The Relation between Credence and Chance: Lewis' "Principal Principle" Is a Theorem of Quantum Probability Theory

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David Lewis' "Principal Principle" is a purported principle of rationality connecting credence and objective chance. Almost all of the discussion of the Principal Principle in the philosophical literature assumes classical probability theory, which is unfortunate since the theory of modern physics that, arguably, speaks most clearly of objective chance is the quantum theory, and quantum probabilities are not classical probabilities. Given the generally accepted updating rule for quantum probabilities, there is a straight forward sense in which the Principal Principle is a theorem of quantum probability theory for any credence function satisfying a suitable additivity requirement. No additional principle of rationality is needed to bring credence into line with objective chance.

1 Introduction

David Lewis (1980) proposed a normative principle, which he dubbed the Principal Principle (PP), linking credence (aka degree of belief) and objective chance. Of course, it is a matter of controversy as to whether there is objective chance in the world and, if so, what it is. Lewis wanted to allow for the possibility that chance exists and, regardless of what "it" is, he thought that it should constrain rational credence in the way specified by PP. Turning this around, PP serves as a functional characterization of chance—chance is what commands rational credence. To give a concrete example of an intended application, suppose that you learn that the objective chance of heads on the next flip of a coin is 1/2, and that you update your credence function to reflect this information. Then (according to PP) in order to count as delivering rational degrees of belief your updated credence function ought to assign degree of belief 1/2 to said outcome, and this is so regardless of other things you might have learned, such as the frequency of heads and tails on past flips of the coin. Lewis intended the regardless clause to apply not just to evidence about past flips of the coin but to any "admissible" evidence; but while examples of inadmissible evidence were given (e.g. knowledge of the outcomes of future flips of the coin) no general criterion of how to parse the admissible/inadmissible distinction was specified. Lewis himself came to believe that there was "bug" in his original formulation of PP, and he sought to reformulate PP in a manner that would avoid the bug while conforming to his desire allow for the Humean supervenience of chance (Lewis 1994).

A fairly sizable and ever growing literature has accreted around these topics.[1] The contributors are mainly analytical metaphysicians who produce ever more nuanced treatments which are sprinkled with interesting insights and clever, and even brilliant, moves. But since the discussion seems to be constrained only by intuitions about how chance ought to work in the actual and other possible worlds it is not surprising that fundamental disagreements have arisen both about how to capture in precise form the idea behind Lewis' PP and about how to justify PP as a principle of rationality of belief. Even more disconcerting is the fact that the literature makes little contact with quantum theory, despite the fact that this is inarguably one of the most successful theories of modern physics and arguably the theory that speaks most clearly of objective chance. This lack of contact is not an oversight that is easily corrected; for the bulk of the philosophical discussions of the issues surrounding Lewis' PP assume classical probability theory, whereas quantum probabilities cannot be construed as probabilities on a classical probability space. In particular, Bayesian conditionalization, used in updating credence functions in classical probability, is inappropriate for credence functions defined over a non-commutative event algebra such as the one encountered in quantum theory.

The purpose of this note is to show that there is a straightforward sense in which no new principle of rationality is needed to bring rational credences over quantum events into line with the events' objective chances—the alignment is guaranteed by as a theorem of quantum probability, assuming the credences satisfy a suitable form of additivity. The normative status the additivity requirement is discussed. More generally, many issues discussed in the literature on Lewis' PP—such as the admissibility of evidence—can be given precise formulations in the context of quantum probability and settled

[1] For a sampling of the literature, see Arntzenius and Hall (2003); Bigelow, Collins, and Pargeter (1993); Black (1998); Haddock (2011); Hall (1994, 2004); Ismael (2008); Meacham (2010); Pettigrew (2012); Roberts (2001, 2013); Strevens (1995); Thau (1994); Vranus (2002, 2004).

by proving appropriate theorems in this context.

2 Quantum probabilities

2.1 Quantum probabilities in the algebraic formulation of quantum physics

In the version of the algebraic approach used here a quantum system is characterized by two objects: a von Neumann algebra $\mathfrak{B}(\mathcal{H})$ of observables acting on Hilbert space $\mathcal{H}$, which may be separable or non-separable[2]; and a set of states $\mathfrak{S}$ on $\mathfrak{B}(\mathcal{H})$, the members of which are regarded as physically realizable for the system at issue. This subsection reviews some basic facts about algebras that serve as the basis for a formulation of quantum probability theory. The following subsection discusses quantum states.

