final manuscript accepted for *Logique et Analyse*

Inferentialism and Structuralism: A Tale of Two Theories

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Abstract

This paper aims to unite two seemingly disparate themes in the philosophy of mathematics and language (and logic) respectively, namely ante rem structuralism and inferentialism. My analysis begins with describing both frameworks in accordance with their genesis in the work of Hilbert. I then draw comparisons between these philosophical views in terms of their similar motivations and similar objections to the referential orthodoxy. I specifically home in on two points of comparison, namely the role of norms and the relation of ontological dependence in both accounts. Lastly, I show that insights from this purported connection can address certain objections to both theories respectively.
Keywords: *Philosophy of language*; *Philosophy of mathematics*; *Structuralism*; *Inferentialism*; *Proof theory*

1 Introduction

The writings of David Hilbert has led to (at least) two major approaches to the foundations of logic and mathematics, one contemporary, another more passe. The latter finitary proof theoretic or Formalist approach has been largely abandoned in its original form as a foundation for mathematics due in no small part to Gödel's incompleteness proof, which showed that a full axiomatisation of a formal system such as arithmetic is not possible i.e. all the truths of arithmetic can never be captured by the axioms (even if we keep adding them piecemeal to them).[1] Despite these circumstances, inferentialism in the philosophy of logic and language has emerged as a strongly proof-theoretic enterprise, drawing strength from Hilbert through the work of Gentzen on sequent calculus and natural deduction.

In the philosophy of mathematics, on the other hand, structuralism has been established as a dominant paradigm. According to one of its most prominent advocates, Stewart Shapiro, the trend toward structuralist thinking in meta-mathematics stems from Hilbert's axiomatics and the introduction of implicit definition (see Shapiro 1997, Shapiro 2005). One of the claims of the present work is that mathematical structuralism and inferentialism in the philosophy of logic and language not only have similar historical motivations but also similar ontological consequences such as the rejection of object-orientated theory. The two main points of comparison into which I will delve are the role of norms and the relation of dependence within both accounts. This connection is deep enough, I will argue, to promote the claim that work from the literature on either theory can open up possibilities for the other in terms of the potential solutions to major objections. I will focus on the notion of norms in inferentialism and a dependence relation in structuralism as points of contact between the two views. Lastly,

[1] Formalism and Hilbert's general philosophy have not however been abandoned. In fact, generalized and relativized versions of Hilbert's programmes have recently received more interest due to reverse mathematics in logic. I thank an anonymous referee for drawing my attention to this development.

since there is a vast variety of specific projects under both the banner of inferentialism and structuralism respectively, I will focus my claims on the similarities between Brandomian inferentialism and ante rem structuralism.

2 Proof Theory and Inferentialism

2.1 Hilbert's Proof Theory

The history of the proof-theoretic approach to mathematics and its foundations is a relatively well-known one. It will not be my purpose here to rehash that account in any detail. I will, however, describe some of the philosophical aspects of the early Hilbert programme and then use these to explain the proof-theoretic underpinnings of the contemporary inferentialist framework.

It might be tempting to think of proof theory itself as a type of inferentialism. This is certainly the case for proof theoretic semantics in which meaning is established purely in terms of rules of inference. In logic, it might take the form of introduction and elimination rules for the logical constants (and perhaps logical "harmony" in the Dummettian sense). Proof theory in mathematics, on the other hand, has a distinct history and philosophical flavour.[2]

Proof theory can be viewed as the mathematical study of the nature of mathematical proofs and arguments, taken to be mathematical objects in themselves. With the advent of logic and the logicist programme in the 20th century, formal proofs and mathematical methodology became increasingly important to the scientific community at the time. The first attempt at such a "meta-mathematical" account was Hilbert's axiomatisation of geometry according to his axiomatic method.

> Every science takes its starting point from a sufficiently coherent body of facts given. It takes the form, however, only by *organizing* this body of facts. This organization takes place through the *axiomatic method*, i.e. one constructs a *logical structure of concepts* so that the relationships between the concepts correspond to relationships between the facts to be organized (Hilbert, [2004]: 540).

Unlike the axiomatics of Euclid which involved explicit definition of geometric terms such as a point being defined as "extensionless" or the like, Hilbert's axioms introduced implicit definition directed toward the goal of divorcing theory from intuition (although intuitions might still play a motivating role for the axioms). As Shapiro puts it "geometry was becoming less the science of space or space-time, and more the formal study of certain structures" (2005: 63). As for the existence of these structures, for Hilbert, consistency provided the guide to existence. Thus, if a system or structure can be shown not to lead to contradiction, then we can be assured of that system or structure's existence. Technically, consistency applies to the theories not the structures or models themselves. The relationship between the theory and the model is then that if the former is shown to be noncontradictory, then we can be assured of the existence of the model or structure emanating from it. This relationship is brought out by the famous Model Existence Lemma. The lemma states that every consistent, countable first-order theory has a countable model. However, the relationship described by the lemma in its pure form fails to pertain for higher order (incomplete) logics (see section 3.1.). Nevertheless, implicit definition proved to be an essential tool of the axiomatic method and proof theory more generally. In section 3, we will return to the axiomatic method. Although proof theory and Hilbert's focus on consistency proofs stemmed from this method, it can be described independently of it in terms of his finitism.

The finitist position aimed to offer a more secure footing for the foundations of mathematics in terms of *signs* or concrete sequences of strokes.[3]

> What is clear in any case is that they are logically primitive, i.e. they are neither concepts (as Frege's numbers are) nor sets. For Hilbert, the important issue is not primarily their metaphysical

[2]In fact, the direct connection between proof theory in mathematics and proof theoretic semantics is not uncontroversial. I address this issue in section 2.2.

[3]Or rather, yet again, formal objects which can be represented by concrete sequences of strokes à la Bernays.

status [...] but that they do not enter into logical relations, i.e. they cannot be predicated of anything (Zach, 2006: 421).

