Philosophers' Imprint

volume 18, no. 9
april 2018

ON CLASSICAL MOTION

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There are two intuitively appealing ways to understand the nature of motion. One is that motion is nothing over and above the occupation, by an object, of different places at different times. This is known as the *at-at theory of motion* (being *at* different places *at* different times) and is generally attributed to Russell (1903, Ch. 54). One is inclined in this case (because of the 'nothing over and above' qualification) to say that facts about motion are grounded in facts about the positions of objects; it is, in short, a reductionist account of motion. The other is that motion essentially involves *moving* in some way, for example, the possession, by an object, of a non-zero quantity of motion — especially a non-zero instantaneous velocity. I will call this the *impetus theory of motion*, in keeping with the terminology used by Arntzenius.[1] As velocity on this account is a basic quantity possessed by objects, it is evidently a non-reductive account of motion.

Note that neither view would preclude the other if the 'nothing over and above' qualification were removed from the statement of the at-at theory. After all, the possession of an instantaneous velocity is not necessarily at odds with being at different places at different times. There is a compelling argument, however, that the at-at theory, qualification or not, implies that there are in fact no *truly* instantaneous velocities (Russell, 1903; Arntzenius, 2000; Albert, 2000). The issue is that the calculus explication of instantaneous velocity makes essential reference to non-instantaneous position developments, and hence this velocity is not truly instantaneous. That is, one may define an object's instantaneous velocity at each instant, but the velocity itself is completely grounded in the object's past and future positions. Velocity is therefore not a basic quantity possessed by objects, as the impetus theory would have it. So, it would seem that the at-at theory of motion and the impetus theory of motion necessarily furnish distinct metaphysical accounts of motion: the former is necessarily reductive and the latter is not.

Particularly important criticisms of these two accounts of motion have been raised by Arntzenius (2000). He argues that the at-at theory precludes the possibility of determinism by the way it defines velocity and concludes that "surely

1. Arntzenius (2000) discusses both of these accounts in relation to Zeno's arrow paradox. He also explores approaches to the paradox that eschew instants of time, which, given my aims, I set aside.

the ‘at-at’ theory is wrong if it entails that” (Arntzenius, 2000, 191). The impetus theory avoids this problem by reifying instantaneous velocities, but it does so at the ontological costs of adding fundamental quantities and necessarily imposing what I will call a kinematical constraint, an additional law of nature. Since the at-at theory is sufficient for describing motion without the super-addition of such instantaneous velocities, the impetus theory should, it seems, by parsimony be rejected. Given these drawbacks, we might find both theories unsatisfactory, as Arntzenius himself does.

Although Arntzenius seems to leave us with a rather damp squib, in my view our prospects are much brighter than it might appear. My aim in this paper is to relate satisfactory versions of both theories of motion, using the criticisms of Arntzenius as a foil. I should note at the outset that there are various other criticisms of the at-at theory of motion, based mainly on the notion that the reductive account fails to do justice to the causal role of velocity (Tooley, 1988; Lange, 2005; Easwaran, 2014). As I think there is a natural extension of what I say here to those concerns (and for the sake of some concision), I leave discussion of these concerns for another occasion and concentrate mainly on Arntzenius’s arguments, for these arguments represent significant challenges to any theory of motion and must be overcome if we are to have a satisfactory account thereof.

The paper runs as follows: In §1 I rebut Arntzenius’s criticisms of the at-at theory of motion, by arguing that the at-at theory only precludes one particularly strong version of determinism. In §2 I take up the impetus theory of motion. I do agree with the standard argument against the most familiar impetus theory of motion, one that merely super-adds instantaneous velocities to the at-at theory of motion, and rehearse this argument for convenience of exposition. There is, however, a subtle way to develop the impetus account to avoid any metaphysical profligacy. Arntzenius (2003) himself in fact moots the basic idea briefly. The thought is to treat velocity as basic (as the impetus account states) but define positions in terms of velocity developments through integration. Arntzenius quickly rejects it, albeit for what I will argue are poor reasons. I take up this nascent proposal, developing it in detail and fully in parallel to the at-at theory, i.e. as an alternative reductive account, one that, however, instead reduces position facts to velocity facts rather than the other way around. Readers familiar with modern physical frameworks for classical motion might suspect that they suggest differing accounts of motion than those discussed thus far, so in §3 I consider two of the best-known frameworks, Hamiltonian and Lagrangian mechanics, concluding that the issues nevertheless remain the same despite initial appearances to the contrary. I briefly conclude in §4.

1. The At-At Theory

The at-at theory of motion holds that what it is to be in motion is nothing over and above being *at* different places *at* different times. The appropriate theoretical setting for such an account of physical motion is classical mechanics. In classical mechanics, motion is conventionally understood to be continuous, and velocity is defined as the instantaneous time rate of change of position. Hence velocity is a derived quantity, as it depends completely on objects’ positions at different times. The standard presentation of classical mechanics is therefore naturally understood in the at-at way.

1.1 Instantaneous Velocity as a Neighborhood Property

Let us begin with the previously mentioned claim that velocity on the at-at theory is not a truly instantaneous quantity (Russell, 1903; Albert, 2000; Arntzenius, 2000), since it is important to understand it in order to appreciate Arntzenius’s objection concerning determinism. The typical calculus-based definition of what is called *instantaneous velocity*, which I will denote $\dot{\mathbf{x}}$, is simply the rate of change of position $\mathbf{x}$ with respect to time t, i.e. the time derivative of position: $\mathrm{d}\mathbf{x}/\mathrm{d}t$.

To compute the derivative of a function, one must determine the limits of particular sequences of functions. In the case of instantaneous velocity, the relevant sequence of functions is the sequence of average velocities over increasingly small temporal intervals approaching each instant of time. The average velocity $\bar{\mathbf{u}}$ over a temporal interval $2\Delta t$ centered on time t is defined as follows:

DEFINITION.

$$\bar{\mathbf{u}}(t,\Delta t) := \frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t-\Delta t)}{2\Delta t}.$$

In words, the average velocity $\bar{\mathbf{u}}$ is just the difference in positions at two times divided by the time interval between them.

Given a particular instant t, consider the sequence of average velocities as the temporal interval Δt decreases to zero. The limiting value of these average velocities as Δt goes to zero defines the time derivative of position $\mathrm{d}\mathbf{x}/\mathrm{d}t$ at time t, i.e. the instantaneous velocity $\dot{\mathbf{x}}$:

DEFINITION.

$$\dot{\mathbf{x}}(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} := \lim_{\Delta t \to 0} \bar{\mathbf{u}}(t, \Delta t) = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t-\Delta t)}{2\Delta t},$$

Consider what these definitions might suggest about the metaphysics of position and velocity: First, it is sometimes assumed that position functions can take any form in classical mechanics. In principle one may even allow discontinuous functions. This is because position is taken to be a truly instantaneous property. It is therefore taken to be a component of an object's instantaneous state, where the state of an object is understood to be the most complete description of all dynamical (changeable) properties of that object at an instant.

Is instantaneous velocity a truly instantaneous property like position? While instantaneous velocity is certainly *defined* for particular times t, Albert and Arntzenius argue that it is not truly a quantity possessed by an object at an instant. As Albert says, "What needs to be kept in mind is just that there is all the difference in the world between being uniquely attachable *to* some particular time and being the component of the *instantaneous physical situation of the world* at that time!" (Albert, 2000, 17, emphasis in original). So, although an instantaneous velocity can be defined for instants when an object's position development satisfies the conditions of differentiability, it would be a mistake, according to Albert and Arntzenius, to say that this velocity is actually a property of an object at an instant.

As this is an essential point to grasp for what follows — and since Albert and Arntzenius only offer a few remarks on why this is so — it is worth expanding on the underlying reasoning. The basic reasoning is clearest in the analogous case of average velocity. Of precisely what is average velocity a property? In general, given some average velocity centered at time t and over an interval $t - \Delta t$ to $t + \Delta t$, an object in motion does not possess an instantaneous velocity at t equal to the given average velocity, except perhaps at isolated instants (unless of course it is in uniform motion). One would surely not want to say that average velocity is a property of an object only at these points, since average velocity is generally well-defined regardless of whether it equals instantaneous velocity. One would also not want to say it is a property of the object at each instant in the given interval, since average velocity in general changes depending on the chosen interval Δt. If one did allow this, then the object would possess inconsistent velocital properties. Indeed, this is why Δt is included explicitly as an argument of the average velocity function. What the definition suggests, then, is simply that average velocity ought to be understood as the property of a temporal stage of an object, i.e. of the object over the specific interval between $t - \Delta t$ and $t + \Delta t$ (that is, insofar as averages should be thought of as properties at all).

