Preprint of a paper published in the Journal of Philosophy 115(5), May 2018
(differs slightly from the published version)
Official published version available at: https://doi.org/10.5840/jphil2018115516

From degrees of belief to binary beliefs: Lessons from judgment-aggregation theory*

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August/September 2014, revised August 2017

Abstract

What is the relationship between degrees of belief and binary beliefs? Can the latter be expressed as a function of the former – a so-called "belief-binarization rule" – without running into difficulties such as the lottery paradox? We show that this problem can be usefully analyzed from the perspective of judgment-aggregation theory. Although some formal similarities between belief binarization and judgment aggregation have been noted before, the connection between the two problems has not yet been studied in full generality. We seek to fill this gap. This paper is organized around a baseline impossibility theorem, which we use to map out the space of possible solutions to the belief-binarization problem. Our theorem shows that, except in limiting cases, there exists no belief-binarization rule satisfying four initially plausible desiderata. Surprisingly, this result is a direct corollary of the judgment-aggregation variant of Arrow's classic impossibility theorem in social choice theory.

1 Introduction

We routinely make belief ascriptions of two kinds. We speak of an agent's *degrees of belief* in some propositions and also of the agent's *beliefs* simpliciter. On the standard

*This paper was presented at seminars and workshops at the Australian National University, 7/2014, the University of Stockholm, 10/2014, the Centre d'Économie de la Sorbonne, 12/2014, New York University Abu Dhabi, 2/2015, the University of Nottingham, 3/2015, the Institute for Logic, Language and Computation, Amsterdam, 3/2015, the University of Delft, 6/2015, and the University of Munich, 4/2016. We are grateful to the participants for their comments. We also thank Geoffrey Brennan, Rachael Briggs, John Broome, David Chalmers, Jake Chandler, Kevin Coffey, Alan Hájek, James Joyce, Hanti Lin, Aidan Lyon, David Makinson, Matthew Silverstein, Daniel Stoljar, Jon Williamson, and two anonymous reviewers for feedback. Christian List further thanks the Australian National University, the Leverhulme Trust, and the Franco-Swedish Programme in Philosophy and Economics for support.

picture, *degrees of belief* (or *credences*) take the form of subjective probabilities the agent assigns to the propositions in question, for example a subjective probability of $\frac{1}{2}$ for the proposition that a coin, which has been tossed but not observed, has landed "heads". *Beliefs* (also known as *full beliefs*, *all-out beliefs*, or *binary beliefs*) take the form of the agent's overall acceptance of some propositions and non-acceptance of others, such as when one accepts that the Earth is round or that 2+2=4, but not that there are trees on Mars. The agent's *belief set* consists of all the propositions that he or she accepts in this all-or-nothing sense. What is the relationship between degrees of belief and binary beliefs? Can the latter be expressed as a function of the former and, if so, what does this function look like, in formal terms? Call this the *belief-binarization problem*.[1]

A widely studied class of belief-binarization rules is the class of *threshold rules*, according to which an agent believes a proposition (in the binary sense) if and only if he or she has a high-enough degree of belief in it. Threshold rules, however, run into the well-known *lottery paradox* (Kyburg 1961). Suppose, for example, that an agent believes of each lottery ticket among a million tickets that this ticket will not win, since his or her degree of belief in this proposition is 0.999999, which, for the sake of argument, counts as "high enough". The believed propositions then imply that no ticket will win. But the agent knows that this is false and has a degree of belief of 1 in its negation: some ticket will win. This illustrates that, under a threshold rule, the agent's belief set may be neither implication-closed (some implications of believed propositions are not believed) nor logically consistent (some beliefs contradict others). The belief-binarization problem has recently received renewed attention (e.g., by Leitgeb 2014, Lin and Kelly 2012a,b, Hawthorne and Bovens 1999, and Douven and Williamson 2006).[2]

In this paper, we reassess this problem from a different perspective: that of judgment-aggregation theory. This is the branch of social choice theory that investigates how we can aggregate several individuals' judgments on logically connected propositions into

[1]The term "belief binarization" captures the idea that we are looking for a function that takes non-binary beliefs (i.e., degrees of belief) as input and delivers binary beliefs as output; it thereby "binarizes" its non-binary inputs. However, readers who do not like the term "belief binarization" may alternatively speak of "belief identification". We thank an anonymous reviewer for raising this point.

[2]Leitgeb (2014) argues that rational belief corresponds to the assignment of a stably high rational degree of belief, where this is a joint constraint on degrees of belief and beliefs, not a reduction of one to the other. Lin and Kelly (2012a, b) use geometric and logical ideas to defend a class of belief-acceptance rules that avoid the lottery paradox, and explore whether reasoning with beliefs can track reasoning with degrees of belief. Hawthorne and Bovens (1999) discuss how to make threshold rules consistent. Douven and Williamson (2006) prove that belief-binarization rules based on a "structural" criterion for the acceptance of any proposition must require a threshold of 1 for belief *or* fail to ensure consistency.

collective judgments.[3] A multi-member court, for example, may have to aggregate several judges' verdicts on whether a defendant did some action (proposition p), whether that action was contractually prohibited (proposition q), and whether the defendant is liable for breach of contract (for which the conjunction $p \wedge q$ is necessary and sufficient). Finding plausible aggregation methods that secure consistent collective judgments is surprisingly difficult. In our example, there might be a majority for p, a majority for q, and yet a majority against $p \wedge q$, which illustrates that majority rule may fail to secure consistent and implication-closed collective judgments. This is reminiscent of a threshold rule's failure to secure consistent and implication-closed beliefs in belief binarization.

We will show that this reminiscence is not accidental: several key results in judgment-aggregation theory have immediate consequences for belief binarization, which follow once the formal apparatus of judgment-aggregation theory is suitably adapted. Although some similarities between the lottery paradox and the paradoxes of judgment aggregation have been discussed before (especially by Levi 2004, Douven and Romeijn 2007, and Kelly and Lin 2011), the focus has been on identifying lessons for judgment aggregation that can be learnt from the lottery paradox, not the other way round. A notable exception is Chandler (2013).[4] So far, there has been no comprehensive study of the lessons that we can learn for belief binarization from the large terrain of aggregation-theoretic impossibility and possibility results. We seek to fill this gap.[5]

We present a "baseline" impossibility theorem, which we use to map out the space of possible solutions to the belief-binarization problem. The theorem says that except in limiting cases, which we characterize precisely, there exists no belief-binarization rule satisfying four formal desiderata:

(i) *universal domain*: the rule should always work;

(ii) *consistency and completeness of beliefs*: beliefs should be logically consistent and complete, as explained in more detail later;

[3]See, e.g., List and Pettit (2002, 2004), Pauly and van Hees (2006), Dietrich (2006, 2007), Dietrich and List (2007a, b, 2013), Nehring and Puppe (2010), Dokow and Holzman (2010a, b), and for a survey List (2012). This work was inspired by legal scholarship on the "doctrinal paradox" (Kornhauser and Sager 1986). Social choice theory in the tradition of Condorcet and Arrow focuses on preference aggregation.

[4]Chandler's paper, which came to our attention as we were revising this paper, derives lessons for belief binarization from "distance-based" judgment aggregation; we return to this in Section 8.2.

[5]Douven and Romeijn (2007, p. 318) conclude their paper with an invitation to conduct the kind of study we embark on here: "given the liveliness of the debate on judgement aggregation, and the many new results that keep coming out of that, it is not unrealistic to expect that at least some theorems originally derived, or still to be derived, within that context can be applied fruitfully to the context of the lottery paradox, and will teach us something new, and hopefully also important, about this paradox."

(iii) *propositionwise independence*: whether or not one believes each proposition should depend only on the degree of belief in it, not on the degree of belief in others; and

(iv) *certainty preservation*: if the degrees of belief happen to take only the values 0 or 1 on all propositions, they should be preserved as the all-or-nothing beliefs.

The upshot is that any belief-binarization rule will satisfy at most three of the four desiderata, and we assess the available possibilities below. For example, if we replace the completeness requirement in desideratum (ii) with the requirement that beliefs be closed under logical implication (but not necessarily complete), then the only possible belief-binarization rule is the one that demands a degree of belief of 1 ("certainty") for belief simpliciter. This is no longer an impossibility, but still a triviality result.

Surprisingly, our main impossibility theorem is a corollary of the judgment-aggregation variant of Arrow's classic impossibility theorem in social choice theory.[6] Originally proved for preference aggregation, Arrow's theorem (1951/1963) shows that there are no non-dictatorial methods of aggregation that satisfy some plausible desiderata. Informally, there is no perfect democratic voting method. One of this paper's lessons is that the Arrovian impossibility carries over to belief binarization and, therefore, that the lottery paradox and the paradoxes of social choice can be traced back to a common source.

What can we learn from this? Just as Arrow's theorem establishes an inconsistency between some plausible requirements of social choice, so our analysis establishes the inconsistency between some desiderata on belief binarization that are, arguably, natural starting points for any investigation of the problem. The tools we import from judgment-aggregation theory allow us to pinpoint the precise (necessary and sufficient) conditions under which this inconsistency arises. Interestingly, it arises not only when the domain of beliefs is an entire algebra of propositions (a standard assumption in formal epistemology), but also for sets of propositions that are much less rich (in a sense made precise in the Appendix). In sum, the conflict between the four desiderata is not just an isolated artifact of a few lottery-paradox examples, but a very general problem.

Furthermore, just as Arrow's theorem can be used to map out the space of possible aggregation methods in social choice theory, so our result yields a very general and novel taxonomy of the space of possible solutions to the belief-binarization problem. As we will see, some of those solutions are more compelling than others, and we suggest that the most palatable (or least unpalatable) solutions involve relaxing propositionwise independence or (if we wish to keep independence) weakening the closure requirements on beliefs.

[6]This variant, discussed in more detail below, was proved by Dietrich and List (2007a) and Dokow and Holzman (2010a), building on results in Nehring and Puppe (2010).

More broadly, our investigation is relevant to some metaphysical, psychological, and epistemological questions. We may be interested, for instance, in whether an agent plausibly has *both* degrees of belief *and* binary beliefs, and/or whether one of the two kinds of belief – say, the binary one – is just a more coarse-grained version of the other and perhaps reducible to it. Furthermore, even if neither kind of belief can be reduced to the other, we may still be interested in whether there is some other systematic connection between the two – such as one of supervenience – or whether they are, in principle, independent of one another. Finally, we may be interested in how *rational* beliefs relate to *rational* degrees of belief, even if, in the absence of rationality, the two could come apart. Our formal analysis of the belief-binarization problem is relevant to all of these questions. It can tell us what conditions the relationship between degrees of belief and binary beliefs could, or could not, satisfy, thereby constraining the substantive philosophical views one can consistently hold on this matter.

2 The parallels between belief binarization and judgment aggregation

To give a first flavour of the parallels between belief binarization and judgment aggregation, we begin with a simple example of a judgment-aggregation problem, which echoes our earlier example of the multi-member court. Suppose a committee of three experts has to make collective judgments on the propositions p, q, r, $p \wedge q \wedge r$, and their negations on the basis of the committee members' individual judgments. These are as shown in Table 1. The difficulty lies in the fact that there are majorities – in fact, two-thirds

Table 1: A judgment-aggregation problem

	p	q	r	$p \wedge q \wedge r$	$\neg(p \wedge q \wedge r)$
Individual 1	True	True	False	False	True
Individual 2	True	False	True	False	True
Individual 3	False	True	True	False	True
Proportion of support	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	1

majorities – in support of each of p, q, and r, but the conjunction of these propositions, $p \wedge q \wedge r$, is unanimously rejected and its negation, $\neg(p \wedge q \wedge r)$, unanimously accepted. Majority voting, or any supermajority rule under which a quota of $\frac{2}{3}$ is sufficient for the collective acceptance of any proposition, yields a set of accepted propositions that is neither implication-closed (it fails to include $p \wedge q \wedge r$ despite the inclusion of p, q, and r) nor consistent (it includes all of p, q, r, and $\neg(p \wedge q \wedge r)$). Pettit (2001) has called such problems *discursive dilemmas*, though they are perhaps best described simply as *majority inconsistencies*. A central goal of the theory of judgment aggregation is to find aggregation rules that generate consistent and/or implication-closed collective judgments while also satisfying some other desiderata (List and Pettit 2002).

A belief-binarization problem can take a similar form. Suppose an agent seeks to arrive at binary beliefs on the propositions p, q, r, $p \wedge q \wedge r$, and their negations, based on his or her degrees of belief. Suppose, specifically, the agent assigns an equal subjective probability of $\frac{1}{3}$ to each of three distinct possible worlds, in which p, q, and r have different truth-values, as shown in Table 2. Each world renders two of p, q, and r true and the other false. The bottom row of the table shows the agent's overall degrees of belief in the propositions. Here the difficulty lies in the fact that while the agent has a

Table 2: A belief-binarization problem

	p	q	r	$p \wedge q \wedge r$	$\neg(p \wedge q \wedge r)$
World 1 (subj. prob. $\frac{1}{3}$)	True	True	False	False	True
World 2 (subj. prob. $\frac{1}{3}$)	True	False	True	False	True
World 3 (subj. prob. $\frac{1}{3}$)	False	True	True	False	True
Degree of belief	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	1

relatively high degree of belief – namely $\frac{2}{3}$ – in each of p, q, and r, his or her degree of belief in their conjunction is 0, and the degree of belief in its negation is 1. Any threshold rule under which a degree of belief of $\frac{2}{3}$ suffices for all-out belief in any proposition (and, *a fortiori*, a rule with a "more-likely-than-not" threshold) yields a belief set that is neither implication-closed nor consistent. On the other hand, if we demand a higher threshold for including a proposition in the agent's belief set, that belief set will include only $\neg(p \wedge q \wedge r)$ and will therefore be incomplete with respect to many proposition-negation pairs – accepting neither p, nor $\neg p$, for instance. Other examples can be constructed in which more demanding threshold rules also lead to inconsistencies.