The purpose of this viewpoint was to restrict the domain of mathematics to "intuitive and immediate" experience thereby obviating the need for abstract concepts (and the various infinities that come with them). This move is similar to that of the introduction of "implicit definition" which characterised the Hilbert-Frege debate on geometry and informs much of the work in contemporary structuralism. The idea is that lines and points are to be defined purely in terms of the axioms of geometry and furthermore anything that fulfills the conditions set by the axioms will do equally well (what Shapiro calls "free-standing"). The idea in both cases is that nothing logico-conceptual is given in advance of theory. Thus, the early proof theory took the shape of providing consistency proofs for parts of elementary arithmetic and analysis *via* the twin goals of establishing the consistency of the axioms (i.e. showing that they do not lead to contradiction)[4] and approaching the decision problem with a determinate answer to any mathematical question. The key to appreciating the proof theory (and its connection to the axiomatics) is stated by Franks (2015), "the fact that the axioms of a formalized theory could be viewed as meaningless inscriptions" (5). Thus, the axioms are construed purely syntactically.

Briefly, a consistency proof proceeded by attempting to establish a metamathematical procedure which was both "contentual" (i.e. epistemic access to its domain did not rely on abstract concepts) and noncontradictory (i.e. there is no derivation that leads both to a formula and its negation in terms of the axioms).[5] The epsilon calculus was meant to provide exactly this procedure for arithmetic and analysis (see Zach 2003b and 2004 for details and historical development). However, detailed discussion of this procedure would take us too far afield for the moment. Suffice to say that the beginnings of proof theory in mathematics incorporated a notion of consistency coupled with existence and a distinct axiomatic approach coupled with the implicit definition of core concepts. Given that the full fruition of this project could not be achieved since the consistency of arithmetic could not be proven through finitary means (thanks to Gödel's result), the latter aspects had to take on another shape, a shape which ultimately led to proof theory as it is today and informs inferentialism more generally.

2.2 Gentzen and Inferentialism

In the wake of the incompleteness theorem, Gentzen developed a proof-theoretic approach without the limitations of the Hilbert programme and finitism. His more specific aim was to prove the consistency of logical deduction within arithmetic. The first move toward this goal was the realisation that proofs (as they are found in actual practice) tend to involve more than just axioms, they also contain assumptions to be exploited and then discharged. This is now part of the familiar apparatus of natural deduction and the sequent calculus (both developed by Gentzen). Consider the (didactic) exposition of the assumption mechanism in a Fitch-style system (for the conditional introduction).

$$
\begin{array}{r|l}
 & \begin{array}{|l} \underline{n.\ \mathsf{A}} \\ \quad\vdots \\ m.\ \mathsf{B} \end{array} \\
\triangleright & \mathsf{A} \to \mathsf{B} \qquad\qquad \to\textbf{Intro: } n\text{–}m
\end{array}
$$

In this case, the assumption used to get us to B (in m.) can be discharged or closed. Similarly for the other connectives. The further proof-theoretic claim of such observations is that the content of the

[4] Hence, the finitist notion of *signs* mentioned above. If elementary number theory could be shown to based on the stable ground of concrete tokens unburdened by logical inference, then no contradictions could arise from this foundation. As Bernays (1930) put it "for this reason every axiomatic theory requires a proof of the *satisfiability*, that is, *consistency*, of its axioms" (7).

[5] The notion of consistency Hilbert was after here is one of "absolute" consistency as opposed to the relative consistency proofs of his earlier work, e.g. that geometry could be reduced to real analysis and the consistency of the latter is enough to ensure the consistency of the former.

logical vocabulary could be provided (exhaustively) by the rules of inference such as the one above (and its eliminative counterpart). Or in other words, "the meaning or significance of logical constants is a matter of the inferential rules, or the rules of proof, that govern them" (Peregrin, 2015: 4).

The insight which initially began with Hilbert as to the seminal place of proofs in mathematics and took shape in Gentzen's proof theory, concerning the introduction and elimination rules of logical constants, has led to the study of the inferential systems and structures themselves within and outside of logic. This is what is called "inferentialism" today. Hilbert's claim related to the idea of using (finitary) proofs as a stable foundation for mathematical practice while the tradition which began with Gentzen and led to proof-theoretic semantics incorporated the idea of the usefulness of proofs in semantics. The connection between proof theory and proof-theoretic semantics can be gleaned from Gentzen's attempts (in full view of Gödel's incompleteness theorem) to establish the consistency of arithmetic. Historically, many of the ideas presented during this research period were picked up by Lorenzen, Popper, Prawitz and others. For instance, Prawitz (1965) adapted the idea of analytic proof used in both Gentzen's natural deduction and sequent calculi, i.e. cut-free proofs, to capture the notion of the value of a proof in terms of its normal form. Eventually, many of these insights which stemmed initially from Hilbert's proof theory found their way (through Gentzen) into what became known, in Schroeder-Heister (1991), as proof-theoretic semantics.

The core idea is that in the same way that the meanings or content of logical constants can be determined by the inferential roles they play in a logical system (*via* intro and elim rules), the meanings of ordinary terms and words in natural language are determined by their inferential roles. This is a radical idea and it challenges a number of tenets of the referential/truth-conditional theories of meaning which have dominated the semantic landscape since the work of Frege and Russell in the early 20th century.

One of the main traditional ideas which inferentialism jettisons is that meanings are somehow individual entities capable of description *in vacuo*. The way in which this idea is usually presented is through the use of representational theories of meaning. On these views, individual words represent or "stand for" actual things. These things are in turn interpreted compositionally (from subsentential elements to complex expressions). As Brandom himself describes it as follows.