Similar reasoning suggests that instantaneous velocities are not truly instantaneous properties on the at-at theory. Refer to the definition of instantaneous velocity above. It makes reference to positions associated with a sequence of intervals (denoted by Δt) neighboring a given instant t. Instantaneous velocity is therefore better thought of as what Arntzenius calls a *neighborhood property*. Neighborhood properties are properties possessed by an object not at an instant but in certain arbitrary *neighborhoods* of an instant.[2] It is crucial to understand that a neighborhood property is not to be thought of as a property of any specific interval around the instant, for it is not to some particular interval which one attributes the property but to the development of the function in neighborhoods of the instant, which can be arbitrarily small but must be nonzero.

Arntzenius's explication of the notion of neighborhood properties is rather compressed, relying on an intimate familiarity with differential calculus. To better understand the concept of neighborhood properties and the unusual kind of reference at work here, one needs to attend carefully to the concept of a limit, so let us examine the usual $\varepsilon - \delta$ definition:

DEFINITION. *The function f* ***approaches the limit*** *l* ***near*** *a means: for every*

2. 'Neighborhood' here is meant in the topological sense: a *neighborhood* of a point in general is a set that contains an open set containing that point. A temporal neighborhood of an instant, then, will simply be any interval containing the instant. No presupposition of size of the neighborhood or distance from the interval's boundary to the point is therefore intended apart from those allowed for in properly topological senses.

$\varepsilon > 0$ *there is some* $\delta > 0$ *such that, for all x, if* $0 < |x - a| < \delta$, *then* $|f(x) - l| < \varepsilon$.

To put this definition to work in the case of instantaneous velocity, replace the symbols in the previous definition with the ones used above in the definition of instantaneous velocity, recalling that the instantaneous velocity is equal to the limit of average velocities in the neighborhood of t, i.e. $\dot{\mathbf{x}}(t) = \lim_{\Delta t \to 0} \bar{\mathbf{u}}(t, \Delta t)$:

DEFINITION. *The average velocity* $\bar{\mathbf{u}}(t, \Delta t)$ ***approaches the limit*** $\dot{x}$ ***near*** t *means: for every* $\varepsilon > 0$ *there is some* $\delta > 0$ *such that, for all* Δt, *if* $0 < |\Delta t| < \delta$, *then* $|\bar{\mathbf{u}}(t, \Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$.

It will help to unpack the limit definition a bit to clarify the conceptual role of neighborhoods. Here is what the definition says: First we pick any positive real number ε we like. If the instantaneous velocity exists at t, then what the definition requires is that there is *some* neighborhood around t, i.e. a temporal interval between $t - \delta$ and $t + \delta$, such that the difference between the instantaneous velocity $\dot{\mathbf{x}}$ and the average velocity $\bar{\mathbf{u}}(t, \Delta t)$ is less than ε, for *every* interval inside of the neighborhood, i.e. for all $\Delta t < \delta$. Since ε is arbitrary, this means that the average velocities must become arbitrarily close to the instantaneous velocity and stay at least that close for all smaller intervals. Indeed, we can require the difference between them to be as small as we like; if the instantaneous velocity exists, there will *always* be some δ for which all temporal intervals inside $t - \delta$ and $t + \delta$ have the required difference (or less). Note that this arbitrariness means that one can always choose a different ε which forces a different δ that "moves inside" of any particular neighborhood with which one might think to associate the property of velocity. Thus the instantaneous velocity does not depend on any particular choice of ε or any particular neighborhood given by a particular δ. But instantaneous velocity does clearly depend on more than the position of the object at t to exist: it depends on the behavior of the position function in certain arbitrary temporal neighborhoods of the instant in question.

To illustrate the application of these concepts, it is useful to consider an example of discontinuous motion. Suppose that there is an object in uniform motion which at some instant t jumps discontinuously to position L before immediately returning to its previous inertial trajectory. The limits of the object's position $\mathbf{x}(t)$ and its instantaneous velocity $\dot{\mathbf{x}}(t)$ are well-defined in this example. Because it is defined by way of t's neighborhood, the limit of its position is where it would have been had it not made the jump, and its velocity is the velocity it was traveling at before and after the jump. The limit of position is thus a neighborhood property here, and the example demonstrates that the limit of position does not necessarily agree with the instantaneous position at time t. Similarly, the instantaneous velocity $\dot{\mathbf{x}}(t)$ is a neighborhood property. Since this is a case of discontinuous motion, however, it is not clear what to say about the object's "actual" state of motion at its "jump" location, except perhaps that it is moving "discontinuously" (which, observe, still requires reference to its behavior at other times).[3] Now suppose, as a second case, that the object jumped to another position $M \neq L$. In both of these cases the instantaneous positions of the two objects at time t differ but their "neighborhood positions" are identical. While it may sound strange to say that the object is at one place (in the usual sense) and in another sense not there, once one recognizes that neighborhood position is just a way of describing the continuity of some object's position development (in the neighborhood of an instant), it is clear that instantaneous velocity too is a way of describing an aspect of the object's position development, namely, how fast it is changing (in the neighborhood of an instant).

1.2 Determinism and Motion

Albert and Arntzenius take it to be an objectionable consequence of a theory of motion if by "definition and logic alone" determinism is rendered impossible. Their objection depends on the main conclusion of the previous section, that instantaneous velocity is not a truly instantaneous property. Let us see first how this objection is raised against the at-at theory of motion. I will then show that Arntzenius and Albert are mistaken: the preclusion of determinism is not at all a logical, definitional consequence of the at-at theory of motion.

Arntzenius and Albert have a particular kind of determinism in mind, usually called *Laplacian determinism*, as it is evoked by Laplace in this oft-quoted passage:

> Given for one instant an intelligence which could comprehend all the

3. See (Jackson & Pargetter, 1988) for related discussion.

> forces by which nature is animated and the respective situation of the beings who compose it ... it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. (Laplace, 1840/1902)

Albert (2000, 10) describes the present state of the world as the instantaneous states of every object in the world at the present instant, a description that accords with Laplace's. It is also the one that Arntzenius adopts. These instantaneous states, along with the appropriate dynamical law (determined by the various classical forces at work), are the things one would need to know in classical mechanics to deduce the trajectories of all the objects in the world, if classical mechanics were deterministic in the Laplacian sense. It would then be in this sense that Laplace's intelligence could know the movements of all objects for all times, past and future.

If, however, the at-at theory is correct and instantaneous velocity is not an instantaneous quantity in classical mechanics, then classical mechanics is not deterministic in the Laplacian sense. As a consequence of Newton's Second Law, both position and velocity are in general required to determine future and past evolution. If velocity is not truly instantaneous, then the present state of the world only includes the positions of all the objects in it. In this case Laplace's intelligence would be left scratching its head looking at its impoverished situational report, unable to determine anything whatsoever about the future or the past on its basis — all this, according to Arntzenius, because instantaneous velocity was simply *defined* to be what it is in classical mechanics. It is for this reason that he says that determinism is rendered impossible by definition (and logic) alone.

The obvious rejoinder to Arntzenius's objection is to say that it is actually our definition of determinism (or alternately, present state of the world) which is mistaken. If one is unperturbed by giving up Laplacian determinism, then a simple solution is obviously at hand. Just allow the Laplacian intelligence to have access to neighborhood properties as well as the truly instantaneous state of all objects. In so doing we modify the notion of "present state of the universe" to include arbitrarily small neighborhoods of the present, in which case determinism comes to depend on both neighborhood and properly instantaneous properties. If we say this, though, then we may seem to find ourselves in an uncomfortable dilemma over definitions. On the one hand, we can insist that the definition of instantaneous velocity is correct and, on the other, that Laplacian determinism (or present state of the world) is defined correctly. Why prefer one definition to the other?