If we identify voters in Table 1 with possible worlds in Table 2, the parallels between our two problems should be evident. In this simple analogy, possible worlds in a belief-binarization problem play the role of individual voters in a judgment-aggregation problem, and the agent's degree of belief in any proposition plays the role of the proportion of individuals accepting it. In fact, the function that assigns to each proposition in a judgment-aggregation problem the proportion of individuals supporting it behaves formally like a probability function over these propositions. Though it is interpretationally different, it satisfies the constraints of probabilistic coherence (assuming individual judgment sets are consistent and complete). This already suggests that belief-binarization and judgment-aggregation problems are structurally similar (for earlier discussions of this similarity, see Levi 2004, Douven and Romeijn 2007, Chandler 2013).

Yet, there is an important difference in format. In a judgment-aggregation problem, we are usually given the entire *profile of individual judgments,* i.e., the full list of the individuals' judgment sets, as in the first three rows of Table 1. In a belief-binarization-problem, by contrast, we are only given an agent's *degrees of belief in the relevant propositions*, i.e., the last row of Table 2, summarizing his or her overall subjective probabilities. The possible worlds underpinning these probabilities are hidden from view. Thus the input to a belief-binarization problem corresponds, not to a full *profile of individual judgment sets*, but to a *propositionwise anonymous profile*, i.e., a specification of the *proportions* of individuals supporting the various propositions under consideration. This gives us, not a full table such as Table 1, but only its last row.[7] Indeed, in our subsequent formal analysis, possible worlds drop out of the picture.

In sum, a belief-binarization problem corresponds to a *propositionwise anonymous* judgment-aggregation problem, the problem of how to aggregate the final row of a table such as Table 1 into a single judgment set. We can view this as an aggregation problem with a special restriction: namely that when we determine the collective judgments, we must pay attention only to the *proportions* of individuals supporting each proposition and must disregard, for example, who holds which judgment set. A belief-binarization problem will then have a solution of a certain kind if and only if the corresponding propositionwise anonymous judgment-aggregation problem has a matching solution.

Of course, the theory of judgment aggregation has primarily focused, not on the aggregation of propositionwise anonymous profiles (final rows of the relevant tables), but on the aggregation of fully specified profiles (lists of judgment sets across all individuals, without the special restriction we have mentioned). We will see, however, that despite the more restrictive informational basis of belief binarization several results from judgment-

[7]The notion of a *propositionwise anonymous profile* should not be confused with that of an *anonymous profile* simpliciter. The former specifies the proportion of individuals supporting each proposition; the latter specifies the proportion of individuals supporting each *combination* of judgments. The example of Table 1 yields an anonymous profile in which the judgment sets $\{p,q,\neg r,\neg(p \wedge q \wedge r)\}$, $\{p,\neg q,r,\neg(p \wedge q \wedge r)\}$, and $\{\neg p,q,r,\neg(p \wedge q \wedge r)\}$ are each supported by $\frac{1}{3}$ of the individuals, which corresponds to a propositionwise anonymous profile in which p, q, and r are each supported by $\frac{2}{3}$ of the individuals, $p \wedge q \wedge r$ is supported by none of them, and $\neg(p \wedge q \wedge r)$ is supported by all. Propositionwise anonymous profiles correspond to equivalence classes of anonymous profiles, which correspond to equivalence classes of full profiles. Degree-of-belief functions are structurally equivalent to propositionwise anonymous profiles.

aggregation theory carry over.[8] We will now make this precise.

3 Belief binarization formalized

We begin with a formalization of the belief-binarization problem. Let X be the set of propositions on which beliefs are held, where propositions are subsets of some underlying set of worlds.[9] We call X the *proposition set*. For the moment, our only assumption about the proposition set is that it is non-empty and closed under negation (i.e., for any proposition p in X, its negation $\neg p$ is also in X). In principle, the proposition set can be an entire *algebra* of propositions, i.e., a set of propositions that is closed under negation and conjunction and thereby also under disjunction.

A *degree-of-belief function* is a function Cr that assigns to each proposition p in X a number $Cr(p)$ in the interval from 0 to 1, where this assignment is probabilistically coherent.[10] A *belief set* is a subset $B \subseteq X$. It is called *consistent* if B is a consistent set, *complete* (relative to X) if it contains a member of each proposition-negation pair $p, \neg p$ in X, and *implication-closed* (relative to X) if it contains every proposition p in X that is entailed by B. Consistency and completeness jointly imply implication-closure.

A *belief-binarization rule* for X is a function f that maps each degree-of-belief function Cr on X (within some domain of admissible such functions) to a belief set $B = f(Cr)$. An important class of binarization rules is the class of *threshold rules*. Here there exists some threshold t in $[0, 1]$, which can be either *strict* or *weak*, such that, for every admissible degree-of-belief function Cr, the belief set B is the following:

$$B = \{p \in X : Cr(p) \text{ exceeds } t\},$$

where "$Cr(p)$ exceeds t" means

[8] Douven and Romeijn (2007) and Kelly and Lin (2011) proceed the other way round and derive some impossibility results for anonymous judgment aggregation from analogous results on belief binarization. These results differ from the canonical "Arrovian" impossibility result on judgment aggregation, on which we focus here (Dietrich and List 2007a, Dokow and Holzman 2010a, Nehring and Puppe 2010). The latter cannot be derived from any belief-binarization results, given the richer informational basis of judgment aggregation. (Belief binarization corresponds to *propositionwise anonymous* judgment aggregation.)

[9] The following definitions apply. Let Ω be some non-empty set of possible worlds. A *proposition* is a subset $p \subseteq \Omega$. For any proposition p, we write $\neg p$ to denote the complement (*negation*) of p, i.e., $\Omega \backslash p$. For any two propositions p and q, we write $p \wedge q$ to denote their intersection (*conjunction*), i.e., $p \cap q$; and $p \vee q$ to denote their union (*disjunction*), i.e., $p \cup q$. A set S of propositions is *consistent* if its intersection is non-empty, i.e., $\bigcap_{p \in S} p \neq \emptyset$; S *entails* another proposition q if the intersection of all propositions in S is a subset of q, i.e., $\bigcap_{p \in S} p \subseteq q$. A proposition p is *tautological* if $p = \Omega$ and *contradictory* if $p = \emptyset$.

[10] Formally, Cr is a function from X into $[0, 1]$ which is extendable to a probability function (with standard properties) on the algebra generated by X (which is the smallest algebra including X).

$$
\begin{aligned}
&Cr(p) > t \text{ in the case of a strict threshold}\\
\text{and}\quad &Cr(p) \geq t \text{ in the case of a weak threshold.}
\end{aligned}
$$

More generally, we can relativize thresholds and their designations as *strict* or *weak* to the propositions in question. We must then replace t in the expressions above with t_p, the threshold for proposition p, where each proposition-specific threshold can again be either *strict* or *weak*. If the threshold, or its designation as *strict* or *weak*, differs across propositions, we speak of a *non-uniform* threshold rule, to distinguish it from the *uniform* rules with an identical threshold for all propositions. Threshold rules are by no means the only possible belief-binarization rules; later, we consider other examples.

We now introduce four desiderata that we might, at least initially, expect a belief-binarization rule to meet; we subsequently discuss their relaxation. The first desideratum says that the belief-binarization rule should always work, no matter which degree-of-belief function is fed into it as input.

Universal domain. The domain of f is the set of all degree-of-belief functions on X.

So, we are looking for a universally applicable solution to the belief-binarization problem. Later, we also consider belief-binarization rules with restricted domains.

The second desideratum says that the belief set generated by the belief-binarization rule should always be consistent and complete (relative to X).

Belief consistency and completeness. For every Cr in the domain of f, the belief set $B = f(Cr)$ is consistent and complete.

Consistency is a plausible requirement on a belief set B (though we consider its relaxation too), but one may object that completeness is too demanding, since it rules out suspending belief on some proposition-negation pairs. Indeed, it would be implausible to defend completeness as a general requirement of rationality. However, for the purpose of characterizing the logical space of possible belief-binarization rules, it is a useful starting point, though to be relaxed subsequently. Note, further, that the present requirement demands completeness only relative to X, the proposition set under consideration.

The third desideratum is another useful baseline requirement. It says that whether or not one believes a given proposition p should depend only on the degree of belief in p, not on the degree of belief in other propositions.

Propositionwise independence. For any Cr and Cr' in the domain of f and any p in X, if $Cr(p) = Cr'(p)$ then $p \in B \Leftrightarrow p \in B'$, where $B = f(Cr)$ and $B' = f(Cr')$.

This rules out a "holistic" relationship between an agent's degrees of belief and his or her binary beliefs, where "holism" means that an agent's belief concerning a proposition p may depend on his or her degrees of belief in other propositions, not just in p. For example, if we sought to "reduce" binary beliefs to degrees of belief, then this would be easiest if an agent's binary belief concerning any proposition p was simply a function of his or her degree of belief in p. A holistic relationship between degrees of belief and beliefs, by contrast, would rule out such a simple reduction. At best, we might achieve a more complicated reduction, expressing an agent's belief concerning each proposition p as a function of his or her degrees of belief in a variety of other propositions. We later discuss examples of holistic belief-binarization rules.

The final desideratum is quite minimal. It says that, in the highly special case in which the degree-of-belief function is already binary (i.e., it only ever assigns degrees of belief 0 or 1 to the propositions in X), the resulting binary beliefs should be exactly as specified by that degree-of-belief function.

Certainty preservation. For any Cr in the domain of f, if Cr already assigns extremal degrees of belief (0 or 1) to all propositions in X, then, for every proposition p in X, B contains p if $Cr(p) = 1$ and B does not contain p if $Cr(p) = 0$, where $B = f(Cr)$.

Note that this desideratum imposes no restriction unless the degree-of-belief function assigns extremal values to *all* propositions in X. So, for instance, if Cr assigns a value of 0 or 1 to some propositions but a value strictly between 0 and 1 to others, then the antecedent condition is not met. We see little reason not to accept this desideratum, though for completeness, we later discuss its relaxation too.

It is easy to see that, in simple cases, our four desiderata can be met by a suitable threshold rule. For example, if the proposition set X contains only a single proposition p and its negation $\neg p$, or if it contains many logically independent proposition-negation pairs, the desiderata are met by any threshold rule that uses a (strict) threshold of t for p and a (weak) threshold of $1 - t$ for $\neg p$, where $0 \leq t < 1$. As we will see below, things become more difficult once the proposition set X is more complex.

4 Judgment aggregation formalized

We now move on to the formal definition of a judgment-aggregation problem (following List and Pettit 2002 and Dietrich 2007). The proposition set X remains as defined in the last section and is now interpreted as the set of propositions on which judgments are to be made. In judgment-aggregation theory, this set is also called the *agenda*. Let there be a finite set $N = \{1, 2, ..., n\}$ of individuals, with $n \geq 2$. Each individual i holds a *judgment set*, labelled J_i, which is defined just like a belief set in the previous section; the name "judgment set" is purely conventional. So J_i is a subset of X, which is called *consistent*, *complete*, and *implication-closed* if it has the respective properties, as defined above. As before, consistency and completeness jointly imply implication-closure. A combination of judgment sets across the n individuals, $\langle J_1, ..., J_n \rangle$, is called a *profile*. An example of a profile is given by the first three rows of Table 1 above, where the relevant proposition set X consists of p, q, r, $p \wedge q \wedge r$, and their negations.

A *judgment-aggregation rule* for X is a function F that maps each profile of individual judgment sets (within some domain of admissible profiles) to a collective judgment set J. Like the individual judgment sets, the collective judgment set J is a subset of X. The best-known example of a judgment-aggregation rule is *majority rule*: here, for each profile $\langle J_1, ..., J_n \rangle$, the collective judgment set consists of all majority-accepted propositions in X, formally

$$J = \{p \in X : |\{i \in N : p \in J_i\}| > \frac{n}{2}\}.$$

As we have seen, a shortcoming of majority rule is that, when the propositions in X are logically connected, the majority judgments may be inconsistent; recall Table 1.

We now state some desiderata that are often imposed on a judgment-aggregation rule. They are generalizations of Arrow's desiderata (1951/1963) on a preference-aggregation rule, as discussed later. The first desideratum says that the judgment-aggregation rule should accept as input any profile of consistent and complete individual judgment sets.

Universal domain. The domain of F is the set of all profiles of consistent and complete individual judgment sets on X.

Informally, the aggregation rule should be able to cope with "conditions of pluralism". It should not presuppose that there is already a certain amount of agreement between different individuals' judgments.

The second desideratum says that the collective judgment set produced by the aggregation rule should always be consistent and complete (again relative to X).

Collective consistency and completeness. For every profile $\langle J_1, ..., J_n \rangle$ in the domain of F, the collective judgment set $J = F(J_1, ..., J_n)$ is consistent and complete.