> For example, singular terms might be associated with functions whose arguments are possible worlds and whose values are objects in those worlds. Then, if an additional step is needed to get there, an account would be offered of what is picked out, referred to, or represented by those subsentential content (2007: 652)

In contrast, inferentialism is top-down or sententialist (semantic analysis starts with sentences) where the content of a sentence is determined by the role the sentence plays in a language game of "giving and asking for reasons" or it is "a matter of being able to play the role both of premise and of conclusion in inferences" (Brandom, 2007: 654). The terminology of language-games is borrowed from the later Wittgenstein.[6]

> Wittgenstein's proposal was that we should see the relation between an expression and its meaning on the model of that between a wooden piece we use to play chess and its role in chess (pawn, bishop...). This was, of course, not a novel proposal (the comparison of language with chess had already been invoked certainly by Frege, de Saussure or Husserl). But Wittgenstein's influence was able to bring the relationship between meaning and the rules of our language games into the limelight of discussion (Peregrin, 2012: 2).

The latter relationship between meaning and the rules of language was picked up more thoroughly in the work of Sellars who emphasised the inferential nature of meaning conference. Under the contemporary Brandomian framework, meanings are no longer to be viewed as individual entities to which our terms stand in certain (representational) relations. Meanings are rather the roles which our words and sentences play according to the inferential rules of language (due to the game of giving

[6]We will return to a related Wittgensteinian line in section 5.

and asking for reasons). As in the logic case, the role of the logical constants is then to make these inferential patterns *explicit*. This is often called expressivism in the literature.

The last aspect of inferentialism which ought to be discussed here is that which separates it from other semantic accounts which go by the same name, namely conceptual-role semantics and its subfield of inferential-role semantics. This distinction lies in the *normative* dimension of the inferentialism project under discussion. In Peregrin (2015) this is precisely the line drawn between what he calls 'normative' inferentialism à la Brandom and the 'causal' inferentialism of Peacocke and Boghossian. The latter view takes the inferential role of meaning to be based in the individual dispositions of language users while the former derives this role from a rich network of interconnected social behaviour. This aspect of normative inferentialism is captured by its focus on inferential *rules* where rules have normative force. The normativity in question is related to the use-theory of meaning on offer with this brand of inferentialism. Speakers in a community, who are playing the game of giving and asking for reasons, are constantly engaging in activities, linguistic and otherwise, which establish *commitments* and corresponding *entitlements*. When I make certain assertions such as *Bill stopped smoking* then I am committed to other sentences such as *Bill was a smoker*.[7] Through score-keeping (and similar devices), my interlocutors keep track of my various statuses in the language-game. Unlike in causal inferentialism, the inferential rules do not establish adherence by any causal or necessary means, rather following the rules is a matter of propriety. I will leave the issue here for now and return to the role of norms in both inferentialism and structuralism in section 5.

As we have seen, contemporary inferentialism shares essential aspects of early proof theory, in that it embodies the attempted development of the proof-theoretic approach which characterised the logical constants (in Gentzen style proof theory for instance) to also include the nonlogical vocabulary of natural language. This is often referred to as "strong inferentialism" in that it does not restrict the inferential analysis to moves from language to language (in Sellar's sense) but also the possibility of inferential rules that govern language-world relationships.

3 Mathematical Structuralism

Mathematical structuralism in the philosophy of mathematics is the view that mathematics is the science of abstract structures, where the objects in these structures are defined purely with relation to each other and the overarching superstructure. Mathematical objects, thus, have no internal or individual natures in themselves. The different structuralist proposals differ on what they take these structures to be, eliminative or object-preserving, and what the background theory looks like, modal or set-theoretic, first-order or higher-order etc. In section 3.2. we will discuss a brand of structuralism that does not eschew individual mathematical objects *per se* but finds a place for them within a structuralist ontology, based on an analogy with *universals* in the history of metaphysics. Before developing that view, however, we will trace certain essential components of the structuralist position to the early Hilbert programme in mathematics.

3.1 Hilbert and Structuralism

In the previous sections we have seen that Hilbert's proof theory inspired the work of Gentzen and in turn the contemporary inferentialist framework in the philosophy of logic and language. In this section, I will briefly allude to a similar exegesis of contemporary structuralism in the philosophy of mathematics.

A natural starting point is, again, Hilbert's notion of implicit definition and the general axiomatic approach with which it came. As previously mentioned, implicit definition assumes no prior definitions or concepts. Rather it incorporates a structural analysis of terms such as "line", "point" or "congruence". For instance, in the case of geometry, once our intuitions have aided us in picking out the structures in which we are interested, they no longer serve any theoretical role. The mathematics

[7] Perhaps entailments such as those from *Bill is a man* to *Bill is a mammal* are more illustrative of these inferential connections.

then pursues only the goal of structural description and analysis. One consequence of this move away from intuition is that "[a]nything at all can play the role of the undefined primitives of points, lines, planes, *etc.*, so long as the axioms are satisfied" (Shapiro, 2005: 64). In this way, the axioms or structures are all there is to the mathematical concepts or domains. The core claim of implicit definition is a holistic one, i.e. we cannot ask for the definition of "line" or "point" or "number" for that matter without looking at the whole structure, which for Hilbert was given by the axioms. As Shapiro mentions, "Hilbert's claim that a concept can be fixed only by its relations to other concepts is a standard motivation for structuralism" (2005: 67).

Returning to the case of geometry, implicit definition was not necessarily such a radical idea. Since the 19th century it was known that definitions of geometric concepts such as parallel line and even triangle receive different treatments under different geometrical theories. For instance, in Euclidean geometry, parallel lines are defined as incapable of intersection, in other words there is only one line parallel to another through any given point not on that line (or some equivalent formulation in terms of equidistance), while in Riemannian geometry this is not the case since parallel lines do not exist in elliptic geometry (imagine Euclidean geometry on the surface of a sphere). In hyperbolic geometry, there are many (or at least two) lines parallel to a line through a given point.