Ordinary QM (sans superselection rules) deals with case where $\mathfrak{B}(\mathcal{H})$ is the Type I factor algebra $\mathfrak{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. To keep this discussion as simple as possible I will concentrate on this case. For most applications of QM it suffices to use an $\mathcal{H}$ that is separable, but it is not hard to conceive idealized cases that require the use of a non-separable $\mathcal{H}$.[3]

A projection $E \in \mathfrak{B}(\mathcal{H})$ is a self-adjoint element such that $E^2 = E$. The range $Ran(E)$ of E is a closed subspace of $\mathcal{H}$, and for $\mathfrak{B}(\mathcal{H})$ the projections are in one-one correspondence with the closed subspaces of $\mathcal{H}$. The projections $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ have a lattice structure that derives from a partial order whereby $E_1 \leq E_2$ iff $Ran(E_1) \subseteq Ran(E_2)$.[4] That $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ forms a lattice means that it is closed under meet $E_1 \wedge E_2$ and join $E_1 \vee E_2$ of $E_1, E_2 \in$

[2] A separable $\mathcal{H}$ has a countable basis. A von Neumann algebra $\mathfrak{N}$ acting on Hilbert space $\mathcal{H}$ is an algebra of bounded operators closed in the weak operator topology. By von Neumann's double commutant theorem the closure condition is equivalent to the condition that $\mathfrak{N}'' := (\mathfrak{N}')' = \mathfrak{N}$, where X' stands for the commutant of X, i.e. the set of all bounded operators that commute with every element of X.

The reader interested in the details of the relevant operator algebra theory can consult Bratteli and Robinson (1987) and Kadison and Ringrose (1997).

[3] Consider, for example, an infinite spin chain consisting of a countably infinite number of sites, each of which can be either spin up or spin down. A Hilbert space of dimension 2^{χ_0} is needed.

[4] This is equivalent to requiring that $E_1 \leq E_2$ iff $E_2 - E_1$ is a positive operator.

$\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ which are defined respectively as the greatest lower bound and the least upper bound. They are respectively the projections corresponding to $Ran(E_1) \cap Ran(E_2)$ and the closure of $Ran(E_1) \cup Ran(E_2)$. Projections E_1 and E_2 are said to be mutually orthogonal iff $Ran(E_1) \cap Ran(E_2) = \varnothing$. When E_1 and E_2 are mutually orthogonal $E_1 \wedge E_2 = E_1 E_2 = E_2 E_1 = E_2 \wedge E_1 = 0$ and $E_1 \vee E_2 = E_1 + E_2$.

The elements of the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are referred to as quantum propositions (also yes-no questions, or quantum events). Quantum probability theory may be thought of as the study of quantum probability functions Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ (see Hamhalter 2003). Pr is required to satisfy the analogs of the basic axioms of classical probability:

(a) $\Pr: \mathcal{P}(\mathfrak{B}(\mathcal{H})) \rightarrow [0, 1]$

(b) $\Pr(I) = 1$, where I is the identity operator

(c) $\Pr(E \vee F) = \Pr(E + F) = \Pr(E) + \Pr(F)$ for all mutually orthogonal pairs $E, F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

The condition (c) of finite additivity may be strengthened to require complete additivity

(c*) $\Pr(\sum_{a \in \mathcal{I}} E_a) = \sum_{a \in \mathcal{I}} \Pr(E_a)$ for any family $\{E_a\} \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ of mutually orthogonal projections.[5]

When $\mathcal{H}$ is separable any family of mutually orthogonal projections is countable, so (c*) reduces to the requirement of countable additivity.