Yet it is surely a mistake to characterize the issue in this way, as a matter of definitional preference, since it is not the case that definitions alone are at issue. As Earman observes, "We cannot begin to discuss the implications of physics for the truth of the doctrine of determinism until we know what determinism is; on the other hand, no precise definition can be fashioned without making substantive assumptions about the nature of physical reality" (Earman, 1986). In other words, we have to make some substantive assumptions about the world even to begin evaluating the notion of determinism. So long as we are making substantive assumptions about the nature of physical reality in defining determinism — and velocity — the issue at hand is not simply a matter of logic and definition alone.

What, then, are the "substantive assumptions" underlying the definitions employed here? To some extent they appear to depend importantly on interpretive matters concerning laws of nature. One who believes that the function of laws is to produce new physical situations from old ones may find any relaxation of the definition of determinism from the Laplacian version objectionable. If one thinks instead that laws are merely a description of regularities that can be gleaned from the physical facts, then relaxing the definition of determinism is unproblematic. The state of the world does not have to be instantaneous on this view; it may be whatever the most physically salient notion of determinism requires. In the case of classical mechanics, the relevant state of the world is then naturally taken to be the neighborhood state of an instant.

That Arntzenius has the former point of view in mind follows from a related objection to the at-at theory he makes. That objection begins with the observation that assuming the standard definition of instantaneous velocity imposes certain non-dynamical constraints on evolution. This is objectionable, he says, because

> ... surely our notion of a physical state is such that being in a particular physical state at some time does not by definition and logic alone put

any constraints on what physical states the system can be in at other times. Physics may impose constraints on the possible developments of the physical states of systems, but surely logic and definition by itself should not do so. And that implies that neighborhood properties and neighborhood states are not physical states, they are features of finite developments of physical states. (Arntzenius, 2000, 195)

If one has the idea that laws act on truly instantaneous states of affairs to produce new states of affairs, this concern appears reasonable. Let us accept this view of laws for the moment. Then for this objection to be sustainable, it must be that Arntzenius is right about two things: that only physics can impose such constraints, and that physics is not responsible for the constraints that instantaneous velocity places on position developments. I maintain that he is wrong on both counts.

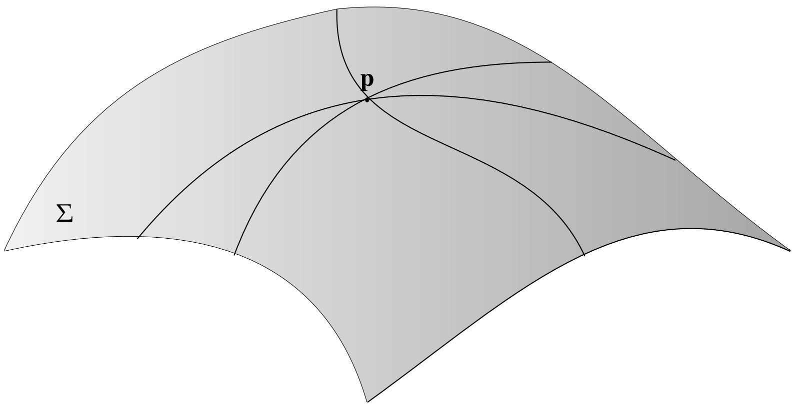

Figure 1: Some trajectories compatible with an object on surface Σ located at position p.

What exactly are these constraints to which he objects? Arntzenius's explication is again compact, and it is necessary to provide a detailed exposition in order to rebut his objections. To first clarify what he has in mind, consider for simplicity the case of some object constrained to move on a surface Σ. At some time let the position of the particle be **p**. If position is the only instantaneous property of the object at that time, then any trajectory on the surface is kinematically possible (see Fig. 1), i.e. possible in advance of consideration of the constraints imposed by forces and the dynamical laws. This is in keeping with the convention noted above, namely that any (continuous) position development is taken to be possible in classical mechanics.

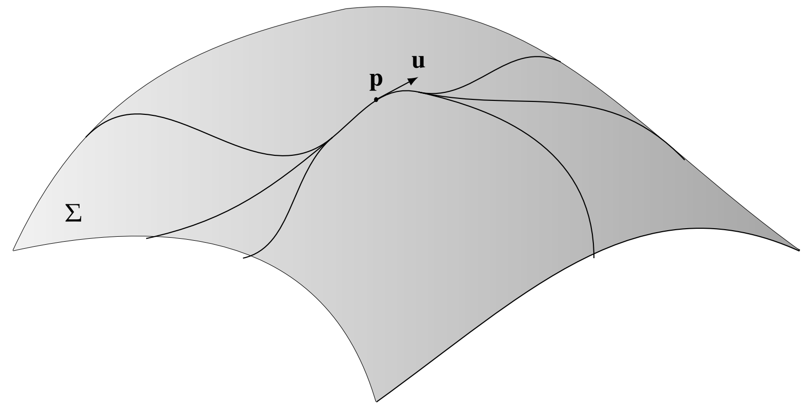

Figure 2: Trajectories compatible with an object on surface Σ located at position p with velocity $\vec{u}$.

If one allows instantaneous velocity to be part of the state of the particle (not necessarily presupposing for the moment that this velocity *is* the time derivative of position $\dot{\mathbf{x}}$), then the possession of a particular velocity, say **u**, restricts the possible position developments to the past and to the future (see Fig. 2). Only those developments which have **u** as their derivative at **p** are kinematically possible when both **p** and **u** are part of the state. Note that velocity imposes this constraint only in the "neighborhood sense" since for any point **q** in some neighborhood of **p** there exists a kinematically possible trajectory that passes through **q** from **p**. It is fair to say that it is a weak constraint, but it is a constraint nonetheless (Butterfield, 2006, 724–725), since certain trajectories are precluded by its imposition, namely those that do not have velocity **u** at **p** (for example, trajectories in Fig. 1).

There are, as I see it, two reasonable ways to respond to Arntzenius's charge that kinematical constraints like this are imposed "by definition and logic alone".

First, we may suppose that the constraint imposed by instantaneous velocity is actually a metaphysical constraint. Note that this constraint need not be imposed in such a way that instantaneous velocity becomes truly instantaneous. Rather it may be imposed in much the same way that dynamical constraints are imposed, namely via laws. For example, the requirement that trajectories be continuous, which classical mechanics generally presupposes, might be considered a metaphysical law, one that is understood to be prior to any physical law. Similarly, the constraint imposed by instantaneous velocity might be considered to be a metaphysical law in this way as well.

Why think that this constraint is a metaphysical law rather than a physical law? Perhaps because it expresses "what we mean" by velocity, or maybe because it is merely a pre-condition on any physics that would describe what we call classical motion. However one explicates the notion of metaphysical law, the mere possibility of this view challenges the notion that only physics can constrain position developments as Arntzenius assumes. Either one accepts this possibility — physics is, after all, obviously committed to many metaphysical presuppositions already — or one has the burden of showing why there cannot be constraints like this.

One objection to taking this course might be based on the popular view of metaphysics where only metaphysical necessity could potentially ground a metaphysical law of the kind I suggest. On this view the exhibition of a possible world where that law does not hold is sufficient to undermine its lawhood. Then one only has to point to worlds where there is discontinuous motion, as Tooley (1988) and Carroll (2002) do, to see that the metaphysical constraint imposed by instantaneous velocity does not hold. If one allows that metaphysical laws do not need to constrain all metaphysical possibilities in this way, then such thought experiments do not militate against this suggestion. Such laws would then be akin to special science laws, which are often considered bona fide laws despite being violated in physically possible worlds.

Nevertheless, there is a second way to respond that to some extent obviates the former, so I will not argue further for it. That second way is to recognize that the kinematical constraint imposed by instantaneous velocity already follows from the dynamical constraints imposed by the laws of classical mechanics (in which case I suppose it ought to be properly called a dynamical constraint). The general form of dynamical constraints in classical mechanics is given by Newton's Second Law:

$$\sum \mathbf{F} = m\mathbf{a};$$

that is to say, the instantaneous sum of forces on an object is equal to the mass of the object times its acceleration. The acceleration $\mathbf{a}$ is defined as the time derivative of the instantaneous velocity $\dot{\mathbf{x}}$; in other words, Newton's Second Law is equivalently expressed as

$$\sum \mathbf{F} = \ddot{\mathbf{x}} := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right),$$

where the acceleration $\ddot{\mathbf{x}}$ is the second time derivative of position. To make acceleration the second time derivative of position, one obviously must have a first time derivative of position, i.e. an instantaneous velocity, to differentiate. Given that Newton's Second Law is a dynamical constraint of the kind that Arntzenius accepts as a physical constraint, it follows as a matter of course that the constraint imposed by instantaneous velocity is a physical constraint as well — it is, from this point of view, a constraint that simply follows from Newton's Second Law.