As previously mentioned, the guide to the existence of mathematical structures, for Hilbert, was consistency. However, as we have seen, consistency cannot provide a guarantee of existence. Certain systems are incomplete such as arithmetic. For instance, as shown in Shapiro (1991), some consistent theories in second-order logic have no models. Thus, the aforementioned Model Existence Lemma fails for higher-order logics (although there is a completeness theorem for Henkin semantics in these logics). Thus, contemporary structuralists are forced to establish the existence of structures *via* alternative means. For Shapiro (1997), this task takes the form of a principle of coherence based on second order logical consequence. For Hellman (1989), structuralism need only posit the existence of possible systems which would be capable of instantiation. Yet others, such as Awodey, suggest that category theory provides a more natural basis for structuralism. On the face of it, category-theoretic structuralism might seem like the best candidate for the full achievement of Hilbert's axiomatic method. After all, a category in category theory is "*anything* satisfying the axioms. The objects need not have 'elements', nor need the morphisms be 'functions' [...] we do not really care what non-categorical properties the objects and morphisms of a given category may have" (Awodey, 1996: 213). This sounds a lot like Hilbert's description of the axiomatic method and its concept of definition, in his reply to Frege.

> [I]t is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, *e.g.*, the system love, law, chimney-sweep [...] and then assume all my axioms as relations between these things, then my propositions, *e.g.*, Pythagoras' theorem, are also valid for these things.

Nevertheless, whichever route we take to the definition of the structures themselves and the characterisation of structuralism generally, the tool of implicit definition, as introduced by Hilbert, seems essential.

It is with this in mind that we move to the specific brand of structuralism, namely ante rem structuralism, which I hold to be the best case for a strong comparison with the inferentialism of the foregoing sections.

3.2 Ante rem Structuralism

As I argued with Brandomian inferentialism, the discussion of ante rem structuralism is not an arbitrary choice. One of the characterising features of ante rem structuralism is that the role of mathematical object is not discarded completely (as in the case of modal structuralism for instance, see Hellman 1989). This view allows us to retain the usual mathematical parlance involving mathematical objects (what Shapiro (1997) calls "realism in ontology") while still maintaining a structuralist interpretation of mathematical practice in general. The strategy employed to achieve this aim consists in appreciating the dual role objects play as positions in structures as both the more abstract "places-are-offices" and the more common "places-are-objects" perspective involved in quantification.

> Clearly, there is an intuitive difference between an object and a place in a structure - between an office and an officeholder. The ante rem structuralist respects this distinction but argues that it is a relative one. What is an office from one perspective is an object—and a potential officeholder—from another. In arithmetic, the natural numbers are objects, but in some other theories natural numbers are offices, occupied by other objects (Shapiro, 1997: 11).[8]

This aspect of the view seems to dovetail more naturally with Hilbert's own position on the foundations of mathematics. Although Shapiro distances his ante rem structuralism from the Platonist conception of independently existing individual abstract objects, Resnik, for instance, is less dismissive on that point (Hellman (2005) describes the view as "hyperplatonism"). Nevertheless, unlike other versions of structuralism (such as *in re* or *post rem*), Hilbert's own axiomatic approach did not necessarily take a negative stand on the issue of the existence of individual mathematical objects (as abstract or nonspatio-temporal etc.). As Bernays describes it,

> Hilbert's theory does not exclude the possibility of a philosophical attitude which conceives of the numbers as existing, non-sensible objects (and thus the same kind of ideal existence would then have to be attributed to transfinite numbers as well, and in particular to the numbers of the so-called second number class). Nevertheless the aim of Hilbert's theory is to make such an attitude dispensable for the foundation of the exact sciences (1923: 226).

For this reason, I argue, ante rem structuralism which contains a notion of a place in a structure as a *bona fide* object over which quantification is possible, in a model-theoretic sense, and a more abstract conception of the role (*qua* office) a position plays is closer to the initial Hilbertian foundations. There are other reasons for the focus on ante rem structuralism as a point of comparison, such as the dependence relation it espouses. This aspect will be covered in the forthcoming section.

On the issue of the identity conditions of structures, there is some discrepancy between accounts. Shapiro (1997) has developed a formal theory of structures along the lines of set theory where an axiom of coherence (modelled in terms of *satisfiability*) guarantees the existence of a structure up to isomorphism. Hellman (2005) takes issue with this axiom in that it merely "mimics Hilbert's idea that consistency suffices for mathematical existence" (546). In so doing, claims Hellman, we have no clearer a notion than that of primitive second-order logical possibility and logical possibility is not a guide to existence. In terms of identity, if indeed structure internal relations exhaustively identify positions or places, then these places should be subject to Leibniz's law. The problem with this is that it suggests rigidity of structures which in turn fails to account for "non-rigid" structures such as those found in "the complex numbers (interchanging *i* and -*i*), the additive group of integers (interchanging +1 and -1), geometric figures with reflectional symmetry" which are not identical (Hellman, 2005: 10). The problem is known as the Burgess-Keränen objection (based on see Burgess (1999) and Keränen (2001)). We will pick up a further objection to structuralism and its interpretive limitations in section 6. Although intra-structural identity might pose a problem for ante rem structuralism, Shapiro does offer an account of inter-structural identity in terms of isomorphism and structure equivalence (borrowed from Resnik).[9]

[8] There is an interesting parallel here between Peregrin's (2015) almost Carnapian distinction between 'insider' and 'outsider' perspectives of the rules of language in which the former allows for "genuine normatives" and the latter offers a more descriptive level of analysis. See chapter 4. Thus within arithmetic, natural numbers could institute certain connections that the set-theoretic perspective merely reports on. More on the role of norms in structuralism in section 5.

[9] For the definition of structure equivalence, "let R be a system and P a subsystem. Then P is a full subsystem of R if they have the same objects (i.e., every object of R is an object of P) and if every relation of R can be defined in terms of the relations of P. Let M and N be systems. Then M and N are structure-equivalent, or simply equivalent, if there is a system R such that M and N are each isomorphic to full subsystems of R" (Shapiro, 1997: 239).

Resnik himself is more sceptical of a precise formal account of identity between structures beyond the intuitive one (of two structures being sub-patterns iff they can be extracted from some larger pattern *via* the deletion of some of its relations).[10] As in the difficulty of identifying geometric points across different figures, we simply lack sufficient information for identification across structures (and from positions in structures to positions in other structures). Nevertheless, he claims the following.

