A PHILOSOPHER LOOKS AT NON-COMMUTATIVE GEOMETRY

NICK HUGGETT

This paper introduces some basic ideas and formalism of physics in non-commutative geometry. It is a draft (written back in 2011) of a chapter of *Out of Nowhere*, a book on quantum gravity that I am co-authoring with Christian Wüthrich.[1] Although it has long been suggested that quantizing gravity – imposing canonical commutations in some way – will lead to the coordinate commutation relations of non-commutative geometry, there is no known formal requirement that this be so. Nevertheless, such relations do show up in theories of quantum gravity, for instance as the result of a possible Planck scale non-locality in the interactions of the D-branes of string theory.

However, our book project has moved on somewhat, leaving this work behind. I am making this draft available because various people have started to get interested in the topic, and I wanted to share some of what I have managed to learn, in case it is useful. Since completing it I have had the opportunity to talk with physicists in the field, and have learned more and developed my thinking further.[2] One day (hopefully soon) I plan to return to this work, and update it (and correct some infelicities); while I am willing to stand by this version, I would certainly appreciate any feedback.

I wrote the following to convey the basic ideas without assuming a substantial technical background: it should be accessible to general philosophers of physics, or anyone with a grasp of linear algebra, differential geometry, and the action principle. To that end §1 outlines the formal ideas in an easy going, conceptual way, with a minimum of formalism; this treatment is repeated with more technical detail and precision (and assuming more background) in the Appendix (§4), for those wanting a deeper understanding of the subject matter. Even this treatment is of course nothing more than a survey of some basic concepts.

§2 uses the material of §1 to argue that the subject matter of non-commutative geometry is less than full spatial, so that physical space would have to be a derived structure if the geometry of fundamental physics (of quantum gravity, for instance) were non-commutative. §3 then asks how such a 'non-commutative field theory' should be interpreted if not as a theory of spacetime physics. In particular, I explain how space may be derived, illuminating (I claim) the 'emergence' of space from the non-spatial; in particular, I identify an

---

Date: December 10, 2018

Contact: huggett@uic.edu

.

[1]I thank Jonathan Bain, Sean Carroll, and Michael Douglas for answering my questions on the subject; John Dougherty and Chris Wüthrich for comments on drafts; and several audiences for valuable feedback.

[2]I particularly thank Fedele Lizzi for his patience, encouragement, and helpful survey (Lizzi (2009)). I have also had a number of interesting discussion on this and related topics with Tushar Menon.

interpretational ansatz that must be appended to the theory in order to identify spatial structures.

My goals are thus three-fold: first to introduce the basic formal and conceptual ideas of non-commutative geometry, and second to raise and address some philosophical questions about it. Third, more generally to illuminate the point that deriving spacetime from a more fundamental theory requires discovering new modes of 'physically salient' derivation. This idea is raised in Huggett and Wüthrich (2013); the ansatz of Chaichian et al, identifying structures defined in terms of non-commutative objects as structures of commutative space, and discussed below is an example. The empirical success of a theory of non-commutative geometry would be evidence for the theory and the ansatz.

1. A pedestrian outline of non-commutative geometry

Consider a manifold of points, p, for instance the (2-dimensional) plane or cylinder. Defined on them are the scalar fields, differentiable functions that assign a real number to each point: the set of such functions is known as C^∞. To understand the following, it is important to distinguish between such fields, which are functions over all space, from their values at a point: the former are complete 'configurations' of individual point-values. The usual notation for fields, e.g. $\Psi(x)$, tempts a conflation here, as the argument could be read as a particular value; but it instead indicates that we have a function over (co-ordinatized) points. Since that is understood, we might leave out the argument, and just denote the field (the function, the configuration) as Ψ. With respect to the set C^∞, the fields are its elements, and their identities depend on the point-values (two fields are the same field, iff all their point values agree).

Two scalar fields, ϕ and $\psi \in C^\infty$, can be multiplied together in an obvious way to obtain a third, $\chi \in C^\infty$ – the value of χ at any point p, is just the ordinary product of the values of ϕ and ψ at p: $\chi(p) = \phi(p) \cdot \psi(p)$. Such 'pointwise' multiplication of fields is in fact so obvious as to almost be invisible: how could there be an alternative? Well, we shall see that there are alternative rules for multiplying fields, and they may even be more physical than pointwise multiplication (what that means is a major topic of this chapter).

Because ordinary multiplication is 'commutative', $a \cdot b = b \cdot a$, so is pointwise multiplication for elements of C^∞: $\phi \cdot \psi = \psi \cdot \phi$. Nonetheless, the algebra of C^∞ contains a great deal of information about the space on which the fields live. For instance, other fields can be decomposed into a weighted sum of periodic waves (their Fourier decomposition): in the plane such waves can have any wavelength, so they are uncountable; while on a cylinder the waves must complete a whole number of periods around the circumference (to avoid jumps), and so are countable. Knowing whether wave decompositions are countable or uncountable tells you about the topology of space.[3]

[3]You might wonder where pointwise multiplication is involved in this picture, for it seems that one is adding not multiplying: for instance, a field might decompose as $\sum_n \alpha_n \cdot \chi_n$ where the χ_n are the waves of different periods, and α_n the weight of each in the decomposition. Well, first the algebra does have addition as well as multiplication; but we don't emphasize this, because addition remains commutative in what follows. Second, there is a hidden pointwise field multiplication in the decomposition, because the weights are themselves fields: $\alpha_n(x)$ is the field which has the same value α_n at every point.

In fact, the C^∞ algebra contains *all* the information that we typically take to characterize a (bare) manifold: not its metrical, distance properties (or even its affine, straightness properties), but all the weaker, 'differential' structure, including the topology.[4] In short, there is a representation theorem, that states the logical equivalence of a manifold's topology and differential structure, and its C^∞ algebra.

