# The Status of Scaling Limits as Approximations in Quantum Theories

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**Abstract**

This paper attempts to make sense of a notion of "approximation on certain scales" in physical theories. I use this notion to understand the classical limit of ordinary quantum mechanics as a kind of scaling limit, showing that the mathematical tools of strict quantization allow one to make the notion of approximation precise. I then compare this example with the scaling limits involved in renormalization procedures for effective field theories. I argue that one does not yet have the mathematical tools to make a notion of "approximation on certain scales" precise in extant mathematical formulations of effective field theories. This provides guidance on the kind of further work that is needed for an adequate interpretation of quantum field theory.

## 1 Introduction

This paper attempts to make sense of a notion of "approximation on certain scales" in physical theories. This broad notion has at least two potential applications. First, this notion plays a role in understanding the way that quantum mechanics explains the success of classical physics through the classical limit, in which one purports to show that classical behavior is reproduced by quantum systems "approximately on certain scales." Second, this notion is required for making sense of effective field theory interpretations of quantum field theories, on which one claims that certain quantum field theories represent the world only "approximately on certain scales." In fact, the mathematics involved in both applications appears similar, although further work is needed to bring them together.

It is my contention in this paper that one can make claims about "approximation on certain scales" precise for the explanation of classical behavior in the classical limit of quantum mechanics. I believe this is of interest in its own right for how it bears on some recent discussions of the classical limit and reduction. But furthermore, I hope to show that this discussion has implications for the interpretation of quantum field theories. I will argue that the mathematical tools that allow one to interpret the classical limit in terms of "approximation on certain scales" are today still missing from the extant mathematical formalism for quantum field theories, even on the effective field theory approach. This leads to some suggestions about what kind of mathematical developments may be helpful, or perhaps even necessary, for giving a philosophical foundation to quantum field theory.

In slightly more detail, I will argue that one can make sense of how classical behavior arises from quantum theories "approximately on certain scales" by formalizing the classical limit in terms of continuous fields of algebras. This mathematical setting allows one to consider a family of quantum theories in which a parameter (Planck's constant, etc.) varies continuously. I will argue that the spectral theorem, which already plays a foundational role in the interpretation of quantum theories, also allows one to reinterpret these varying parameters in terms of varying scales, represented by the magnitude of an error bound relative to a choice of units. The majority of the paper is devoted to making this precise and drawing out some consequences of this interpretation for our understanding of the classical limit.

I will then argue that this understanding of "approximation on certain scales" has implications for effective field theories. I will argue that the classical limit and the scaling limits used in effective field theories are relevantly similar so that we should expect the same mathematical tools and interpretations to apply in each case. But I will note that in rigorous versions of effective field theories, one does not have a version of the spectral theorem, which precludes transferring the interpretation of the classical limit in terms of "approximation on certain scales" to this setting. This is not in principle a problem for effective field theories because it is possible that we need to understand the "scaling limits" involved in effective field theories in a fundamentally different way from the classical limit. But this does at least point to further work to be done to establish the adequacy of effective field theory interpretations. I will argue that *either* one needs to develop different methods for interpreting quantum field theories beyond the ones I describe here, which rely on the spectral theorem; *or else* one needs to develop a mathematical framework in which one can employ these methods involving the spectral theorem to make sense of "approximation on certain scales."

The structure of the paper is as follows. In §2, I will lay out the mathematical tools for understanding the classical limit of quantum theories in terms of continuous fields of algebras, which gives rise to the theory of so-called strict deformation quantization. In §3, I will then use the mathematical background to spell out how one can use the spectral theorem to interpret the classical limit in terms of "approximation on certain scales" in a way reminiscent of discussions of scaling behavior in renormalization. I will argue that such an interpretation sheds some light on philosophical issues surrounding reduction and explanation through the classical limit. In §4, I will then discuss the implications of this work for philosophical approaches to interpreting quantum field theory, where some appear to invoke similar interpretive methods even though the mathematical tools I use to underlie those methods for interpreting the classical limit are lacking. Finally, in §5, I will conclude with some further discussion.

# 2 The classical limit

Although historically the term "quantization" has been reserved for the process of constructing a quantum theory, modern mathematical theories of *deformation quantization* are understood instead to provide tools for the "inverse" or "dual" process of taking the classical limit of quantum theories (Landsman, 2006, 2017). This is accomplished by constructing a family of algebras, each representing a quantum theory of the same "form" (i.e., with the capacity to represent the same physical systems), but with a different value for Planck's constant $\hbar$. As I will attempt to make precise later, each of these algebras can be thought of as representing the same system on a different "scale". One provides additional structure to "glue" these algebras together into a *continuous field*, which allows one to specify continuous limits of states and quantities in the limit as $\hbar \to 0$. The result of this construction is a partial explanation of the success of classical physics in approximately describing quantum systems. Of course, there are other limits one might analyze to recover classical behavior from quantum theories (e.g., the limit $N \to \infty$ of increasing number of particles), but I choose to focus on the $\hbar \to 0$ limit for my purposes.

How is the $\hbar \to 0$ limit supposed to explain classical behavior? One imagines the procedure going as follows. Start with a fully quantum theory in which Planck's constant takes the value $\hbar = 1$ in natural units. Then "zoom out" from the quantum description, looking at larger and larger scales by letting Planck's constant get smaller and smaller until $\hbar \approx 0$. The theory one obtains is an approximate classical description of the same system on the appropriate scales.

This is a nice outline, but there are many missing pieces. What does it mean to have the same theory of a physical system but "zoom out"? What exactly is a "scale" in this context, and what is the notion of approximation involved? Does the $\hbar \to 0$ limit capture this process in a way that has explanatory force? Other accounts of this limiting procedure or similar examples[1] (Rohrlich, 1989; Batterman, 1995, 1997; Rosaler, 2015a,b) have challenged the status of the putative explanations. It is my goal in §3 to attempt a partial answer to these questions that defends the explanatory status of the $\hbar \to 0$ limit. In order to do so, I will need to use some details of the mathematical tools surrounding deformation quantization, which I now present in this section.

It is worth a remark already at this stage to ward off a potential objection. My goal later in this paper is to use lessons about the classical

[1] For more on limiting explanations and reduction, see Nickles (1975). See also Batterman (2002) for many examples from other areas of physics.

limit to draw conclusions about scaling limits in effective field theories. But one might object: the "scaling limits" involved in renormalization are different from the $\hbar \to 0$ classical limit in important ways, so why should we expect to learn anything about renormalization and effective field theories from looking at the classical limit? Of course, the $\hbar \to 0$ limit is not the one taken during the renormalization process. However, I hope to show that the $\hbar \to 0$ limit can indeed be understood as a particular "scaling limit", and so I hope to demonstrate that it *is* at least in some respects similar to the limits taken during renormalization. Still, one might not be satisfied that this is the right kind of similarity. We will need to look to the details to see how continuous fields of C*-algebras might be used to represent both kinds of limits.

However, one should already expect the analogy to hold roughly as follows: the classical theory in the $\hbar \to 0$ limit is an effective theory for an underlying quantum theory just as an effective field theory in a "scaling limit" is an effective theory for an underlying, higher-energy (perhaps "more fundamental") field theory. Thus, one expects to be able to give at least a similar analysis of what it means for each of these effective theories to hold "approximately on certain scales". Below, I will gesture at connections between the $\hbar \to 0$ limit in deformation quantization and the "scaling limits" involved in renormalization for effective field theories as I introduce continuous fields of algebras.

I am perfectly happy to admit, though, that the tools of continuous fields of C*-algebras may not be appropriate for capturing the "scaling limits" involved in renormalization. I hope to demonstrate at least that these tools can make explanations given in the classical limit precise. It is striking how similar this explanation is to some of the rhetoric surrounding "scaling limits" in effective field theories, which is the reason I believe the analogy is worth discussing. But there may be better ways of capturing the "scaling limits" in quantum field theories. If this is the case, then I hope that others will take up the charge of making "scaling limits" precise in some other fashion and demonstrating any relevant disanalogies with the classical limit.

## 2.1 Continuous fields of C*-algebras

My goal in this section is to describe the mathematical tools used to represent the classical limit of a quantum theory. I take a quantum theory to be given by a (non-commutative) C*-algebra[2] representing the bounded physical quantities of a physical system including perhaps generalized position and momentum quantities that satisfy some canonical (anti-)commutation relations (See Petz, 1990; Clifton and Halvorson, 2001; Ruetsche, 2011). Physicists and philosophers[3] have debated whether C*-algebras are an appropriate starting point for quantum theories or whether we need the additional structure of a Hilbert space representation. I will ignore these debates in what follows and simply take as an assumption that we start with a C*-algebra for a quantum theory; this should be uncontroversial because all parties to these debates agree that we require *at least* the structure of a C*-algebra.[4]

A C*-algebra already carries enough structure to define a number of topologies (e.g., norm, weak, etc.) that provide different notions of the limit of a net of physical quantities *within an algebra* (See Feintzeig, 2018b). However, to understand the classical limit, one requires resources for taking the limit of a family of algebras, where each algebra is understood to represent a full quantum theory. Such tools are provided in the theory of *strict deformation quantization*.

In a strict deformation quantization, one has a family of algebras $\mathfrak{A}_h$ for each possible numerical value $h \in [0, 1]$ of Planck's constant $\hbar$ (more on the significance of this in §3).[5] The algebra $\mathfrak{A}_0$ at the value

[2]Due to constraints of space, I assume some familiarity with the theory of C*-algebras in this paper, although all relevant results will be stated explicitly. For mathematical background, see Sakai (1971); Dixmier (1977); Takesaki (1979); Fell and Doran (1988); Kadison and Ringrose (1997). For algebraic quantum theory, see Emch (1972); Haag (1992); Baez et al. (1992); Bratteli and Robinson (1987, 1996); Landsman (1998, 2017). For a philosophical introduction, see Halvorson (2006).

[3]See, e.g., Segal (1959); Haag and Kastler (1964); Robinson (1966); Arageorgis (1995); Lupher (2008); Ruetsche (2002, 2003, 2006, 2011); Feintzeig (2016, 2018b).

[4]It will be important to my argument later on, however, that some mathematical physicists approach quantum field theory using only *-algebras of formal power series rather than C*-algebras. For the purposes of dealing with the classical limit of ordinary quantum mechanics with finitely many degrees of freedom, I hope it is uncontroversial that C*-algebras suffice.

[5]One can more generally allow $h \in I$ for some locally compact $I \subseteq \mathbb{R}$ containing

$h = 0$ will represent a classical theory; so one requires that $\mathfrak{A}_0$ contains as a norm dense subset a (complex) Poisson algebra $(\mathcal{P}, \{\cdot, \cdot\})$. This Poisson algebra arises from a Hamiltonian formulation of a classical theory, where the elements of $\mathcal{P}$ are smooth functions on a phase space with the structure of a Poisson (or even symplectic) manifold, and hence can be thought of as physical magnitudes in a natural way (Landsman, 1998). On the other hand, each algebra $\mathfrak{A}_h$ for $h \neq 0$ will be a non-commutative algebra representing a quantum theory.