The consistency requirement is easy to justify: most real-world collective decision-making bodies – ranging from expert committees and courts to legislatures and the boards of organizations – are expected, at a minimum, to avoid inconsistencies in their collective judgments. Furthermore, in many (though not all) judgment-aggregation problems, completeness is a reasonable requirement as well, insofar as propositions are put on the agenda (i.e., included in the set X) precisely because they are supposed to be adjudicated. We also consider relaxations of this requirement below.

The third desideratum says that the collective judgment on any proposition p should depend only on the individual judgments on p, not on the individual judgments on other propositions.

Propositionwise independence. For any profiles $\langle J_1, ..., J_n \rangle$ and $\langle J'_1, ..., J'_n \rangle$ in the domain of F and any p in X, if $p \in J_i \Leftrightarrow p \in J'_i$ for every individual i in N, then $p \in J \Leftrightarrow p \in J'$, where $J = F(J_1, ..., J_n)$ and $J' = F(J'_1, ..., J'_n)$.

This captures the idea that when we aggregate judgments, we should consider each proposition independently. Although this requirement is often challenged and we relax it later, there are at least two familiar arguments in its support. First, propositionwise independence can be viewed as a requirement of informational parsimony in collective decision making: if an aggregation rule satisfies it, then we can determine the collective judgment on any proposition p by considering only the individual judgments on p. There are no holistic interaction effects, whereby the collective judgment on p may change due to a change in individual judgments on other propositions, with the individual judgments on p remaining equal. Such holistic interaction effects would complicate the relationship between individual and collective judgments and thereby make the aggregation rule potentially less transparent. Second, an aggregation rule that violates propositionwise independence is vulnerable to strategic voting: individuals may strategically influence the collective judgments on some propositions by misrepresenting their judgments on others. If one cares about non-manipulability, one has a *prima facie* reason to endorse independence as a requirement on judgment aggregation (Dietrich and List 2007c).

The final desideratum says that if all individuals hold the same individual judgment set, this judgment set should become the collective one.

Consensus preservation. For any unanimous profile $\langle J, ..., J \rangle$ in the domain of F, $F(J, ..., J) = J$.

Since consensus preservation imposes restrictions only when there is a universal consensus on all propositions on the agenda – not when there is a consensus only on some propositions without a consensus on others – it is rather undemanding (especially when the set X is large) and therefore hard to challenge.

As in our discussion of the four baseline desiderata on a belief-binarization rule, it is important to note that, in simple cases, the present desiderata can easily be met. For example, if the proposition set X contains only a single proposition p and its negation $\neg p$, or if it contains many logically independent proposition-negation pairs, then majority rule satisfies all four desiderata, as does a suitable super- or sub-majority rule.

5 The correspondence between belief binarization and judgment aggregation

We can now describe the relationship between belief binarization and judgment aggregation more precisely. Let f be a belief-binarization rule for the proposition set X. We show that, for any group size n, we can use f to construct a corresponding judgment-aggregation rule F for X. The construction is in two steps.

In the first step, we convert any given profile of consistent and complete individual judgment sets into the corresponding propositionwise anonymous profile, i.e., the specification of the proportion of individual support for each proposition in X. Formally, for each profile $\langle J_1, ..., J_n \rangle$, let $Cr_{\langle J_1,...,J_n \rangle}$ be the function from X into $[0, 1]$ that assigns to each proposition p in X the proportion of individuals accepting it:

$$Cr_{\langle J_1,...,J_n \rangle}(p) = \frac{|\{i \in N : p \in J_i\}|}{n}.$$

Although the function $Cr_{\langle J_1,...,J_n \rangle}$ is a "proportion-of-support" function on X, it behaves formally like a degree-of-belief function and can thus be mathematically treated as such a function. In particular, it is probabilistically coherent, since each individual judgment set in $\langle J_1, ..., J_n \rangle$ is consistent and complete.

In the second step, we apply the given belief-binarization rule f to the constructed proportion function $Cr_{\langle J_1,...,J_n \rangle}$ so as to yield a binary belief set, which can then be reinterpreted as a collective judgment set. As long as $Cr_{\langle J_1,...,J_n \rangle}$ is in the domain of f, the judgment set $J = f(Cr_{\langle J_1,...,J_n \rangle})$ is well-defined, so that $\langle J_1, ..., J_n \rangle$ is in the domain of the judgment-aggregation rule that we are constructing.

These two steps yield the judgment-aggregation rule F which assigns to each admissible profile $\langle J_1, ..., J_n \rangle$ the collective judgment set

$$F(J_1, ..., J_n) = f(Cr_{\langle J_1,...,J_n \rangle}).$$

Call this the *judgment-aggregation rule induced by the given belief-binarization rule.* Simply put, it aggregates any given profile of individual judgment sets by binarizing the proportion function which corresponds to that profile.

Proposition 1. The judgment-aggregation rule F induced by a belief-binarization rule f is *anonymous*, where anonymity is defined as follows.

Anonymity. F is invariant under permutations (relabellings) of the individuals. Formally, for any profiles $\langle J_1, ..., J_n \rangle$ and $\langle J'_1, ..., J'_n \rangle$ in the domain of F which are permutations of one another, $F(J_1, ..., J_n) = F(J'_1, ..., J'_n)$.

Proposition 1 is a consequence of the fact that the proportion of individuals accepting each proposition is not affected by permutations of those individuals. Formally, we have $Cr_{\langle J_1,...,J_n \rangle} = Cr_{\langle J'_1,...,J'_n \rangle}$ whenever the profiles $\langle J_1, ..., J_n \rangle$ and $\langle J'_1, ..., J'_n \rangle$ are permutations of one another. Furthermore, the following result holds:

Proposition 2. If the binarization rule f satisfies universal domain, belief consistency and completeness, propositionwise independence, and certainty preservation, then, for any group size n, the induced aggregation rule F satisfies universal domain, collective consistency and completeness, propositionwise independence, and unanimity preservation.

To show this, we suppose that the binarization rule f satisfies the relevant desiderata, and we take F to be the induced aggregation rule for a given group size n. Then:

(i) F satisfies universal domain because, for every profile $\langle J_1, ..., J_n \rangle$ of consistent and complete individual judgment sets, the function $Cr_{\langle J_1,...,J_n \rangle}$ is in the domain of f, and so $F(J_1, ..., J_n) = f(Cr_{\langle J_1,...,J_n \rangle})$ is well-defined.

(ii) F satisfies collective consistency and completeness because, for every profile $\langle J_1, ..., J_n \rangle$ in its domain, $f(Cr_{\langle J_1,...,J_n \rangle})$ is consistent and complete.

(iii) F satisfies propositionwise independence because, for any profiles $\langle J_1, ..., J_n \rangle$ and $\langle J'_1, ..., J'_n \rangle$ in its domain, if $p \in J_i \Leftrightarrow p \in J'_i$ for every individual i in N, then $Cr_{\langle J_1,...,J_n \rangle}(p) = Cr_{\langle J'_1,...,J'_n \rangle}(p)$, and so $p \in J \Leftrightarrow p \in J'$, where $J = f(Cr_{\langle J_1,...,J_n \rangle})$ and $J' = f(Cr_{\langle J'_1,...,J'_n \rangle})$ (by propositionwise independence of f).

(iv) F satisfies consensus preservation because, for any unanimous profile $\langle J, ..., J \rangle$ in its domain, $Cr_{\langle J,...,J \rangle}$ assigns extremal degrees of belief (0 or 1) to all propositions in X (namely 1 if $p \in J$ and 0 if $p \notin J$), and so we must have $f(Cr_{\langle J,...,J \rangle}) = J$ (by certainty preservation of f).

In sum, the existence of a belief-binarization rule satisfying our baseline desiderata guarantees, for every group size n, the existence of an anonymous judgment-aggregation rule satisfying the corresponding aggregation-theoretic desiderata. In the next section, we discuss the consequences of this fact.

6 An impossibility theorem

As noted, when the proposition set X is sufficiently simple, such as $X = \{p, \neg p\}$, we can indeed find belief-binarization rules for X that satisfy our four desiderata. Similarly, for such a set X, we can find judgment-aggregation rules satisfying the corresponding aggregation-theoretic desiderata. We now show that this situation changes dramatically when X is more complex. In this section, we state and prove the simplest version of our impossibility result. To state this result, call a proposition set X a *non-trivial algebra* if, in addition to being closed under negation, it is closed under conjunction (equivalently, under disjunction) and it contains more than one contingent proposition-negation pair (where a proposition p is *contingent* if it is neither tautological, nor contradictory).

Theorem 1. For any non-trivial algebra X, there exists no belief-binarization rule satisfying universal domain, belief consistency and completeness, propositionwise independence, and certainty preservation.

To prove this result, suppose, contrary to Theorem 1, there exists a belief-binarization rule satisfying all four desiderata for some non-trivial algebra X. Call this binarization rule f. Consider the judgment-aggregation rule F induced by f via the construction described in the last section, for some group size $n \geq 2$. By Proposition 1, F satisfies anonymity. By Proposition 2, since f satisfies the four baseline desiderata on belief binarization, F satisfies the corresponding four aggregation-theoretic desiderata. However, the following result is well known to hold, as referenced and explained further below:

Background Result 1. For any non-trivial algebra X, any judgment-aggregation rule satisfying universal domain, collective consistency and completeness, propositionwise independence, and consensus preservation is *dictatorial*: there is some fixed individual i in N such that, for each profile $\langle J_1, ..., J_n \rangle$ in the domain, $F(J_1, ..., J_n) = J_i$.

So, there could not possibly exist an *anonymous* (and thereby non-dictatorial) aggregation rule satisfying all four conditions. Hence the belief-binarization rule f on which the aggregation rule F was based could not satisfy our four desiderata on belief binarization, contrary to our supposition. This completes the proof of Theorem 1.

Since subjective probability functions are normally defined on algebras, Theorem 1 shows that our four baseline desiderata are mutually inconsistent when we wish to binarize a full-blown subjective probability function, except in trivial cases. In the Appendix, we present a more general version of this impossibility result, derived from a more general version of Background Result 1 (due to Dietrich and List 2007a, Dokow and Holzman 2010a, building on Nehring and Puppe 2010). The more general theorem and background result are exactly like their simplified counterparts stated here, except that they replace the assumption that the proposition set X is a non-trivial algebra with the less demanding assumption that X satisfies a combinatorial property called "strong connectedness". A non-trivial algebra is just one instance of a "strongly connected" proposition set. Other proposition sets, which fall short of being algebras, qualify as "strongly connected" too, and so the impossibility result applies to them as well.

7 A sibling of Arrow's impossibility theorem

The significance of Background Result 1 lies in the fact that – in its general form – it is the judgment-aggregation variant of Arrow's classic impossibility theorem in social choice theory. This, in turn, means that our impossibility theorem on belief-binarization and Arrow's theorem are siblings in logical space: they can be derived from a common parent theorem. To explain this point, it is useful to revisit Arrow's original result (1951/1963).[11]

As already noted, Arrow considered the aggregation of preferences, rather than judgments. Let $N = \{1, 2, ..., n\}$ be a finite set of individuals, with $n \geq 2$, each of whom holds a preference ordering, P_i, over some set $K = \{x, y, ...\}$ of options. Interpretationally, the elements of K could be electoral candidates, policy proposals, or states of affairs, and each P_i ranks them in some order of preference (e.g., from best to worst). A combination of preference orderings across the n individuals, $\langle P_1, ..., P_n \rangle$, is called a *profile* of preference orderings. We are looking for a *preference-aggregation rule*, $\mathcal{F}$, which is a function that maps each profile of individual preference orderings (within some domain of admissible profiles) to a collective preference ordering P. Arrow imposed four conditions on a preference-aggregation rule, which were the initial inspiration for the four baseline requirements on judgment aggregation that we have already discussed.

Universal domain. The domain of $\mathcal{F}$ is the set of all profiles of rational individual preference orderings. (We here call a preference ordering *rational* if it is a transitive, irreflexive, and complete binary relation on K; for expositional simplicity, we thus restrict our attention to indifference-free preference orderings.)

Collective rationality. For every profile $\langle P_1, ..., P_n \rangle$ in the domain of $\mathcal{F}$, the collective preference ordering $R = \mathcal{F}(P_1, ..., P_n)$ is rational.

Pairwise independence. For any profiles $\langle P_1, ..., P_n \rangle$ and $\langle P'_1, ..., P'_n \rangle$ in the domain

[11] We here follow the analysis in Dietrich and List (2007a). For a precursor, see List and Pettit (2004).

of $\mathcal{F}$ and any pair of options x and y in K, if P_i and P'_i rank x and y in the same way for every individual i in N, then P and P' also rank x and y in the same way, where $R = \mathcal{F}(P_1, ..., P_n)$ and $R' = \mathcal{F}(P'_1, ..., P'_n)$.

The Pareto principle. For any profile $\langle P_1, ..., P_n \rangle$ in the domain of $\mathcal{F}$ and any pair of options x and y in K, if P_i ranks x above y for every individual i in N, then P also ranks x above y, where $R = \mathcal{F}(P_1, ..., P_n)$.

Arrow's original theorem (1951/1963) now asserts the following:

Arrow's theorem. For any set K of three or more options, any preference-aggregation rule satisfying universal domain, collective rationality, pairwise independence, and the Pareto principle is *dictatorial*: there is some fixed individual i in N such that, for each profile $\langle P_1, ..., P_n \rangle$ in the domain, $\mathcal{F}(P_1, ..., P_n) = P_i$.