2.2 Quantum states and quantum probabilities.

Quantum states $\mathfrak{S}$ on $\mathfrak{B}(\mathcal{H})$ are normed positive linear functional mapping elements of $\mathfrak{B}(\mathcal{H})$ to $\mathbb{C}$. A particularly important subclass of quantum states that will play an outsized role in what follows are the normal states $\mathcal{N}$, states with a density operator representation, i.e. there is a trace class operator ϱ on $\mathcal{H}$ with $Tr(\varrho) = 1$ such that $\omega(A) = Tr(\varrho A)$ for all $A \in \mathfrak{B}(\mathcal{H})$.[6] The standard practice of QM has baked into it the presumption that only normal

[5]If $\mathcal{I}$ is infinite, the convergence of $\sum_{a \in \mathcal{I}} E_a$ is taken in the weak operator topology.
[6]This is what physicists call the Born rule for calculating probabilities see below.

states are physically realizable. There are good, but not conclusive, reasons to support this presumption (see Arageorgis et al. 2017 and Ruetsche 2011). For present purposes it will be taken on board.

For future reference, some additional nomenclature should be noted. A pure state ω is defined by the property that there are no distinct states ω_1 and ω_2 and real numbers λ_1 and λ_2, $\lambda_1 + \lambda_2 = 1$, such that $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$. Impure states are also referred to as mixed states. A vector state is a state ω such that there is a vector $\psi \in \mathcal{H}$ with $\omega(A) = \langle\psi|A|\psi\rangle$ for all $A \in \mathfrak{B}(\mathcal{H})$. Vector states are normal, and for the algebra $\mathfrak{B}(\mathcal{H})$ the vector states coincide with the pure states.[7]

Quantum states induce quantum probability functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$: it is easy to verify that for any $\omega \in \mathfrak{S}$

$$\Pr^{\omega}(E) := \omega(E) \text{ for } E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$$

satisfies the requirements (a)-(c) for a quantum probability function. Furthermore, if $\omega \in \mathcal{N}$ then $\Pr^{\omega}$ is completely additive, whereas if $\omega \notin \mathcal{N}$ then $\Pr^{\omega}$ will be merely countably or merely finitely additive depending on the dimension of $\mathcal{H}$.

3 Downward and upward: objective vs. subjective interpretations of quantum probabilities

3.1 Objectivist take: top down

The top-down approach to quantum probabilities lends itself to an objectivist reading of probabilities. On this reading quantum states codify objective, observer-independent physical features of quantum systems and, thus, the probabilities states induce are objective physical probabilities—chances, if you like. The objectivist reading of quantum probabilities has more going for it than merely postulating theoretical entities and sticking the label 'objective' on them; for it is supported by infrastructure of the theory which gives an account of state preparation, at least for the normal pure states.

[7] When the algebra of observables is a Type III von Neumann algebra, as with the local algebras in the algebraic formulation of relativistic QFT, all normal states are vector states and all vector states are impure. This introduces a number of complications in the discussion of quantum probabilities which will be ignored here.

A basic interpretational tenet of QM holds that the elements of the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are in principle verifiable/refutable by appropriate 'yes-no' experiments. The theory itself does not provide a manual for how to construct a laboratory device for carrying out the experiment—that belongs to the experimental practicum of QM. Among the elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is the support projector S_φ for a normal state, defined as the smallest projection in $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ to which φ assigns probability 1. Since normal pure states on $\mathfrak{B}(\mathcal{H})$ are coextensive with the vector states and $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ contains all projectors, the support projector S_φ for a normal pure state φ is the projector onto the ray spanned its vector representative. As a result, S_φ for a normal pure state serves as a filter for φ in the set $\mathcal{N}$ of all normal states, viz. for any $\omega \in \mathcal{N}$ (pure or impure) such that $\omega(S_\varphi) \neq 0$, $\omega(S_\varphi A S_\varphi)/\omega(S_\varphi) = \varphi(A)$ for all $A \in \mathfrak{B}(\mathcal{H})$ (see Ruetsche and Earman 2011).

By the von Neumann projection postulate, when a measurement of $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is made on a system initially in state ω and the measurement returns a Yes answer, the new state of the system is $\omega_F(\bullet) := \omega(F \bullet F)/\omega(F)$. It follows from this and filter property of S_φ for a normal pure state φ that, whatever the initial state ω, as long as $\omega(S_\varphi) \neq 0$, a Yes answer to a measurement of S_φ for a normal pure state φ ensures that the new state ω_{S_φ} is φ. A normal impure (or mixed) state does not have a filter, and so there is no preparation procedure in this sense for impure states (see Ruetsche and Earman 2011).