The core idea of taking the limit of a collection of C*-algebras is to gather them into a structure known as a *continuous field of algebras*.

**Definition 1.** A *continuous field of C*-algebras*[6] $((\mathfrak{A}_h)_{h\in[0,1]}, \mathcal{K})$ consists in a family of C*-algebras $\mathfrak{A}_h$ for each value of $h \in [0, 1]$ and a C*-subalgebra $\mathcal{K}$ of $\prod_{h\in[0,1]} \mathfrak{A}_h$ (i.e., each element $K \in \mathcal{K}$ is a map that sends each value $h \in [0, 1]$ to an element of $\mathfrak{A}_h$). For each $h \in [0, 1]$, the set $\{K(h) \mid K \in \mathcal{K}\}$ must be norm dense in $\mathfrak{A}_h$ and $\mathcal{K}$ must satisfy:

1. For each $K \in \mathcal{K}$, the map $h \mapsto \|K(h)\|_h$ is continuous. The elements of $K$ are called *continuous sections* of the field.

2. For each $K \in \mathcal{K}$, the norm in $\mathcal{K}$ is given by $\|K\| = \sup_{h\in[0,1]} \|K(h)\|_h$.

3. For each $f \in C([0, 1])$ and each $K \in \mathcal{K}$, the map $h \mapsto f(h)K(h)$ is a continuous section in $\mathcal{K}$.

Those familiar with fiber bundles in differential geometry should recognize some concepts here. The topological space $[0, 1]$ can be understood as a base space, with the fiber $\mathfrak{A}_h$ above $h \in [0, 1]$. Continuous sections are sections of the resulting bundle with the additional (pointwise) algebraic structure induced by the algebraic structure of the fibers.

Notice that this definition of a continuous field of C*-algebras is not specific to quantization and the classical limit—the algebra $\mathfrak{A}_0$ *need not* represent a classical theory.[7] As such, these tools could be applied to other limits of quantum theories—namely, the scaling limits used in

0 as an accumulation point (Landsman, 1998).

[6]See also Rieffel (1994); Landsman (1998, 2006, 2017). Our presentation follows closely that of Binz et al. (2004b).

[7]See, e.g., (Landsman, 2006) for examples of the $N \to \infty$ limit for which the limit system is an infinite quantum system.

renormalization. There, $\mathfrak{A}_0$ would be an effective theory, while $h$ would represent a scaling parameter (not necessarily Planck's constant) and $\mathfrak{A}_h$ a higher-energy theory.[8] In this case, one does not yet actually have C*-algebras adequate to fill the role of $\mathfrak{A}_h$, although one can imagine constructing such algebras somehow from the Lagrangians defining the field theories by appropriately implementing canonical commutation relations on the field and momentum quantities.

I will discuss the possible use of such structures for accounting for the limits taken in quantum field theories in §4. For the remainder of §2-3, however, I will focus on the specific case where a continuous field is used to represent the classical limit of a quantum theory. In this case, the effective theory represented by $\mathfrak{A}_0$ is a classical theory, and $\mathfrak{A}_h$ represents a quantum theory for a particular value $h \in [0, 1]$ of Planck's constant. The classical limit has further structure:

**Definition 2.** A *continuous quantization* $((\mathfrak{A}_h)_{h\in[0,1]}, \mathcal{K}, \mathcal{Q})$ of the (complex) Poisson algebra $(\mathcal{P}, \{\cdot, \cdot\})$ consists in a continuous field of C*-algebras $((\mathfrak{A}_h)_{h\in[0,1]}, \mathcal{K})$ and a linear, *-preserving map $\mathcal{Q} : \mathcal{P} \to \mathcal{K}$ such that the maps $\mathcal{Q}_h : \mathcal{P} \to \mathfrak{A}_h$ defined by

$$\mathcal{Q}_h(A) := \mathcal{Q}(A)(h)$$

for all $A \in \mathcal{P}$ satisfy:

> **(Dirac's Condition)** The $h$-scaled commutator, defined for $X, Y \in \mathfrak{A}_h$ by $[X, Y]_h := \frac{i}{h}(XY - YX)$, approaches the Poisson bracket in norm as $h \to 0$, i.e.,
>
> $$\lim_{h\to 0} \|[\mathcal{Q}_h(A), \mathcal{Q}_h(B)]_h - \mathcal{Q}_h(\{A, B\})\|_h = 0$$

The map $\mathcal{Q}_h : \mathcal{P} \to \mathfrak{A}_h$ will be called the *quantization map* for the value $h \in [0, 1]$ of Planck's constant. Dirac's Condition enforces the

[8] See, e.g., Brunetti et al. (2009) for an application that comes quite close to this. They fail to construct a continuous field of C*-algebras for precisely the reason outlined in §4 that they deal with non-convergent formal power series, which do not form a C*-algebra, but rather a *-algebra over an ordered ring. However, it seems clear that they aim at a structure on a family of algebras closely related to that described here.

canonical commutation relations as they arise from the classical Poisson bracket, at least in the limit. The structure $(\mathfrak{A}_h, \mathcal{Q}_h)_{h\in[0,1]}$ is also called a *strict quantization* of $(\mathcal{P}, \{\cdot,\cdot\})$, the name "strict" signifying that $h$ is a number rather than a formal parameter. A strict quantization is called a strict *deformation* quantization if $\mathcal{Q}_h$ is injective for each $h \in [0,1]$ and $\mathcal{Q}_h[\mathcal{P}]$ is closed under the product in $\mathfrak{A}_h$.

The deformation condition is important because it allows one to "pull back" the non-commutative product on $\mathfrak{A}_h$ to the original Poisson algebra by defining a product

$$A \cdot_h B := \mathcal{Q}_h^{-1}(\mathcal{Q}_h(A)\mathcal{Q}_h(B))$$

for all $A, B \in \mathcal{P}$. This has inspired attempts to directly define the non-commutative product $\cdot_h$ on $\mathcal{P}$ in terms of formal power series in $h$ (Bordemann, 2008; Waldmann, 2015). This approach, known as *formal deformation quantization* has been useful for making renormalization and effective field theories rigorous (Costello, 2011; Rejzner, 2016). However, since one employs formal power series, which in general do not converge, one does not have a C*-algebra over the field $\mathbb{C}$ for each value of $h$, but instead a single *-algebra over an ordered ring. It is in this sense that, since $h$ is treated as a formal parameter, formal deformation quantization differs from strict deformation quantization. The claim I will eventually argue for is that the strict quantization approach (in terms of C*-algebras) can make sense of a notion of "approximation at certain scales" using mathematical resources that the formal power series approach (in terms of *-algebras) does not have available.

Note that for my purposes, quantization maps are particular tools for constructing continuous fields of C*-algebras. Namely, specifying a quantization map uniquely determines a collection of continuous sections that "glues" a family of C*-algebras together into a continuous field (See Landsman (1998)). So the traditional role of quantization in the construction of quantum theories is irrelevant for my purposes. In what follows, I will ignore the important role that quantization plays in theory construction in both ordinary quantum mechanics and quantum field theories. Instead, I will only be interested in how one can use continuous fields of C*-algebras to represent "scaling limits". Again, it is worth emphasizing that one does not expect the scaling limits in quantum field theories to be constructed through a quantization map because those scaling limits are not $\hbar \to 0$ limits, but rather limits of some other parameter. Still, it seems that those scaling limits resemble the $\hbar \to 0$ limit as understood in the next section and so could possibly be modeled using continuous fields of algebras.

In preparation for the next section, let us specify a notion of equivalence that allows one to compare continuous quantizations constructed with seemingly different interpretive motivations. I will say that two strict quantizations $(\overset{1}{\mathfrak{A}}_h, \overset{1}{\mathcal{Q}}_h)_{h\in[0,1]}$ and $(\overset{2}{\mathfrak{A}}_h, \overset{2}{\mathcal{Q}}_h)_{h\in[0,1]}$ of $(\mathcal{P}, \{\cdot,\cdot\})$ are *equivalent* just in case for each $h \in [0,1]$, there is a *-isomorphism $\alpha_h : \overset{1}{\mathfrak{A}}_h \to \overset{2}{\mathfrak{A}}_h$ such that the following diagram commutes:[9]

$$\begin{array}{ccc} & \mathcal{P} & \\ {\scriptstyle \overset{1}{\mathcal{Q}}_h}\swarrow & & \searrow{\scriptstyle \overset{2}{\mathcal{Q}}_h} \\ \overset{1}{\mathfrak{A}}_h & \xleftrightarrow{\;\alpha_h\;} & \overset{2}{\mathfrak{A}}_h \end{array}$$

Two continuous quantizations are equivalent just in case the fibers $\mathfrak{A}_h$ over each point $h \in [0,1]$ are *-isomorphic, in a way that allows one to identify the continuous sections of each resulting continuous field. If one accepts that *-isomorphic C*-algebras have the capacity to represent the same physical systems,[10] then it follows that equivalent continuous quantizations have the capacity to represent the classical limits for the same systems. Later, I will construct two different continuous quantizations corresponding to conceptually different motivations and physical interpretations. However, I shall demonstrate that the resulting continuous quantizations are equivalent, thus allowing us to transfer the natural notion of approximation on one interpretation to the mathematical structure in any of its equivalent instantiations.

[9]This definition of equivalence is different than that given in Landsman (1998), who considers continuous fields of C*-algebras with identical fibers that agree asymptotically. I drop the condition that the fibers are identical and instead only require them to be *-isomorphic; however, the notion of equivalence here forces more than just asymptotic agreement.

[10]This view is related to that expressed in Feintzeig (2015); Weatherall (2016, 2017); Fletcher (2019) concerning isomorphisms and representational capacities. I simply take on this understanding of isomorphisms and representational capacities as an assumption in what follows without further argument. This assumption is controversial, but I set the issue aside for present purposes.

## 2.2 Example: the Weyl algebra

I now define a particular C*-algebra, known as the *Weyl algebra* that is often used to represent certain quantum systems (See Petz, 1990; Dubin et al., 2000; Clifton and Halvorson, 2001).[11] I will use this example to illustrate the concept of a strict deformation quantization. I will then draw upon this example in §3 to illustrate an interpretation of the classical limit as a "scaling limit" involving a precise notion of "approximation on certain scales."

Start with a classical theory of a system with finitely many degrees of freedom that is topologically "simple" in the sense that the phase space of the system is $\mathbb{R}^{2n}$. Such a system might consist of a finite number $n$ of particles, each moving in one-dimension. Then, each point $x = (q_1, ..., q_n, p_1, ..., p_n) \in \mathbb{R}^{2n}$, understood in some canonical coordinate system, lists the familiar position $q_i$ and momentum $p_i$ of each of the $n$ particles.