To confirm that Background Result 1 (in its fully general form) is indeed a generalization of Arrow's theorem, we note that the latter can be derived from the former. The key observation is that, setting aside interpretational differences, we can represent any preference-aggregation problem formally as a special kind of judgment-aggregation problem. The representation is surprisingly simple. Let the set X of propositions on which judgments are made – the agenda – consist of all pairwise ranking propositions of the form "x is preferable to y", abbreviated xPy, where x and y are options in K and P represents pairwise preference. Formally,

$$X = \{xPy : x, y \in K \text{ with } x \neq y\}.$$

Call this proposition set the *preference agenda* for K. Under the simplifying assumption of irreflexive preferences, we can interpret yPx as the negation of xPy, and so the set X is negation-closed. We call any subset Y of X *consistent* if Y is a consistent set of binary ranking propositions relative to the rationality constraints on preferences introduced above (transitivity etc.).[12] For example, the set $Y = \{xPy, yPz, xPz\}$ is consistent, while the set $Y = \{xPy, yPz, zPx\}$ is not, as it involves a breach of transitivity.

Since any preference ordering P over K is just a binary relation, it can be uniquely represented by a subset of X, namely the subset consisting of all pairwise ranking propositions validated by P. In this way, rational preference orderings over K stand in a one-to-one correspondence with consistent and complete judgment sets for the preference agenda X. Further, preference-aggregation rules (for preferences over K) stand

[12] Technically, Y is consistent if and only if there exists at least one rational (here: transitive, irreflexive, and complete) preference ordering over K that validates all the binary ranking propositions in Y.

in a one-to-one correspondence with judgment-aggregation rules (for judgments on the associated preference agenda X). Now, applied to X, the judgment-aggregation desiderata of universal domain, collective consistency and completeness, and propositionwise independence reduce to Arrow's original desiderata of universal domain, collective rationality, and pairwise independence. Consensus preservation reduces to a weaker version of Arrow's Pareto principle, which says that if all individuals hold the same preference ordering over all options, this preference ordering should become the collective one.[13]

Of course, the proposition set X that we have just constructed is not an algebra: it is not closed under conjunction or disjunction. However, when K contains more than two options, X can be shown to be "strongly connected", in the sense defined in the Appendix, and so Background Result 1 in its general form can be applied, yielding Arrow's original theorem as a corollary.

Corollary of the judgment-aggregation variant of Arrow's theorem. For any preference agenda X defined for a set K of three or more options, any judgment-aggregation rule satisfying universal domain, collective consistency and completeness, propositionwise independence, and consensus preservation is dictatorial.

Figure 1 displays the logical relationships between (i) the judgment-aggregation variant of Arrow's theorem, (ii) Arrow's original theorem, and (iii) our baseline impossibility theorem on belief binarization. In short, Arrow's theorem and our result on belief binarization, which are at first sight very different from one another, can both be derived from the same common impossibility theorem on judgment aggregation.

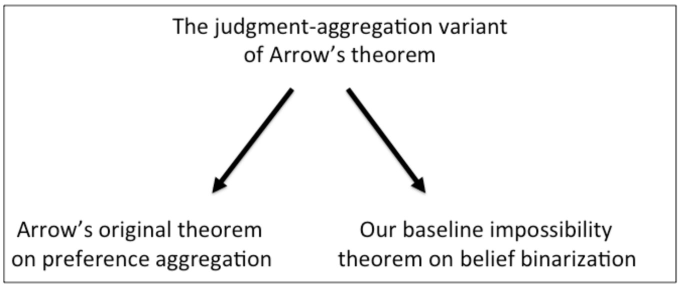

Figure 1: The common source of two distinct impossibility results

[13]This desideratum is implied by Arrow's Pareto principle (given collective rationality), but does not generally imply it. The converse implication holds under universal domain and pairwise independence.

8 Escape routes from the impossibility

If we wish to avoid the impossibility of belief binarization, we must relax at least one of the baseline desiderata we have introduced. We suggest the following tentative order of how plausible the desiderata are. (In this list, we split the consistency-and-completeness desideratum into its components.)

(1) Certainty preservation is very plausible as well as extremely undemanding: it only ever applies when the entire degree-of-belief function is already binary, meaning that it assigns no values other than 0 or 1 to any propositions. Not preserving an agent's beliefs in this special case would be hard to defend.

(2) Universal domain seems non-negotiable if our aim is to find a universally applicable belief-binarization rule. That said, it is common to study judgment aggregation in the context of certain domain restrictions, for instance by assuming that the amount of pluralism in individual judgments is limited. Analogously, one might ask whether we can find plausible belief-binarization rules if we restrict the admissible degree-of-belief functions. This suggests the theoretical possibility of relaxing universal domain.

(3) The consistency requirement on beliefs – part of the consistency-and-completeness desideratum – is also very plausible and familiar. Nonetheless, if full consistency is too difficult to achieve, one might opt for a less idealistic requirement, which demands only the avoidance of "blatantly inconsistent" beliefs, as discussed below. Perhaps real people do not have fully consistent beliefs and only manage to avoid "blatant" inconsistencies.

(4) Implication-closure, which is a consequence of the consistency-and-completeness desideratum, is another standard requirement on beliefs. The idea that beliefs should be implication-closed is responsible for the intuitive force of the lottery paradox. Implication-closure seems plausible when implication relations between propositions are transparent, for instance when the proposition set X is not very complex. However, when X is large and complex, requiring implication-closure is tantamount to requiring logical omniscience, which is no longer realistic, and thus relaxing it may sometimes be warranted.

(5) Propositionwise independence is arguably a stronger candidate for relaxation. As noted, it rules out a holistic relationship between degrees of belief and binary beliefs, by requiring the binary belief concerning each proposition to depend only on the degree of belief in that proposition, not on the degrees of belief in others. Since propositions form an interconnected web, however, some propositions are relevant to others, for instance by standing in premise–conclusion relations. So, we may plausibly let the belief on a proposition depend on the degrees of belief in all propositions relevant to it. Furthermore, the case against relaxing propositionwise independence is weaker in the context of belief binarization than in the context of judgment aggregation, where aggregation rules violating independence are vulnerable to strategic voting. There is presumably no such strategic vulnerability in belief binarization. Even in the context of judgment aggregation, the fact that some propositions are relevant to others is often seen as a reason to give up propositionwise independence. In light of our impossibility result, we may well conclude that an agent's binary belief on each proposition cannot be a function of his or her degree of belief in that proposition alone.

(6) Finally, completeness of beliefs – another part of the consistency-and-completeness desideratum – is the most natural candidate for relaxation. As noted, we introduced this requirement mainly for analytic purposes, and unlike in judgment aggregation, where a definitive adjudication of every agenda item is often needed, completeness is not a general requirement on binary beliefs.

In what follows, we discuss the escape routes from our impossibility result that open up if we relax these desiderata. We consider them in the reverse order of the list just given, beginning with the desiderata that seem most natural to give up.[14]

8.1 Relaxing completeness of beliefs

As noted, the initially most obvious response to our impossibility result is to argue that the completeness requirement on beliefs is too strong. There is nothing irrational about suspending belief on some proposition-negation pairs: neither believing the proposition, nor believing its negation. This suggests relaxing completeness, while retaining the familiar requirement that beliefs should be consistent and closed under logical implication (within the set X):

Belief consistency and implication-closure. For every Cr in the domain of f, the belief set $B = f(Cr)$ is consistent and implication-closed (relative to X).

This permits suspending belief on some proposition-negation pairs in X. (Indeed, even an empty belief set is consistent and implication-closed, assuming X contains no tautology.) Surprisingly, however, the use of this weaker desideratum does not get us very far if we insist on the other desiderata. Only a single, extremely conservative binarization rule becomes possible, namely a uniform threshold rule with threshold 1 for all propositions. This can be viewed as a triviality result, along the lines of other triviality results in the literature (e.g., Douven and Williamson 2006).

[14]For simplicity, we assume that the proposition set X is finite in Sections 8.2 to 8.5.

Theorem 2. For any non-trivial algebra X (more generally, any "strongly connected" proposition set), any belief-binarization rule satisfying universal domain, belief consistency and implication-closure, propositionwise independence, and certainty preservation is a threshold rule with a uniform threshold of 1 for the acceptance of any proposition, i.e., for any degree-of-belief function Cr in the domain, $f(Cr) = \{p \in X : Cr(p) = 1\}$.[15]

This result, too, is a consequence of a result on judgment aggregation, though the proof is a bit longer than that of Theorem 1. Consider a proposition set X with the specified properties, and suppose f is a belief-binarization rule satisfying the desiderata listed in Theorem 2. As before, for any group size n, f induces an anonymous judgment-aggregation rule F. By the analogue of Proposition 2, F satisfies universal domain, collective consistency and implication-closure, propositionwise independence, and consensus preservation. The following result holds:

Background Result 2. For any "strongly connected" proposition set X, any judgment-aggregation rule satisfying universal domain, collective consistency and implication-closure, propositionwise independence, and consensus preservation is *oligarchic*: there is some fixed non-empty set M of individuals in N such that, for each profile $\langle J_1, ..., J_n \rangle$ in the domain, $F(J_1, ..., J_n) = \bigcap_{i \in M} J_i$ (Dietrich and List 2008, Dokow and Holzman 2010b).

The set M of "oligarchs" could be any non-empty subset of N, ranging from a singleton set, where $M = \{i\}$ for some individual i, to the set of *all* individuals, where $M = N$. In the first case, the aggregation rule is dictatorial; in the last, it is the *unanimity rule*. Since any aggregation rule induced by a belief-binarization rule is anonymous, and an anonymous aggregation rule can be oligarchic *only if* it is the unanimity rule, Background Result 2 immediately implies that the induced rule F is the unanimity rule. So, no proposition is collectively accepted under F with less than 100% support.

Could the belief-binarization rule f on which F is based still differ from a threshold rule with threshold 1? A slightly more technical argument shows that if f were distinct from such a rule, this would contradict what we have just learnt from Background Result 2.[16] And so f must be a threshold rule with a uniform threshold of 1 for the

[15] Unlike Douven and Williamson's result, our result applies not only to algebras, but to all "strongly connected" proposition sets, and it does not presuppose that the sufficient condition for belief acceptance is what Douven and Williamson call "structural"; rather, our desideratum of propositionwise independence allows, in principle, the use of different acceptance criteria for different propositions.

[16] Suppose f is *not* a threshold rule with threshold 1. Then there must exist a proposition q in X and a degree-of-belief function Cr with $Cr(q) < 1$ such that $q \in B$, where $B = f(Cr)$. (Here q must be contingent: if q were tautological, we could not have $Cr(q) < 1$; if it were contradictory,

acceptance of any proposition, as stated by Theorem 2. The bottom line is that relaxing the requirement of completeness of beliefs alone, while retaining all other desiderata, does not open up a very strong escape route from our impossibility result.

8.2 Relaxing propositionwise independence

A more promising escape route from the impossibility involves giving up the requirement that the binary belief on any proposition p depend exclusively on the degree of belief in p, not on the degrees of belief in other propositions. Instead, we may admit a more "holistic" dependence of beliefs on degrees of belief, by taking an agent's belief on p to be a function of his or her degrees of belief across several propositions – in the limit, an entire "web" of propositions. The "units of binarization" will then no longer be individual propositions in isolation, but suitable *sets* of propositions.

How might one argue for such a more holistic approach to belief binarization? One natural thought is that beliefs in the all-or-nothing sense pick out "salient peaks" in the "credence landscape", such as propositions to which the agent assigns high credence compared to their salient alternatives.[17] Relatedly, it is plausible to suggest that the binary belief concerning any proposition p should be formed upon considering the degrees of belief in all those propositions that are *relevant* to it.

To formalize these ideas, it is helpful to introduce the notion of a *relevance relation* between propositions (another import from judgment-aggregation theory; see Dietrich 2015). Formally, this is a binary relation $\mathcal{R}$ on the proposition set X, where $q\mathcal{R}p$ is interpreted to mean that q is "relevant" to p. For any proposition p in the set X, we write $\mathcal{R}(p)$ to denote the set of all propositions q in X that are relevant to p, formally

$$\mathcal{R}(p) = \{q \in X : q\mathcal{R}p\}.$$

The key idea, now, is that the binary belief on p may depend on the degrees of belief in all propositions that are relevant to p. This suggests the following desideratum:

we could not have $q \in B$, given B's consistency.) In the Appendix, we show that, under the present conditions, f must be *monotonic*: if $q \in f(Cr)$, then $q \in f(Cr')$ for any other credence function Cr' with $Cr'(q) > Cr(q)$. Consider what this implies for any induced aggregation rule F. Pick two consistent and complete judgment sets $J, J' \subseteq X$ such that $q \in J$ and $q \notin J'$, and construct a profile $\langle J_1, ..., J_n\rangle$ (for a sufficiently large n) such that a proportion of more than $Cr(q)$ of the individuals in N, but fewer than all, have the judgment set J and the rest have the judgment set J'. By the construction of F, we have $F(J_1, ..., J_n) = f(Cr_{\langle J_1,...,J_n\rangle})$, where for each proposition p in X, $Cr_{\langle J_1,...,J_n\rangle}(p) = \frac{|\{i \in N: p \in J_i\}|}{n}$. Since $Cr_{\langle J_1,...,J_n\rangle}(q) > Cr(q)$ and f is monotonic, we must have $q \in f(Cr_{\langle J_1,...,J_n\rangle})$ and hence $q \in F(J_1, ..., J_n)$, despite the lack of unanimous support for q. This contradicts the fact that F is the unanimity rule.

[17] We are indebted to an anonymous reviewer for suggesting this point.