> The exclusion of patterns themselves from the field of identity is consistent with the practice of mathematics itself. Number theory quantifies over just the numbers but not over the number theoretic structure, set theory quantifies over sets but not over the set theoretic hierarchy (Resnik, 1981: 538).

The last aspect of ante rem structuralism with which I want to deal here is that of the relationship between actual collections of objects or *systems* as it is called by Shapiro (1997) and the structures which they exemplify. The purported relationship between systems of objects and structures is often said to be captured by the process of "Dedekind abstraction" (in honour of another forefather of modern structuralist thinking). The idea is that a structure is an abstract form of a system which homes in on only the structural (relational) properties of the objects of the system and nothing else. Parsons (1990) suggests that we take a system such as the von Neumann ordinals as instantiating the concept of a simply infinite structure, i.e. the natural number structure.[11] As we know, the Zermelo numerals would do just as well for the instantiation of the natural number structure. In fact, this a point upon which structuralism can be seen as a marked improvement on Platonist conceptions of mathematics. The two systems (von Neumann ordinals and Zermelo numerals) are not equivalent set-theoretically, yet they can still be said to exemplify the same structure. With Platonism, no such manoeuvre is permitted since the two theories pick out different abstract objects and there is a fact of the matter as to which objects the natural numbers correspond, e.g. either $2 \in 4$ or $2 \notin 4$ (as Benacerraf (1965) convincingly argued).[12] It is important to note, that this notion of abstraction has ontological import. Unlike in the natural sciences, where abstraction is a tool used mostly for tractability or simplicity, mathematical structures (the target of mathematics as a science) *are* purely structural in nature, i.e. they only have structural properties, on this view.

4 Dependence

So far we traced the influence of Hilbert's proof theory to contemporary inferentialism and his axiomatics coupled with implicit definition to contemporary structuralism. Neither of these points are particularly novel and the general line in each case has been well-established in the literature (albeit controversially in the former). In this section, however, I will argue that one core aspect of the structuralist programme in mathematics, namely the specific notion of dependence (as described by Linnebo 2008), can be found in the inferentialist picture of logic and natural language as well.

Structuralism is often characterised in contrast with Platonism, especially in terms of its treatment of objects. For a Platonist, abstract objects are analogous to ordinary physical ones in that they are ontologically independently of one another. My toaster no more relies on my backpack than my carpet relies on my desk chair for its existence. Mathematical objects, on the other hand, have no such independent existence according to structuralists. These objects *qua* positions in structures depend on other positions for their very existence and on the structures as a whole. For this reason, Linnebo (2008) distinguishes between two notions of dependence.

ODO Each object in D [domain of some mathematical structure] depends on every other object in D (67).

[10] It should be noted that Shapiro too is sceptical of a formal definition of coherence. "I take "coherence" to be a primitive, intuitive notion, not reduced to something formal, and so I do not venture a rigorous definition" (Shapiro, 1997: 135).

[11] Although Parsons switches the terminology used by Shapiro.

[12] Of course, the Platonist might not be committed to there being a fact of the matter. For instance, Wright (1983) takes Benacerraf's dilemma to be an instance of indeterminacy of reference.

ODS Each mathematical object depends on the structure to which it belongs (68).

The difference between ODO and ODS is that the former just says that the objects in a structure depend on other objects such as some natural numbers depending on other natural numbers. While the latter adds that the existence of one object in a structure ensures that the structure itself exists or is ensured by the existence of the structure as a whole. Another way to think of ODS is that structures are ontologically prior to positions (some structuralists like to say that there would be structures even if there were no objects fulfilling the various roles). Linnebo goes on to argue that requirements such as non-circularity (cashed out in terms of well-foundedness) militate against ODO straightforwardly and (perhaps) ODS to a lesser extent. Notwithstanding various difficulties with either or both of these dependence relations, it is important for my purposes that (ante rem) structuralism incorporates a strict notion of *upwards dependence*.[13] Upwards dependence is the relation in which objects depend on the overarching structures as opposed to depending on their own constituents.[14]

As mentioned in section 2.2, one of the main points of departure of inferentialism from the standard semantic theories is that it possesses a distinctly "top-down" approach to semantic composition. The ubiquitous and dominant definition (or family of definitions) of compositionality in philosophy of language and linguistics is constituent-based. It usually takes the form of something like the following.

> The meaning of a complex expression is determined by the meaning of its constituents and their method of combination.

Inferentialism challenges this claim. Instead of an atomistic view of compositionality as the one cited above, it proposes a holistic view essentially based on the concept of implicit definition. Atomism (and standard compositionality) presupposes that individual constituents have meaning independently and these meanings combine to yield the meanings of the complex expressions in which they are contained. This is directly analogous to the mathematics case. In fact, standard compositionality is based on the compositionality of formal languages such as propositional and predicate logic (which have straightforward homomorphisms between the the algebras constituted by the rules of the syntax and semantics respectively). However, the compositionality of inferentialism is different. In a particularly illuminating passage, Peregrin describes an important aspect of inferential rules.

> Thus roles are given merely through an 'implicit definition', and just as Quine (1969, p. 45) claims that 'there is no saying absolutely what the numbers are, there is only arithmetic', we can claim that *there is no saying absolutely what inferential roles are, there are only rules of inference (and compositionality)* (2015: 53).

Here we see that, just as in the mathematical structuralist case, the inferential roles which determine the meanings of sentences and words are *upwards dependent* on the linguistic structures (or social-normative networks) in which they are found. Thus, ODS holds. It seems that the top-down or sentential approach of which Brandom speaks is equivalent to the upwards dependence of ante rem structuralism.