It's worth emphasizing the strength of this point, by reflecting on what is meant by an 'algebra': nothing but a pattern of relations – a structure – with respect to some abstract operations. One might, for instance fully characterize an algebra by saying that there are two elements, $\{a, b\}$, and an operation $\circ$, such that $a \circ a = b \circ b = b$ and $a \circ b = b \circ a = a$ (and specifying that the operation is associative). What the elements are is not relevant, neither is the meaning of $\circ$; all that matters is how many elements, and what function on pairs of elements $\circ$ is. Of course, an algebra can have different concrete *representations*: concretely, a might be represented by the set of true propositions, and b by the set of false propositions, in which case $\circ$ is represented by the boolean not-biconditional connective. But there are other representations: addition mod-2 for instance (and perhaps a could be represented by the presence of a 30kg hemisphere of uranium 235, b by the absence, and $\circ$ by the operation of putting together – the critical mass of U^{235} is 52kg!). These are different representations of a *single* algebra, which captures their common structure.

And the same point applies to the representation theorem. It is not relevant that the concrete elements of C^∞ are fields over the manifold, all that need be specified are their relations with respect to a binary operation. However, the scalar fields on a particular manifold define a specific C^∞ algebra, and, according to the representation theorem, no other manifold has scalar fields with the same algebra. The point is that the algebra does all the work: there is nothing smuggled in about the manifold simply because we realize the algebra with fields over it.

Moreover, the standard framework of differential geometry – the mathematics of classical spacetime theories – can be developed in parallel, in an algebraic form. For instance, the operation of differentiation is characterized (in part) by the Leibniz, or product rule:

$$\frac{\mathrm{d}\{f(x) \cdot g(x)\}}{\mathrm{d}x} = f(x) \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x} + \frac{\mathrm{d}f(x)}{\mathrm{d}x} \cdot g(x). \tag{1}$$

In ordinary differential geometry, derivatives simply are functions from scalar fields to scalar fields, that respect the Leibniz rule (and a couple of other conditions). But that characterization is essentially algebraic: derivatives, ∇, are functions that map C^∞ to itself ('*automorphisms*'), and satisfy the Leibniz law: for any $\phi, \psi \in C^\infty$, $\nabla(\phi \cdot \psi) = \phi \cdot \nabla\psi + \nabla\phi \cdot \psi$. But derivatives are identified with vector fields (indicating the direction in which the derivative is taken), which in turn allow the definition of covector fields, tensors, metric, and all the machinery of differential geometry (including fibre bundles over the manifold): all of which can be taken over, and expressed in purely algebraic terms. Indeed, the power of the frame work was exploited in (Geroch (1972)), which demonstrated that GR could be recapitulated in terms of the C^∞ algebra.

[4]See the Appendix for a more precise statement.

Because standard geometries can thus be given an equivalent formulation in terms of a *commutative* geometry, they are also 'commutative geometries'. Writing, in the usual way, $[\phi, \psi] \equiv \phi \cdot \psi - \psi \cdot \phi$, we have $[\phi, \psi] = 0$. For a mathematician, always looking to expand the frontiers of Platonic heaven, such a situation leads inevitably to the question of what would happen if the algebra was replaced by some kind of non-commutative algebra, in which fields do not commute: $[\phi, \psi] \neq 0$.

Of course, in that case we do not have a representation of the algebra by scalar fields over a manifold, and pointwise multiplication: because pointwise multiplication is commutative. But do we still have algebraic versions of geometric objects? To what extent? It's easy to see that much geometric structure remains in algebraic form: for instance, the Leibniz law still makes perfect sense when fields do not commute[5], so automorphisms on the non-commutative algebra that satisfy the law, are identified as derivatives. And so on in parallel for all the familiar objects.

For example, in the simple case of the ordinary plane, the two co-ordinates commute, $[x, y] = 0$, and hence so do all fields that are a product of xs and ys: $[x^m y^n, x^{m'} y^{n'}] = 0$. (And hence so do all fields that are a weighted sum of such products: such polynomials in x and y are a very large class of differentiable functions.) One way to specify a non-commutative geometry is to take two base elements, $\hat{x}$ and $\hat{y}$, specify $[\hat{x}, \hat{y}] = i\theta$ for some constant, and take the algebra of polynomials, which is known as $\mathcal{R}^d_\theta$. For instance, because we have $\hat{x}\hat{y} = \theta + \hat{y}\hat{x}$, it is easy to verify that $[\hat{x}^2, \hat{y}] = \hat{x}\hat{x}\hat{y} - \hat{y}\hat{x}\hat{x} = i\theta(1 + \hat{x}) \neq 0$. Moreover, derivatives satisfying the Leibniz law have the algebraic form $\nabla_x \phi \equiv \theta^{-1}[\phi, i\hat{y}]$ (it's easy to check, for instance, that $\nabla_x \hat{x} = 1$ and $\nabla_x \hat{y} = 0$). The example is explained in more detail in the appendix. Obviously, because the algebra is non-commutative, but we still have geometrical structure (if expressed algebraically), one speaks of a 'non-commutative geometry'.

All of this might be of only abstract interest, if it weren't the case that the framework of modern physics, also survives the transition to non-commutativity. The geometry and calculus involved, we have already seen, can be given algebraic form, and survive. Specifically, the Lagrangian that characterizes any theory, can be fully rendered in algebraic terms, even in the non-commutative case: so we have physics in a non-commutative geometry, specifically, 'non-commutative field theory' (though note carefully that it is the geometry that the fields 'inhabit' that is non-commutative, as the fields themselves have not been quantized at this stage). Moreover, other important pillars of modern physics, especially Noether's Theorem, also survive (it only requires that the algebra be associative): hence the central importance of conserved currents remains. (And gauge fields also exist, though importantly the distinction between 'internal' gauge symmetries, and 'external' spatial symmetries is blurred.)