Physical magnitudes of this system can be represented as complex-valued functions on phase space $f : \mathbb{R}^{2n} \to \mathbb{C}$. I will focus on functions of the form $W_0(x) : \mathbb{R}^{2n} \to \mathbb{C}$ for each $x \in \mathbb{R}^{2n}$ defined by

$$W_0(x)(y) := e^{ix \cdot y}$$

where $\cdot$ is the standard inner product on $\mathbb{R}^{2n}$. The *classical Weyl algebra*, denoted $\mathcal{W}_0$, is defined as the C*-algebra containing all norm limits of polynomials of functions of the form $W_0(x)$ for $x \in \mathbb{R}^{2n}$, endowed with the algebraic structure of pointwise addition, multiplication, and complex conjugation, and with the standard supremum norm.[12]

One constructs the quantum Weyl algebra by starting from the same generating magnitudes $W_0(x)$ and deforming the commutative pointwise multiplication relation to obtain a non-commutative algebra. I

[11] Feintzeig (2018a, 2019), Feintzeig et al. (2019), and Feintzeig and Weatherall (2019) argue against the use of the Weyl algebra for representing quantum theories. Those arguments, however, do not bear on the present issues because the claims of the current paper about approximation can also be recovered with other algebras favored by those authors. I employ the Weyl algebra in this paper because it provides a clear illustration of the notion of approximation in the classical limit through the way it encodes position and momentum quantities.

[12] This C*-algebra $\mathcal{W}_0$ is known as the algebra $AP(\mathbb{R}^{2n})$ of almost periodic functions on $\mathbb{R}^{2n}$. See, e.g., Anzai and Kakutani (1943); Hewitt (1953); Rudin (1962); Gamelin (1969).

denote by $W_h(x)$ the element $W_0(x)$ understood now as an element of the quantum Weyl algebra. Define the non-commutative product on the quantum Weyl algebra by

$$W_h(x)W_h(y) := e^{\frac{ih}{2}\sigma(x,y)}W_h(x+y)$$

for all $x, y \in \mathbb{R}^{2n}$. Here $\sigma$ is the standard symplectic form on $\mathbb{R}^{2n}$:

$$\sigma((q,p),(q',p')) := q' \cdot p - q \cdot p'$$

for $q, p, q', p' \in \mathbb{R}^n$, where $\cdot$ in the above expression is now the standard inner product on $\mathbb{R}^n$. The *quantum Weyl algebra* $\mathcal{W}_h$ is the C*-algebra obtained as the completion of the collection of all polynomials (now with respect to the non-commutative multiplication operation) of magnitudes of the form $W_h(x)$ for $x \in \mathbb{R}^{2n}$ in the so-called minimal regular norm (Manuceau et al., 1974; Binz et al., 2004a,b).

One can use these algebraic tools to construct a strict deformation quantization, which can be used to represent the classical limit of the quantum systems represented by the Weyl algebra. Binz et al. (2004b) show that there is a Poisson algebra $(\mathcal{P}, \{\cdot,\cdot\})$, norm dense in $\mathcal{W}_0$, containing all "suitably smooth" magnitudes. Here, $\{\cdot,\cdot\}$ is just the usual Poisson bracket determined by the standard symplectic form $\sigma$. The quantization maps $\mathcal{Q}_h : \mathcal{P} \to \mathcal{W}_h$ for each $h \in [0,1]$ are defined as the unique continuous linear extension of

$$\mathcal{Q}_h(W_0(x)) := W_h(x)$$

for all $x \in \mathbb{R}^{2n}$. With the quantization maps so defined, the family $(\mathcal{W}_h, \mathcal{Q}_h)_{h\in[0,1]}$ is a strict deformation quantization. Furthermore, one can define a collection of continuous sections $\mathcal{K}$ as the smallest C*-subalgebra of $\prod_{h\in[0,1]} \mathfrak{A}_h$ containing the maps $[h \mapsto \mathcal{Q}_h(A)]$ for each $A \in \mathcal{P}$. Then with the global quantization map $\mathcal{Q} : \mathcal{P} \to \mathcal{K}$ defined by

$$\mathcal{Q}(A)(h) := \mathcal{Q}_h(A)$$

for all $A \in \mathcal{P}$, the structure $((\mathcal{W}_h)_{h\in[0,1]}, \mathcal{K}, \mathcal{Q})$ becomes a continuous quantization. Thus, one can encode the classical limit of a quantum system represented by the Weyl algebra in a continuous field of C*-algebras. I will spell out in more detail in the next section how one can interpret the mathematical structure specified by this continuous field of algebras as I use this example to illustrate the notion of "approximation on certain scales" at play in the classical limit.

# 3 A notion of approximation

Now that we have some familiarity with continuous quantizations, it is the purpose of this section to interpret the mathematical tools they provide. Specifically, in this section I will argue that continuous quantizations provide tools for interpreting the classical limit of quantum theories through a notion of "approximation on certain scales". I will argue that the *spectral theorem* (Reed and Simon, 1980; Kadison and Ringrose, 1997), which already plays a central role in the interpretation of quantum theories, is also essential to this notion of approximation.

## 3.1 The Spectral Theorem and Numerical Values

The spectral theorem states that for every real-valued physical magnitude modeled by a self-adjoint element $A$ of a C*-algebra $\mathfrak{A}$, there is a projection valued measure $E : B(sp(A)) \to \mathfrak{A}^{**}$ with the following properties. Here, $B(sp(A))$ is the Borel $\sigma$-algebra of the spectrum $sp(A)$ of $A$, and $E$ takes values in the universal enveloping W*-algebra $\mathfrak{A}^{**}$ of $\mathfrak{A}$. The relationship between $E$ and $A$ is given by:

$$A = \int_{sp(A)} \lambda \, dE_\lambda$$

where $\lambda$ is understood as the identity function on $sp(A)$. In this section, I use the spectral theorem to aid the development of a notion of approximation suitable for the physical interpretation of the $\hbar \to 0$ limit. After attending to the essential role played by the spectral theorem, I will remark in the next section that there is no version of the spectral theorem available in the existing formalisms for effective field theories. Thus, the notion of approximation developed in this section cannot straightforwardly be carried over to effective field theories, at least not without significant conceptual and mathematical work.

Before embarking on the primary task of this section to develop a notion of approximation to understand the $\hbar \to 0$ limit, I review the traditional interpretive role the spectral theorem plays already in ordinary quantum mechanics without attention to the classical limit. Recall that while a general self-adjoint operator $A \in \mathfrak{A}$ can be used to represent a real-valued physical magnitude, one can say more about the representational capacities of a projection $E \in \mathfrak{A}^{**}$. Projections are capable of representing *propositions*. The reason is that $sp(E) = \{0, 1\}$, so one can think of the two possible values of $E$ as true (1) and false (0). The usual rules for calculating the expectation value of a projection give one the probability that the proposition $E$ is true (a number in $[0, 1]$), which is in agreement with the Born rule for calculating the probability of a particular outcome for a self-adjoint operator.[13]

Notice that the interpretations of projections and general self-adjoint operators already differ in that while the value of a physical magnitude represented by a self-adjoint operator may depend on a choice of units, the value of a proposition represented by a projection will not. For example, if $A$ is a self-adjoint operator representing the position magnitude for a particle,[14] then changing units of distance from $m$ to $cm$ changes the possible values $A$ can take on, scaling the numerical values by 100. Thus, one may need to use different self-adjoint operators to represent the same physical magnitude understood in different systems of units (or at least pay close attention to how we compare these self-adjoint operators and keep in mind the appropriate scale factor).

On the other hand, if $E_O$ is a spectral projection associated with $A$ for a Borel set $O \subseteq sp(A)$, then $E_O$ can be interpreted as representing the proposition "The particle is located in the region represented by $O$". If the particle is located in the region represented by $O$, then the proposition is true and $E_O$ takes the value 1, whereas if the particle is outside of the region represented by $O$, then the proposition is false and $E_O$ takes the value 0. Even if one changes units for distance from $m$ to $cm$, if we hold fixed the proposition that we take the projection $E_O$ to represent (e.g., by holding fixed the physical region associated with the projection), then the value of this proposition should not change. The particle is still either located within or outside of the region represented by $O$, independent of our choice of units. Hence, one need not use a

[13]For the purposes of this paper, I remain agnostic about how to interpret probability assignments and measurements.

[14]A C*-algebra will contain only bounded operators capable of representing only bounded physical quantities, and hence will not contain a position operator, generally. However, analogous interpretations can be given for bounded functions of position. Additional complications arise for unbounded operators *affiliated* with a C*-algebra, which I ignore. One also has a spectral theorem for unbounded self-adjoint operators (Kadison and Ringrose, 1997).

different operator to represent this proposition when we change units. It is in this sense that values of physical magnitudes represented by general self-adjoint operators may depend on a choice of units while the values of propositions represented by projections will not.[15]

The spectral theorem can inform our understanding of how physical magnitudes (self-adjoint operators) relate to propositions (spectral projections). Physical magnitudes assign numerical values to certain propositions. For example, the position magnitude assigns to the proposition "The particle is $4m$ from the origin" the numerical value 4, when understood in units of $m$.[16] So if one changes units, one can leave fixed the projections representing propositions while changing the numerical value a given physical magnitude assigns to each proposition. For example, if one changes units from $m$ to $cm$, the position magnitude then assigns to the very same proposition "The particle is $4m$ from the origin" the new numerical value 400. In other words, a change of units induces a change $A \mapsto A' \in \mathfrak{A}$ given by

$$A' = \int_{sp(A)} f(\lambda)\ dE_\lambda$$

where $E$ is the projection valued measure for $A$, and the Borel function $f : sp(A) \to \mathbb{R}$ is in general not the identity, but some rescaling function governing how the numerical values of $A$ in $sp(A)$ change under the scale transformation $A \mapsto A'$. (In the example of the unit change from $m$ to $cm$, the function $f$ is $f(\lambda) = 100\lambda$ for all $\lambda \in \mathbb{R}$.) This makes

[15]It is possible that one could give an alternative interpretation of changes of units as leaving general self-adjoint operators representing physical quantities fixed while changing the projection associated with any given physical proposition. While this may be a viable alternative, it is not the one I choose in this paper. It seems to me that interpreting changes of units as affecting self-adjoint operators and leaving projections fixed is at least feasible, and provides a clear illustration of the conceptual structure of explanations in the classical limit, so I will stick with this interpretation in what follows. Moreover, on either interpretation, one relies on the spectral theorem for relating projections and general self-adjoint operators, so the remarks in §4 still hold on the alternative view.

[16]Projection valued measures assign projections to (even extended) *regions* of the spectrum, not just particular points. The simplified interpretation mentioned here still makes sense, however, because the projections play essentially this role when integrated, as in the spectral theorem.

sense because the new collection of possible numerical values of $A'$ is $sp(A') = f[sp(A)]$. Thus, we induce the change $A \mapsto A' = f(A)$.