Independence of irrelevant propositions. For any Cr and Cr' in the domain of f and any p in X, if $Cr(q) = Cr'(q)$ for all $q \in \mathcal{R}(p)$, then $p \in B \Leftrightarrow p \in B'$, where $B = f(Cr)$ and $B' = f(Cr')$.

We think this desideratum is hard to contest, provided we have specified the relevance relation $\mathcal{R}$ adequately. Indeed, the demandingness of the present desideratum depends entirely on the specification of $\mathcal{R}$. The smaller the set $\mathcal{R}(p)$ of relevant propositions for each proposition p, the more restrictive the desideratum becomes. By contrast, the larger the set $\mathcal{R}(p)$ for each p, the more permissive it is.

Replacing propositionwise independence with independence of irrelevant propositions makes explicit the need to specify which propositions are relevant to which others, so that the binary beliefs on the latter may depend on the degrees of belief in the former. This move also makes clear how restrictive the original independence desideratum is. It seems justified only if no two distinct propositions are ever relevant to one another – a very restrictive assumption. Formally, propositionwise independence is the special case of independence of irrelevant propositions where $\mathcal{R}(p) = \{p\}$ for all p in X.

As we will now see, for plausible specifications of the relevance relation $\mathcal{R}$, we can avoid our impossibility result and find possible binarization rules. Again, they have counterparts in judgment-aggregation theory. We here give the most salient examples.

Premise-based rules. A premise-based rule exploits the fact that there may be certain premise–conclusion relationships between propositions. Specifically, we designate a subset Y of X as a set of "premises" (taking Y to be closed under negation), and assume that the premises are relevant to all other propositions, while those other propositions are not relevant to the premises. Formally, we assume that $\mathcal{R}(p) = Y$ for each p outside Y while $\mathcal{R}(p) = \{p\}$ for each p in Y. We now (i) form the binary beliefs on all premises by means of a suitable propositionwise independent binarization rule such as a threshold rule, and (ii) derive the binary beliefs on all other propositions by logical inference. Formally, for every degree-of-belief function Cr in the domain, we have

$$f(Cr) = \{p \in X : g(Cr|_Y) \text{ entails } p\},$$

where Y is the set of premises, g is the binarization rule (for Y) used to determine the beliefs for those premises, and $Cr|_Y$ is the restriction of the degree-of-belief function Cr to the premise set Y.[18] As long as the set Y and the rule g are chosen so as to guarantee

[18] For any admissible degree-of-belief function Cr, $Cr|_Y$ is the function from Y into the interval from 0 to 1 such that $Cr|_Y(p) = Cr(p)$ for each p in Y. Now g is a (propositionwise independent) function that assigns to each such restricted degree-of-belief function $Cr|_Y$ a belief set $B \subseteq Y$ for the premises.

a consistent set $g(Cr|_Y)$ (e.g., by making sure that the different proposition-negation pairs in Y are logically independent from one another), the premise-based rule will always yield consistent and implication-closed belief sets. Furthermore, given the way the relevance relation $\mathcal{R}$ has been specified, the rule satisfies independence of irrelevant propositions. Premise-based rules have been studied extensively in judgment-aggregation theory.[19] In premise-based aggregation, a group makes its collective judgments by taking majority votes only on some logically independent premises (e.g., "Did the defendant do a particular action?", "Was he or she contractually obliged not to do that action?") and deriving its judgments on all other propositions by logical interference (e.g., "Is the defendant liable for breach of contract?"). The downside of a premise-based aggregation rule is that a proposition can end up being collectively accepted by logical inference even if only a minority, or in the extreme case none, of the individuals accept it. Similarly, in premise-based belief binarization, a proposition could be included in the belief set despite the assignment of a very low, or even zero, degree of belief to it. In Table 2 above, taking p, q, and r to be the premises and applying an acceptance threshold of $2/3$ to them would lead to the acceptance of p, q, r, and $p \wedge q \wedge r$, despite the assignment of a zero degree of belief to $p \wedge q \wedge r$. To be sure, the output of a premise-based rule depends on what the specified set of premises is. Premise-based belief binarization, like premise-based aggregation, is plausible to the extent that we have a non-arbitrary way of selecting the premises and are prepared to generate our overall beliefs or judgments on the basis of considering those premises alone.

Sequential priority rules. Sequential priority rules are generalizations of premise-based rules. Here, we specify some *order of priority* among the propositions in X, representable by a relevance relation $\mathcal{R}$ that constitutes a linear order on X (a complete, transitive, and anti-symmetric binary relation). For each degree-of-belief function Cr, we then construct the belief set $B = f(Cr)$ as follows. The propositions in X are considered in the given order of priority, and the belief set B is built up sequentially. For each proposition under consideration, say p, we begin by asking whether p is entailed by propositions that we have included in B in earlier steps. If the answer is yes, we embrace this entailment and include p in B. If the answer is no, we apply some propositionwise binarization criterion to $Cr(p)$, such as a suitable threshold, and include p in B *if and only if* that binarization criterion recommends the acceptance of p *and* this acceptance does not yield an inconsistent belief set. By construction, the resulting belief set is

[19] See, for example, Kornhauser and Sager (1986) and Kornhauser (1992), Pettit (2001), List and Pettit (2002), Chapman (2002), Bovens and Rabinowicz (2006), List (2006), Dietrich (2006); for more recent generalizations, see Dietrich and Mongin (2010).

always consistent. Furthermore, the belief on each proposition p depends only on the degrees of belief in propositions that are ahead of p in the given order of priority (including p itself), which are precisely the propositions that are deemed relevant to p. So, the present binarization rule satisfies independence of irrelevant propositions. Whether the rule also generates implication-closed beliefs depends on the propositionwise binarization criterion and the order of priority. The downside of a sequential priority rule, like that of a premise-based rule, is that it sometimes mandates the inclusion of a proposition in the belief set B even when the agent's degree of belief in it is very low or zero. This can happen when that proposition is entailed by other accepted propositions. Sequential priority rules for belief binarization are analogous to sequential priority rules for judgment aggregation (List 2004, Dietrich and List 2007b). The only difference lies in the use of a propositionwise *binarization* criterion (such as a propositionwise threshold) instead of a propositionwise *aggregation* criterion (such as propositionwise majority voting). In both belief binarization and judgment aggregation, sequential priority rules may be *path-dependent*: their output is not generally invariant under changes of the order of priority among the propositions. This means that the defensibility of such a rule depends, in part, on our ability to specify that order non-arbitrarily. However, if we take the desideratum of independence of irrelevant propositions seriously, then we should accept that different specifications of the relevance relation may give rise to different binary beliefs.

Generalized priority rules. Sequential priority rules can be further generalized by replacing a linear priority order with a more general priority graph, formally represented by an acyclic but not necessarily complete relevance relation $\mathcal{R}$.[20] For example, consider the proposition set X consisting of p, $p \to q$, q, $q \to r$, r, and their negations.[21] Here the priority graph might deem (i) r relevant to itself, (ii) each of q and $q \to r$ relevant to r and to itself, and (iii) each of p and $p \to q$ relevant to q and r and to itself, as illustrated in Figure 2. (In that figure, one proposition is relevant to another whenever there exists a directed path along the dotted arrows from the first proposition to the second.) The overall belief set is then defined recursively, beginning with the propositions at the "top" of the graph (those to which no other propositions are relevant), where a propositionwise binarization criterion is applied, such as a propositionwise credence threshold. The belief on any "non-top" proposition (i.e., one to which some other propositions are relevant) is formed either by logical entailment from beliefs on propositions that are "prior" to it

[20]By "acyclicity", we here mean that there is no priority cycle in which $p_1 \mathcal{R} p_2$, $p_2 \mathcal{R} p_3$, ..., $p_{k-1} \mathcal{R} p_k$, and $p_k \mathcal{R} p_1$, where p_1, p_2, ..., p_k all belong to distinct proposition-negation pairs $\{p_j, \neg p_j\}$.

[21]There are a number of possible ways of interpreting the implication arrow $\to$ in the present context. The easiest is to define $p \to q$ as $\neg p \vee q$.

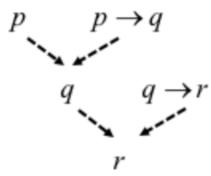

Figure 2: A priority graph

in the graph or – if the beliefs on those "prior" propositions leave the given proposition unconstrained – by applying a propositionwise binarization criterion. One can show that a generalized priority rule yields consistent beliefs, while satisfying independence of irrelevant propositions, provided the priority graph is "well-behaved". Well-behavedness, in turn, means that $\mathcal{R}$ is transitive (i.e., if $p\mathcal{R}q$ and $q\mathcal{R}r$, then $p\mathcal{R}r$), negation-invariant (i.e., if $p\mathcal{R}q$ then also $\neg p\mathcal{R}q$ and $p\mathcal{R}\neg q$), and there are never any logical interdependencies between the "relevance ancestors" of pairwise mutually irrelevant propositions (where a proposition's "relevance ancestors" are all those propositions that are relevant to it).[22]

Distance-based rules. We have already encountered three classes of non-propositionwise-independent judgment-aggregation rules that have direct analogues for belief binarization: premise-based, sequential priority, and generalized priority rules. A fourth class consists of the distance-based rules. Their application to belief binarization was first investigated and defended by Chandler (2013). Let us begin by defining a distance-based *aggregation* rule. To do so, we need to introduce a *distance metric* over judgment sets, which specifies how "distant" any two judgment sets are from one another. Formally, a *distance metric* assigns to each pair of judgment sets a non-negative number, interpreted as the distance between them. For each profile of individual judgment sets, we then select a collective judgment set that minimizes the sum of the distances from the individual judgment sets, according to that distance metric. Some distance-based aggregation rules require more information than what is contained in a propositionwise anonymous profile, and hence have no counterpart in the case of belief binarization, but others naturally carry over to belief binarization. The best-known distance-based aggregation rule is the *Hamming rule* (e.g., Konieczny and Pino Pérez 2002, Pigozzi 2006). Here the distance between any two judgment sets is given by the number of propositions in X on which the two judgment sets disagree (which means the proposition in question is contained in one set but not in the other). For each profile of individual

[22]This is the binarization analogue of Theorem 1 in Dietrich (2015) (a result concerning general priority rules in judgment aggregation), here stated informally.

judgment sets $\langle J_1, ..., J_n\rangle$, we select a consistent and complete (or perhaps consistent and implication-closed) collective judgment set J which minimizes

$$\sum_{i \in N} |\{p \in X : p \in J \nLeftrightarrow p \in J_i\}|.$$

Such a judgment set need not be unique; so we must either define the Hamming rule as a multi-function (under which more than one collective judgment set can be assigned to any given profile of individual judgment sets), or introduce a tie-breaking criterion. The details need not concern us here. Note that minimizing the total Hamming distance is equivalent to minimizing

$$\sum_{p \in X} |\{i \in N : p \in J \nLeftrightarrow p \in J_i\}|.$$

This, in turn, is equivalent to minimizing

$$\sum_{p \in X} |J(p) - Cr_{\langle J_1,...,J_n\rangle}(p)|,$$

where, for each p in X,

$$J(p) = \begin{cases} 1 & \text{if } p \in J \\ 0 & \text{if } p \notin J \end{cases}$$

and $Cr_{\langle J_1,...,J_n\rangle}$ is the function that assigns to each proposition p in X the proportion of individuals accepting p within the given profile $\langle J_1, ..., J_n\rangle$, as defined earlier. This suggests the following definition of a *Hamming rule for belief binarization*: for each degree-of-belief function Cr, let $f(Cr)$ be a consistent and complete (or alternatively, consistent and implication-closed) belief set B which minimizes

$$\sum_{p \in X} |B(p) - Cr(p)|,$$

where $B(p)$ is defined in exact analogy to $J(p)$. Informally speaking, the Hamming rule binarizes any given degree-of-belief function by selecting a belief set that is "minimally distant" from it, subject to the constraints of consistency and completeness (or alternatively, consistency and implication-closure).[23] The Hamming rule is more "holistic" than the premise-based and priority rules discussed earlier. Unlike those rules, the Hamming rule assumes, in effect, that all propositions are relevant to one another; the relevance relation $\mathcal{R}$ with respect to which the Hamming rule is guaranteed to satisfy independence of irrelevant propositions is the total relation, under which $\mathcal{R}(p) = X$ for all p in X.

[23] As in judgment aggregation, we must either define the Hamming rule as a multi-function, since there may be more than one distance-minimizing belief set, or introduce some tie-breaking criterion.