And so we have a promising model of how spacetime might emerge from a theory of non-spatial degrees of freedom. On the one hand, prima facie we have no spatial degrees of freedom: no point-valued fields, but instead the state-values are simply elements of the algebra $\mathcal{R}^d_\theta$. While on the other, the theory apparently contains enough structure that one

[5] Once we specify the order of multiplication.

successfully might connect it (in some kind of limit) with familiar physics, in a classical, phenomenal, spacetime. Of course, the situation raises many questions: how should the theory be interpreted? Is the algebra strictly abstract, or is there some non-spatial concrete representation? Does that distinction make sense? Is there, after all, a different spatial representation? If so, does that mean we have a theory of a classical spacetime after all? How, formally, are spatiotemporal quantities derived? Are such derivations 'physically salient', in the sense we have discussed? These are the questions to which the rest of the chapter is addressed.

2. (How) is NCG 'non-spatial'?

Having seen some of the basic ideas of NCG we turn to the question of the emergence of spacetime in the theory. In fact, to keep our feet on firm ground formally speaking, we will continue to focus on the case in which only spatial dimensions are non-commuting, and ask what that means.[6] (Let me note that in this section we consider *classical* NCG; we turn to the quantum version below.)

The first point is the rather obvious one that NCG – specifically, θ-space ($\mathcal{R}^d_\theta$) – is not non-spatial in the fullest sense. After all, the basic quantities of the theory – the non-commuting coordinates, the $\hat{x}^i$s – are intended to somehow represent familiar spatial quantities. (Coordinates aren't the most basic kind of spatial quantities, but they are functions of more basic measurable quantities; we can give the coordinates of a point in terms of its distance and angle relative to a physical something.) So on the one hand, NCG does not exemplify the purest kind of emergence that we are looking for, a theory in which no elements are spatial in any way. On the other, since one can see where to start to find space in the theory, NCG is a useful warm up to purer examples of emergence. But more than that, non-commutivity makes NCG profoundly different from the ordinary conception of geometry – hence its algebraic formulation. Thus space in the ordinary conception is emergent, even if the formulation of NCG shows the way to the spatial. However one wants to think about space, we are all agreed that there is an important, perplexing gap between commutative and non-commutative 'geometries': so, without making any substantive claims, let us use the word in the ordinary, *commutative* sense.

The starting point for the interpretive work here will be the algebraic theory presented: we will take it that it somehow captures the structure, ontology and (given a lagrangian) laws of a world, and ask how best to think about such a world, especially in relation to the way we think about space (in the ordinary sense, remember) and spatial theories of the world. There is an interesting antecedent of this work, which is worth noting, based on the fact that one can formulate commutative geometries algebraically, as we saw in the previous section.

John Earman (Earman (1989), §9.9) proposed using such a formulation to advance the substantivalist-antisubstantivalist debate in the foundations of spacetime.[7] (Broadly,

---

[6]Non-commuting time seems to lead to pathological theories. See Huggett et al. (2012) for a discussion.

[7]Note that Earman's presentation neglects to mention that the representation theorems require that an (algebraic) metric be given.

substantivalists takes the manifold to have 'independent reality'; one alternative is the view that the manifold merely *represents* the spatial relations of matter.) In particular, he was considering the various issues that come up because of symmetries that allow the contents of spacetime to be 'rearranged' without any observable difference: Leibniz argued that God could not choose between this universe and its 180^o rotation in absolute space; Einstein worried that the symmetries of GTR would lead to indeterminism (the 'hole argument'). Formally, such symmetries are realised by 'diffeomorphisms', nice maps of the manifold onto itself, which say how a world should be rearranged in spacetime. But an Einstein algebra fixes a spacetime only up to diffeomorphism, so it seems that the symmetric situations get the very same algebraic descriptions – Earman suggested that an interpretation that takes the algebra as fundamental would thus avoid such problems (specifically, Einstein's).

But what is the interpretation of such a theory? Earman described it as substantival 'at a deeper level'; but if substantivalism is something like the doctrine of the reality of spacetime points this is a perplexing remark. In the algebraic formulation spacetime points can be thought of as sets of fields: specifically, a point p is a 'maximal ideal' (in the Weyl representation of the algebra discussed just below, this is the set of functions that vanish at p). So points are not at a 'deep' level at all, but arise from something deeper, the elements of the algebra. Indeed, that claim is the stated conclusion of the work (Geroch (1972)) which Earman takes as a starting point. I suspect that what Earman had in mind was not really substantivalism in that sense, but rather the view that in such a formulation the geometrical structure of spacetime was given independently of the *spatial* relations of material systems – not so much substantivalism, as anti-relationism. After all, it is fixed instead by the *algebraic* structure of the fields, including the algebraic metric, in the way described.

I shall have more to say about the interpretation of such a theory below; let's just note that the case in front of us takes matters one step further, not just to an algebraic formulation, but to a non-commutative one. For now I want to discuss an important reformulation of NCG – in terms of a commutative space!

It turns out that there is a map, the 'Weyl transformation', from the θ-space field algebra, $\mathcal{R}^d_\theta$, to that of smooth functions on a commutative space $\mathcal{R}^d$ (and vice versa) – figure (1).[8] This fact greatly facilitates extracting the physical consequences of the theory because the usual methods of the calculus (and hence of standard field theory, including QFT) can then be applied. Not surprisingly, as we shall see, one of the main ways of exploring the empirical consequences of a NCFT are through its Weyl transform. I will not give details of the transformation, but rather just give some of the more important features, the first of which is that $\hat{x} \to x$ (and similarly for the other coordinates); there are some more details in the technical section, but for a clear, concise but more complete account of the map see (Szabo (2003)).