So much for the formalism. It is now time for Nature to weigh in. Prepare a quantum system in a normal pure state φ. Conduct a yes-no experiment for some $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, and record the result. Reset the system (or a similar system) in the same state φ, and repeat the yes-no experiment for E. The probability for a Yes outcome on any trial is $\varphi(E)$ regardless of the outcomes on any other trials, which is to say the experimental protocol gives i.i.d. trials. The strong law of large numbers thus implies that as the number of trials tends to infinity the frequency of Yes responses almost surely tends to $\varphi(E)$. In actual realizations of such experiments expectations are fulfilled in that there is rapid apparent convergence to the value supplied by the quantum formalism. Of course, the inductive skeptic will caution that the apparent convergence may disappear if trials are continued into the indefinite future, but such skepticism if pushed too far would undermine all scientific inquiry. In short, it is hard to resist the notion that the probabilities delivered by the quantum formalism latch on to objective features of the physical world.

3.2 The personalist take: bottom up

Quantum Bayesians (QBians as they style themselves) reject the top-down approach to quantum probabilities; in particular, they reject the idea that quantum probabilities on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ arise from quantum states on $\mathfrak{B}(\mathcal{H})$ construed as codifying objective features of physical systems.[8] Instead they propose a bottom-up perspective from which the starting point is from probability functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ construed as the credence functions of actual or potential Bayesian agents. States are seen as bookkeeping devices used to keep track of credence functions. This alternative point of view is given an initial foothold from Gleason's theorem, the fundamental theorem of quantum probability theory, which shows that, with a mild restriction, one can also move in the upward direction from probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ to states on $\mathfrak{B}(\mathcal{H})$:

> *Gleason's theorem.* Let $\mathcal{H}$ be a separable Hilbert space with $\dim(\mathcal{H}) \geq 3$. Then any quantum probability function Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ has a unique extension to a state $\omega^{\Pr}$ on $\mathfrak{B}(\mathcal{H})$. Further, if Pr is countably additive (respectively, merely finitely additive) then $\omega^{\Pr}$ is normal (respectively, non-normal).

When $\mathcal{H}$ is non-separable it is still true that any quantum probability function Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ has a unique extension to a state $\omega^{\Pr}$ on $\mathfrak{B}(\mathcal{H})$; but the the last clause of the theorem has to be emended to: if Pr is completely additive (respectively, non-completely additive) then $\omega^{\Pr}$ is normal (respectively, non-normal). Gleason's theorem has also been extended to cover quite general von Neumann algebras, but these developments will not be of concern here.

This is not the place to discuss the problems and prospects for QBian program.[9] For present purposes I accept the less contentious part of QBism while rejecting the more interesting and controversial part: specifically, I assume that quantum physicists are to be construed as if they were Bayesian agents who have degrees of belief about quantum events that are codified in quantum probability functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$; but I reject the notion that all quantum probabilities are to be given a personalist reading, and I assume that normal pure states induce objective chances on elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Perhaps in the end the QBians will prevail on both parts; but if so the issue

[8] See von Baeyer (2016) for a readable overview of QBism.

[9] See Timpson (2008) and Earman (2018a) for a critical assessment.

of the relation between credence and chance is a dead issue as far as QM is concerned. Here I assuming that it is a live issue and, on the basis of this assumption, I will argue that QM itself, without any help from philosophers, provides a resolution.

4 Updating credences for quantum events

4.1 Motivating a quantum updating rule

As already mentioned, the typical discussion of Lewis' PP is conduced in the context of classical probability theory where the generally accepted rule for updating personal probabilities is Bayesian conditionalization: if an agent's initial probability function is pr and the agent learns that F is true then agent's new probability ought to be $pr_F(\bullet) := pr(\bullet \cap F)/pr(F)$, provided that $pr(F) \neq 0$. In order to discern the relation between credence and chance for quantum events the quantum-probability analog of the classical updating rule is needed. Fortunately, others have already done the work for us.