The point is especially transparent when one rewrites Newton's Second Law as two first-order differential equations:

$$\sum \mathbf{F} = m\dot{\mathbf{u}};$$
$$\mathbf{u} = \dot{\mathbf{x}},$$

where $\mathbf{u}$ is the instantaneous velocity. Here the "kinematical" constraint that is part of Newton's Second Law is made explicit in the second line.

Let me summarize this section. I have argued that the objections related to determinism raised against the at-at theory of motion are unfounded. Although I do agree with Arntzenius and Albert that instantaneous velocity is not truly instantaneous according to the at-at theory and is better understood as a neighborhood property, I disagree with the claim that instantaneous velocity threatens the determinism of classical mechanics. If one does insist on holding onto the idea that only truly instantaneous properties can figure into the physical state of the world (or a system), then it is true that one must give up on the idea that the laws produce future states on this basis alone (in keeping with Laplacian determinism). It is indeed a fundamental fact of classical physics that the world cannot be deterministic in this sense. But it is not true that this is a matter of logic and definition alone, as Arntzenius insists. If one allows that the state of the world include neighborhood properties (and arguably one may do this however one chooses to interpret laws of nature), then there is a clear and substantive sense in which classical mechanics is deterministic.[4] This idea is motivated by attending to the kinematical constraints imposed by instantaneous velocities. If they are seen as either falling under a metaphysical law or following simply from Newton's Second Law, as I claim they should, then instantaneous velocities are rightly considered a part of the physically relevant state of the world. This move renders the at-at theory satisfactory as an account of motion, at least in view of Arntzenius's challenges. Although there are other challenges to the theory, especially related to causation, I believe similar considerations to those raised here can be adapted to address them. Pursuing this line would perhaps be an interesting exercise, but it would be a distraction from the larger aim of this paper. Satisfying that requires moving now to the impetus theory of motion.

2. Impetus Theory of Motion

The impetus theory of motion holds that what it is to be in motion is to possess a non-zero quantity of motion, i.e. a non-zero instantaneous velocity. In the context of classical mechanics this velocity should be a truly instantaneous velocity — in the same way as position is taken to be truly instantaneous. It follows that it should be considered a component of an object's physical state. However, since it is customary in classical dynamics to call the time rate of change of position, i.e. $\dot{\mathbf{x}}$, the instantaneous velocity, I will call the *truly* instantaneous quantity of motion of an object its *impetus* and denote it $\mathbf{v}$. To further avoid confusion I will also prefer to call $\dot{\mathbf{x}}$ the *kinematical velocity*. In this section I investigate the prospects of the impetus theory of motion in the context of classical motion. First, in agreement with Arntzenius (2000), I reject rescuing Laplacian determinism by simply supplementing classical dynamics with impetuses and quickly rehearse the basic argument to provide relevant context. I then raise a new challenge to the at-at theory of motion by re-conceiving classical mechanics in a way that takes velocity rather than position as basic and makes position a derived quantity.

2.1 Super-addition of Impetus to At-At Theory of Motion

I showed in the previous section that the Laplacian picture of determinism — a state of the universe at an instant evolving deterministically forward by the laws of motion — is untenable on the at-at theory of motion. The issue was that velocities are non-instantaneous on the at-at theory of motion while the laws of motion require truly instantaneous velocities in order to be deterministic in the Laplacian sense. An obvious solution to this problem is to supplement the instantaneous properties of objects with a truly instantaneous velocity, i.e. an impetus. Then one may ostensibly maintain the intuitive definitions of determinism and physical state preferred by, among others, Albert and Arntzenius.

This kind of impetus theory, which super-adds impetuses to the account of motion given by the at-at theory, is, however, an unacceptable theory of motion (as Arntzenius himself argues). Insofar as objects follow continuous trajectories in classical mechanics, kinematical velocity $\dot{\mathbf{x}}$ exists and correctly describes the time rate of change of position (it is a velocity). Now, if impetuses are to describe these actual motions, then they have to agree exactly with the corresponding kinematical velocities, i.e. it must be the case that $\mathbf{v} = \dot{\mathbf{x}}$. As the kinematical velocity and the impetus are both meant to be velocities describing motion, such a relation introduces a kinematical constraint between them. Such constraints are to be considered, following the discussion in the previous section, as either metaphysically or physically necessary in classical mechanics. Yet, given the necessity of this kinematical constraint, impetuses appear to be incapable of doing any additional "work" in the theory over and above kine-

4. That is, in physically significant cases, setting aside well-known failures of determinism like Norton's dome (Norton, 2008) and "space invaders" (Earman, 1986).

matical velocity (which, again, is given in the theory since position is). They are physically idle: one cannot say anything more with them than what one can already say with position developments.

Although impetus is necessarily constrained to equal kinematical velocity for differentiable trajectories, perhaps one might persist with the super-additive impetus theory by supposing that the conceptual independence of the two velocities becomes manifest in a wider context, viz. one where trajectories are not continuous. This is not so. If position is not a differentiable function of time at some instants, then the impetuses must likewise be undefined at these instants, else they would be irrelevant to the description of actual motion. To see why, let us suppose that impetuses could disagree with the kinematical velocities. In the case of differentiable position functions from above, the impetuses would not function as quantities of motion as intended, since they would give the wrong dynamical evolution of the objects' positions. Now, in cases where the position function has undefined kinematical velocities at some instants, if impetuses were to possess a defined value, then they would also give the wrong dynamical evolution of the object, as they would indicate a future motion that does not occur. Impetus is therefore entirely expendable in this wider context as well. Both velocities must agree when kinematical velocity describes motion, yet the kinematical velocity is not beholden to impetus in any way.

It will simply not do to add physically superfluous properties and constraints solely in order to preserve a particular metaphysics of laws. We should eschew impetuses (rendered thus), along with their associated kinematical constraints.

2.2 Position Reduced to Velocity Developments

Insofar as one makes the usual assumption that position is included in an object's instantaneous physical state, impetuses are therefore superfluous and instantaneous velocity is not really instantaneous. If the arguments in §1 that the at-at theory of motion does not preclude determinism are correct, then one has reason to believe that the at-at theory should be our preferred theory of classical motion. Although I do believe that the at-at theory gives us a satisfactory account of the metaphysics of motion, I also believe that there exists an unappreciated impetus alternative to the at-at theory. Moreover, I claim that this alternative, taken at face value, seriously underdetermines our interpretation of classical motion, for only epistemically inaccessible facts could possibly decide between them.

The incipient idea is to take velocity as a quantity of an object's instantaneous state and position as a quantity derived from velocity by integration: $\mathbf{x}(t) = \int \mathbf{v}(t)\,\mathrm{d}t$. Whereas the at-at theory considered above reduces velocity developments to position developments, this theory reduces position developments to velocity developments. It gives rise naturally to a kind of impetus theory, since velocity is now taken as a truly instantaneous quantity, although in this case $\mathbf{v} = \dot{\mathbf{x}}$ not because they are equal by a kinematical constraint but because the fundamental theorem of calculus applied to the definition of position just given informs us that the terms on both sides simply refer to the same property.

Arntzenius in fact briefly raises the possibility of such a theory of motion in his response to (Smith, 2003) but quickly overrules it since there are "velocity developments that are incompatible with calculus" — in particular the "calculus definition of velocity" (Arntzenius, 2003, 282). In essence, he claims that one cannot allow arbitrary velocity functions (as one allows arbitrary position functions in the at-at view), because time derivatives of position functions cannot recover a (very) large class of these arbitrary velocity functions, viz. those functions which are not the derivatives of any function. Since the definition of velocity (as time derivatives of position) would limit which functions could be used to describe velocity,"logic and definition alone would still imply constraints between instantaneous states at different times" (Arntzenius, 2003, 282).