[8]Here we continue to assume flat, infinite space, but the claim also holds for more complex cases. We also restrict attention to fields that vanish smoothly at infinity (so that the 'physicists fundamental theorem of the calculus' reads $\int \mathrm{d}x\ \mathrm{d}f(x)/\mathrm{d}x = 0$). This restriction is a common but notable assumption in physics: on the one hand it is justified locally by the assumption that arbitrarily distant differences are irrelevant; on the other hand it raises questions about the universality of physical theories.

$$C^\infty \rightarrow \mathcal{R}^d_\theta \leftrightarrow_{Weyl} C^\infty$$

FIGURE 1. A schematic to help keep track of the moves being made. First, we consider the algebraic formulation of differential geometry, based on the algebra of smooth fields, C^∞; this we generalize to a formulation in terms of a non-commutative 'fields' algebra $\mathcal{R}^d_\theta$ (or more generally, as in the technical appendix, $\mathcal{A}$); the elements of this algebra can be mapped back into commutative fields, by the Weyl transform – with multiplication according to a non-commutative product, the 'Moyal star'.

Given this equivalence, one may wonder whether NCG isn't, after all, a theory of perfectly ordinary space, and hence one from which space cannot emerge. However, if we look a little closer at the Weyl transformation, we shall see that such a conclusion would be too hasty. For we haven't yet addressed the fact that while $\mathcal{R}^d_\theta$ is non-commutative, the smooth fields on space commute with respect to ordinary (pointwise) multiplication: $\phi(x) \cdot \psi(x) = \psi(x) \cdot \phi(x)$. Therefore, to preserve the algebra, products of Weyl transforms are not formed using ordinary multiplication, but using a new 'multiplication operation' known as the 'Moyal star product':

$$\phi(x) \star \psi(x) = \phi(x) \cdot \psi(x) + \sum_{n=1}^{\infty} (\frac{i}{2})^n \frac{1}{n!} \theta^{i_1 j_1} \ldots \theta^{i_n j_n} \partial_{i_1} \ldots \partial_{i_n} \phi(x) \cdot \partial_{j_1} \ldots \partial_{j_n} \psi(x). \tag{2}$$

Clearly the $\star$-product contains new terms in addition to ordinary multiplication. (2) is given here for completeness: the details aren't really important for the non-technical reader. One just needs to observe that the new terms form an (infinite) sum of derivatives with respect to the coordinates, weighted by the elements of θ. For instance,

$$\hat{x}\hat{y} \rightarrow x \star y = xy + \frac{i}{2}\theta^{xy} \neq xy. \tag{3}$$

That is, the Weyl transform of $\hat{x}\hat{y}$ is $x \star y$, *not* the ordinary product of the commuting coordinates, but their Moyal product. The very same point holds for the other elements of $\mathcal{R}^d_\theta$ – *the Weyl transforms have the algebraic relations of* $\mathcal{R}^d_\theta$ *with respect to Moyal multiplication.* Hence equation (2) allows the computation of the transform of any element of the algebra, since all are sums of products of the coordinates.

We can use (3) to verify the relevant commutator in the Weyl transform:

$$[x, y]_\star \equiv x \star y - y \star x = xy - yx + (\frac{i}{2})\theta^{xy} - (\frac{i}{2})\theta^{yx} = i\theta^{xy}, \tag{4}$$

by the antisymmetry of θ^{xy}. Thus commuting coordinates do indeed have the algebra of the non-commutative theory, with respect to the Moyal product.

So working in the Weyl transformed theory involves multiplying physical quantities, not in the usual way, but with the $\star$-product: an area is $x \star y$ not $x \cdot y$; fields given as series expansions of the x^is are to be understood in terms of expansions using the $\star$-product (for instance, exponentials); and terms in the equations of motion involve $\star$-multiplication. In other words, for every physical purpose, the $\star$-product is the relevant operation, and ordinary multiplication is only relevant insofar as it is involved in the definition (2) of 'real' multiplication. In yet other words, the physical facts don't care about the commutivity of space, and it is thus natural to see it just as a convenient way of representing, the real non-commutative nature of space.

To sharpen this point up, note that among the predictions of NCG is the violation of Lorentz invariance (e.g., Carroll et al. (2001); Anisimov et al. (2001)); again, θ^{ij} is an invariant area, so such violations are to be expected. Now, the commutative spacetime in which our Weyl transformed theory lives is just Minkowski spacetime. But the Lorentz invariance of a theory simply means that it can be formulated in terms of Minkowski geometry alone. Put the other way around, the violation of Lorentz invariance by the Weyl transform of NCFT means that it *cannot* be formulated in terms of Minkowski geometry alone; of course that's because the $\star$-product involves invariant areas, which cannot be defined in terms of Minkowski geometry.[9] Thus, despite its formulation in Minkowski spacetime, NCG is not properly thought of as describing physics of that spacetime, since the physics of NCG simply doesn't care about the commutative geometry in the right way. One of the clear lessons of the development of relativity (and spacetime theories more generally) is that geometric structure should respect the laws.

It may sound paradoxical that a theory can at once be formulated in a geometry but not be a theory of physics in that geometry, but it is not; Minkowski spacetime merely gives a way of formally *representing* NCG, in conjunction with additional, non-fundamental structure. In that case, we should start with the algebraic formulation when interpreting the theory, to avoid confusing merely representational structure from the real commitments of the theory.

It's worth stressing at this point that elements of the algebra are represented by *functions* over the commuting space, $\phi(x)$ – the value at any point, or over some proper subspace does not correspond to anything in the algebraic formulation, so to anything fundamental. Normally we think of the value of a field at a given point as conveying some physical meaning, such as the electric field strength, but in the Weyl transform, this is not the case: again, only the full function corresponds to anything fundamental. Obviously this situation presents a puzzle, for most familiar physical quantities are associated with points and finite regions. We turn to this puzzle below.