Other non-independent binarization rules. The relationship between anonymous judgment-aggregation rules and belief-binarization rules can be used not only to derive binarization rules from aggregation rules but also to derive aggregation rules from binarization rules that have been proposed in the literature. In this way, some insights from belief-binarization theory carry over to judgment-aggregation theory (recall Levi 2004). Given space constraints, we here discuss only one class of rules for which this reverse translation is possible: Leitgeb's *P-stability-based rules*. Leitgeb (2014) offers a method of constructing, for each degree-of-belief function Cr (defined on some algebra of propositions), a specific acceptance threshold such that the set of all propositions for which the agent's degree of belief exceeds the threshold is consistent and implication-closed. Crucially, the threshold may differ for different degree-of-belief functions. The key idea is to identify a so-called *P-stable* proposition; this is a proposition p for which $Cr(p|q)$ exceeds $\frac{1}{2}$ for any proposition q consistent with p. The agent then accepts all those propositions in which he or she has a degree of belief greater than or equal to $t = Cr(p)$, where p is the identified P-stable proposition. If we assign to each degree-of-belief function the belief set generated through this process, we obtain a well-defined belief-binarization rule. Since the acceptance threshold may differ for different degree-of-belief functions, the present binarization rule does not satisfy propositionwise independence and thus gives rise to a holistic relationship between degrees of belief and binary beliefs. Leitgeb acknowledges that one feature of his proposal is "a strong form of sensitivity of belief to context" (p. 168) and defends this holism. In the present terms, independence of irrelevant propositions is generally satisfied only with respect to the total relation (as in the case of the Hamming rule discussed above). Using the construction presented in Section 5, we can use Leitgeb's belief-binarization rule to define a corresponding anonymous judgment-aggregation rule. To the best of our knowledge, this aggregation rule has not been investigated in the literature on judgment aggregation (however, for a recent independent discussion of this idea, see Cariani 2014). It inherits its interest-value from the arguments that Leitgeb has offered in support of the underlying belief-binarization rule. A similar translation is possible for Lin and Kelly's *camera-shutter rules* for belief binarization (2012a, 2012b). These, too, are non-independent rules that could be used to generate corresponding judgment-aggregation rules (see also Kelly and Lin 2011).

8.3 Relaxing implication-closure of beliefs

A third escape route from our impossibility result is to relax not only the completeness requirement on beliefs – as already discussed – but also the requirement of implication-closure. As noted, this requirement might be challenged for large and complex sets of propositions, where implication-closure amounts to a requirement of logical omniscience. Instead, we might require only the consistency of beliefs and perhaps their closure under implication by singletons (compare Kyburg 1961). Call a belief set B *closed under implication by singletons* (relative to X) if it contains every proposition p in X for which there is some proposition q in B that entails p.[24]

Belief consistency (and closure under implication by singletons). For every Cr in the domain of f, the belief set $B = f(Cr)$ is consistent (and, if we add the closure requirement, closed under implication by singletons).

Replacing implication-closure with this weaker requirement opens up some non-trivial possibilities of belief-binarization, even in the presence of the other desiderata. We need one preliminary definition. Call a set of propositions *minimally inconsistent* if it is inconsistent but all its proper subsets are consistent. The following result holds (a version of which has been independently proved by Easwaran and Fitelson 2015):

Theorem 3. Let k be the size of the largest minimally inconsistent subset of the proposition set X. Any threshold rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition satisfies universal domain, belief consistency (and closure under implication by singletons), propositionwise independence, and certainty preservation.[25]

It is worth explaining the significance of k, the size of the largest minimally inconsistent subset of X. This parameter can be interpreted as a simple measure of the interconnectedness between propositions in X. If X contains only one or several unconnected proposition-negation pairs, then the largest minimally inconsistent subsets of X are of the form $\{p, \neg p\}$, so k is 2.[26] If X contains only p, q, $p \wedge q$, and their negations, then the largest minimally inconsistent subset is $\{p, q, \neg(p \wedge q)\}$, so k is 3. If X contains only p, q, r, $p \wedge q \wedge r$, and their negations, as in our example in Section 2, then the largest minimally inconsistent subset is $\{p, q, r, \neg(p \wedge q \wedge r)\}$, so k is 4. In consequence, the binarization threshold $\frac{k-1}{k}$ required in Theorem 3 increases with the complexity of these cases, from $\frac{1}{2}$ to $\frac{2}{3}$ to $\frac{3}{4}$.[27]

[24] In other words, for any p and q in X, if q is in B and q entails p, then p is also in B.

[25] The highest admissible threshold in this theorem is a weak threshold of 1 (under which the acceptance criterion for any proposition is a degree of belief of 1).

[26] This assumes that some proposition p in X is contingent. If X contains no contingent propositions, then the largest minimally inconsistent subset of X is the singleton set consisting of the contradiction.

[27] Formulas similar to $\frac{k-1}{k}$ have been used by Bovens and Hawthorne (1999), though without explicitly invoking the notion of minimally inconsistent sets of propositions.

To prove Theorem 3, let f be a threshold rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition. It is easy to see that f satisfies universal domain, propositionwise independence, certainty preservation, and closure under implication by singletons.[28] Suppose, for a contradiction, that $B = f(Cr)$ is inconsistent for some degree-of-belief function Cr in the domain. Then B must have at least one minimally inconsistent subset Y, whose size, in turn, is at most k. For any proposition p in Y to be accepted by f, we must have $Cr(p) > \frac{k-1}{k}$. Since Cr is probabilistically coherent, it is extendable to a probability function Pr on the algebra generated by X. This algebra contains the conjunction of all propositions in Y. Since Y is an inconsistent set, this conjunction is a contradiction and must be assigned probability 0 by Pr. But we now show that this contradicts the fact that $Pr(p) > \frac{k-1}{k}$ for every p in Y (which holds because $Pr(p) = Cr(p)$ for any p in X, as Pr is an extension of Cr). First note that the probability of the conjunction of any two propositions – no matter how negatively correlated – must exceed 0 when each proposition has a probability greater than $\frac{1}{2}$. Similarly, the probability of the conjunction of any three propositions must exceed 0 when each has a probability greater than $\frac{2}{3}$. Generally, the probability of the conjunction of any k propositions that each have a probability greater than $\frac{k-1}{k}$ must exceed 0, a contradiction. This completes the proof.

Again, this result has a counterpart in judgment-aggregation theory. To state it, we require one definition. A *qualified majority rule* is a judgment-aggregation rule which assigns, to each profile $\langle J_1, ..., J_n \rangle$, the collective judgment set

$$J = \{p \in X : |\{i \in N : p \in J_i\}| \text{ exceeds } qn\},$$

where "exceeds" can be read either strictly (as ">") or weakly (as "$\geq$"), and q is some acceptance threshold between $\frac{1}{2}$ and 1. It is a *supermajority rule* when the acceptance threshold requires more than a simple majority of the individuals ("50%+1"). The following result holds:

Background Result 3. Let k be the size of the largest minimally inconsistent subset of the proposition set X. Any qualified majority rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition satisfies universal domain, collective consistency, propositionwise independence, and unanimity preservation (Dietrich and List 2007b).[29]

It should be evident that the qualified majority rule in Background Result 3 is simply the judgment-aggregation rule induced by the binarization rule in Theorem 3.

[28]The latter excludes a strict threshold of 1 for any non-contradictory proposition, but permits a weak threshold of 1.

[29]The highest admissible threshold in this theorem is a weak threshold of 1 (under which the acceptance criterion for any proposition is unanimous support).

8.4 Relaxing consistency of beliefs

A fourth response to our impossibility result is to argue that beliefs need not be consistent. Of course, if outright inconsistency of beliefs is permitted, the impossibility we have identified goes away immediately. Even a binarization rule as simple as the one that employs the "more-likely-than-not" criterion for belief will then be feasible. But since inconsistent beliefs violate standard requirements of rationality, one might wonder whether the present response is a non-starter. However, there is a notion of less-than-fully-consistent belief which captures the idea that some inconsistencies are less "blatant" than others, so that we might reasonably opt for a belief-binarization rule that avoids "blatant" inconsistencies despite not securing full consistency. Indeed, typical human beings are unlikely to hold fully consistent beliefs, and so the avoidance of blatant inconsistencies may be viewed as a plausible requirement of bounded rationality.

To introduce the relevant notion of less-than-full consistency (drawing on List 2014), we begin with a few intuitive observations. If someone believes a proposition that is self-contradictory, such as $p \wedge \neg p$, he or she is rather blatantly inconsistent. If someone believes two propositions, neither of which is self-contradictory, but which are jointly inconsistent, such as p and $\neg p$, he or she is still fairly inconsistent, but less so than in the previous case. If someone believes three jointly inconsistent propositions, any two of which are mutually consistent, such as p, $p \rightarrow q$, and $\neg q$, his or her belief set is still relatively inconsistent, but not as much as in the two previous cases. If someone's belief set contains ten jointly inconsistent propositions, any nine of which are mutually consistent, this is nowhere near as bad as the previous inconsistencies. Now the key idea is to interpret the size of the smallest inconsistent set of believed propositions as a measure of the agent's inconsistency.

Formally, let us say that a belief set B is *k-inconsistent* if it has an inconsistent subset of size less than or equal to k. In our examples, a belief set that includes the proposition $p \wedge \neg p$ is 1-inconsistent; a belief set that includes the propositions p and $\neg p$ is 2-inconsistent, and so on. Similarly, we say that a belief set B is *k-consistent* if it is free from any inconsistent subsets of size up to k. As the value of k increases, k-consistency becomes more demanding, and any residual inconsistencies become less "blatant". Full consistency is the limiting case of k-consistency as k goes to infinity. Suppose we replace the requirement of belief consistency with the following:

Belief k-consistency (for some fixed value of k). For every Cr in the domain of f, the belief set $B = f(Cr)$ is k-consistent.

We then obtain a possibility result:

Theorem 4. Any threshold rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition satisfies universal domain, belief k-consistency, propositionwise independence, and certainty preservation.[30]

The proof of this theorem, which we omit for brevity, is very similar to that of Theorem 3 above. The key point is that a probabilistically coherent function Cr could never assign a degree of belief greater than $\frac{k-1}{k}$ to each of k or fewer mutually inconsistent propositions; the implied subjective probability of their conjunction would then have to be greater than 0, which would violate probabilistic coherence. Like our other results, Theorem 4 has an analogue in judgment-aggregation theory.

Background Result 4. Any qualified majority rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition satisfies universal domain, collective k-consistency, propositionwise independence, and unanimity preservation (List 2014).[31]

In sum, agents who are prepared to settle for less-than-full consistency in their beliefs, while still avoiding "blatant" inconsistencies, can safely use threshold rules with a sufficiently high threshold.[32]

8.5 Relaxing universal domain

A fifth, more theoretical escape route from our impossibility result opens up if we relax the desideratum of universal domain. Recall that universal domain requires the belief-binarization rule to work for every well-defined degree-of-belief function. As the name of the desideratum suggests, this ensures the binarization rule's universal applicability. If, however, we suitably restrict the domain of admissible degree-of-belief functions, we can find a belief-binarization rule satisfying the other desiderata.

Suppose, for example, that a degree-of-belief function Cr is deemed admissible only if it has the property that, for every minimally inconsistent subset Y of X, there is at least one proposition in p in Y with $Cr(p) \leq \frac{1}{2}$. It then follows that even a permissive binarization rule such as a "more-likely-than-not rule" (a threshold rule with a strict threshold of $\frac{1}{2}$ for all propositions) will never generate an inconsistent belief set B. If B were inconsistent for some Cr in the restricted domain, then B would have to have some minimally inconsistent subset Y, which, in turn, would have to contain at least one

[30] As before, the highest admissible threshold in this theorem is a weak threshold of 1.

[31] Once again, the highest admissible threshold in this theorem is a weak threshold of 1.

[32] For a further critical discussion of the consistency requirement on belief, see Briggs, Cariani, Easwaran, and Fitelson (2014).

proposition p for which $Cr(p) \leq \frac{1}{2}$ (as Cr is in the restricted domain). But then p would not be accepted under a threshold rule with a strict threshold of $\frac{1}{2}$. More generally, if we admit only credence functions with the property that, for every minimally inconsistent subset Y of X, there is at least one proposition p in Y with $Cr(p) \leq t$, then any threshold rule with a strict threshold of t or above will guarantee consistency.

When translated into restrictions on admissible profiles of judgment sets in judgment-aggregation theory, the domain restrictions just mentioned match the domain restrictions required for the consistency of majority rule and supermajority rule with threshold t, respectively. In judgment-aggregation theory, domain restrictions are often associated with situations in which the group of individuals whose judgments are aggregated is reasonably "cohesive": disagreements among the individuals are limited. For example, if a group engages in collective deliberation before voting, this might reduce any disagreements, even if it does not produce a full consensus, and an aggregation rule with a restricted domain may become applicable. This is in fact a much-discussed idea of deliberative democracy. In belief-binarization theory, it is harder to justify the required domain restrictions in a non-ad-hoc way. Still, it is worth acknowledging the theoretical possibility of satisfying our other desiderata (apart from universal domain) if the agent's degree-of-belief function falls into a sufficiently restricted domain.

8.6 Relaxing certainty preservation

As we have noted, a final logically possible escape route from our impossibility result is to relax certainty preservation. However, this escape route is of little interest. First, certainty preservation is a very undemanding and plausible requirement and thus hard to relax. Second, even if we were prepared to give it up, this would not get us very far. For a large class of proposition sets X, we would still be faced with an impossibility result. To state this result, call a proposition p an *atom* of X if, for every proposition q in X, p entails exactly one of q or $\neg q$. Further, call the proposition set X *atom-closed* if it contains an exhaustive set of atoms.[33] It is easy to see that any finite proposition set X that forms an algebra is atom-closed. The following result holds:

Theorem 5. For any atom-closed proposition set X that contains more than one contingent proposition-negation pair, any belief-binarization rule satisfying universal domain, belief consistency and completeness, and propositionwise independence is *constant*: it delivers as its output the same fixed belief set B, no matter which degree-of-belief function Cr is fed into it as input.

[33] Formally, X is *atom-closed* if the set $\{\neg p \in X : p \text{ is an atom of } X\}$ is inconsistent.

Of course, such a binarization rule is totally useless. According to it, the agent's binary beliefs are completely unresponsive to his or her degrees of belief. Like our earlier results, Theorem 5 is a corollary of an analogous theorem on judgment aggregation.