5 Norms, Rules and Proofs

In this section a more radical connection between the two frameworks under discussion will be attempted. It could be argued that the dependence of the previous section is merely suggestive and could be applied to many different theories and frameworks. For instance, it could be argued that contextualism in epistemology respects ODO. The claim of the present section, however, is much bolder, namely that structuralism, like its inferentialist counterpart, has a distinctively normative element.

[13] As Linnebo suggests, this might be a fundamental difference between the realm of the mathematical and the realm of the physical.

[14] Of course, ODO might be deemed necessary in cases in which the dependence on the entire structure might lead to contradiction such as ordinal set theory, in which the dependence on the totality of sets is notoriously problematic.

In order to appreciate the normativity of the ante rem structuralist programme, we need to delve somewhat into Wittgenstein's (positive) thoughts on mathematics. For Wittgenstein, in an almost inferentialist move, mathematics concerns a rich "network of norms" (RFM, VII §67).[15] Once again, *contra* Platonism, mathematics is not the study of objective facts or facts of the matter but rather the study of norms such as those that constitute the rules of language. "Let us remember that in mathematics we are convinced of *grammatical* propositions; so the expression, the result, of our being convinced is that we *accept a rule*" (RFM, III §27). Thus, on this view, mathematical statements or sentences are primarily normative in nature.

This view goes against the mainstream views in the philosophy of mathematics concerning proofs and theorems (in which proofs operate in a truth-preserving way in order to establish objective theorems).[16] For Wittgenstein, mathematical sentences constitute conceptual norms which among other things establish conventions for using non-mathematical objects, e.g. geometry aids us in spatial arrangement and manipulation. Mathematicians, like language users, are playing conceptual games in which certain rules are established (an intuitionist mathematician is playing a different game to a classical mathematician and so on). Proofs and theorems are just some of the tools used in the (language) game of mathematics.

How this picture connects with Hilbert's axiomatics is the key to appreciating its further connection with structuralism. Friederich (2011) claims that Wittgenstein's notion of mathematical sentences as norms can be specifically defended in terms of Hilbert's notion of axioms as implicit definitions. The idea is that axioms defined implicitly do not report objective mathematical facts (as in Peregrin's "outsider" perspective) but rather constitute normative conventions for the use of mathematical concepts ("insider" perspective). Consider an axiom of arithmetic such as "every natural number has a unique successor". Instead of stating a fact or a primitive of the system, this definition partly defines both 'natural number' and 'successor', i.e. it establishes a connection (or dependence) between objects. In other words, "we may say that whenever the axioms are used as implicit definitions, to accept and endorse them is a necessary condition for using the concepts defined through them" (Friederich, 2011: 10).

One way of appreciating the normative element of the axioms as implicit definitions is by noticing the equivalent "grammatical form" of the axioms in terms of a covert *Let* in front of them. For example, the sentence/axiom "every natural number has a unique successor" should be understood as "*Let* every natural number have a unique successor". Thus, "the grammatical form of this sentence makes it clear that its role is that of stating a norm for the usage of the concepts 'natural number' and 'successor' and not that of describing anything" (Friederich, 2011: 10).

This latter idea is linked in an obvious way to Hilbert's notion of an axiom as a schema or schematic sentence which only serves to specify structural relations between objects. As before, the structure is freestanding which means that it can be exemplified by any system of objects whatsoever. This aspect of Hilbert's programme is picked up in structuralism under the property of *algebraic* structures or structures which are "schematic, applying to any system of objects that meets certain conditions" (Shapiro, 2005: 67).

> [T]he schematicity of the axioms emphasised by Hilbert and the structuralists can be seen as an aspect of their normativity [...] Conceiving of the axioms as schematic and hence algebraic - the structuralist view - and conceiving of them as conceptual norms - the view defended here - are two perspectives on the axioms as implicit definitions which differ only in emphasis (Friederich, 2011: 12).

The algebraic or schematic nature of structures act like the axioms of Hilbert's system in that they limit or delimit the space of appropriate moves in the "laying down" of conditions. Rules in inferentialism are described as doing exactly the same thing, namely they limit the space of assertability of

[15] A precise definition of "norm" is a tendentious matter. One way we can think of norms is as patterns of behaviour or regulations which govern social behaviour. For Sellars (1954), the "realm of the normative" is constituted by propositions and concepts (as opposed to the constituents of the "realm of the causal").

[16] It is not, however, completely out of sync with certain views in the philosophy of mathematical practice. See Rav (1999).

expressions and their usage. Furthermore, Hilbert's notion of *signs* as an intuitive conceptual basis for mathematics is additional evidence for the normativity of the early programme. On that view, mathematics does not start with objective facts but rather with immediate concepts conceived of non-logically. These concepts have a certain normative force. For instance, the counting of strokes, which lays the foundations for arithmetic, are not established by logical inference but intuitive perhaps even conventional usage. Resnik (1982) tells a similar epistemological story of the origins of mathematical concepts. The picture starts with a primitive community of people who establish conventions for the use of objects and manipulating spaces around them based on concrete instances of what will eventually be recognised as patterns. The requirements of language and description beyond the visible immediate environment introduce limits (in the mathematical sense) which require more and more complex patterns for characterisation.

Now the alleged normativity of the contemporary structuralism is not uncontroversial. Despite the fact that certain prominent structuralists, such as Hellman (2005), admit to the compatibility of the view with Wittgenstein's thoughts on the normativity of mathematical statements, ante rem structuralism is usually not described in terms of normativity. There is, however, good reason to prefer this reading. For one thing, it serves to further distance the view from Platonism and avoid certain charges of structuralism being Platonism in disguise or "hyperplatonism". Furthermore, ante rem structuralists such as Shapiro and Resnik do take pains to argue for a sound epistemological footing for mathematics in terms of physical tangible reality or mental processes (as in intuitionism). Social normativity offers a useful means of grounding mathematical reality and the enterprise in general. In addition, it retains some of the Hilbertian foundations which inform the structuralist programme as a whole. At the very least, there is nothing incompatible with conceiving of ante rem structuralism through the conceptual role of implicit definition which is normative in nature.