The Weyl transform of a NCFT is formally a 'representation' of it, in the sense of consisting of objects faithfully obeying the algebra of the NCFT. So from now on I will generally talk of the 'Weyl representation' of a NCFT instead of its 'Weyl transform'. I have avoided this use up to now to avoid equivocation in the argument just given: that the Weyl transform is a formal representation of $\mathcal{R}^d_\theta$ is a mathematical result, but that it is *nothing but* such a representation, and not a faithful image of the theory's ontology

[9]For a detailed exploration of this issue see Chaichian et al. (2004).

(because it introduces a non-fundamental, commuting spacetime) is something argued for by the preceding.

(Parenthetically I would add that even if my argument failed, NCG would still be interesting to our current project. Even if the Weyl transform was plausible as a fundamental formulation, it would still be the case that the algebraic formulation was a coherent, interesting theory without standard space. So it would still be well-worth studying to see what it shows about emergence. Because of non-commutivity, it is a richer subject of investigation than the commutative algebras discussed by Earman – and even they are interesting to consider.)

Finally, I want to note that there is another technique, using a commutative field theory, that can be used to extract predictions from NCFT (indeed, this is the method used in the papers cited above showing violations of Lorentz invariance). This technique is based on the Seiberg-Witten correspondence, between non-commutative gauge theories into ordinary gauge theories (Seiberg and Witten (1999), §3). I will not discuss this map at length because it is not one that preserves full physical equivalence; some solutions that are non-singular on one side are singular on the other. For example, see (Douglas and Nekrasov (2001), §II.D.5 and §III.D). Thus the correspondence allows one to derive perturbative physics, but cannot be relied on non-perturbatively.[10]

3. Interpretation

The work on understanding NCG, and how (commutative) space 'emerges' from it will proceed in two directions. First, we will work 'down', asking what kind of world, what kind of ontology, could be described by the formalism. Second, we will work 'up' from the theory to phenomena, investigating the 'empirical content' of the theory, how spatial phenomena are derived from it, how they are represented within the theory. The first kind of work focusses on what a non-spatial metaphysics might be like, to get clear on the two sides of the relation; the second kind addresses the relation of 'emergence' between the two sides.

3.1. **Ontology.** In Book IV, Chapter 1 of his *Physics* Aristotle offers the view that existence requires being somewhere: everything that is, has a place. (He is setting up the question, attributed to Zeno, of where places are, if they exist.) This idea is intuitive: the world seems fundamentally spatial, and it starts to capture the idea that 'real' things can be interacted with, by traveling to them. If one accepts such a view of existence, then it becomes impossible to take the algebraic formulation of the theory as giving an ontology for the world, because it does away with space as a fundamental object. (Ok, in principle one could allow Aristotle non-commutative or emergent spaces, but let's say that's against the intuition.) But of course, since you are reading this book, you know better: philosophers are more likely to take a logical view of existence, perhaps adopting Quine's view that to be is just to be the value of quantifier variables in a true theory. At any rate, it is unlikely that you will reject the algebraic formulation out of hand because

[10] Thanks to Michael R. Douglas for some discussion of this point.

of a view of existence. On the other hand, space seemingly provides a very useful handle for investigating ontology, insofar as individuals can be separately localized, and the parts of space give a way of distinguishing parts of individuals. (Of course, even non-relativistic quantum non-locality makes this road to ontology treacherous at best.) Or again, *physical* reality is often tied to causal connectedness, which in turn is most readily understood in terms of effects propagating in space. So despite our metaphysical sophistication it is still puzzling to know where to start in talking about the ontology of a theory like NCG, in which the familiar spatial handles are missing. What, then is there? How can we discern a coherent ontology from the theory?

Related to the idea that ties existence to space, is the idea that comprehension requires the spatial. In the introduction we addressed Maudlin's concern that theories without fundamental spatiotemporal quantities could not be connected to experiments, which immediately concern local beables. But a vaguer objection that only spatiotemporal theories can be properly 'understood' perhaps remains; useful predictions might be possible, but otherwise a theory can only be an instrument, not comprehendible by us. Kant had a view like this of course, which influenced Maxwell in the construction of electromagnetism; and such claims were made by Schrödinger in his arguments with Heisenberg. See (de Regt (2001)) for some history of the topic. But, like the spatial view of ontology, it does not pose any obstacles here. First, one can make the same kind of move to logic as before. Suppose a theory is algebraic rather than geometric, then it may not give a 'picture' that is easily visualized by the human mind, but it still can provide understanding in the sense of systematizing the connections between different parts of nature; between the quantum and gravitational realms, ideally. That is, our ability to understand formal systems that aren't spatial does give us the ability to understand non-spatial physical systems. Second, there is a sense of 'understanding' that indicates facility with a theory rather than seeing the bigger picture. For instance, (de Regt (2001)) develops Feynman's view that understanding a formalism is a matter of seeing what the solution to a problem will be without having to compute it explicitly. But as de Regt points out, while our geometric intuitions are a fruitful resource for 'seeing solutions', they need not be the only one; again, familiarity with an algebra also allows one to anticipate when algebraic relations hold without explicit calculation. (Moreover, if a theory turned out to be 'incomprehensible' in this second sense of facility of use, that wouldn't seem to be a barrier to the kind of comprehension we're seeking here – the 'what kind of world' sort.)