Background Result 5. For any atom-closed proposition set X that contains more than one contingent proposition-negation pair, any judgment-aggregation rule satisfying universal domain, collective consistency and completeness, propositionwise independence, and consensus preservation is either dictatorial or constant (Dietrich 2006; for a related result, see Pauly and van Hees 2006).

It is fair to conclude, then, that the relaxation of certainty preservation offers no compelling escape route from our impossibility result.

9 Concluding remarks

We have investigated the relationship between degrees of belief and binary beliefs through the lens of judgment-aggregation theory. We have proved a baseline impossibility theorem, which turns out to be a sibling of Arrow's classic impossibility theorem on preference aggregation. The two results are each corollaries of a single, mathematically more general impossibility theorem on judgment aggregation, as illustrated in Figure 1 above.

The message of our analysis is that the possibilities of expressing beliefs as a function of degrees of belief are very limited. Any such possibility requires the relaxation of at least one of our four baseline desiderata, and in each case this comes at a cost:

(i) If we relax universal domain, we must use a binarization rule that does not work for all possible degree-of-belief functions and hence is not universally applicable.

(ii) If we relax belief consistency and completeness, then, depending on whether or not we still retain implication-closure, we must either use an extremely conservative acceptance criterion for every proposition – namely a degree-of-belief threshold of 1 – or live with violations of implication-closure or consistency. Suitable threshold rules can, however, satisfy belief consistency and closure under implication by singletons.

(iii) If we relax propositionwise independence, we must accept a holistic relationship between degrees of belief and beliefs, whereby an agent's belief in one proposition may be affected by changes in his or her degrees of belief in others. We suggest that embracing this holism (which Leitgeb describes as "a strong form of sensitivity of belief to context") is still the most palatable way to avoid our impossibility result, and it is supported by the existence of certain relevance relations betweens propositions. Indeed, several established proposals on belief binarization give up independence (e.g., Chandler 2013, Leitgeb 2014, and Lin and Kelly 2012a, 2012b).

(vi) If we relax certainty preservation, finally, we face another negative result: for a large class of proposition sets, the only binarization rules satisfying the other three desiderata are constant rules, under which the agent's beliefs are not responsive at all to his or her degrees of belief.

If pressed to choose one of these escape routes from our impossibility result, we would opt for (iii) or (ii). However, for those who are reluctant to embrace any of these possibilities, a natural conclusion may be that the search for a simple formal relationship between degrees of belief and binary beliefs is futile.

The most radical version of this conclusion would be the denial that agents really have both kinds of belief. Extreme Bayesians, for instance, might hold that agents have no binary beliefs. On that view, beliefs always come in degrees, and "full belief" is, at most, the limiting case in which the degree of belief is 1. The opposite view would be that degrees of belief are theoretical constructs of probability theory and that, in reality, agents have *only* binary beliefs. On this picture, degrees enter at most in the *content* of a belief. An agent might have a full belief in the proposition that the probability of another proposition is x. Here, the attitude towards the "outer" proposition is an all-or-nothing attitude, which does not come in degrees; it just so happens that that proposition asserts a probability assignment to another proposition – the "inner" one.

A less radical conclusion would be that agents have degrees-of-belief as well as binary beliefs, but that the two kinds of belief may come apart and are not related in any simple way: they may be two distinct aspects of an agent's credal state, neither of which is determined by the other. To defend that picture, one would have to say more about what such a multi-faceted credal state would look like – a topic well beyond the scope of this paper. (For a recent relevant discussion, see Easwaran and Fitelson 2015.)

What we can conclude is that, if one is unwilling to relax the standard consistency and closure requirements on binary beliefs, then the prospects for expressing those beliefs as a *propositionwise function* of degrees of belief are extremely slim. The binary belief on a proposition may not generally supervene on the degree of belief in that proposition alone. Our aim has been to lay out some salient impossibilities and possibilities of belief binarization, and to offer a systematic analysis of the relevant logical space, in the hope that this exercise will inspire further exploration.

References

Arrow, K. (1951/1963) *Social Choice and Individual Values.* New York: Wiley.

Bovens, L., and W. Rabinowicz (2006) "Democratic answers to complex questions—an epistemic perspective." *Synthese* 150(1): 131–153.

Briggs, R., F. Cariani, K. Easwaran, and B. Fitelson (2014) "Individual Coherence and Group Coherence." In J. Lackey (ed.), *Essays in Collective Epistemology.* Oxford: Oxford University Press.

Cariani, F. (2014) "Local Supermajorities." Working paper, Northwestern University.

Chandler, J. (2013) "Acceptance, Aggregation and Scoring Rules." *Erkenntnis* 78(1): 201-217.

Chapman, B. (2002) "Rational Aggregation." *Politics, Philosophy and Economics* 1: 337–354.

Dietrich, F. (2006) "Judgment Aggregation: (Im)Possibility Theorems." *Journal of Economic Theory* 126: 286–298.

Dietrich, F. (2007) "A generalised model of judgment aggregation." *Social Choice and Welfare* 28: 529–565.

Dietrich, F. (2015) "Aggregation theory and the relevance of some issues to others." *Journal of Economic Theory* 160: 463-493

Dietrich, F., and C. List (2007a) "Judgment aggregation by quota rules: majority voting generalized." *Journal of Theoretical Politics* 19: 391–424.

Dietrich, F., and C. List (2007b) "Arrow's theorem in judgment aggregation." *Social Choice and Welfare* 29: 19–33.

Dietrich, F., and C. List (2007c) "Strategy-proof judgment aggregation." *Economics and Philosophy* 23(3): 269–300.

Dietrich, F., and C. List (2013) "Propositionwise judgment aggregation: the general case." *Social Choice and Welfare* 40(4): 1067–1095.

Dietrich, F. and P. Mongin (2010) "The premise-based approach to judgment aggregation." *Journal of Economic Theory* 145: 562–582.

Dokow, E., and R. Holzman (2010a) "Aggregation of binary evaluations." *Journal of Economic Theory* 145: 495–511.

Dokow, E., and R. Holzman (2010b) "Aggregation of binary evaluations with abstentions." *Journal of Economic Theory* 145: 544–561.

Douven, I., and J.-W. Romeijn (2007) "The Discursive Dilemma as a Lottery Paradox." *Economics and Philosophy* 23(3): 301–319.

Douven, I., and T. Williamson (2006) "Generalizing the Lottery Paradox." *British Journal for the Philosophy of Science* 57(4): 755–779.

Easwaran, K., and B. Fitelson (2015) "Accuracy, Coherence and Evidence." In T. Szabó Gendler and J. Hawthorne (eds.), *Oxford Studies in Epistemology* 5.

Hawthorne, J., and L. Bovens (1999) "The preface, the lottery, and the logic of belief." *Mind* 108(430): 241–264.

Kelly, K. T., and H. Lin (2011) "Judgment aggregation: A geometrical impossibility proof." Presentation at the 2011 Episteme Conference, Carnegie Mellon University, June 2011.

Konieczny, S., and R. Pino Pérez (2002) "Merging Information Under Constraints: A Logical Framework." *Journal of Logic and Computation* 12: 773–808.

Kornhauser, L. A. (1992) "Modeling collegial courts. II. Legal doctrine." *Journal of Law, Economics and Organization* 8: 441–470.

Kornhauser, L. A., and L. G. Sager (1986) "Unpacking the Court." *Yale Law Journal* 96: 82–117.

Kyburg, H. E. (1961) *Probability and the Logic of Rational Belief.* Middletown: Wesleyan University Press (1961)

Leitgeb, H. (2014) "The Stability Theory of Belief." *Philosophical Review* 123(2): 131–171.

Levi, I. (2004) "List and Pettit." *Synthese* 140(1/2): 237–242.

Lin, H., and K. T. Kelly (2012a) "Propositional Reasoning that Tracks Probabilistic Reasoning." *Journal of Philosophical Logic* 41(6): 957–981.

Lin, H., and K. T. Kelly (2012b) "A Geo-logical Solution to the Lottery Paradox." *Synthese* 186: 531–575.

List, C. (2004) "A Model of Path-Dependence in Decisions over Multiple Propositions." *American Political Science Review* 98: 495–513.

List, C. (2006) "The Discursive Dilemma and Public Reason." *Ethics* 116: 362–402.

List, C. (2012) "The Theory of Judgment Aggregation: An Introductory Review." *Synthese* 187: 179–207.

List, C. (2014) "When to defer to supermajority testimony – and when not." In J. Lackey (ed.), *Essays in Collective Epistemology.* Oxford: Oxford University Press.

List, C., and P. Pettit (2002) "Aggregating Sets of Judgments: An Impossibility Result." *Economics and Philosophy* 18(1): 89–110.

List, C., and P. Pettit (2004) "Aggregating Sets of Judgments: Two Impossibility Results Compared." *Synthese* 140(1/2): 207–235.

Nehring, K., and C. Puppe (2010) "Abstract Arrovian Aggregation." *Journal of Economic Theory* 145: 467–494.

Pauly, M., and M. van Hees (2006) "Logical Constraints on Judgment Aggregation." *Journal of Philosophical Logic* 35: 569–585.

Pettit, P. (2001) "Deliberative Democracy and the Discursive Dilemma." *Philosophical Issues* 11: 268–299.

Pigozzi, G. (2006) "Belief merging and the discursive dilemma: an argument-based account to paradoxes of judgment aggregation." *Synthese* 152: 285–298.

Appendix

The judgment-aggregation variant of Arrow's theorem and its corollary for belief binarization

We now state Background Result 1 and Theorem 1 in full generality. As noted, the significance of the fully general background result lies in the fact that it is the judgment-aggregation variant of Arrow's impossibility theorem. While in the simplified exposition in Section 6 we required the proposition set X to be a non-trivial algebra, we now merely require it to be what we call "strongly connected". A proposition set X is *strongly connected* if it has the following two combinatorial properties, which are jointly weaker than the previous requirement that X be a non-trivial algebra:[34]

Path-connectedness. For any two contingent propositions p, q in X, there exists a path of conditional entailments from p to q (as explicated in a footnote).[35]

Pair-negatability. There exists a *minimally inconsistent* subset Y of X which contains two distinct propositions p and q such that replacing p and q with $\neg p$ and $\neg q$ renders Y consistent.[36]

Many different proposition sets are strongly connected in this sense. A simple example is the set X consisting of p, q, $p \wedge q$, $p \vee q$, and their negations.[37] Another example is a set X consisting of *binary ranking propositions* of the form "x is preferable to y", "y is preferable to z", "x is preferable to z", and so on, where x, y, z, ... are three or more electoral options, as discussed in Section 7. A third example, familiar from the main text, is a set X which constitutes an algebra with more than one contingent proposition-negation pair. The judgment-aggregation variant of Arrow's theorem, in full generality, can now be stated as follows:

Background Result 1 (fully general version). For any strongly connected proposition set X, any judgment-aggregation rule satisfying universal domain, collective consistency and completeness, propositionwise independence, and consensus preservation is dictatorial (Dietrich and List 2007a, Dokow and Holzman 2010a, building on Nehring and Puppe 2010).[38]

[34] The first property was originally introduced by Nehring and Puppe (2010) in a different formalism under the name "total blockedness". The second was introduced, in different variants, by Dietrich (2007), Dietrich and List (2007a), and Dokow and Holzman (2010a), sometimes under the name "non-affineness".

[35] Formally, a proposition p *conditionally entails* a proposition q if there exists some subset Y of X, consistent with each of p and $\neg q$, such that $\{p\} \cup Y$ entails q. A *path of conditional entailments* from p to q is a sequence of propositions $p_1, p_2, ..., p_k$ in X with $p_1 = p$ and $p_k = q$ such that p_1 conditionally entails p_2, p_2 conditionally entails p_3, ..., and p_{k-1} conditionally entails p_k.

[36] Formally, $Y \backslash \{p, q\} \cup \{\neg p, \neg q\}$ is consistent.

[37] To prove path-connectedness, note that, from any one of these propositions, we can find a path of conditional entailments to any other (e.g., from p to q via $p \vee q$: p entails $p \vee q$, conditional on the empty set; and $p \vee q$ entails q, conditional on $\{\neg p\}$). To prove pair-negatability, note that the minimally inconsistent set $Y = \{p, q, \neg(p \wedge q)\}$ becomes consistent if we replace p and q with $\neg p$ and $\neg q$.

[38] This theorem was proved independently, in different formal frameworks, by Dietrich and List (2007a) and Dokow and Holzman (2010a). The latter proved in addition that, if X is finite, the two combinatorial properties of X are not only sufficient, but also necessary for the theorem's conclusion to hold. Both of

Put differently, whenever X is strongly connected, there exists no non-dictatorial aggregation rule satisfying the four desiderata. It should be clear from our earlier discussion that this result has a direct corollary for belief binarization, which can be derived in exact analogy to the main-text version of Theorem 1 above. The non-existence of any non-dictatorial judgment-aggregation rule for X satisfying the four desiderata implies the non-existence of any *anonymous* such rule. Since any belief-binarization rule for X satisfying our four binarization desiderata would induce such an aggregation rule, there cannot exist a binarization rule of this kind. In sum, the following theorem holds:

Theorem 1 (fully general version). For any strongly connnected proposition set X, there exists no belief-binarization rule satisfying universal domain, belief consistency and completeness, propositionwise independence, and certainty preservation.

In what follows, we present a further refinement of this result, namely a characterization of the *minimal* (necessary and sufficient) conditions on the proposition set X under which Theorem 1's negative conclusion – the non-existence of any belief-binarization rule satisfying the four desiderata – holds. Theorem 1 itself gives only a sufficient condition on X for that conclusion to hold, not a necessary condition.