Arntzenius again does not spell out the details of his objection, so, to see how his objection is supposed to work, let us consider his example, the pathological function known as the Dirichlet function. It is defined as follows: let $\mathbf{v}(t)$ be defined such that $\mathbf{v}(t) = 1$ for rational t and $\mathbf{v}(t) = 0$ for irrational t. The particular details of this function are not visualizable at any scale — if one tries to plot it, the best depiction is two apparently unbroken lines at $\mathbf{v} = 1$ and $\mathbf{v} = 0$. (The Thomae function is a similar alternative function that is better able to be visualized.)

Assume that there is actually a position development $\mathbf{x}(t)$ whose derivative

$\dot{\mathbf{x}}(t)$ is this just defined velocity development $\mathbf{v}(t)$. Recalling the definition of instantaneous velocity in §1, it must then be the case that $\bar{\mathbf{v}}(t, \Delta t)$, i.e. $(\mathbf{x}(t + \Delta t) - \mathbf{x}(t - \Delta t))/2\Delta t$, approaches the limit $\mathbf{v}$ near t. This cannot be the case, however, since for any ε such that $1 > \varepsilon > 0$ the inequality $|\bar{\mathbf{v}}(t, \Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$ will not hold. This is because, as one moves closer to t, the Dirichlet function repeatedly jumps between 0 and 1 as t goes from rational to irrational numbers. There is never an interval which "traps" the sequence of positions such that a limit is approached. Since velocity functions with characteristics similar to the Dirichlet function cannot be recovered by differentiating position functions in this way, it seems that this definition of velocity imposes constraints on possible velocity developments merely by "logic and definition".

I say this is a bad argument. Before explaining in full why it is, though, some preliminaries are needed. Suppose that we do take pathological functions like the Dirichlet function seriously (for the moment) as physically possible velocity developments. In the impetus view under consideration, position is a derived property, as it is defined via integration. It therefore becomes important precisely which notion of integration we use. The notion of integration familiar from basic calculus, Riemann integration, cannot be applied to the Dirichlet function (the function has no Riemannian integral). Thus one might conclude that an object with the Dirichlet velocity development, paradoxically, has no position development.

Surely this makes for a serious problem with the proposed account of motion. One cannot allow the potential absence of well-defined positions! Yet this is not a defect of the impetus theory alone, if a defect it even is. Suppose that we allow everywhere continuous but nowhere differentiable position developments in the at-at theory of motion. Then an object whose position development is described by the Weierstrass function, the most well-known function of this kind, has no velocity development. This is because the position development is nowhere differentiable; hence the object with this position development has a velocity nowhere! Such a case, I submit, is neither more nor less strange than the previous. If pathological functions are a problem for the impetus theory, then equally they are for the at-at theory as well.

Perhaps one might object to the cases being treated equipollently, for example by saying that it is possible to imagine an object moving along a trajectory with the functional form of the Weierstrass function but not possible to imagine an object failing to have a position. This objection, however, begs the question against the impetus view under discussion. We are of course used to thinking of objects moving in space. But we should not allow a mere prejudice or lack of imagination to preclude an alternative theory of motion. Only if the theory suffers from legitimate defects, for example an inability to recover empirical content or to provide a coherent account of classical motion, should it be disfavored. This, I claim, cannot be shown.

As it happens, Arntzenius's own objection is not that the mere lack of a position development is problematic. Rather it is that there exist velocity developments which are not the derivatives of any position development according to the previously given definition of derivative. But why should we assume *this* definition of velocity in the present impetus theory of motion? Velocity is not and should not be defined in the impetus theory; it is rather taken as basic, like position is in the at-at theory. Position is derived and hence must be defined in the impetus theory. Arntzenius's objection depends on duplicitously treating velocity as both basic and derived. And that is not right.

If one were to do the same in the context of the at-at theory, i.e. treat position as both fundamental and derived, then one would have precisely the same problem: there exist position developments which are not the integrals of any velocity development. A position development following the Weierstrass function would be precisely such a case. In this case there would be position developments that are incompatible with calculus, namely the "calculus definition of position" (via integration). Arntzenius's objection cuts both ways (if it cuts at all).

That said, it is perhaps worth pointing out that there are alternative definitions of integration according to which functions like the Dirichlet function are integrable. The well-known Lebesgue integral, for example, generalizes the notion of Riemann integration by utilizing a measure with respect to which integration is performed. Given a set X and a measure μ on the measurable subsets of X, the Lebesgue integral of a function f over the set $A \subset X$ is written $\int_A f \, \mathrm{d}\mu$. If we take the f to be the Dirichlet function, X as the set of real numbers $\mathbf{R}$, $A = [0, 1]$, and μ the standard Lebesgue measure associated with the real numbers, then

$$\int_A f \, \mathrm{d}\mu = 1.$$

When one makes use of the Lebesgue integral, one finds that the integral of the Dirichlet function over this finite interval is just a constant.

If we adopt this alternative definition of integration, then we have what appears to be an unpalatable consequence: the usual derivative of a constant is zero, which is obviously not the same as the Dirichlet function. When previously we used Riemannian integration, we could rely on the fundamental theorem of calculus to guarantee a certain duality between differentiation and integration, such that a function could be equated with the derivative of its antiderivative. This kind of duality holds for Lebesgue integration as well when one introduces a suitable definition of derivative. The simplest way to do this is just to define differentiation in terms of Lebesgue integration: the derivative of a function f at a point x is

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_B f(y) \, \mathrm{d}\mu,$$

where $B(x,r)$ is the open ball centered at x of radius r. The Lebesgue differentiation theorem guarantees that this derivative exists and, more importantly, equals f at almost every point in X. If we take f to be the Dirichlet function, the derivative of its antiderivative is *almost* the Dirichlet function, i.e. is equal to the Dirichlet function everywhere except for a set of measure zero. Although one does not recover the Dirichlet function, one does obtain a function which is "almost" the same, but this is, as expected, still the zero function.

The mere fact that the derivative of the Dirichlet function's antiderivative is only almost the Dirichlet function should not be cause for much concern. According to the reductive impetus theory of motion under discussion, it is velocity which is basic, not position. From a metaphysical point of view it does not matter whether one can recover the velocity development by differentiation. That one cannot in this case is nothing more than a technical defect. Nevertheless, recalling some concerns raised earlier, it does seem puzzling why an object with a constant position development (derived via integration of the Dirichlet function) would have such a complicated velocity development (the Dirichlet function), as the deviations in that velocity development appear to have no physical effect on the object's position development.

I expect by this point it should seem that this deepening excursion into niceties of mathematical analysis is becoming rather strained. It has not further clarified motion much at all (although I would insist that it was not entirely pointless, since it does serve to address several spurious objections that might be raised against the reductive impetus theory). Indeed, I think the discussion of pathological functions *is* rather strained in the context of classical motion. For the purposes of mechanics and motion, we would, I think, do well to heed the words of Poincaré:

> Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.
>
> In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that.
>
> If logic were the sole guide of the teacher, it would be necessary to begin with the most general functions, that is to say with the most bizarre. It is the beginner that would have to be set grappling with this teratologic museum. (Poincaré, 1952, 125)

The central lesson which the tortuous course so far followed in this section is intended to evince — and what I believe Poincaré's pleads in this quotation — is that logical and mathematical possibility do not correspond neatly to metaphysical and physical possibility. When one reasonably restricts attention to suitably well-behaved functions to describe the motion of objects, one is able to keep the relevant physical notions in plain view. In so doing, it is clear that the reductive impetus theory of motion proposed here is permissible just as much as the reductive at-at theory of motion discussed in the previous section is. The wider context of functions in mathematical analysis is, from the point of view of (meta)physics, a distracting teratology — for the at-at theory as much as for the impetus theory.

Perhaps the foregoing discussion belabors a banal point, but it seems necessary given that many arguments in the literature, including Arntzenius's dismissal of the reductive impetus theory, have been based on the "dishonest" possibility of objects following bizarre paths. Many authors discussing motion have relied on examples involving various physical pathologies, e.g. discontinuous (or not differentiable) position developments, in order to conjure intuitions about the nature of motion (Jackson & Pargetter, 1988; Tooley, 1988; Carroll, 2002). Insofar as we wish to understand classical motion, it is surely enough to focus on the relevant kinds of motion treated by successful theories describing it — and that kind of motion is (at the very least) continuous.