Since we are considering a theory that replaces differential geometry with algebra it will be useful to bear in mind the kind of interpretational moves made in the former case. There are two main questions at stake: to what extent, if any, are the points of a differential manifold real, physical objects, akin to material systems? What aspects of spatiotemporal structure, such as topological and geometrical relations, are fundamental (capable, for instance, of providing 'deep' explanations)? Especially following the (re-)introduction of the 'hole argument' (Earman and Norton (1987)), the locus of philosophical debate was on the first question: the 'manifold substantivalist', who holds the points to be physically real, is faced with versions of Leibniz's shift arguments, in which one imagines the material content of the universe rearranged in spacetime. Earman and Norton's argument makes the point especially sharp in theories with dynamical geometries, such as GTR, since then the problem of indeterminism can (arguably) be added to that of underdetermination. One kind of response to such arguments is of course to play with the identity conditions of spacetime points, so that points before and after rearrangement can't be identified by anything but their contents, and hence things are in fact just as they started (e.g., Butterfield (1989)).

For better and worse, the hole argument has rather played itself out in recent years, and interest has turned to the second question. The idea of 'dynamical interpretations' of spacetime theories is that certain spatiotemporal structures (particularly affine and metrical ones) are not fundamental, but merely represent, say, the symmetries of the laws of material systems; hence 'real' explanations (of time dilation, for instance) are in terms of how systems behave according to physical laws, not geometry (for instance, (Brown (2005)); I discuss such interpretations in (Huggett (2009)).

In the formalism of NCG, instead of points and their relations, we have elements of an algebra and their relations; this observation suggests that the elements could be thought of, metaphysically, along the lines of points. To pursue this idea more concretely let's take the algebra to be $\mathcal{R}^d_\theta$. 'Algebraic substantivalism' then attributes to the elements of $\mathcal{R}^d_\theta$ the same kind of 'physical reality' that manifold substantivalism attributes to points. To be a little more careful, just as the latter view takes mathematical points to represent, more-or-less literally, physical points, so algebraic substantivalism takes the elements of the mathematical algebra to represent, more-or-less literally, physical objects, which we shall continue to call 'fields'. (Though these are algebraic, not spatial objects; their connection to 'fields' in the ordinary sense is through the Weyl representation, or their place in algebraic geometry. Elsewhere in the chapter, where the distinction is not important, we also call the elements of the mathematical algebra 'fields'.) To be clear, the mathematical representation of the NCG is not itself something physical, but, according to substantivalism, what it represents is.

The idea that the points of a mathematical spacetime manifold could represent points of physical spacetime seems to be a natural one; at least philosophers (including Newton and Leibniz) have taken it (or something like it) to be a view worth defending or disputing. I have observed (in myself and others) that applying parallel reasoning in the parallel case of NCG is feels less natural. However, as far as I can tell the only differences between the two cases lies in the non-spatiality of the fields. But that is no reason to reify in one case and not the other: as far as existence goes, we have already rejected spatiality as a condition. And while non-spatiality makes the fields less immediately connected to objects of experience, we shall see below how (quantum) NCG does connect with experience. In other words, however manifold substantivalism views points, algebraic substantivalism views elements of the algebra; understand one and you understand the other.

I don't intend here to defend any interpretation of NCG, but rather to lay out some options. For example, suppose one has a NCFT of a scalar field: suppose that the dynamical object of the theory is a scalar field, as in electromagnetism one has a theory of an anti-symmetric tensor field. (For instance, suppose the physics is given by (36).) For want of a better term, call this a 'material field', to distinguish it from the fields of the algebraic geometry (though it is only 'material' in the same sense that the electromagnetic is). Then, the (algebraic) fields correspond to the *possible* states of the material field, and can be interpreted to *be* such. This move is reminiscent of the anti-substantivalist proposal that points are merely possible locations, not physical objects themselves; the typical response to this suggestion is that it simply introduces new entities with all the troubling features of points, and so the difference is too small to generate a truly distinct view in the interpretation of spacetime theories. But I will note that in scalar NCFT things are potentially more promising in that no new possibilia are proposed, because the material field already possesses the possible states; the only addition is the algebraic relations of the states. Of course one might say that locations are also possible states of spacetime objects, but in that case there is the option of taking *relative positions* to capture locations. There seems to be no corresponding move for the states of a non-commutative scalar field.[11]

On the other hand, the suggestion does require that the theory contain a scalar field (since elements of $\mathcal{R}^d_\theta$ are scalar), so what is to be said about theories of tensor or gauge fields (as defined in the technical appendix)? Such objects can generally be used to define a scalar (through contraction say), so perhaps one would be willing to take the fields to be possible states of that. (Things would presumably be more complicated if the field algebra were other than $\mathcal{R}^d_\theta$, say that of matrix-valued fields.)

Another option for interpretation is suggested by (**?**, Bain (Unpublished)) in the similar context of the Einstein algebra formulation of GTR. In that case (too) diffeomorphisms correspond to automorphisms of the algebra, and so Leibniz shifts have an algebraic counterpart: there are maps of the algebra onto itself that preserve the algebra, and hence the geometry. As Earman (Earman (1989), 193) points out, there are two distinct issues raised by such shiftiness: first, there are problems associated with underdetermination – why should god create this world rather than that? How can we tell which world do we live in? And so on. Second, in GTR, there is the problem of indeterminism: there are diffeomorphic universes that agree up to a given Cauchy surface, but diverge after (and since they are diffeomorphic both are allowed by the Einstein field equations). Earman is dismissive of the first problem as an unavoidable consequence of non-categoricity. Determinism he claims is essentially a doctrine about ordinary spacetime theories, and that the proper way to pose it for an algebraic formulation is in terms of the *classes* of spacetime representations; for any two algebraic models whose representations, restricted to $t < T$ are the same, then are they the same set of representations. Given the representation theorems about Einstein algebras, the answer is 'yes'. Bain, following (Rynasiewicz (1992)), is unconvinced by this response. He proposes that the fields only have identities in virtue of their algebraic relations to one another – a version of 'structuralism', since the fields become bare relata for the essential, algebraic, content of the theory. In this case, since the structure is preserved by automorphisms, such shifts make no difference; the analogy to similar moves in the hole argument – or in response to other issues arising from shifts

[11](Dieks (2001)) proposes an approach along these lines for spacetime in ordinary QFT, in which points are identified with sets of local observables, themselves taken as possible properties of systems.