I will use this understanding of the relationship between self-adjoint operators, spectral projections, and numerical values in different systems of units as I spell out a notion of approximation inherent in strict quantization next.

### 3.2 Units, Scales, and (Counter)Factual Limits

When interpreting limits of physical constants like $\hbar \to 0$, there are two different approaches one can take. Following Fletcher (2018) (who draws on Rohrlich (1989)), I will call these the *counterfactual* and *factual* approaches[17] to interpreting the classical limit. The counterfactual interpretation attempts to answer the question, "How would the world be different if Planck's constant $\hbar$ were to take a different value in *the same system of units*?" Hence, the counterfactual interpretation only answers questions about other possibilities besides the actual world (or so it is claimed). On the other hand, the factual interpretation attempts to answer the question, "In the actual world, how do quantities behave in *different systems of units* in which Planck's constant $\hbar$ takes different values?" Thus, the two interpretations differ on whether they are concerned with modeling the actual world with the actual observations and experiments (factual interpretation) or alternative physical possibilities (counterfactual interpretation).[18]

One might think only the factual interpretation can answer the explanatory questions about approximate classical behavior outlined §2. So if one thought that there only existed the mathematical resources to spell out a counterfactual interpretation, then this would signify a real conceptual and explanatory gap. In fact, it does seem that this is how at least some philosophers approach quantization and the classical limit. For example, Rosaler (2015a) claims that the mathematical tools involved in the quantization procedures outlined above don't capture

[17]Fletcher (2018) works out these approaches in the context of the Newtonian limit $c \to \infty$ for general relativity. Here, I adapt the analysis to quantum theories.

[18]Note the similarity here between the factual/counterfactual limit distinction and the active/passive transformation distinction. For the latter, see, e.g., (Redhead, 2003; Norton, 2018).

the empirical content necessary for explanations in the classical limit. And Batterman (1995) argues more generally that the classical limit of quantum mechanics does not yield a genuine reduction because of the "singular" nature of the limits. However, it is the goal of the current section to show that the very same mathematical resources used to spell out the counterfactual interpretation of the $\hbar \to 0$ limit can be used to give a factual interpretation as well. The reason is that when one develops the mathematical tools for quantization under each interpretation, one finds that both approaches lead to *equivalent* continuous quantizations. Using the resulting continuous field of algebras and interpretation of scaling behavior, I will argue that one avoids problems with so-called "singular" limits and achieves a genuine explanation of classical behavior. The remainder of this section builds the mathematical tools for the factual and counterfactual interpretations and illustrates the equivalence in the case of the quantization of the Weyl algebra. At the end of this section, I return to a comparison of the current approach with other philosophical views of the classical limit.

**Counterfactual Limit**

First, I will develop the counterfactual approach by constructing the following continuous quantization of the Weyl algebra. On the counterfactual approach, for each value of $\hbar$ one represents a different "world" in which the physical quantities of the system are represented by a distinct algebra $\overset{CF}{\mathfrak{A}}_h := \mathcal{W}_h$. One keeps fixed the classical "world" in which the quantities of the system are represented by $\overset{CF}{\mathfrak{A}}_0 := \mathcal{W}_0$. One then uses a quantization map to identify the "same" or "counterpart" magnitudes in distinct worlds by defining $\overset{CF}{\mathcal{Q}}_h : \mathcal{P} \to \overset{CF}{\mathfrak{A}}_h$ for any $h \in [0, 1]$ as the unique linear, continuous extension of

$$\overset{CF}{\mathcal{Q}}_h(W_0(x)) := W_h(x)$$

for all $x \in \mathbb{R}^{2n}$. According to this definition, the different "worlds" specified by the counterfactual quantization map are *really* different because they give rise to different commutation relations

$$W_h(x)W_h(y) = e^{\frac{ih}{2}\sigma(x,y)}W_h(x+y)$$

for all $x, y \in \mathbb{R}^{2n}$, which clearly depends on the numerical value $h$ of Planck's constant in the "world" considered. Thus, each "world" can be interpreted as being governed by distinct physical laws (even though the "form" of the laws in each world is in some sense the same).[19] This definition gives rise to precisely the continuous quantization of the Weyl algebra I defined in §2.2, which we now denote by $((\overset{CF}{\mathfrak{A}}_h)_{h\in[0,1]}, \overset{CF}{\mathcal{K}}, \overset{CF}{\mathcal{Q}})$. As before, define the collection of counterfactual continuous sections $\overset{CF}{\mathcal{K}}$ to be the smallest C*-subalgebra of $\prod_{h\in[0,1]} \overset{CF}{\mathfrak{A}}_h$ containing the maps $[h \mapsto \overset{CF}{\mathcal{Q}}_h(A)]$ for all $A \in \mathcal{P}$. Define the counterfactual global quantization map $\overset{CF}{\mathcal{Q}} : \mathcal{P} \to \overset{CF}{\mathcal{K}}$ by

$$\overset{CF}{\mathcal{Q}}(A)(h) := \overset{CF}{\mathcal{Q}}_h(A)$$

Since this structure is identical with the structure $((\mathcal{W}_h)_{h\in[0,1]}, \mathcal{K}, \mathcal{Q})$, it is a continuous quantization.

### Factual Limit

To make precise the factual interpretation, I will take a somewhat different approach. Recall that on the factual approach, one wants to specify algebras $\overset{F}{\mathfrak{A}}_h$ that represent physical quantities in *the actual world*, and use the index $h$ only to investigate how these quantities change in different systems of units. Since one want to model quantities in *only the actual world*, one can start with the constraint that all of the algebras $\overset{F}{\mathfrak{A}}_h$ be identical. So let us fix our world in units where $h = 1$ and define $\overset{F}{\mathfrak{A}}_h := \mathcal{W}_1$ for all $h \in (0, 1]$. Of course since one wants to explain classical behavior, one still needs to let $\overset{F}{\mathfrak{A}}_0 := \mathcal{W}_0$. One then uses a quantization map to identify how a physical quantity changes in the actual world when one changes units so that the numerical value of Planck's constant changes as $h' \mapsto h$.

To think about these changes of units, let us start by analyzing the classical quantities. It will be helpful to specify a canonical coordinate

[19]Notice here that I am not talking about dynamical laws, but only the kinematic relations governing quantum systems like the uncertainty principle.

system $(q_1, ..., q_n, p_1, ..., p_n)$ on the classical phase space $\mathbb{R}^{2n}$ and expand the operators $W_0(x)$, where $x = (a_1, ..., a_n, b_1, ..., b_n)$:

$$W_0(a_1, ..., a_n, b_1, ..., b_n)(q_1, ..., q_n, p_1, ..., p_n) = e^{i(a_1 q_1 + ... + a_n q_n + b_1 p_1 + ... + b_n p_n)}$$

For concreteness, suppose that the unit change that induces the change in the numerical value of Planck's constant $h' \mapsto h$ is a change of distance units (i.e., $m$ to $cm$, $km$ to $mm$, etc.).[20] To understand how this affects the numerical values of quantities involved, recall that Planck's constant has units $[\frac{(mass)\cdot(distance)^2}{time}]$, each position quantity $q_j$ has units $[(distance)]$, and each momentum quantity $p_j$ has units $[\frac{(mass)\cdot(distance)}{(time)}]$. Since Planck's constant involves units of $(distance)^2$, while position and momentum involve units only of $(distance)$, if one changes units of distance in a way that induces a change in numerical values $h' \mapsto h$, this will change the numerical values of position and momentum by

$$\begin{aligned} q_j &\mapsto \sqrt{\frac{h}{h'}} \cdot q_j \\ p_j &\mapsto \sqrt{\frac{h}{h'}} \cdot p_j \end{aligned} \qquad (*)$$

Now, let us shift focus to the magnitudes $W_1(a_1, ..., a_n, b_1, ..., b_n)$ for the corresponding quantum system. In this case, at least in the (regular) Schrödinger representation $(\pi, L^2(\mathbb{R}^n))$ of the Weyl algebra[21] the Weyl unitaries take the form:

$$\pi(W_1(a_1, ..., a_n, b_1, ..., b_n)) = e^{i(a_1 Q_1 + ... + a_n Q_n + b_1 P_1 + ... + b_n P_n)}$$

for self-adjoint unbounded operators $Q_1, ..., Q_n, P_1, ..., P_n$ representing the quantized position and momentum magnitudes. Although generally $Q_j$ and $P_j$ will not belong to any C*-algebra $\overset{F}{\mathfrak{A}}_h$ because they are unbounded, one may restrict attention to the Schrödinger representation in which the spectral theorem still applies.

[20] One can in an exactly analogous way define factual continuous quantizations for mass and time unit changes. These are equivalent to the one defined here.

[21] See Feintzeig (2017, 2018a, 2019); Feintzeig et al. (2019); Feintzeig and Weatherall (2019) for reasons to focus on regular representations of the Weyl algebra, of which the Stone-von Neumann theorem tells us the Schrödinger representation is the unique irreducible one.

So suppose $Q_j$ is a position magnitudes affiliated with the quantum system. Letting $E$ denote the projection valued measure in the Schrödinger representation associated with $Q_j$, one has

$$Q_j = \int_{sp(Q_j)} \lambda \, dE_\lambda$$

A unit change that induces the change in numerical values of Planck's constant $h' \mapsto h$ leaves each projection $E$ fixed but change the value $Q_j$ assigns to $E$ according to the scale factor $\sqrt{\frac{h}{h'}}$ as per Eq. (*). This yields a change[22]

$$Q_j \mapsto Q'_j = \int_{sp(Q_j)} \sqrt{\frac{h}{h'}} \cdot \lambda \, dE_\lambda = \sqrt{\frac{h}{h'}} \cdot Q_j$$

Similarly, suppose $P_j$ is a momentum magnitude affiliated with the quantum system, understood as the standard momentum operator in the Schrödinger representation of the Weyl algebra associated with projection valued measure $F$:

$$P_j = \int_{sp(P_j)} \lambda \, dF_\lambda$$

A change of units that induces the change in numerical values of Planck's constant $h' \mapsto h$ leaves each projection $F$ fixed but change the value $P_j$ assigns to $F$ according to the scale factor $\sqrt{\frac{h}{h'}}$ as per Eq. (*). This yields a change

$$P_j \mapsto P'_j = \int_{sp(P_j)} \sqrt{\frac{h}{h'}} \cdot \lambda \, dF_\lambda = \sqrt{\frac{h}{h'}} \cdot P_j$$

Now, to use this change of units to define the factual quantization map $\overset{F}{\mathcal{Q}}_h : \mathcal{P} \to \overset{F}{\mathfrak{A}}_h$, recall that we wish to model changing units from $\overset{F}{\mathfrak{A}}_h = \mathcal{W}_1$ in which the numerical value of Planck's constant is $h' = 1$

[22]In a moment, I will specify this unit change directly in terms of the Weyl unitaries. It is only for clarity of presentation that I present the unit change as it affects first and foremost the position and momentum magnitudes.

so this unit change induces the change in numerical values $1 \mapsto h$ with scaling factor $\sqrt{\frac{h}{h'}} = \sqrt{h}$. Now, this motivates defining the factual quantization map as the unique linear norm continuous extension of

$$\overset{F}{\mathcal{Q}}_h(W_0(a_1, ..., a_n, b_1, ..., b_n)) := W_1(\sqrt{h} \cdot a_1, ..., \sqrt{h} \cdot a_n, \sqrt{h} \cdot b_1, ..., \sqrt{h} \cdot b_n)$$

for all $(a_1, ..., a_n, b_1, ..., b_n) \in \mathbb{R}^{2n}$. This appears in the Schrödinger representation as

$$\pi(W_1(\sqrt{h} \cdot a_1, ..., \sqrt{h} \cdot a_n, \sqrt{h} \cdot b_1, ..., \sqrt{h} \cdot b_n)) = e^{i(a_1\sqrt{h}\cdot Q_1 + ... + a_n\sqrt{h}\cdot Q_n + b_1\sqrt{h}\cdot P_1 + ... + b_n\sqrt{h}\cdot P_n)}$$

with each $Q_j$ and $P_j$ scaled by the fact $\sqrt{h}$ as desired. More succinctly, one can write

$$\overset{F}{\mathcal{Q}}_h(W_0(x)) = W_1(\sqrt{h} \cdot x)$$

for all $x \in \mathbb{R}^{2n}$.