That said, I can now turn to explicating this reductive impetus account of motion and its take on the "honest" kind of classical motion described by classical physics. Recall, for purpose of contrast, that the at-at theory of motion takes position as basic. It is therefore naturally bound up with the existence of space and time: objects have a position in virtue of their embedding in space (à la substantivalism) or in virtue of their spatial relations with other objects (à la relationalism). It is worth stressing that objects have an *absolute* position solely in virtue of the existence of absolute space, as is supposed in the doctrine of substantivalism. This supposition is, strictly speaking, an additional metaphysical posit above and beyond the metaphysics required for the at-at theory of motion.

What does the reductive impetus theory suggest? It takes velocity as basic, so, in analogy to the at-at theory, it is naturally understood as depending on the existence of some sort of "velocity space" (whether understood in a way analogous to relationist or substantivalist space). Objects have a velocity, one would say, in virtue of their embedding in velocity space or in virtue of their velocity differences with other objects. Although one could also posit absolute "locational" space in addition to the assumed velocity space, it would seem to go completely against the idea of taking velocities as basic. Thus "locational space" is naturally taken to be a derivative concept (and perhaps in some sense emergent) according to this theory of motion.

Granted, as spatial reasoning is so familiar, this velocital view can seem rather strange and unintuitive. I insist that it is not thereby implausible — at least it is no less plausible than the at-at theory. A simple imaginative exercise may be helpful to see the dualistic relationship of the two accounts of motion. Suppose that we are out along the Bruntsfield Links watching a chipped golf ball in flight. Suppose that we were able, for a moment, to perceive its instantaneous physical situation from our vantage on the adjoining walk. Look at the ball. Is it moving? On the at-at theory, we would certainly be able to say *where* it is (in relation to ourselves) but not whether it is *moving* (in relation to ourselves), since it has no instantaneous property of velocity. If we only knew the positional facts about the temporal neighborhood of that instant, by allowing time to pass, say, then we could say for sure that it is in fact moving. On the impetus theory, the situation is just the opposite. In this case we would certainly be able to say that the ball is indeed *moving* (with respect to ourselves), but we could not at all say *where* it is (with respect to ourselves). Yet if we knew the velocital facts about the temporal neighborhood of that instant, by allowing time to pass, say, then we could say well enough where the ball is (with respect to us). Of such facts we are of course normally in possession (whereas we obviously never perceive instantaneous physical situations).

On the face of it these two accounts are metaphysically distinct, since each acknowledges facts that the other does not: they allege different things about the world at instants of time, even if they do not in ordinary circumstances. It might seem, then, that they are not metaphysically intertranslatable, given their different verdicts in the example. Supposing that metaphysical accounts which are not intertranslatable are not equivalent (Miller, 2005; Hirsch, 2009), the possibility of the reductive impetus account of motion would underdetermine our account of classical motion. This conclusion might be too quick, however.[5] There may, for example, be a subtle translation between the accounts such that facts in one which appear to be denied in the other actually have a suitable translation. One might also suppose that the instantaneous facts on which the two accounts differ, the untranslatable differences, are merely pseudo-differences

5. I am grateful to Neil Dewar and a reader for this journal for drawing my attention to alternate possibilities.

which do not depend on facts in the world or which do not reflect reality (Miller, 2005).[6] Or perhaps there is a third perspective from which one can derive the purported facts of each, in this case a unified account of motion, as suggested by McSweeney (2016) in a discussion of metaphysical equivalence.[7] It is not clear to me how to effect such a translation or whether the at-at account and impetus account should be considered metaphysically equivalent, so for the present I choose to rely on the thought that the reductive impetus account at least purports to differ from the at-at account on the metaphysical facts at instants.

I suspect it will have already occurred to some readers, however, that there is at least one further salient difference which distinguishes it from the at-at theory and which seemingly gives rise to a serious objection. That difference is the one secondary students learn between derivatives and integrals: the derivatives of well-behaved functions are determined, whereas (indefinite) integrals of well-behaved functions are determined only up to an additive constant, the so-called constant of integration. Thus, on the at-at theory, given a temporal neighborhood, the velocities would be fully determined, whereas on the impetus theory, given a temporal neighborhood, the positions would be determined only up to a constant.

Something must be said to defuse worries which arise from this well-known difference. As it happens, the most salient objections rest merely on a misunderstanding of mechanics. It is especially worth seeing how so, since it helpfully illustrates important facets of the impetus account and classical motion in general.

The first thing to note is that the fundamental dynamical law in Newtonian mechanics, Newton's Second Law, does not by itself force the problem upon us. Re-expressing $\mathbf{F} = m\mathbf{a}$ in the impetus theory's basic terms yields $\mathbf{F} = m\dot{\mathbf{v}}$, which makes no reference to position, so no reference to derived quantities, so no reference to integrals which could give rise to undetermined integration constants. It is therefore only when the forces themselves depend on positions, i.e. $\mathbf{F} = \mathbf{F}(\mathbf{x})$, that the Second Law necessarily becomes a second-order ordinary differential equation. Then, since $\mathbf{x} = \int \mathbf{v}\,\mathrm{d}t$, there will be derived quantities determined by integrals and, hence, constants of integration which potentially become physically significant.

Let us consider a few examples to see what comes of these constants of integration. Suppose first that there is but a single inertially moving particle in the universe. In this case, $\mathbf{v}$ is constant and $m\dot{\mathbf{v}} = 0 = \mathbf{F}$. What is the position of the particle? Since the particle experiences no forces, one is not forced to say anything at all about space and position. If we like, however, we can integrate the velocity to yield a position function, i.e. $\int \mathbf{v}\,\mathrm{d}t = \mathbf{x}(t) + \mathbf{X}$, where X is a constant. What is X? If there were absolute space, then X would be the difference between the relative position $\mathbf{x}$ and the absolute position of the object. But there are no absolute positions from the point of view of the reductive impetus theory of motion. So in this case the constant of integration should be neglected, just as it would be in the relationist version of the at-at theory of motion.

Suppose next that there are two free particles in the universe. In this case the velocities of both particles are constants and both particles experience no forces. Again, we are not forced to say anything about space and position, but we can integrate the two independent velocities to yield two position functions with two constants of integration. Now it would seem that there *is* a problem, since even if we rid ourselves of one spurious constant of integration, because of the absence of absolute positions, there still remains one which ought to represent the relative positions of the two particles. But our assumptions do not indicate any spatial relation at all between the two particles, as they are free particles which experience no forces. The only facts in such a universe are the velocity developments, from which we can infer the "distances" they travel, understanding that these distances should be measured out in fully independent positional spaces. This circumstance, i.e. of independent locational spaces, is somewhat analogous to the tangent (velocity) spaces of curved manifolds (spaces) in geometry, which cannot be automatically identified as they are in Euclidean space without the addition of some way to parallel transport tangent vectors, i.e. a connection. It is of course usual to think that there is only one unified "locational" space in which motion takes place, but that intuition is unsupported in this case since the particles are free. Fortunately for our intuitions, the weirdness of independent locational spaces of motion depends on the unrealistic idealization

6. This circumstance might even incline one towards a "no instants" view of time, although see (Arntzenius, 2000) for criticisms of such views.
7. One might think that Lagrangian or Hamiltonian mechanics provides this perspective, but that is not so; see §3 below.

of force-free motion.

So let us next consider the more realistic example of motion within the scope of Newton's Law of Universal Gravitation, which of course depends on relative positions to determine motion. For simplicity, consider the gravitational system of two objects and the law applied to one:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3},$$

where $\mathbf{F}_{ij}$ is the force on object i due to object j, G is Newton's gravitational constant, m_i is the mass of the object, and $\mathbf{x}_i$ is the position of the object i. In the impetus theory of motion the positions are derived quantities, so we would want to re-express Newton's Law of Universal Gravitation as follows:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t + \mathbf{X}}{|\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t + \mathbf{X}|^3},$$

where $\mathbf{X}$ is a constant equal to the difference of the two constants of integration introduced by re-expressing the relative positions as indefinite integrals. What the gravitational force at a particular instant is clearly depends on what $\mathbf{X}$ is. Presumably this is the kind of circumstance that the impetus theory's detractor envisions as fatal for the view.