6 From Related Problems to Potential Solutions

The last aspect of the comparison between structuralism and inferentialism takes the form of an investigation into some of the common objections presented against each theory. I will argue that these objections can be unified and thus that insights from each theory can be used to approach solutions to the other (although not in every case).

Before moving on to this task, there is a further possibility which should be addressed, namely that inferentialism is a form of structuralism. In Peregrin (2008), it is argued that inferentialism offers a new way of capturing many of the insights of the structuralism of de Saussure, Quine and Sellars in the philosophy of language.[17] He argues that meaning is a "purely structural" matter exemplified through inferential structure.

> Now it is important to realize that structuralism, broadly construed, can be seen as embracing answer (4) ["there are no such entities as meanings, talk about them is a metaphor"], as its message can be interpreted as there are no meanings, there is only semantic structure (Peregrin, 2008: 1212).

It is important to note that both Peregrin's target and analysis differ from mine. Although the broader perspective can be related in certain ways, I do not make the claim that inferentialism is a form of structuralism directly in the present work. Below is a survey of potential areas in which the connection I propose can be fruitfully explored.

6.1 Norms and Identity

The first issue has already been mentioned above in the context of identity in structuralism. It was argued, by Burgess (1999) and Keränen (2001), that intra-structural identity, i.e. identity between

[17] De Saussure's linguistic structuralism is the view characterised by the claim that language is a system of values induced by elementary oppositions (such as the voiced and unvoiced distinction in phonology etc.). This perspective can perhaps be related quite naturally to the contemporary literature on bilateralism in proof-theoretic semantics. Hjelmslev's structuralist "Glossematics" was also an interesting revival of many of de Saussure's ideas.

positions-as-objects, poses a distinct problem for ante rem structuralism. This problem forced a rigidity reading of mathematical objects through Leibniz' law of the indiscernibility of identicals. This reading rendered the theory unable to account for many non-rigid mathematical structures. Simply put "structures with certain symmetries are not adequately captured by ante rem structuralism" (Räz, 2014: 117).

As before, the idea is that non-rigid structures admit for non-identical yet *structurally indiscernible* places (e.g. i and its additive inverse $-i$ in the complex numbers). Following Ketland (2006), Shapiro (2008) counters that identity has to be taken as primitive as it is presupposed in mathematical practice (and a non-circular definition is impossible). This is one way out of the conundrum. Another is that although inverses and the like of non-rigid structures might be *structurally* indiscernible, they are certainly not *inferentially* so. For instance, the role of the concept of an additive inverse involves reducing any number a to zero when added to it. This is enough to demarcate the structurally indiscernible places in structures within mathematical practice without resorting to taking identity as primitive, i.e. the suggestion is that places in structures should additionally be *inferentially* indiscernible in order to qualify as identical.

6.2 Reference

Inferentialism is equally beset with interpretation problems. For one thing, by rejecting the referential orthodoxy, inferentialism leaves itself open to charges of failing to account for singular reference. This is a problem Brandom himself recognises.

> [The phenomenon of our] talking about something (characterizing an object) [...] is too central to our understanding of what we are doing when we think and talk simply to be ignored. Unless it accounts for the possibility of representing particular objects, a semantic theory will not address the concerns that many have taken to define its topic (1994: 338).

The problem is essentially that of showing how inferentialism, in its rejection of representationalism, can account for certain seemingly representational aspects of natural language, such as singular reference (and other phenomena which depend on it such as anaphora). Brandom proposes that singular reference, unlike other locutions in language, involves "symmetric inferential relations". The way in which explains this notion out is through the idea of proprieties associated with inferential substitution (which are always symmetric in the case of singular reference). This is a structural solution. It is uncertain whether this solution works and some, such as McCullargh (2005), think it only does so if one presupposes a denotational explanation to begin with. Nevertheless, the important point for our purposes is that the inferentialist attempts to solve the problem of singular reference by means of structural properties of discourse.

Both inferentialism and structuralism make the move toward purely structural description but this means that reference to individual objects or the linguistic mechanism of singular terms (which seems to be ubiquitous in mathematics and natural language) is potentially left by the wayside. As we have seen, on this point, Resnik demurs from identifying individual positions within structures.

Räz (2014) describes a similar reference problem for ante rem structuralism. He calls it the no-name problem. He claims that finite cardinals represent structures in which places are too homogeneous to be picked out. "Certain mathematical structures with symmetries have the property that we cannot name or refer to the objects, or places, in these structures because they are too homogeneous" (Räz, 2014: 118). However, within mathematical practice, he argues, naming finite cardinal structures is not a problem (by means of permutations and cycle types defined non-structurally). If ante rem structuralism is meant to be conceived of as truly antirevisionist, i.e. interpreting mathematics as it is actually practised (as its chief proponents claim), then something's got to give.

One possible diagnosis is that reference in structuralism and inferentialism is tied up with the dependence relation (either ODO or ODS). The problem for inferentialism is that singular terms do not seem to be perfectly characterisable in terms of dependence. Similarly, objects within the finite cardinal structure have no structural properties and are thus indiscernible in purely structural terms.

Linnebo (2008) himself argues that even sets pose a problem for structuralism. If the sets are the structures and their elements are the positions, then the dependence relation seems to go in the reverse direction. Sets are dependent on their elements (extensionality), "[t]he relation between a set and its elements is thus asymmetric [...] [a] set thus appears to depend on its elements in a way in which the elements do not depend on the set" (Linnebo, 2008: 72). The clearest case of this asymmetry is interestingly brought out by the singleton set in which a single element needs to be specified (picked out or referred to) in order for the structure to be characterised.