Notice that the operators $W_1(x)$ for $x \in \mathbb{R}^{2n}$ obey the standard commutation relation

$$W_1(x)W_1(y) = e^{\frac{i}{2}\sigma(x,y)}W_1(x+y)$$

for all $x, y \in \mathbb{R}^{2n}$. This relation remains fixed in each of the algebras $\overset{F}{\mathfrak{A}}_h = \mathcal{W}_1$, indicating that the "physical laws"[23] remain the same in the one "world" we represent. However, the rescaled operators $\overset{F}{\mathcal{Q}}_h(W_0(x))$ give the appearance of different physics because their commutation relation is

$$\overset{F}{\mathcal{Q}}_h(W_0(x))\overset{F}{\mathcal{Q}}_h(W_0(y)) = e^{\frac{ih}{2}\sigma(x,y)}\overset{F}{\mathcal{Q}}_h(W_0(x+y))$$

for all $x, y \in \mathbb{R}^{2n}$ due to the fact that $\sigma(\sqrt{h} \cdot x, \sqrt{h} \cdot y) = h \cdot \sigma(x, y)$. This expression depends on the numerical value of $h$ and thus on the system of units chosen.

With this factual quantization map, one can follow the procedure above to construct a continuous quantization $((\overset{F}{\mathfrak{A}}_h)_{h\in[0,1]}, \overset{F}{\mathcal{K}}, \overset{F}{\mathcal{Q}})$. Define the collection of factual continuous sections $\overset{F}{\mathcal{K}}$ as the smallest

[23]That is, at least the kinematical laws. See fn. 19.

C*-subalgebra of $\prod_{h\in[0,1]} \overset{F}{\mathfrak{A}}_h$ containing the maps $[h \mapsto \overset{F}{\mathcal{Q}}_h(A)]$ for all $A \in \mathcal{P}$. Define the factual global quantization map $\overset{F}{\mathcal{Q}} : \mathcal{P} \to \overset{F}{\mathcal{K}}$ by

$$\overset{F}{\mathcal{Q}}(A)(h) := \overset{F}{\mathcal{Q}}_h(A)$$

This structure also provides a continuous quantization of $\mathcal{P}$.

**Equivalence**

Finally, now that we have constructed two continuous quantizations with quite different motivations and physical interpretations—the factual interpretation for our world and the counterfactual interpretation for alternative possible worlds—I will establish that both mathematical structures have the same representational capacities. That is, I will show that the factual $((\overset{F}{\mathfrak{A}}_h)_{h\in(0,1]}, \overset{F}{\mathcal{K}}, \overset{F}{\mathcal{Q}})$ and counterfactual $((\overset{CF}{\mathfrak{A}}_h)_{h\in[0,1]}, \overset{CF}{\mathcal{K}}, \overset{CF}{\mathcal{Q}})$ continuous quantizations are *equivalent*. For each $h \in (0,1]$, define $\alpha_h : \overset{CF}{\mathfrak{A}}_h \mapsto \overset{F}{\mathfrak{A}}_h$ as the unique linear, norm continuous extension of[24]

$$\alpha_h(W_h(x)) := W_1(\sqrt{h} \cdot x)$$

It suffices to notice that $\alpha_h$ is a *-isomorphism for each $h \in (0,1]$ and

$$\alpha_h \circ \overset{CF}{\mathcal{Q}}_h = \overset{F}{\mathcal{Q}}_h$$

Hence, the two quantizations are equivalent.

Thus, the counterfactual continuous quantization, which is exactly the original continuous quantization of the Weyl algebra presented in §2.2, can be given an interpretation in terms of a notion of "approximation at certain scales". That is, one can import the notion of "approximation at certain scales" from the factual continuous quantization, which I interpret as follows.

In the factual continuous quantization, suppose we are given arbitrary classical quantities $A, B \in \mathcal{P}$ and a chosen numerical error bound

[24] Clearly, we can take $\alpha_0$ to be the identity on $\overset{CF}{\mathfrak{A}}_0 = \overset{F}{\mathfrak{A}}_0$.

$\epsilon > 0$. It follows that there is a choice of units (i.e., for some combination of distance, time, and mass) in which Planck's constant $\hbar$ takes on a numerical value $h$ such that in this system of units the behavior of $\mathcal{Q}_h(A)$ and $\mathcal{Q}_h(B)$ is "within $\epsilon$" of the behavior of $A$ and $B$:

$$\begin{aligned} \big| \|\mathcal{Q}_h(A)\|_h - \|A\| \big| &< \epsilon \\ \|\mathcal{Q}_h(A)\mathcal{Q}_h(B) - \mathcal{Q}_h(AB)\|_h &< \epsilon \\ \|[\mathcal{Q}_h(A), \mathcal{Q}_h(B)] - \mathcal{Q}_h(\{A, B\})\|_h &< \epsilon \end{aligned}$$

Similarly, for every $h' < h$, the above inequalities will hold with $h$ replaced by $h'$. Notice this implies that for any state $\omega$ on $\mathfrak{A}_h$,

$$\begin{aligned} |\omega(\mathcal{Q}_h(A)\mathcal{Q}_h(B)) - \omega(\mathcal{Q}_h(AB))| &< \epsilon \\ |\omega([\mathcal{Q}_h(A), \mathcal{Q}_h(B)]) - \omega(\mathcal{Q}_h(\{A, B\}))| &< \epsilon \end{aligned}$$

and so the expectation values of relevant physical quantities are within $\epsilon$ of one another. Thus, $\mathcal{Q}_h(A)$ and $\mathcal{Q}_h(B)$ approximate $A$ and $B$ within the error bound $\epsilon$ at all scales above the one where Planck's constant $\hbar$ takes the numerical value $h$.

The notion of scale here corresponds to the physical magnitude of the error bound that the number $\epsilon$ determines. The very same numerical error bound $\epsilon$ determines a physically larger magnitude as we change units. Even when the numerical values of the physical quantities $\mathcal{Q}_h(A)\mathcal{Q}_h(B)$ and $\mathcal{Q}_h(AB)$ differ wildly in one choice choice of units, they are very close together in another. Thus we recover the intuitive notion that as $\hbar \to 0$, we "zoom out" from the quantum system by caring less and less about the microscopic details. The way one coarse-grains from the microscopic details is by fixing a numerical error bound $\epsilon$ and changing units so that $\epsilon$ represents a physically larger and larger magnitude of allowable error.

Notice that, as claimed, the spectral theorem plays a crucial role in this interpretation. In order to understand the $\hbar \to 0$ limit under this notion of "approximation at certain scales", one understands a change in the numerical value $h$ to correspond to a change of units. This can be accomplished precisely because one has spectral projections for each physical magnitude, which are held fixed even as units change. Thus, the spectral projections and the relations among them represent "invariant" (under unit changes) physical content of the theory. In a context without the spectral theorem (like the formalism of effective field theories) it becomes increasingly unclear how to interpret mathematical objects as physical quantities with numerical values at all. But the point I wish to emphasize is furthermore that it is unclear how to make sense of units, scales, and approximation without the spectral theorem. These are precisely the notions that effective field theories rely on for their interpretation. I will discuss this issue in §4.

It is worth a number of small remarks about the explanatory status of the classical $\hbar \to 0$ limit before proceeding. First, in response to the claims of Rosaler (2015a) that explanations of classical behavior must have some empirical content, I simply note that empirical content is indeed encoded in the way quantum magnitudes approximate classical magnitudes within a given error bound relative to a choice of units. The error bound may be chosen according to the empirical context at hand—e.g., the error associated with a particular measuring device. If one is in an empirical situation where one requires more precise descriptions than the coarse-grained ones specified by the error bound, then classical physics will *not* suffice to capture the empirical phenomena. If, on the other hand, one is in an empirical situation where one only requires coarse-grained precision within the specified error bound, then classical physics *will* suffice to capture the empirical phenomena. Another way of putting this point is that if the error bound appropriately captures, say, the coarse-grained precision of a measuring device employed (which defines a kind of "empirical situation"), then it will suffice to use classical mechanics to capture the empirical phenomena displayed by that measuring device. Thus, according to the notion of "approximation on certain scales" I have provided for interpreting continuous quantizations, quantum physics explains the success of classical physics by referencing empirical contexts that we associate with particular error bounds.[25]

Second, in response to the claims of Batterman (1995) that the classical limit is "singular", I note that all of the mathematical tools described above allow for rigorous specification of continuous limits. So

[25]See also (Rosaler, 2018) for discussion of the relation between his views and deformation quantization. There, Rosaler works with formal deformation quantizations and also requires the specification of a particular continuous field of states whose position and momentum uncertainties vanish as $\hbar \to 0$.

there is at least a sense in which *a* relevant $\hbar \to 0$ limit is *not* "singular". However, the continuous field of algebras I used only specifies limits relative to one particular kind of structure, so if one looks at limits that correspond to sections that are *not* continuous in the field of algebras (i.e., if one considers a map $K : h \in [0,1] \mapsto A \in \mathfrak{A}_h$ such that $K \notin \mathcal{K}$), then the limits of these quantities may appear "singular". I believe this is an interesting feature of any limiting process, but it does not appear to me to undermine the explanatory status of the classical limit for the quantities that *can* be described by continuous sections. The upshot is that one can recover a sense in which the $\hbar \to 0$ limit is explanatory by virtue of the structures I have provided, while allowing for the phenomenon Batterman (1995) points to in which some quantities appear to have "singular" limits.