Plainly, if one were to have the view that laws act on instantaneous states, then the integration constants would be essential for securing determinism, since different choices of $\mathbf{X}$ — essentially different choices of relative distance between the two particles — would entail different forces, accelerations, and future motions. The reductive impetus theory of motion, however, like the at-at theory of motion, does not rescue Laplacian determinism. Instantaneous velocities in concert with Newton's Second Law cannot fully determine future and past states alone, just as instantaneous positions in concert with Newton's Second Law cannot, as argued at length above. So, while the desire to save this kind of determinism motivates some to consider super-added impetuses, it is certainly not a motivation which would lead one to consider the reductionist impetus theory.

If one does not expect the impetus theory to rescue Laplacian determinism, then I claim that there is in fact no underdetermination of the physically relevant integration constants. An extended example would perhaps usefully illustrate why, but the following general argument is far more straightforward: Consider that classical mechanics (in the abstract) is a theory of possible motions, where a motion is a specification of each object's position and velocity at each instant. A natural way of conceiving a mechanics problem is to begin with the set of kinematically possible motions; a motion in this set is a collection of trajectories of the objects in the system. What dynamical laws then do is select from this set the dynamically possible motions — dynamical laws, that is, are constraints on the set of kinematically possible motions. In physically reasonable cases, dynamical evolution is deterministic, in the specific sense that these dynamically possible motions never "cross" (do not ever share all the same positions and velocities for all trajectories). It is not enough to pick out a particular motion to specify just the velocities of each object at a particular instant, since many motions have the same velocities at an instant but different position specifications. This is why there is the constant in the gravitational force function above: specifying the velocities at a time alone is insufficient for picking out the forces at that time. It is a well known fact, however, that it *is* enough to specify the positions of each object at two instants to pick out a particular motion. This is essentially because Newton's Second Law is a second-order differential equation. But there is nothing special about positions here: it is just as well to specify the velocities of each object at two instants to pick out a particular motion. It is also enough to specify the positions and velocities of all the objects at a single instant. But to do this one needs neighborhood properties on both the at-at and impetus theory, since velocities and positions, respectively, are reduced to position and velocity developments (respectively).

Returning now to the example of Newtonian gravitation, if we express Newton's Law of Universal Gravitation with definite integrals (which represent position displacements by integrating velocity developments in the neighborhood of an instant), i.e.

$$\mathbf{F}_{12} = Gm_1m_2\frac{\int_{t-\Delta t}^{t+\Delta t}(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t}{|\int_{t-\Delta t}^{t+\Delta t}(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t|^3},$$

then the constant of integration disappears (it is, after all, a definite integral) and the gravitational forces are fully determined by the given velocity developments. This is because we are in effect using neighborhood properties of the objects to determine their trajectories and, hence, the actual motion of the system.

Since some integration constants are physically significant, i.e. are required to fully specify the position developments (these being the antiderivatives of the velocity developments), it might appear that some additional information is still required in the impetus theory in comparison with the at-at theory. I stress that this appears so only when one confines attention to a single instant. Since picking out a particular motion (by, for example, specifying velocities at two instants or even during some small temporal neighborhood of the velocity development) determines an entire position development, whatever physically relevant integration constants there are will be included automatically in those determined position developments. There is in fact no conceptual problem created by the presence of integration constants at all in the reductive impetus account. Indeed, only persisting in believing the myth that classical motion is Laplacian deterministic will lead one to find a problem with it (or the at-at theory).

My hope is that the foregoing discussion has made it plausible that the reductive impetus view I have developed is coherent and, moreover, dual to the at-at theory of motion in such a way as to underdetermine our choice between them. To rehearse the story, recall that the at-at theorist denies that kinematical velocity is truly instantaneous, giving a reductive account of kinematical velocity in terms of position developments. One adds impetuses to this account at the cost of ontological redundancy. In the reductive impetus theory, however, the roles of position and velocity are exactly reversed. By the same arguments, one finds that velocities are truly instantaneous and that positions are not; the latter are instead reduced to velocity developments. One could add truly instantaneous positions back to try to regain Laplacian determinism, but the argument against them would mirror the argument against truly instantaneous velocities being added to the at-at theory of motion. Thus we are left with two reasonable alternatives: the at-at theory of motion and the reductive impetus theory of motion (where, note, the reduction mentioned here is of position developments to velocity developments, not of motion).

I claim that there is no available reason to prefer one over the other as an account of classical motion, due to their differences lying in epistemically inaccessible facts. Already Carroll has argued that the at-at theory is irreproachable as a theory of motion because of our epistemological limitations. As he says, knowing that the at-at theory is false would "require powers of discrimination well beyond us" (Carroll, 2002, 64), since any counter-example concerns goings-on at infinitesimal temporal intervals. If he is right, and it certainly seems to be hard to dispute it, then the impetus theory of motion that I have presented is similarly irreproachable. The impetus version of reality describes classical motion of the same kind as the at-at theory would, differing only on what is happening in epistemically inaccessible infinitesimal temporal intervals. So far as I can see, due to the symmetry of the two views, there are no obvious reasons to favor one theory over the other; only long custom and convenience motivates a preference for the at-at theory. Recognizing this, that there are actually two dual, viable interpretations of classical motion, is, it seems to me, of considerable interest, especially for what it suggests about the possibilities for the metaphysical grounds of motion (particularly vis-à-vis the spaces in which motion occurs). Yet there remain difficult questions about how to interpret the significance of this apparent underdetermination and whether the two accounts are indeed metaphysically inequivalent.

3. Analytical Mechanics

Although classical mechanics's most familiar presentation is through Newton's Laws, most physicists understand the subject in terms of the more sophisticated Hamiltonian or Lagrangian frameworks. As philosophers have become acquainted with these frameworks through recent work that makes use of them, e.g. (North, 2009; Curiel, 2014), I offer some further remarks in this section on the preceding arguments by discussing motion in them.[8] Although the frame-

8. It would also be worth including some remarks on the Hamilton-Jacobi approach, as it provides a distinctive picture of mechanics (Butterfield, 2005). Indeed, it even strongly suggests the equivalent defensibility of the two accounts of motion considered in this paper, as it furnishes a neat demonstration that only one half of the kinematical quantities (position, velocity) are actually required to completely describe motion in classical mechanics. However it is best left as a topic for another time, particularly due to its

works are suggestive of different interpretations, I claim that the issues raised in the previous sections nevertheless remain entirely the same.

First, Hamiltonian mechanics. The basic quantities in Hamiltonian mechanics are (generalized) positions $\mathbf{q}$ and (generalized) momenta $\mathbf{p}$. These quantities (for each object) are taken to compose the state of the system. The equations of motion are given by solving the following two first-order ordinary differential equations, Hamilton's equations:

$$\dot{\mathbf{p}} = -\frac{\mathrm{d}H}{\mathrm{d}\mathbf{q}} \qquad \dot{\mathbf{q}} = \frac{\mathrm{d}H}{\mathrm{d}\mathbf{p}}, \tag{1}$$

where H is the Hamiltonian, a function defined on the phase space (the space of states) of a system. The phase space of Hamiltonian mechanics is $6n$-dimensional, where n is the number of objects in the system (three for the position and three for the momentum of each object). On the face of it, the quantities $\mathbf{q}$ and $\mathbf{p}$ are on a par: they are both basic quantities in the framework, Hamilton's equations are nearly identical for each (they differ only by a minus sign), and there are no apparent kinematical constraints that break the equal footing on which they stand.

How can Hamiltonian mechanics give the same classical motions as the simpler Newtonian approach, since it seemingly allows non-classical possibilities, i.e. motions that do not satisfy the kinematical constraint $\dot{\mathbf{q}} = \mathbf{p}/m$, which links momentum and change in position? The only place where constraints like this can be imposed in the framework is in the Hamiltonian. Indeed, only Hamiltonians that satisfy a couple requirements will result in classical motions: H must be a function that is second-order in the momenta and otherwise solely a function of the positions. In systems treated by classical mechanics, the Hamiltonian can then be understood as the total energy of the system.

Arntzenius (2000) mentions Hamiltonian mechanics as a modern example of an impetus theory. Given appearances alone, this would be the natural classification, since both position and momentum are taken as basic quantities. Yet one should not be too easily taken in by formalism — supposing that the Hamiltonian respects the requirements mentioned in the previous paragraph, the second of Hamilton's equations above is actually just $\dot{\mathbf{q}} = \mathbf{p}/m$, i.e. merely equates the product of mass and velocity with momentum. It is in essence just a physically motivated definition of momentum, not a dynamical law per se.