Minimal conditions for the impossibility result on belief binarization

In the case of the judgment-aggregation variant of Arrow's theorem, it is known that the "strong connectedness" requirement on the proposition set X is not only sufficient, but also necessary for the theorem's negative conclusion, as long as X is finite (Dokow and Holzman 2010a, building on Nehring and Puppe 2010). In other words, if the set X is either not path-connected or not pair-negatable, there exist non-dictatorial judgment-aggregation rules satisfying the four desiderata; we no longer face an impossibility.

Interestingly, the same combinatorial properties, while sufficient for our impossibility result on belief binarization, are not necessary for it. We can derive the impossibility even under weaker assumptions about the proposition set X. This is because, as we have seen, belief binarization corresponds to a particularly restrictive case of judgment aggregation: the case of propositionwise anonymous aggregation. In the presence of this special restriction, judgment aggregation also runs into an impossibility more easily.

Consider the following combinatorial property, which is weaker than path-connectedness (Nehring and Puppe 2010).

the cited papers build on earlier work by Nehring and Puppe, reported in Nehring and Puppe (2010). Their version of the theorem imposes an additional monotonicity desideratum on the aggregation rule, but does not require pair-negatability of X.

Blockedness. There is at least one proposition p in X such that there exists a path of conditional entailments from p to $\neg p$ and also a path of conditional entailments from $\neg p$ to p. (Paths of conditional entailments are defined as before.)

For example, the set consisting of p, q, $p \leftrightarrow q$, and their negations is blocked (with $\leftrightarrow$ understood as a material biconditional), while the set consisting of p, q, $p \wedge q$, and their negations is not.[39] The following background result holds:

Background Result 6. There exist judgment-aggregation rules satisfying universal domain, collective consistency and completeness, propositionwise independence, consensus preservation, and anonymity for all group sizes n if and only if the proposition set X is not blocked (Dietrich and List 2013, building on Nehring and Puppe 2010).[40]

We now use this result to the derive the following:

Theorem 6. For any proposition set X that is blocked, there exists no belief-binarization rule satisfying universal domain, belief consistency and completeness, propositionwise independence, and certainty preservation. Conversely, for any finite proposition set X that is *not* blocked, there exists such a belief-binarization rule.

Before we prove this result, it is worth commenting on its significance. The present theorem establishes exact minimal conditions on X for the impossibility of belief binarization to hold. To illustrate, even for a proposition set as simple as the one consisting of p, q, $p \leftrightarrow q$, and their negations, there exists no belief-binarization rule satisfying our four baseline desiderata (because the proposition set is blocked). By contrast, for the proposition set consisting of p, q, $p \wedge q$, and their negations, there exists such a rule (because the set is not blocked), although, as we explain below, the rule is fairly "degenerate". It would accept any proposition among p, q, and $p \wedge q$ if and only if the agent assigns degree of belief 1 to it, and would accept its negation otherwise.

To prove Theorem 6, let us first assume that the proposition set X is blocked, and suppose, contrary to Theorem 6, there exists a belief-binarization rule for X satisfying all four desiderata. Call it f. We have seen that, for any group size n, f induces an anonymous judgment-aggregation rule F, via the construction described in Section 5,

[39] To show that the first set is blocked, it suffices to observe that there exist paths of conditional entailments from p to $\neg p$ and back. To see the former, note that $\{p\} \cup \{p \leftrightarrow q\}$ entails q, and $\{q\} \cup \{\neg(p \leftrightarrow q)\}$ entails $\neg p$. To see the latter, note that $\{\neg p\} \cup \{p \leftrightarrow q\}$ entails $\neg q$, and $\{\neg q\} \cup \{\neg(p \leftrightarrow q)\}$ entails p. To show that the second set is not blocked, it suffices to observe that there is no path of conditional entailments from $\neg p$ to p, from $\neg q$ to q, or from $\neg(p \wedge q)$ to $p \wedge q$.

[40] The "if" claim assumes that the proposition set X is finite.

and this aggregation rule satisfies all four aggregation-theoretic desiderata. However, the existence of such a judgment-aggregation rule contradicts Background Result 6. So, there cannot exist a belief-binarization rule of the specified kind. This completes the negative part of the proof.[41]

Conversely, let us assume that the proposition set X is finite and not blocked. The following result holds:

Background Lemma 1. If the proposition set X is finite and not blocked, there exists at least one consistent and complete subset $B^* \subseteq X$ which has at most one proposition in common with every minimally inconsistent subset $Y \subseteq X$ (Nehring and Puppe 2010).

To illustrate, recall that the set X consisting of p, q, $p \wedge q$, and their negations is not blocked. Indeed, we can find a subset $B^* \subseteq X$ which has at most one proposition in common with every minimally inconsistent subset $Y \subseteq X$. Take $B^* = \{\neg p, \neg q, \neg(p \wedge q)\}$. The minimally inconsistent subsets of X are firstly all the proposition-negation pairs, with which B^* obviously has only one proposition in common, secondly the sets $\{\neg p, p \wedge q\}$ and $\{\neg q, p \wedge q\}$, with which B^* again has only one proposition in common, and finally $Y = \{p, q, \neg(p \wedge q)\}$, with which B^* also has only proposition in common.

To establish the existence of a belief-binarization rule satisfying all four desiderata, we construct one such rule. Define f as follows. For every degree-of-belief function Cr on X, let $f(Cr) = B$, where

$$B = \{p \in X : Cr(p) = 1 \text{ or } [p \in B^* \text{ and } Cr(p) \neq 0]\},$$

where B^* is as specified in Background Lemma 1. To give an intuitive flavour of this binarization rule, let us interpret B^* as a "default belief set". The present binarization rule then "accepts" a proposition p if and only if the agent assigns degree of belief 1 to that proposition or the proposition belongs to the default set and the agent does not assign degree of belief 0 to it. (This is in fact a special kind of non-uniform threshold rule, with a strict threshold of 0 for all propositions in B^* and a weak threshold of 1 for all other propositions.) Why does this binarization rule satisfy the four desiderata? Let us begin with the desiderata that are easy to check:

- f satisfies universal domain because it is well-defined for every degree-of-belief function on X.

[41] Alternatively, some of the additional formal results in the second part of the Appendix could be used to give a direct proof of this negative result.

- f satisfies propositionwise independence because, under its definition, the all-or-nothing belief in any proposition p (i.e., whether or not p is in B) depends only on the degree of belief in p (i.e., $Cr(p)$), not on the degree of belief in other propositions.

- f satisfies certainty preservation because, for any degree-of-belief function Cr that assigns extremal degrees of belief (0 or 1) to all propositions in X, the definition of f ensures that B contains all propositions p in X for which $Cr(p) = 1$ and does not contain any for which $Cr(p) = 0$.

- f satisfies belief completeness because, for every degree-of-belief function Cr and every proposition-negation pair p, $\neg p$ in X, one of the two propositions always satisfies the criterion for membership in $B = f(Cr)$.

Finally, to see that f satisfies belief consistency, suppose, for a contradiction, that $B = f(Cr)$ is inconsistent for some degree-of-belief function Cr on X. Then B, being a finite inconsistent set of propositions, has at least one minimally inconsistent subset Y. By Background Lemma 1, at most one proposition in Y occurs in B^*. Consider first the case in which there is no such proposition. For all propositions q in Y, it then follows immediately that $Cr(q) = 1$; otherwise those propositions could not have met the membership criterion for B (of which Y is a subset). But since Cr is probabilistically coherent, the fact that $Cr(q) = 1$ for all q in Y contradicts the inconsistency of Y. So let us turn to the alternative case in which exactly one proposition in Y occurs in B^*. Call it p. For all propositions q in Y distinct from p, it follows again that $Cr(q) = 1$; otherwise those propositions could not have met the membership criterion for B (of which Y is a subset). But since Y is inconsistent and Cr is probabilistically coherent, the fact that $Cr(q) = 1$ for all q in Y distinct from p implies that $Cr(p) = 0$, and so p cannot meet the membership criterion for B, a contradiction. This completes our proof that f does indeed satisfy the four desiderata on belief binarization.

To emphasize, we do not claim that f is a substantively interesting binarization rule. The point of its construction is merely to show that, if the set X violates the property of blockedness, the impossibility result of Theorem 6 no longer goes through.

Some additional formal results

In this final section, we prove some additional formal results to give some further insights into the consequences of our desiderata on the belief-binarization rule f. We have already relied on one of these insights (about the *monotonicity* of f) in one of our earlier proofs (in Section 8.1). Moreover, the present results allow us to establish some facts about belief binarization "directly", i.e., not as corollaries of background results on judgment aggregation.

Let f be a belief-binarization rule satisfying propositionwise independence. This implies that, for every proposition p in X, the question of whether or not p is included in the belief set B depends only on the degree of belief in p. For each proposition p, let C_p be the set of those credence values in $[0,1]$ for which p is accepted into B, i.e.,

$$C_p = \{x \in [0,1] : p \in f(Cr) \text{ for some admissible } Cr \text{ with } Cr(p) = x\}.$$

Call the elements of C_p the *acceptance credences* for p. We can then represent f in terms of the family $(C_p)_{p \in X}$ of sets of acceptance credences:

$$\text{for any admissible } Cr,\ f(Cr) = \{p \in X : Cr(p) \in C_p\}.$$

Claim 1. If f satisfies, in addition, universal domain and certainty preservation, then, for every proposition p in X (where $p \neq \emptyset$), we must have $1 \in C_p$.

To see this, consider any $\{0,1\}$-valued degree-of-belief function Cr such that $Cr(p)$=1. Since $p \neq \emptyset$, a well-defined (i.e., probabilistically coherent) such degree-of-belief function exists, and since f satisfies universal domain, Cr is in the domain of f. Certainty preservation then implies that p must be contained in $B = f(Cr)$, and hence $1 \in C_p$.

Claim 2. If f satisfies, in addition, implication-closure, then, for any two propositions p, q in X, if p conditionally entails q, then every acceptance credence for p is also an acceptance credence for q, and thus $C_p \subseteq C_q$.

To show this, suppose p conditionally entails q, and x is an acceptance credence for p. Because of this conditional entailment, there exists a subset $Y \subseteq X$, consistent with each of p and $\neg q$, such that $\{p\} \cup Y$ entails q. It is easy to see that $\{p, q\} \cup Y$ and $\{\neg p, \neg q\} \cup Y$ are each consistent sets. For this reason, there exist well-defined degree-of-belief functions Cr' and Cr'' such that Cr' assigns credence 1 to all the propositions in the first set, and Cr'' assigns credence 1 to all the propositions in the second set. Since any linear average of well-defined degree-of-belief functions is probabilistically coherent, the degree-of-belief function $Cr = xCr' + (1 - x)Cr''$ is well-defined. Let $B = f(Cr)$. Note the following. First, we have $p \in B$, because $Cr(p) = x$, and x is an acceptance credence for p. Second, we have $Y \subseteq B$, because $Cr(r) = 1$ for every $r \in Y$, and 1 is an acceptance credence for every proposition (by Claim 1). Finally, since $\{p\} \cup Y \subseteq B$ and

$\{p\} \cup Y$ entails q, we must have $q \in B$, because B is implication-closed. But $Cr(q) = x$; so x is an acceptance credence for q too.

Claim 3. If f is as in Claim 2, then for any two propositions p, q in X that are connectable, in both directions, by a path of conditional entailments, we have $C_p = C_q$.

This follows immediately from Claim 2. Note further that, if all propositions connectable, in both directions, by a path of conditional entailments, then the binarization rule f is representable by a single set $C \subseteq [0, 1]$ of acceptance credences, which are applied to every proposition in X.

Claim 4. If f is representable by a single C and X is pair-negatable, then f is monotonic, meaning that, for any x, y in [0,1] with $y > x$, if x is in C, then y is also in C. This implies that f is a threshold rule with $t = \inf(C)$. The threshold is weak if $\inf(C) \in C$ and strict otherwise.

To prove this, consider some x in C, and consider any $y > x$. Suppose X is pair-negatable. Then X has a minimally inconsistent subset Y in which we can find two distinct propositions p, q such that $Y \backslash \{p, q\} \cup \{\neg p, \neg q\}$ is consistent. Since the sets $Y \backslash \{q\} \cup \{\neg q\}$, $Y \backslash \{p, q\} \cup \{\neg p, \neg q\}$, and $Y \backslash \{p\} \cup \{\neg p\}$ are each consistent (the first and last because of Y's minimal inconsistency), there exist well-defined degree-of-belief functions Cr', Cr'', and Cr''' that assign credence 1 to all propositions in the first set, to all propositions in the second, and to all propositions in the third set, respectively. Now consider the function $Cr = xCr' + (y - x)Cr'' + (1 - y)Cr'''$. Because Cr is a linear average of three well-defined degree-of-belief functions, Cr is itself a well-defined degree-of-belief function. Let $B = f(Cr)$. Since $x \in C$, we must have $p \in B$. Since all elements of $Y \backslash \{p, q\}$ are assigned credence 1 by Cr and $1 \in C$ (by Claim 1), we must have $Y \backslash \{p, q\} \subseteq B$. Since Y in its entirety is inconsistent, $Y \backslash \{q\}$ entails $\neg q$, and hence B entails $\neg q$. By implication-closure of B, $\neg q$ must be in B. But $Cr(\neg q) = y$, and hence y is in C, as required.