Third, I note that Landsman (2017) gives a different interpretation of the same mathematical formulation of the classical limit through strict quantization in terms of small values of some *dimensionless* ratio of Planck's constant to other magnitudes of the system at hand. This is in contrast to my interpretation of $h$ as representing a *dimensionful* version of Planck's constant. Landsman's idea seems to be that certain equations governing the behavior of the system will hold approximately if the dimensionless ratio considered is small (e.g., if the system has large temperature).[26] I do not know whether this interpretation can be made more precise. One might wonder, for example, how to understand the value of the dimensionless parameter varying when it depends on physical magnitudes associated with the system. These physical magnitudes are represented by the elements of the C*-algebras in the continuous field, which can take on many different expectation values in different states. It is difficult for me to see how a strict quantization represents such a dimensionless ratio varying, when the values that go into the dimensionless ratio (e.g., energy, temperature) are not yet fixed by the specification of the algebra. Or similarly, one might ask why the commutation relations, which may be understood as kinematical laws, change when the value of this dimensionless parameter varies. One does not usually understand changing a parameter like energy or temperature to change the kinematics one describes.

It seems to me that in order to make such a "dimensionless ratio"

[26] Although see also the discussion of rescaling on Landsman (2017, p. 247).

interpretation more precise, one would need to specify a continuous field of states on the continuous field of C*-algebras one is using to represent the classical limit. One could take the physical magnitudes in the C*-algebras to have a fixed interpretation and then specify a continuous field of states whose expectation values appropriately represent increasing values of energy, action, temperature, etc. Then one could make sense of the relevant dimensionless parameter in terms of the expectation values of these states, and one would have a sense in which the dimensionless parameter varies between different states in this continuous field. After specifying such states, one would still need to show that the expectation values of some relevant physical magnitudes will approximately satisfy the equations of classical physics. (One might need to pick different families of states to represent the variation of different dimensionless ratios relevant to each aspect of classical behavior.) If one could accomplish this, one would certainly have shown a relevant sense in which quantum mechanics explains part of the success of classical physics, and perhaps this is what Landsman has in mind.

However, it is worth some care concerning how Landsman's "dimensionless ratio" interpretation differs from my "systems of units" interpretation of the $\hbar \to 0$ limit. One difference comes in what the interpretations are intended to explain. The "dimensionless ratio" interpretation can explain why *some equations* of classical physics can be used to adequately represent a particular system *relative to a particular continuous field of states*. Moreover, as far as I can tell, to even make sense of such an interpretation of a continuous field of states as having increasing values of action, energy, temperature, etc., one needs a prior interpretation of the physical magnitudes that the states assign expectation values to. On the other hand, the "systems of units" interpretation I have given of the classical limit aims at a different explanandum. It aims to explain why the *kinematical framework* of classical physics can be used to represent the physical magnitudes and the space of physically possible states of the system at all. As such, my explanation does not depend on the specification of a continuous field of states. Furthermore, it seems to me that this explanation of the success of the kinematical framework of classical physics can even contribute to Landsman's explanatory task by helping us interpret the physical magnitudes that a continuous field of states is assigning expectation values to. So I do not believe my "systems of units" interpretation is in conflict with the "dimensionless ratio" interpretation.

On my view, then, it would be all the better if there were multiple ways of interpreting the mathematical structures used to model the classical limit. I only claim that the interpretation I have given above of the classical limit as a kind of scaling limit is one possibility. This interpretation seems quite useful for capturing the explanatory utility of the classical limit, although it may not be the only interpretation or even the best one for all purposes.

I hope it is clear that I do not necessarily disagree with the analyses that Rosaler, Batterman, and Landsman give of the classical limit. I do, however, believe that value is added when one pays attention to the interpretation of the classical limit in terms of approximation on scales with different systems of units. I believe this interpretation helps specify the extent to which the classical limit can be understood as explanatory. Moreover, I believe this interpretation is useful because of the analogy it gives rise to between the classical limit and scaling limits in effective field theories, which is the topic of the next section.

# 4 Scaling limits in effective field theories

In the previous section, I focused on interpreting one particular limit—the $\hbar \to 0$ limit—with a precise notion of "approximation on certain scales." Now, I turn to scaling limits in renormalization and effective field theories.[27] Effective field theory interpretations have played a prominent role in a number of recent philosophical discussions of quantum field theory.[28] I will argue that the scaling limits in effective field theories are analogous to the classical limit, at least enough so that it is interesting and problematic to notice that the mathematical tools I used above to interpret the classical limit are not present in contemporary

[27] For standard physics references, see Kadanoff (1966); Wilson (1971b,a); Wilson and Kogut (1974); Polchinski (1984). For philosophical discussions, including in the context of condensed matter physics, see Butterfield and Bouatta (2015); Batterman (2017); Franklin (2018).

[28] The line of philosophical inquiry I have in mind comes from the debate between Wallace (2006, 2011) and D. Fraser (2009, 2011), which has continued with Li (2013); Williams (2018); J. Fraser (2018a,b).

formulations of effective field theories. I will argue that this difference in the mathematical structures of the theories has implications for work on interpreting quantum field theories. I will give a rough classification of two types of strategies for addressing this difference. Importantly, the taxonomy I offer is *orthogonal* to a classification that has become prominent in philosophy of physics, which distinguishes *rigorous* and *non-rigorous* approaches to quantum field theory. I will suggest that we have reason to forgo the old distinction.

### Analogies between the classical limit and scaling limits

The claims I will make depend on the idea that the classical limit and the scaling limits taken in effective field theories are relevantly similar. So before proceeding, I make a case for this similarity. I concede up front that if one finds a way of understanding scaling limits in effective field theories that makes them relevantly *dis*analogous from my interpretation of the classical limit, then the rest of my argument will not apply. I do not know of an interpretation of the scaling limits that would make them relevantly disanalogous from the classical limit, but that does not mean such an interpretation is impossible. On the other hand, I believe there is positive reason to see the scaling limits involved in renormalization as relevantly similar to the classical limit. The discussion of §3 already makes some of these similarities apparent. Still, it is worth making the analogy more explicit.

As mentioned in §2, the scaling limits in effective field theories, by virtue of being limits of entire theories, appear conducive to domestication through continuous fields of C*-algebras, which seem up to precisely this task of modeling limits of entire kinematic frameworks. So one might hope to draw not only on physical analogies between the classical limit and scaling limits, but also to fit effective field theories into the same mathematical framework we have used.

Here is an outline of how this could work. Some standard approaches to renormalization in physics (e.g., Polchinski, 1984) use families of Lagrangians indexed by energy scales. The Lagrangian of a quantum field theory is supposed to determine an algebra of physical magnitudes by virtue of determining canonically conjugate field and momentum quantities on which commutation relations are defined. So one could imagine the family of Lagrangians giving rise to a family of algebras of physical magnitudes indexed by the energy scales of the Lagrangians employed. Suitably "gluing" these algebras together (i.e., specifying continuous sections) could give rise to a continuous field of algebras. And identifying elements of these algebras that correspond to the same physical magnitude at different scales (i.e., magnitudes that are related by some scaling transformation) could give rise to a structure analogous to a strict deformation quantization, in which the quantization map plays this role. In effective field theories, one would use these tools to represent not the $\hbar \to 0$ limit, but the limit as energy becomes relatively small (or, as a "cutoff" disappears).[29]

Moreover, when changing a "cutoff" or applying the renormalization group, effective field theorists want to understand themselves as analyzing the *same* physical theory (presumably with the *same* physical magnitudes for the *same* system) at a different scale. So beyond the analogy in the mathematical formulations of these scaling limits, the intended interpretations of the classical limit and other scaling limits are analogous. Hence, one should be worried if one cannot find the projections we used to identify "scale-invariant" content in the classical limit. These projections were essential for understanding why other quantities changed under rescaling as they did. Next, we will see that these mathematical tools indeed *are* missing from the extant formalism for effective field theories.

### Differences between the classical limit and scaling limits

Given these analogies between the classical limit and other scaling limits, I am now in a position to state how the two cases differ. Let us suppose that we try to interpret the scaling limits in effective field theories just as we interpreted the classical limit. That is, suppose we try to make precise the notion of "approximation on certain scales" involved in those limits using the methods of §3. Then, we would look at the operators we identify in the different theories at different scales as representing the same physical magnitude. We would need to under-

[29]Instead of requiring Dirac's condition, as in the classical limit, one would presumably require some other compatibility condition to hold for the scaling limits of effective field theories.

stand ourselves as making this identification of magnitudes by virtue of invariant spectral projections representing the same physical propositions. The existence of these invariant projections would justify the scaling transformations that make other physical magnitudes vary from one scale to another. Recall this was the crucial move to understand "approximation on certain scales" in §3.

The "hopefulness" of our analogy between the classical limit and scaling limits in effective field theories now falls apart because these crucial projections are nowhere to be found. One does *not* have a spectral theorem in the formalism for effective field theories. In fact, even in the most recent and rigorous versions of effective field theories, one does *not* have C*-algebras of physical magnitudes. Instead, one works with so-called "perturbative expansions" of physical magnitudes that together form only *-algebras of formal power series[30] (See, e.g., Rejzner, 2016). The algebraic tools involved in this approach to effective field theories are similar to, and indeed inspired by, the old-fashioned algebraic quantum field theory originated by Haag and Kastler (1964), but they are *not* the same.[31] Even though discussions of renormalization have these mathematical physicists working with families of algebras (Brunetti et al., 2009), these structures do not generally form a continuous field of C*-algebras. Without the spectral theorem and the crucial projections that supply invariant physical content under scale transformations, one does not have the mathematical tools to give an interpretation of scaling limits in effective field theories using the notion of "approximation on certain scales" spelled out in §3.[32]

If one could establish that the formal power series expansions employed in these effective field theories converged in a relevant sense when

[30]Even Polchinski (1984), a standard physics reference, is very clear about working with only formal power series expansions.

[31]See Dütsch and Fredenhagen (2001); Brunetti and Fredenhagen (2009); Fredenhagen and Rejzner (2013, 2015) for a sampling of applications of these methods in algebraic quantum field theory. See also Costello (2011) and Hairer (2015) for further mathematical approaches to renormalization that are somewhat different from, but appear related to, the algebraic framework for quantum field theory.

[32]It is noteworthy that Bordemann and Waldmann (1998) list as an open problem the development of a "formal spectral theory" for formal deformation quantizations. Furthermore, Waldmann (2005) investigates some properties of matrices over formal deformation quantizations that mimic results from spectral theory.

the formal parameter is assigned a numerical value, then one could use them to construct a C*-algebra. However, it is well known that these formal power series do not converge both because of so-called "large-order divergences" that prevent one from summing the terms of the series while maintaining a finite result and because of so-called "ultraviolet divergences" that prevent one from even assigning the integrals involved in individual terms of the series a finite value.[33] The methods of renormalization are meant to allow one to deal with the ultraviolet divergences even without a solution to the problems of large-order divergences. Thus, the scaling limits involved in effective field theories are intended to produce theories that hold "approximately on certain scales" even when it is agreed that one does not have a C*-algebra, a spectral theorem, or invariant projections in one's mathematical toolkit.