It should be clear now that the interpretive situation is not so simple. One may insist that the Hamiltonian framework affords the possibility of a complete instantaneous state, one including both position and momentum (velocity). But the second of Hamilton's equations suggests two interpretations. One is that it imposes a kinematical constraint, in which case we have once more the problems of the super-additive impetus theory of motion of §2.1. The other is that it merely reveals momentum as velocity in disguise, i.e. merely as a neighborhood property, in which case we have once more the at-at theory of motion. Integrating Hamilton's equations leads one to the dual possibility of the impetus theory of motion of §2.2. Thus Hamiltonian mechanics affords precisely the same two metaphysical possibilities discussed in the previous sections: the at-at theory of motion and the reductive impetus theory of motion.

Second, Lagrangian mechanics. Here the relevant quantities are called the generalized coordinates (positions) and generalized velocities, commonly labeled $\mathbf{q}$ and $\dot{\mathbf{q}}$. The Lagrangian, L, is a function defined as the difference between kinetic energy and potential energy of the system. Now, it may seem as if Lagrangian mechanics builds in the usual kinematical connection between position and velocity at the start, since (notationally at least) $\dot{\mathbf{q}}$ is the time derivative of $\mathbf{q}$. There is a slight subtlety, however. The two different classes of variables (the velocities and positions) are on par in how they are initially treated in the Lagrangian framework. When one derives the equations of motion from the Lagrangian analog of Newton's Second Law, i.e. the Euler-Lagrange equation —

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\mathbf{q}}} - \frac{\mathrm{d}L}{\mathrm{d}\mathbf{q}} = 0 \tag{2}$$

— one takes an independent derivative of the Lagrangian with respect to each of $\mathbf{q}$ and $\dot{\mathbf{q}}$. In the derivation of the equations of motion of a particular system, they are treated as if they had no dependence at all on one another (whereas normally the chain rule of calculus would have to be applied). It is only after the equations of motion are derived that $\dot{\mathbf{q}}$ becomes understood as the time

unfamiliarity among philosophers.

derivative of **q**. Thus, as with Hamiltonian mechanics, the Lagrangian framework ostensibly allows non-classical motions, i.e. ones which do not satisfy the kinematical constraint between time derivatives of position and velocity. Only those motions that do satisfy the Euler-Lagrange equation end up satisfying this constraint, which is why the constraint is understood to be enforced after the derivation of the equations of motion. So one has, once more, an at-at theory of motion. Naturally, one could re-interpret that constraint in terms of integration rather than differentiation, in which case one has, once more, the reductive impetus theory of motion.

Although there is therefore no novelty in these frameworks vis-à-vis a theory of motion, there is perhaps a novel issue to be found here with respect to the status of the non-classical motions which is worth mentioning. These motions do not enforce the relation $\mathbf{v} = \dot{\mathbf{x}}$, i.e. the relation that velocity is the time rate of change of position (or the impetus theory analog). Are they mathematical fictions, introduced merely for the purpose of formulating the powerful analytic techniques of Hamiltonian and Lagrangian mechanics? Or should we countenance these as genuine metaphysical possibilities? My inclination is towards the former, since it is difficult to understand what **v** is meant to represent, if it is not representing the velocity determined by the object's trajectory. Could it perhaps be, though, some sort of unactualized "causal power"? It just so happens, of course, that in our world — insofar as it is classical — all such powers are fully actualized. One would like to know why *that* would be so. That they should be is enshrined in the principle of least action (or Hamilton's principle), but little philosophical work has been done to explain why the principle should be necessary (and not just true).

4. Conclusion

I have defended two theories of motion, the at-at theory and a novel impetus theory of motion. I claim that the choice between these two theories is epistemically underdetermined. It may indeed simply be a matter of convention whether one understands motion in one way or the other given the epistemic irreproachability of each theory. Assuming that there is some fundamental "space" that grounds the basic properties of the two theories, it would also perhaps then be a matter of convention whether one sees the world as fundamentally spatial or as fundamentally velocital, as they differ only in their instantaneous ontologies. It may also be possible, however, to see these two metaphysical accounts as intertranslatable and hence equivalent, but this would require a subtle account of translation to handle the contrary facts concerning instants in the two views.

To review the main arguments, I began in the first part of the paper by defending the at-at theory of motion against the criticism that it makes determinism impossible by fiat. The impossibility of the Laplacian version of determinism is indeed a feature of classical mechanics, but it is not so merely by how velocity is defined. While I agree with the critics that velocities should not be thought of as really instantaneous in the at-at theory of motion, that they are still reasonably considered part of the physical state of a system follows from the nature of the laws, whether these laws be metaphysical laws or physical laws. The kind of determinism relevant to mechanics is thus not Laplacian determinism, but determinism based on states including instantaneous and neighborhood properties.

In the second part of the paper, I considered two impetus theories of motion. I rehearsed the argument against an impetus theory of motion that makes velocity truly instantaneous by fiat. Kinematical velocity is a derived quantity in the at-at theory of motion, but one that fully accounts for motion without reference to some super-added impetus. Thus we are led to conclude that this impetus theory should be discarded. The second, novel impetus theory of motion turns the at-at theory on its head by assuming that velocity is basic and truly instantaneous and position is derived (reduced to velocity developments). I defended this view from two major criticisms, namely that it is at odds with the standard definition of velocity and that integration introduces undetermined constants of integration. I concluded that this impetus theory of motion and the at-at theory are dual theories of motion, sufficiently symmetric so that there is no evident way to decide between them as accounts of motion, at least on any epistemic grounds.

Finally, I wish to remark that it is not inconceivable that people could have naturally conceived of motion in the velocital way which I suggest, although admittedly the possibility does seem remote to us, since we are so used to thinking of the world in spatial terms. As it happens, Borges provides an intriguing report of the people of Tlön whose perception of motion can perhaps be interpreted in a manner friendly to the impetus view (although his favored analysis of their philosophy is a kind of idealism):

> It is no exaggeration to state that the classic culture of Tlön comprises only one discipline: psychology. All others are subordinated to it. I have said that the men of this planet conceive the universe as a series of mental processes which do not develop in space but successively in time. Spinoza ascribes to his inexhaustible divinity the attributes of extension and thought; no one in Tlön would understand the juxtaposition of the first (which is typical only of certain states) and the second — which is a perfect synonym of the cosmos. In other words, they do not conceive that the spatial persists in time. The perception of a cloud of smoke on the horizon and then of the burning field and then of the half-extinguished cigarette that produced the blaze is considered an example of association of ideas. (Borges, 1961, 116)

It might be thought that it is the perception of motion which is direct and foremost from the point of view of psychology; space is, as it were, something inferred, "typical only of certain states". It would be natural, from this point of view, to suppose that recourse to a concept of space be gained only by specifying particular forces of interaction which could give rise to a unified spatial description — and thus only in particular (although perhaps ubiquitous) states. If one were to think so, then the impetus account might be natural to adopt as a way of conceiving the world around us; although of course, as argued above, to do so would be no more justified in the epistemic sense than adopting the at-at account is from our favored spatial point of view. Nevertheless, I do think it worth emphasizing that the impetus picture does give a conceptually coherent way of proceeding in theorizing about the world, albeit one that has so far not found favor, neither in physics nor in everyday life. Still, there is perhaps an intriguing hint of the duality of motion emphasized here in quantum physics, between the "position representation" and the "momentum representation" of wave functions, for example, to which these accounts of classical motion might relate.[9]

9. Thanks to Craig Callender, John Dougherty, Gil Hersh, Nat Jacobs, Wayne Myrvold, Matt Pead, David Schroeren, Christian Wüthrich, and two referees for helpful comments or conversations on the topic of this paper. Special thanks go to Natalja Deng for commenting on the paper at the 2016 Society for the Metaphysics of Science Conference in Geneva. I also wish to thank audiences at that conference, as well as at the 90th Joint Session of the Aristotelian Society and the Mind Association in Cardiff and the 2012 Philosophy of Logic, Mathematics, and Physics Graduate Conference at the University of Western Ontario for their comments and questions.

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