Truth*. Springer.

[Scott, 1977] Scott, D. (1977). Foreword to *boolean-valued models and independence proofs*. In *[Bell, 2011]*, pages xiii–xviii. Oxford University Press.

[Steel, 2014] Steel, J. (2014). Gödel's program. In Kennedy, J., editor, *Interpreting Gödel*. Cambridge University Press.

[van Heijenoort, 1967] van Heijenoort, J., editor (1967). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press.

[Viale, 2016] Viale, M. (2016). Useful axioms. *ArXiv e-prints*.

[Welch, 2017] Welch, P. (2017). *Obtaining Woodin's cardinals*, pages 161–170. Contemporary Mathematics. American Mathematical Society, United States.

[Woodin, 2011] Woodin, W. H. (2011). The transfinite universe. In Baaz, M., editor, *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, page 449. Cambridge University Press.

[Woodin, 2017] Woodin, W. H. (2017). In search of ultimate-l the 19th midrasha mathematicae lectures. *The Bulletin of Symbolic Logic*, 23(1):1–109.