An elegant presentation of the argument for the probabilities on the projection lattice $\mathcal{P}(\mathfrak{N})$ of a general von Neumann algebra $\mathfrak{N}$ is provided by Cassinelli and Zanghi (1983). Here attention is restricted to the case of ordinary QM where $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$.[10] The argument starts by motivating updating classical probability by the following proposition:

> *Prop 1.* Let (Ω, Σ, pr) be a classical probability space, and let $F \in \Sigma$ be such that $pr(F) \neq 0$. Then there is a unique functional $pr_F(\bullet)$ on Σ such that (a) $pr_F(\bullet)$ is a probability measure on Σ, and (b) for all $E \in \Sigma$ such that $E \subseteq F$, $pr_F(E) = pr(E)/pr(F)$.[11]

And, no surprise, this unique functional $pr_F(\bullet)$ is just the familiar classical conditional probability functional $pr(\bullet \cap F)/pr(F)$.

Next, the argument shows that there is a parallel, but more qualified, result for quantum probability:

[10] The special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ was treated in Bub (1977).

[11] Ω is the sample space; Σ (the measurable sets) consists of a set of subsets of Ω; and $\Pr : \Sigma \to [0,1]$ satisfies (i) $\Pr(\Omega) = 1$ and (ii) $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$ for any $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \emptyset$.

Prop. 2. Let Pr be a countably additive quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ for separable $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$, and let $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ be such that $\Pr(F) \neq 0$. Then there is a unique functional $\Pr_F(\bullet)$ on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that (a) $\Pr_F(\bullet)$ is a quantum probability, and (b) for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $E \leq F$, $\Pr_F(E) = \Pr(E)/\Pr(F)$.

The unique $\Pr_F(\bullet)$ is called Lüders updating. Prop. 2 can be extended to cover the case of a non-separable $\mathcal{H}$ if Pr is completely additive.

Lüders updating can also be motivated by adapting the Lewis-Teller diachronic Dutch book argument for classical Bayesian updating (see Teller 1976) to show that agents who update their credence functions for quantum events by some rule other than Lüders updating can be Dutch booked if their updating rule is known to a canny bookie (see Earman 2018b).

4.2 What is Lüders updating?

Unlike classical probability where the updated credence function can be defined in terms of the starting credences, Lüders updating requires the use of Gleason's theorem. Suppose that $\dim(\mathcal{H}) > 2$ and the agent's initial credence function Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is countably additive (respectively, completely additive) if $\mathcal{H}$ is separable (respectively, non-separable). Then by Gleason's theorem there is a unique normal state ω that extends Pr to $\mathfrak{B}(\mathcal{H})$. For any $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\omega(F) \neq 0$, $\omega_F(E) := \omega(FEF)/\omega(F)$, $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, defines a new normal state ω_F. Hence, ω_F induces a completely additive probability given by $\Pr^{\omega_F}(E) := \omega_F(E)$, $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. This $\Pr^{\omega_F}$ is the Lüders updating of Pr (denoted above by $\Pr_F$ above) picked out by Prop. 2.

When $E, F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ commute $FEF = EF = FE = E \wedge F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and, Lüders updating can be expressed as $\Pr_F(E) = \Pr(E \wedge F)/\Pr(F)$, which agrees with classical conditionalization. However, when E and F don't commute $FEF \notin \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and Lüders updating cannot be written as $\Pr_F(E) = \Pr(FEF)/\Pr(F)$ since $\Pr(FEF)$ is undefined. The fact that the updating of personal probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ requires the use of states should be an embarrassment for the QBians; for if quantum states are merely bookkeeping devices for tracking personal probabilities these devices should be dispensable. But apparently they are not. Since I am not concerned here with the strong form of QBism I pass on to the key issue at hand.

5 Credence and chance for quantum events

5.1 A simple corollary of Gleason's theorem

A key feature of the relation between quantum credence and chance is given by a corollary of Gleason's theorem:

> *Cor*: Suppose that $\dim(\mathcal{H}) > 2$ and that Pr is a countably additive (respectively, completely additive) probability function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ if $\mathcal{H}$ is separable (respectively, non-separable). Let φ be a normal pure state on $\mathfrak{B}(\mathcal{H})$ and let S_φ be its support projection. Then $\Pr_{S_\varphi}(E) = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, provided that $\Pr(S_\varphi) \neq 0$.