6.3 Instantiation and the Quasi-concrete

Another problem which I argue faces both structuralism and inferentialism is that of quasi-concrete objects. Here inferentialism is faced with a specific case of the more general quasi-concrete objection initially levelled at structuralism by Charles Parsons (1990). Parsons argues that structuralism neglects or is unable to account for a specific genus of abstract objects which is "directly 'represented' or 'instantiated' in the concrete" (1990: 304). Sets, geometric figures, and Hilbert's *signs* are meant to be examples of these sorts of objects. We have already seen that sets might pose a problem for those possessed by the structuralist persuasion. The problem is that these sorts of objects require a representation (or instantiation) relation, i.e. in re as opposed to ante rem which cannot be accommodated in the purely structural picture which involves nothing in addition to intra-structural relations (namely, ODO and ODS). Or rather as Parsons puts it,

> What makes an object quasi-concrete is that it is of a kind which goes with an intrinsic, concrete "representation," such that different objects of the kind in question are distinguishable by having different representations (Parsons, 1990: 34).

How this problem presents itself within inferentialism is via the type-token problem or what Wanderer (2008) calls "the challenge of token repeatability". Importantly, the relation between linguistic types and their physical tokens is also quasi-concrete, according to Parsons. A token is representative of a type and we can individuate types by their tokens. Linguistic tokens are concrete and the idea is that the way in which they represent their types is something intrinsic to the tokens themselves. Thus, the type-token relation is supposedly determined in terms of the intrinsic features of the tokens and not external structural or relational ones. The problem manifests itself by the fact that standard inferentialism does not seem to be able to explain the repeatability of certain tokens of expressions.

> On the one hand, two tokening performances of the sentence "Mandela is a lawyer" may have different semantic content if, for example, the sentence is uttered in different contexts (e.g. a context involving Nelson Mandela, the former president of South Africa, and a context involving Mandla Mandela, his grandson and current chief of the Mvezo traditional council). On the other hand, one tokening of "Mandela is a lawyer" may have the same semantic content as another tokening (such as "he is a lawyer"), even when the two are not lexically cotypical (Wanderer, 2008: 127).

Thus, the type-token distinction proves difficult to deal with on purely inferentialist grounds in the same way that quasi-concrete objects more generally prove difficult for structuralism to deal with. The latter issue seems to be a subspecies of the former.

6.4 Aberrant Structures and *Tonk*

Lastly, I argue that the classical problem of the capricious connective "tonk", first introduced by Arthur Prior to challenge inferentialism, can be given an analysis and potential solution in terms of structuralism. The objection goes that if inferential roles (in logic) were truly constitutive of the meaning of logical constants, then how can we deal with inferential patterns like the following which, if introduced into the language, would result in contradiction?

$A \vdash A\ tonkB$

$A \; tonk B \vdash B$

An inferential rule such as the one above licenses the introduction of any expression whatsoever and renders any language which contains such a connective contradictory. There have been a plethora of proposals aimed at solving this problem, from conservativity (Belnap 1962) and harmony principles (Dummett 1991) to normalisibility (Prawitz 1965). What is important for my purposes is that *tonk* and similar connectives do not only present a problem for inferentialism, i.e. this debate can essentially be restaged on structuralist grounds.

The problem can then be recast as the introduction of pernicious patterns or structures and a need for principled reasons for rejecting them. The second-order coherence axiom of Shapiro (1997) won't do since it does not correspond to consistency. Rather it corresponds to possibility as Hellman pointed out or some "intuitive, primitive notion" as Shapiro claims. A language with a *tonk* connective is logically possible or rather not necessarily incoherent. If coherence (*qua* possibility) entails existence, then the *tonk* language is an existent structure. This claim is made clear by Cook's (2005) introduction of the *tonk* connective into a suitably modified version of a non-transitive logic called 'First Degree Entailment' (Anderson and Belnap 1975). In this logic, *tonk* is stable, i.e. does not reduce the logic to triviality. The core idea is that logical connectives should not be evaluated in isolation from the background or "antecedently given context of deducibility" such that "a rejection of tonk as illegitimate depends on a prior (at least partial) account of what constitutes a legitimate notion of logical consequence" (Cook, 2005: 223). In other words, the background logical structures can mitigate the viciousness of certain inferential patterns, e.g. structures incorporating transitive logical consequence coupled with the *tonk* connective lead to contradiction or triviality. Similarly for languages or mathematics based on paraconsistent rules of inference where the law of noncontradiction or bivalence does not hold. We are left in the same boat of finding additional constraints in order to limit legitimate structures or banish illegitimate operators. Thus, we have to set boundaries on what kind of mathematical structures, in accordance with which rules of inference, we want to study: conservative, harmonic or normal ones or based on transitive, classical logics versus the alternatives.[18] This is a structurally analysis of the problem.

7 Conclusion

In this paper, I have suggested an historical connection between ante rem structuralism and Brandomian inferentialism. Inferentialism was taken to follow from Hilbert's finitary proof theory while structuralism drew its origins from his axiomatics characterised in terms of implicit definition. In the biological sense, one could think of these two frameworks as homologous.

I then attempted to show that these theories are more alike that we might think. I showed that there is a distinct place for norms within the ante rem structuralist programme and that inferentialism incorporates the same dependence relation as does structuralism. Finally, I argued that many of the problems or objections which the theories face are closely related and can be interpreted as similar in structure and motivation. More importantly, I suggested a number of possibilities for how the appreciation of this connection might led to the resolution of certain common problems faced by both theories.

Acknowledgements

I am indebted to Jarda Peregrin, Stephen Read and Fenner Tanswell for looking at early drafts of this research and offering insightful comments and corrections on its content. In addition, I am grateful to an anonymous reviewer of this journal for useful corrections on certain content. I would also like

[18] Of course, one could deny that "tonk" is a problem for inferentialism, in that the position is not committed to the claim that every inferential pattern constitutes a useable constant. Nevertheless, if this strategy is good for inferentialism it should be good for structuralism too, i.e. not all structures are useable ones for mathematics. I thank Jarda Peregrin for this suggestion.

to thank participants and fellow speakers at the Arché Inferentialism Conference at the University of St Andrews in November 2015, especially Greg Restall, Olé Hjortland and David Ripley for their very useful suggestions.

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