– should be clear. One even (in principle) could go a step further, and treat the elements of the algebra as purely formal, not representing any physical 'fields', even with weakened identity conditions. (Bain seems pulled in this direction, but I am not sure whether he would fully embrace it.) What would be left would be pure structure.

It seems that all of these structuralist moves could be applied in the interpretation of NCG; except, one could argue that determinism is an issue, despite Earman's comments, if time commutes with the spatial dimensions, as we have assumed. (I.e., do algebraic worlds that agree up to a time agree after?)

Algebraic substantivalism and structuralism, and the idea that fields are states of a scalar field focus on the ontology of the elements of the algebra, as similar views concern the status of points in standard spacetime theories, but perhaps one could also attempt a 'dynamical' interpretation of NCFT. Such a view would take the 'material' non-commuting field as the fundamental thing, and view the algebra (certainly the algebraic relations, but potentially also its elements) as merely representing something about its equations of motion. How such a view might be developed, how plausible it might be, and how it might relate to the others described here, are questions that will have to be addressed at a later time.

3.2. **Finding Spacetime.** The Weyl transformation provides a way of extracting empirical consequences from NCFT (in what follows I will assume familiarity with the basics of quantum field theory (QFT), as covered in §**??** of our *Primer*). First, one can take the equations of motion for a classical field theory in ordinary commuting space, and rewrite them replacing all ordinary multiplication with Moyal star multiplication: the result is the equations of motion for a NCFT expressed in its Weyl representation. (More precisely, as discussed in the technical section, one rewrites the action in this way.) One obtains a more complex, but otherwise formally standard field theory – though of course non-commutation of the coordinates undermines a straight-forward interpretation. Then standard methods allow one to formulate a 'second quantized' version, a non-commutative QFT; in particular, the machinery of the path integral formalism can be brought to bear. Finally, those standard methods allow the derivation of empirical results, especially probabilities for particles to scatter off one another in different ways – scattering 'cross-sections'.

As we discussed in the *Primer*, a scattering cross-section gives, for fixed incoming particles, the probability density for finding other given particles outgoing at given angles (and energies). We argued there that one can reasonably take a spacetime as given in terms of the spatiotemporal predictions of a QFT. Moreover, although a QFT ought to have other spatiotemporal predictions (concerning bound states, for instance) cross sections do constitute a sufficiently central and rich collection of such predictions to be taken to give spacetime.

In particular, consider a world in which the cross-sections of a NCFT turn out to be correct. To understand the place of phenomenal space in such a NCFT it therefore makes good sense to focus on these cross-sections; there are other empirical aspects, and other ways in which space can be found, but cross-sections exemplify both perfectly. So I will proceed (and indeed speak) as if to understand the meaning of cross-sections in NCFT *is* to understand phenomenal spacetime – more could be said, but we expect it to be more of the same, and not to add to the central issue. That is, understanding the place of cross-sections in NCFT is to understand the place of familiar space in the theory – how it emerges. This question has been addressed by Chaichian, Demichev and Presnajder in a very interesting paper (Chaichian et al. (2000), 8); what follows is based on their analysis, though it suggests a somewhat different solution.

The problem of finding scattering cross-sections can be further reduced to the calculation of '2-point functions': squared, these represent the probability that, left to itself, a quantum at x in space and time would be 'found' at y, the simplest kind of 'scattering'.[12] These, along with interaction terms, are the ingredients of the Feynman method for calculating cross-sections, so they can be taken as giving the empirical spatial content of a QFT – and hence of NCFT. The 2-point functions make the problem of giving a spatial interpretation very clear, for they are functions of x and y, coordinates in phenomenal, commuting spacetime – and so have no immediate significance in NCFT, in which the coordinates cannot be ordinary number-valued, since they don't commute! That is, finally, the question of the meaning of phenomenal cross-sections – so of *space* – in NCFT narrows to the question of the significance of the commuting coordinate arguments of the 2-point functions.

Let's take a closer look at the 2-point functions: as we discussed in the *Primer*, these are the vacuum state expectation values of a product of field operators – equation (??). In the quantization of NCFT it is the Weyl transforms that are second quantized – I will denote the transforms $\phi_W(x)$ and the corresponding quantum operators $\hat{\phi}_W(x)$, so that the corresponding 2-point function, $G_W(x_1, x_2)$, can be written

$$G_W(x_1, x_2) = \langle 0|\hat{\phi}_W(x_1)\hat{\phi}_W(x_2)|0\rangle. \tag{5}$$

In the path integral formalism, such quantities are given by field integrals over the classical fields, weighted by the action:

$$= \int D\phi_W \; \phi_W(x_1)\phi_W(x_2) \cdot e^{i\int d^n x \mathcal{L}(\phi_W(x), \dot{\phi}_W(x))}. \tag{6}$$

This expression makes the interpretational issue very clear, for the dependence on the field at x_1 and x_2 is explicit in the (functional) integral. But the value of the Weyl transform at a point has no fundamental meaning – only the field configuration over the whole space represents anything in the algebraic formulation, namely an element of the algebra. So the same is true of the 2-point function: it can have some significance as a function over $\mathcal{R}^d \times \mathcal{R}^d$, but its point-values do not. But its point-values are exactly what we would like to take as scattering amplitudes, the empirical content of the theory.