Proof: By Gleason's theorem Pr extends uniquely to a normal state ω on $\mathfrak{B}(\mathcal{H})$ such that $\omega(S_\varphi) \neq 0$ provided that $\Pr(S_\varphi) \neq 0$. By the filter property of S_φ, $\omega_{S_\varphi}(E) = \dfrac{\omega(S_\varphi E S_\varphi)}{\omega(S_\varphi)} = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. But the Lüders updated $\Pr_{S_\varphi}$ just is the probability induced by the state ω_{S_φ}.

The intended interpretation of this Corollary should be obvious. Suppose that an agent whose initial credence function on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is Pr learns that a Yes-No experiment for S_φ has been performed and that a Yes answer has been returned. Since she is rational she Lüders updates her credence function to $\Pr_{S_\varphi}$. On the objectivist interpretation of quantum probabilities, the returning of a Yes answer implies that the normal pure state φ has been prepared and, hence, that the objective chance of an event $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is $\varphi(E)$, which is the same as the agent's updated credence $\Pr_{S_\varphi}(E)$. Credence and chance have been brought into alignment without the cudgel of any extra normative principle.

There are three caveats to this happy conclusion. The first concerns the case of $\dim(\mathcal{H}) = 2$ where Gleason's theorem does not apply. Here there are probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that do not extend to any quantum states. The objectivist will brush aside such rogue probability functions as not being capable of representing physically realizable probabilities since they are not induced by a quantum state.

The second caveat concerns the proviso that $\Pr(S_\varphi) \neq 0$. All that I can say at present about this matter is that there is a general problem for updating of personal probabilities—whether over classical or quantum event spaces—when the updating concerns events with zero prior probabilities. Various remedies have been offered for classical probabilities, including the use of Popper functions and probabilities taking infinitesimal values. Whether or not such remedies can be extended to quantum probabilities remains to be seen.

The third caveat concerns the requirement that Pr be countably or even completely additive. I will say more about this concern in the following section.

5.2 Additivity requirements

The probabilities induced on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ by a normal state on $\mathfrak{B}(\mathcal{H})$ are countably additive when $\mathcal{H}$ is separable and completely additive whatever the dimension of $\mathcal{H}$. In sofar as these probabilities codify objective chances a PP-like principle can hold for credences over $\mathfrak{B}(\mathcal{H})$ only if the credence function is countably additive or completely additive as the case may be. For those who want to maintain the idea that PP is a normative principle, a vestige of normativity may be seen in the requirement that a rational credence function ought to be countably or even completely additive.

Perhaps, however, such a requirement is redundant. Bruno de Finetti, the patron saint of the personalist interpretation of probability, used a Dutch book argument to justify finite additivity. He disliked countable additivity and despised complete additivity (de Finetti 1972, 1974). But it is well known that the Dutch book argument can be rewired to justify countable additivity as a rationality constraint on degrees of belief (see, for example, Howson 2008). Complete additivity is more problematic since, as noted by Skyrms (1992), if an agent is only required to stand ready to accept any bet she regards as favorable (as opposed to merely fair) then it does not follow that a violation of completely additivity necessarily makes the agent vulnerable to Dutch book.

However, the need for complete additivity in ordinary QM is ameliorated by the result that countably additive and completely additive probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ coincide if and only if $\dim(\mathcal{H})$ is less than the first measurable cardinal (see Eilers and Horst 1975 and Drish 1979 for details). Since there are no known applications of QM that require a Hilbert space of such gigantic dimension, the result means that FAPP countable additivity suffices. Note however that this result does not hold for the Type III von Neumann algebras encountered in relativistic QFT (see Arageorgis et al.

2016).

In any case, finite vs. countable vs. complete additivity remains a contentious issue in the statistics literature (see Kadane et al. 1986 and Seidenfeld 2001). Those who wish to locate the normativity of PP for quantum probabilities in an additivity requirement on personal probabilities cannot avoid entering this fray.