It is worth a small remark concerning *why* one only has *-algebras of formal power series and not C*-algebras in effective field theories because this will help us ward off a potential confusion. Difficulties dealing with non-linear interactions in field theories with infinitely many degrees of freedom leads to the use of *formal* deformation quantizations to construct those theories rather than the *strict* deformation quantizations we used above.[34] As mentioned in §2, formal deformation quantizations are similar to the tools of strict deformation quantization we have used, *except* that they employ formal power series in order to define the non-commutative products between physical magnitudes that form the core of quantum theories. So the fact that the projections we need to make sense of scaling limits (as I have argued in §3) are missing in some sense traces back to difficulties in constructing the quantum field theories that are at issue.

However, I want to emphasize that the difficulties in *constructing* interacting quantum field theories (e.g., through these quantization procedures) are *not* the ones I am pointing to in this paper. One might think that it is possible to employ and interpret renormalization meth-

[33]For references and discussion, see Butterfield and Bouatta (2015); Miller (2017).

[34]For some discussion of the differences between strict and formal deformation quantization, see Bordemann (2008); Waldmann (2015); Landsman (2017). Foundational works in formal deformation quantization include Kontsevich (2003), who establishes the existence of formal deformation quantizations for theories with finitely many degrees of freedom, but those methods cannot be extended to field theories with infinitely many degrees of freedom.

ods and effective field theories even without addressing potential issues about the construction of quantum field theories; indeed this seems to be the position taken by Wallace (2006, 2011), Williams (2018), and J. Fraser (2018b). For example, one might think that understanding quantization is only important for the $\hbar \to 0$ limit, which does not typically play any role in renormalization procedures. What I am arguing is that interpretive problems in the construction of quantum field theories (and conversely, their classical limit) trickle down to interpretive problems with other limiting procedures taken in those theories, including the scaling limits one may have thought to be safe. I claim that the move to *-algebras of formal power series as employed in formal deformation quantization for interacting quantum field theories leads to a framework in which the mathematical resources are missing for interpreting scaling limits, even though these scaling limits are distinct from the $\hbar \to 0$ limit of the quantum field theory.

### Implications for interpreting quantum field theories

Suppose one accepts my argument that the current mathematical framework for effective field theories lacks the mathematical resources to make sense of the notion of "approximation on certain scales" used in the interpretation of renormalization and scaling limits. Or more specifically and more modestly, suppose one accepts that the current mathematical framework for effective field theories does not have the mathematical resources that were crucial to the notion of "approximation on certain scales" presented for the classical limit in §3. Then what is to be done? I see two routes to address this problem:

(i) First, one could keep the mathematical tools currently employed for scaling limits in effective field theories—namely, formal power series—and attempt to develop methods for interpreting these limits with a notion of "approximation on certain scales" that does not depend on spectral projections.

(ii) Second, one could attempt to implement the same mathematical tools for scaling limits in effective field theories that I used to make sense of the classical limit—namely, continuous fields of C*-algebras, or other tools with an analog of the spectral theorem.

One might take approach (i) if one does not find my argument for the similarity of the classical limit and other scaling limits convincing and instead believes there are alternative, distinct methods for interpreting quantum field theories from those described in §3. One might take approach (ii) if one believes that the classical limit is analogous to the scaling limits in effective field theories and that the classical limit helps identify the mathematical tools necessary for interpreting scaling limits.

I will not take a stance concerning the relative merits of approaches (i) and (ii); however, I do wish to point out that one can use this distinction to classify some contemporary research programs. For example, Miller (2017) appears to take approach (i), arguing that quantum field theory gives us reason to use an alternative to the standard semantics for physical theories. On the other hand, some mathematical physicists appear to take approach (ii), at least implicitly. For example, this attitude seems implicit in work that attempts to prove convergence results for the formal power series constructions in some models of quantum field theories (e.g., Bahns and Rejzner, 2017).[35] In that case, one uses perturbative methods as a stepping stone toward the construction of a non-perturbative theory. Another avenue that I believe also falls under (ii) is to attempt to construct the non-perturbative theories directly and make sense of scaling limits with them. This route is taken by researchers who use so-called "scaling algebras" to construct a C*-algebraic framework for renormalization (Buchholz, 1996a,b; Buchholz and Verch, 1995, 1998). On either of these strategies for approach (ii), I believe further philosophical attention could be of value for interpreting the resulting mathematical structures. So there is a real opportunity here for philosophical work that engages with the cutting edge of research in mathematical physics.

Perhaps most importantly for the philosophical literature, the distinction between (i) and (ii) has nothing to do with whether one requires quantum field theories to be rigorous. The issue of rigor was the primary focus of the debate between Wallace (2006, 2011) and D. Fraser (2009, 2011), which led to a number of recent philosophical discussions of the issue. But my perspective suggests that rigor is beside the point

[35]cf. also fn. 32, which references the possibility of formal approaches to spectral theory. If those tools were developed, they could provide a different route to the kinds of interpretations one has in the presence of convergence.

because both the non-perturbative C*-algebraic approaches and the perturbative formal power series approaches are mathematically rigorous. Researchers who choose approach (i) could be satisfied with the existing rigorous approaches to quantum field theory and aim only at interpretive developments. And researchers who choose approach (ii) need not aim for more rigor, but rather to apply existing rigorous tools to realistic quantum field theories.

I am not the only one to challenge the view that rigor is the central issue in the interpretation of quantum field theories. D. Fraser (2011) herself and Koberinski (2016) have already noticed that the recent philosophical literature has been misleading about whether rigorous and non-rigorous approaches to quantum field theory are "competing research programs." Li (2015) also makes this point by noticing that applications of the "exact" renormalization group are perfectly rigorous, so rigor cannot be the main concern in the interpretation of quantum field theory. However, Li makes this point in the context of non-perturbative approaches to constructive field theory, which do not involve effective field theory interpretations like the one originally given by Wallace (2006). So the current paper has gone further by showing that rigor is not the issue even within effective field theory interpretations. Finally, J. Fraser (2018a) also argues that in the perturbative approach to effective field theories, rigor is not the central interpretive issue. I agree with J. Fraser's diagnosis, but we differ on one further point. J. Fraser describes effective field theories as approximate theories for which one does not yet know the underlying theory to be approximated. If my argument in this paper is correct, then to the contrary one does not yet have the resources to interpret effective field theories as approximations. So the comparison I have undertaken between the classical limit and other scaling limits leads to a new and precise way identifying a problem for interpreting quantum field theories.

# 5 Conclusion

In this paper, I have argued that one has mathematical resources in continuous fields of C*-algebras for making sense of a notion of "approximation on certain scales" in the interpretation of the classical limit of quantum mechanics. I argued that because these mathematical tools are not available in perturbative quantum field theories, further work is needed to make sense of the effective field theory approach, which relies on an interpretive notion of "approximation on certain scales." The advocate of effective field theories either needs to develop new interpretive methods, or do further mathematical work to connect the existing mathematical tools for effective field theories with the resources that exist in continuous fields of C*-algebras.

I believe the analysis offered here of approximation in the interpretation of the classical limit is interesting in its own right. But it gains special significance because of the comparison it allows with scaling limits in effective field theories.

This comparison between the classical limit and other scaling limits leads to an important conclusion for contemporary philosophical discussions of quantum field theory. I argued that once one sees the problem, one sees that there are two possible broad strategies for dealing with the lack of mathematical resources in effective field theories: either develop more mathematical resources or different interpretive resources. This classification shows that previous philosophical discussions that classify approaches to quantum field theory as rigorous or non-rigorous seem to miss the real issue. Moreover, I believe this discussion is important because on either of the approaches for interpreting effective field theories that I outlined, there is room for philosophical investigation that can make a difference to ongoing work in mathematical physics.

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# References

Anzai, H. and S. Kakutani (1943). "Bohr Compacifications of a Locally Compact Abelian Group I." *Proceedings of the Imperial Academy*, *19*, 476–480.

Arageorgis, Aristidis. *Fields, particles, and curvature: Foundations and philosophical aspects of quantum field theory in curved spacetime.* PhD thesis, University of Pittsburgh, (1995).

Baez, J., I. Segal, and Z. Zhou (1992). *Introduction to Algebraic and Constructive Quantum Field Theory*. Princeton, NJ: Princeton University Press.

Bahns, D. and K. Rejzner (2017). "The Quantum Sine-Gordon Model in Perturbative AQFT." *Communications in Mathematical Physics*, *Online first*.

Batterman, R. (1995). "Theories between theories: Asymptotic limiting intertheoretic relations." *Synthese*, *103*(2), 171–201.

Batterman, R. (1997). "'Into a Mist': Asymptotic theories on a caustic." *Studies in the History and Philosophy of Modern Physics*, *28*(3), 395–413.

Batterman, R. (2002). *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence*. Oxford: Oxford University Press.

Batterman, R. (2017). "Autonomy of Theories: An Explanatory Problem." *Nous*, *Online first, DOI* `10.1111/nous.12191`.

Binz, E., R. Honegger, and A. Rieckers (2004a). "Construction and uniqueness of the C*-Weyl algebra over a general pre-symplectic space." *Journal of Mathematical Physics*, *45*(7), 2885–2907.

Binz, E., R. Honegger, and A. Rieckers (2004b). "Field-theoretic Weyl Quantization as a Strict and Continuous Deformation Quantization." *Annales de l'Institut Henri Poincaré*, *5*, 327–346.

Bordemann, M. (2008). "Deformation Quantization: A Survey." *Journal of Physics: Conference Series*, *103*, 012002.

Bordemann, M. and S. Waldmann (1998). "Formal GNS Construction and States in Deformation Quantization." *Communications in Mathematical Physics*, *195*, 549–583.

Bratteli, O. and D. Robinson (1987). *Operator Algebras and Quantum Statistical Mechanics.* Volume 1. New York: Springer.

Bratteli, Ola and Derek Robinson (1996). *Operator Algebras and Quantum Statistical Mechanics.* Volume 2. New York: Springer.

Brunetti, R., M. Dütsch, and K. Fredenhagen (2009). "Perturbative Algebraic Quantum Field Theory and the Renormalization Groups." *Advances in Theoretical and Mathematical Physics*, *13*(5), 1541–1599.

Brunetti, R. and K. Fredenhagen (2009). "Quantum Field Theory on Curved Backgrounds." *Lecture Notes in Physics*, *786*, 129–155.

Buchholz, D. (1996a). "Phase space properties of local observables and structure of scaling limits." *Annales de l'Institut Henri Poincaré*, *64*(4), 433–459.

Buchholz, D. (1996b). "Quarks, gluons, colour: facts or fiction." *Nuclear Physics B*, *469*, 333–353.

Buchholz, D. and R. Verch (1995). "Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory." *Reviews in Mathematical Physics*, *7*(8), 1195.