6 Admissibility of evidence

In the quantum context the question of the admissibility of evidence is not a matter to be submitted to the intuitions of wise analytical metaphysicians but rather a matter of proving results in quantum probability theory. When does an agent's learning that the support projection S_φ for a normal pure state φ obtains trump additional evidence F acquired along side of S_φ in the sense that her credences updated on both S_φ and F line up with the the φ-chances? Because of the non-commutative nature of quantum propositions, the "along side" has to be treated with care. When needed, I will use the notation $\Pr_{X,Y}$ to indicate the updating of Pr on X followed by the updating on Y. It is assumed in what follows that $\dim(\mathcal{H}) > 2$ and that the agent's initial credence function is countably additive (or completely additive if need be) so that Gleason's theorem and the Corollary can be invoked.

First, that F is learned "along side" S_φ could mean that F and S_φ are learned simultaneously. In this case standard quantum doctrine on simultaneously measurability requires that F and S_φ commute. In addition, the proviso (which needs to be attached to a PP-like principle) that $\Pr(S_\varphi F) = \Pr(FS_\varphi) \neq 0$ requires that $S_\varphi F = FS_\varphi \neq 0$. Since S_φ is a minimal projection for $\mathfrak{B}(\mathcal{H})$ the upshot is that $S_\varphi F = FS_\varphi = S_\varphi$ and F is entailed by S_φ. Thus, $\Pr_{FS_\varphi}(E) = \Pr_{S_\varphi F}(E) = \Pr_{S_\varphi}(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. And by the Corollary, $\Pr_{S_\varphi}(E) = \varphi(E)$, with the upshot that F is admissible.

Now suppose that F is learned before S_φ. The first updating of Pr to $\Pr_F$ invokes the proviso that $\Pr(F) \neq 0$. The second updating to $\Pr_{F,S_\varphi}$ invokes the proviso that $\Pr_F(S_\varphi) \neq 0$, in which case the Corollary gives $\Pr_{F,S_\varphi}(E) = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Thus, F is admissible provided that $\Pr(F) \neq 0$ and $\Pr_F(S_\varphi) \neq 0$, the latter of which entails the former.

Finally, consider the case where S_φ is learned before F. Provided that $\Pr(S_\varphi) \neq 0$, the first updating gives $\Pr_{S_\varphi}(E) = \varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

The second updating gives $\Pr_{S_{\varphi},F}(E) = \dfrac{\varphi(FEF)}{\varphi(F)}$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, provided that $\varphi(F) \neq 0$. Thus, under the provisos, F will count as admissible just in case $\varphi(FEF) = \varphi(F)\varphi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. When E and F commute (which is always the case classically), the condition of admissibility of F reduces to $\varphi(FE) = \varphi(EF) = \varphi(F)\varphi(E)$, which is to say that, relative to φ, F is stochastically independent of all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

7 Conclusion

On the account of quantum chance on offer here Lewis' PP takes on a character quite different from what is found in treatments in the philosophical literature: to the extent that PP-like principles are valid in quantum probability they are valid because they are theorems of quantum probability for Bayesian agents whose credence functions are defined over the lattice of quantum propositions and satisfy the same form of additivity as do quantum chances. If there is a legitimate normative component to PP-like principles in the quantum setting it lies in the demand that rational credence should have the same form of additivity as chance.

The account of quantum chance on offer is based on several interpretational moves in quantum theory, all of which can be disputed. The basic existential challenge comes from QBism which maintains that the relation of credence and chance is a pseudo-topic since all probabilities in quantum theory are to be given a personalist/subjectivist reading. Less immediately existential but still serious is the challenge that comes from non-collapse interpretations of quantum measurement. If post-measurement probabilities are to be calculated by Lüders conditionalization—as the empirical evidence indicates—but the von Neumann projection postulate is rejected, then post-measurement probabilities are not those induced by the post-measurement quantum state, and the account on offer is undermined. What this means for the existence of quantum chance and its relation to rational credence depends on the details of the various non-collapse interpretations. Evaluating these challenges is a large and demanding project. But it can be undertaken in the confidence that it is more apt to produce understanding of the relation between credence and chance than the arm chair speculations of analytical metaphysicians.

Acknowledgment. David Lewis was one of the great philosophical talents

of the 20th century. If he is looking down from Philosophers' Heaven I hope he would smile on the direction in which I have taken his Principal Principle.

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