Buchholz, D. and R. Verch (1998). "Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory. II. Instructive Examples." *Reviews in Mathematical Physics*, *10*(6), 775–800.

Butterfield, J. and N. Bouatta (2015). "Renormalization for Philosophers." *Metaphysics in Contemporary Physics*. Ed. T. Bigaj and C. Wütrich. Brill.

Clifton, Robert and Hans Halvorson (2001). "Are Rindler Quanta Real?: Inequivalent Particle Concepts in Quantum Field Theory." *British Journal for the Philosophy of Science*, *52*, 417–470.

Costello, K. (2011). *Renormalization and Effective Field Theory*. Providence, RI: American Mathematical Society.

Dixmier, Jacques (1977). *C*-Algebras.* New York: North Holland.

Dubin, D.A., M.A. Hennings, and T.B. Smith (2000). *Mathematical Aspects of Weyl Quantization and Phase*. Singapore: World Scientific Publishing.

Dütsch, M. and K. Fredenhagen (2001). "Perturbative Algebraic Quantum Field Theory, and Deformation Quantization." *Proceedings of the Conference on Mathematical Physics in Mathematics and Physics, Siena*. .

Emch, Gérard (1972). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. New York: Wiley.

Feintzeig, B. (2015). "On broken symmetries and classical systems." *Studies in the History and Philosophy of Modern Physics*, *52, Part B*, 267–273.

Feintzeig, B. (2016). "Unitary Inequivalence in Classical Systems." *Synthese*, *193*(9), 2685–2705.

Feintzeig, B. (2018a). "On the Choice of Algebra for Quantization." *Philosophy of Science*, *85*(1), 102–125.

Feintzeig, B. (2019). "The classical limit of a state on the Weyl algebra." *https://arxiv.org/abs/1711.04357*.

Feintzeig, B., J. Manchak, S. Rosenstock, and J. Weatherall (2019). "Why be regular? Part I." *https://arxiv.org/abs/1805.05568*.

Feintzeig, B. and J. Weatherall (2019). "Why be regular? Part II." *https://arxiv.org/abs/1805.05567*.

Feintzeig, Benjamin (2017). "On Theory Construction in Physics: Continuity from Classical to Quantum." *Erkenntnis*, *forthcoming*.

Feintzeig, Benjamin (2018b). "Toward an Understanding of Parochial Observables." *British Journal for the Philosophy of Science*, *69*(1), 161–191.

Fell, G. and R. Doran (1988). *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles*. Boston: Academic Press.

Fletcher, S. (2018). "On the Reduction of General Relativity to Newtonian Gravitation." *Unpublished*.

Fletcher, S. (2019). "On Representational Capacities, with an Application to General Relativity.." *Unpublished Manuscript.*

Franklin, Alexander (2018). "On the Renormalization Group Explanation of Universality." *Philosophy of Science*, *forthcoming*.

Fraser, D. (2009). "Quantum Field Theory: Underdetermination, Inconsistency, and Idealization." *Philosophy of Science*, *76*, 536–567.

Fraser, D. (2011). "How to take particle physics seriously: A further defence of axiomatic quantum field theory." *Studies in the History and Philosophy of Modern Physics*, *42*, 126–135.

Fraser, J. (2018a). "The Real Problem with Perturbative Quantum Field Theory." *British Journal for Philosophy of Science*, forthcoming.

Fraser, J. (2018b). "Renormalization and the Formulation of Scientific Realism." *Philosophy of Science*, forthcoming.

Fredenhagen, K. and K. Rejzner (2013). "Batalin-Vilkovisky Formalism in Perturbative Algebraic Quantum Field Theory." *Communications in Mathematical Physics*, *317*(3), 697–725.

Fredenhagen, K. and K. Rejzner (2015). "Perturbative Construction of Models in Algebraic Quantum Field Theory." *Advances in Algebraic Quantum Field Theory.* . Springer, 31–74.

Gamelin, T. (1969). *Uniform Algebras.* Englewood Cliffs, NJ: Prentice Hall.

Haag, R. and D. Kastler (1964). "An Algebraic Approach to Quantum Field Theory." *Journal of Mathematical Physics*, *5*, 848–861.

Haag, Rudolf (1992). *Local Quantum Physics.* Berlin: Springer.

Hairer, M. (2015). "Introduction to regularity structures." *Brazilian Journal of Probability and Statistics*, *29*(2), 175–210.

Halvorson, Hans (2006). "Algebraic Quantum Field Theory." *Handbook of the Philosophy of Physics*. Ed. J. Butterfield and J. Earman. New York: North Holland, 731–864.

Hewitt, E. (1953). "Linear Functionals on Almost Periodic Functions." *American Mathematical Society*, *74*(2), 303–322.

Kadanoff, L. (1966). "Scaling Laws for Ising Models Near $T_c$." *Physics*, *2*(6), 263–272.

Kadison, Richard and John Ringrose (1997). *Fundamentals of the Theory of Operator Algebras*. Providence, RI: American Mathematical Society.

Koberinski, A. (2016). "Reconciling axiomatic quantum field theory with cutoff-dependent particle physics." *Unpublished*.

Kontsevich, M. (2003). "Deformation Quantization of Poisson Manifolds." *Letters in Mathematical Physics*, *66*(3), 157–216.

Landsman, N. P. (2017). *Foundations of Quantum Theory: From Classical Concepts to Operator Algebras*. Springer.

Landsman, Nicolaas P. (1998). *Mathematical Topics Between Classical and Quantum Mechanics*. New York: Springer.

Landsman, Nicolaas P. (2006). "Between Classical and Quantum." *Handbook of the Philosophy of Physics*. Ed. J. Butterfield and J. Earman. New York: North Holland, 417–553.

Li, B. (2013). "Interpretive Strategies for Deductively Insecure Theories: The Case of Early Quantum Electrodynamics." *Studies in the History and Philosophy of Modern Physics*, *44*(4), 395–403.

Li, B. (2015). "Coarse-Graining as a Route to Microscopic Physics: The Renormalization Group in Quantum Field Theory." *Philosophy of Science*, *82*(5), 1211–1223.

Lupher, T. *The Philosophical Significance of Unitarily Inequivalent Representations in Quantum Field Theory*. PhD thesis, University of Texas, (2008).

Manuceau, J., M. Sirugue, D. Testard, and A. Verbeure (1974). "The Smallest C*-algebra for the Canonical Commutation Relations." *Communications in Mathematical Physics*, *32*, 231–243.

Miller, M. (2017). "Mathematical Structure and Empirical Content." *http://philsci-archive.pitt.edu/12678/*.

Nickles, T. (1975). "Two Concepts of Intertheoretic Reduction." *Journal of Philosophy*, *70*, 181–201.

Norton, J. (2018). "The Hole Argument." *The Stanford Encyclopedia of Philosophy*. Ed. Edward N. Zalta. Summer 2018 edition. Metaphysics Research Lab, Stanford University.

Petz, Dénes (1990). *An Invitation to the Algebra of Canonical Commutation Relations*. Leuven: Leuven University Press.

Polchinski, J. (1984). "Renormalization and Effective Lagrangians." *Nuclear Physics B*, *231*, 269–295.

Redhead, M. (2003). "The interpretation of gauge symmetry." *Symmetries in Physics: Philosophical Reactions*. Ed. K. Brading and E. Castellani. Cambridge University Press, 125–139.

Reed, Michael and Barry Simon (1980). *Functional Analysis*. New York: Academic Press.

Rejzner, K. (2016). *Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians*. New York: Springer.

Rieffel, M. (1994). "Quantization and C*-Algebras." *Contemporary Mathematics*, *167*, 67–97.

Robinson, D. (1966). "Algebraic Aspects of Relativistic Quantum Field Theory." *Axiomatic Field Theory*. Ed. M. Chretien and S. Deser. New York: Gordon and Breach, 389–509.

Rohrlich, F. (1989). "The logic of reduction: The case of gravitation." *Foundations of Physics*, *19*(10), 1151–1170.

Rosaler, J. (2015a). ""Formal" Versus "Empirical" Approaches to Quantum-Classical Reduction." *Topoi*, *34*(2), 325–338.

Rosaler, J. (2015b). “Local reduction in physics.” *Studies in the History and Philosophy of Modern Physics*, *50*, 54–69.

Rosaler, J. (2018). “Generalized Ehrenfest Relations, Deformation Quantization, and the Geometry of Inter-model Reduction.” *Foundations of Physics*, *48*, 355–385.

Rudin, W. (1962). *Fourier Analysis on Groups*. Wiley & Sons, Inc.

Ruetsche, Laura (2002). “Interpreting Quantum Field Theory.” *Philosophy of Science*, *69*(2), 348–378.

Ruetsche, Laura (2003). “A matter of degree: Putting unitary inequivalence to work.” *Philosophy of Science*, *70*(5), 1329–1342.

Ruetsche, Laura (2006). “Johnny’s So Long at the Ferromagnet.” *Philosophy of Science*, *73*(5), 473–486.

Ruetsche, Laura (2011). *Interpreting Quantum Theories*. New York: Oxford University Press.

Sakai, Shôichirô (1971). *C*-algebras and W*-algebras*. New York: Springer.

Segal, I. (1959). “The Mathematical Meaning of Operationalism in Quantum Mechanics.” *The Axiomatic Method*. Ed. L. Henkin, P. Suppes, and A. Tarski. Amsterdam: North Holland, 341–352.

Takesaki, M. (1979). *Theory of Operator Algebras*. New York: Springer.

Waldmann, S. (2005). “States and representations in deformation quantization.” *Reviews in Mathematical Physics*, *17*(1), 15–75.

Waldmann, S. (2015). “Recent developments in deformation quantization.” *Proceedings of the Regensburg Conference on Quantum Mathematical Physics*. .

Wallace, D. (2006). “In Defence of Naiveté: The Conceptual Status of Lagrangian Quantum Field Theory.” *Synthese*, *151*(1), 33–80.

Wallace, D. (2011). "Taking particle physics seriously: A critique of the algebraic approach to quantum field theory." *Studies in the History and Philosophy of Modern Physics*, *42*(2), 116–125.

Weatherall, J. (2016). "Regarding the Hole Argument." *British Journal for Philosophy of Science*, forthcoming.

Weatherall, J. (2017). "Categories and the Foundations of Classical Field Theories." *Categories for the Working Philosopher*. Ed. E. Landry. forthcoming.

Williams, Porter (2018). "Scientific Realism Made Effective." *British Journal for Philosophy of Science, forthcoming.*

Wilson, K. (1971a). "Renormalization Group and Critical Phenomena. I. Renormalization Group and Kadanoff Scaling Picture." *Physical Review B*, *4*(9), 3174–3183.

Wilson, K. (1971b). "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior." *Physical Review B*, *4*(9), 3184–3205.

Wilson, K. and J. Kogut (1974). "The Renormalization Group and the $\epsilon$ Expansion." *Physics Reports*, *12*(2), 75–200.