A first response would be to more-or-less ignore this situation. One simply takes the coordinates in the Weyl representation to correspond to phenomenal coordinates – the ones by which we label points of phenomenal space. However, while this approach might be expected to produce decent predictions over distances above greater than the $\sqrt{\theta}$s, it

[12] Saying what 'found' means in such contexts is to propose a solution to the measurement problem, something we are trying to sidestep as far as possible.

is conceptually incoherent. At first glance it looks as if it simply throws away the non-commutative spacetime and views the theory as one with unusual equations of motion; if the coordinates are just those of phenomenal commuting space, then we just have a QFT in that space with a standard lagrangian modified by use of the Moyal star. In other words, it looks as if the field simply does not live in a non-commutative space at all, and the question of emergence is moot. But that cannot be quite right, if one investigates a little more carefully, because the Moyal product in the action means – as we saw – that the laws don't properly respect the geometry of the commutative spacetime: θ^{ij} is a non-invariant area; physical quantities are $\star$-multiplied, so co-ordinates don't commute; Lorentz violation and spacelike correlations are to be expected. The fundamental theory doesn't really live in the commutative spacetime.

The second response is that based on the work of Chaichian et al, and is one in which we can (finally!) get a handle on emergent spacetime. To keep things simple, we work in three spatial dimensions, which means that only two can be non-commutative (since non-commutative dimensions can always be organized in pairs), so we can pick coordinates in which the field algebra is defined by a single parameter, θ:

$$[\hat{x}_i, \hat{x}_j] = \begin{cases} i\theta & i = 1, j = 2 \\ 0 & i, j = 3. \end{cases} \tag{7}$$

We will focus on the two non-commuting dimensions and so work in the plane.

As we saw, because of this non-commutivity, the Weyl field operators do not have the usual interpretation as localized quantities, but that doesn't mean that the same is true for other operators in the theory. Indeed, we should expect that some other observables do represent phenomenal quantities.[13] So let's make a guess: any non-commutivity is of order θ, so if we smear the Weyl fields over that scale, we can get commutative fields that should describe physics above that scale – physics insensitive to the non-commutivity. More specifically, let's propose that the following (which simply smears the fields over a gaussian) maps Weyl fields into commuting fields living in phenomenal spacetime (i.e., fields describing scattering phenomena):

$$\phi_P(x) \equiv \int \mathrm{d}^2x' \frac{e^{-(x-x')^2/\theta}}{\pi\theta} \phi_W(x').^{14} \tag{8}$$

This proposal is a 'guess', a hypothetical part of the theory, subject to testing, and potentially to replacement by some other ansatz; but it is based on the most natural way of relating non-commuting and commuting space. (Note too that the form of the

[13]Leaving open, for now that they don't represent exactly in the usual way, via expectation values or the eigenvector-eigenvalue link: perhaps, for instance, only expectation values in a certain sector of the statespace can be thought to correspond to phenomenal spacetime quantities. That doesn't seem to be the case in the proposal pursued here.

[14]Technically, these fields are the 'normal' symbols of the non-commuting fields.

smearing is the simplest, rotationally invariant form one can have.) But we can take the x-arguments of these 'new' fields to be those of phenomenal space: x' is the coordinate of the space in which the Weyl transforms live. That is, (8) can be read as a map from points of phenomenal space and Weyl field configurations (the integral means that the map depends on the full configuration) into the reals: $x \times \phi_W(x') \to \mathcal{R}$. Thus it is part of the proposal here that the point-values of $\phi_P(x)$ have physical meaning: the value of a phenomenal field at that phenomenal point. It is this understanding that solves the problem raised in this section – we will attempt to rationalize it further below.

But given (8) and the interpretation of $\phi_P(x)$, the 2-point function for the phenomenal fields is given by the path integral prescription:

$$G_P(x_1, x_2) \equiv \langle 0|\hat{\phi}_P(x_1)\hat{\phi}_P(x_2)|0\rangle \tag{9}$$

$$= \int D\phi_W \ \phi_P(x_1)\phi_P(x_2) \cdot e^{i\int d^n x \mathcal{L}(\phi_W(x), \dot{\phi}_W(x))} \tag{10}$$

which is simply the smeared version of the Weyl 2-point function:

$$= \int \mathrm{d}^2 x_1' \mathrm{d}^2 x_2' \frac{e^{-(x-x_1')^2/\theta}}{\pi\theta} \cdot \frac{e^{-(x-x_2')^2/\theta}}{\pi\theta} G_P(x_1', x_2'). \tag{11}$$

Note that it is at this point that I diverge from Chaichian et al's proposal. Their idea is that the action in the path integral should be rewritten in terms of the phenomenal field ϕ_P. Their approach amounts to treating the phenomenal field as the true degrees of freedom. Instead, what I suggest is that we treat the Weyl fields as the true degrees of freedom, as we should if we take the non-commuting spacetime seriously: we simply recognize that the canonical degrees of freedom are not those we experience as phenomenal fields – those are represented by ϕ_P. Again, that hypothesis (in conjunction with the rest of the theory) is testable, and links the fundamental theory to experiment.

One could certainly complain at this point that the *theoretical* meaning of the phenomenal field is unclear – the $\phi_P(x)$ can formally be defined according to (8), but can we get a clearer insight into their place in the theory? In particular, do the phenomenal xs have an interpretation in the theory, since they are not the non-commuting coordinates? Since they label points of phenomenal space, an answer will illuminate how phenomenal space is found in a NCFT. Chaichian et al suggest an answer (Chaichian et al. (2000), 8): they note that the phenomenal fields are equal to the expectation values of the non-commuting fields in so-called 'coherent' states, $|\xi_x\rangle$[15]:

$$\phi_P(x) = \frac{\langle \xi_x|\hat{\phi}_W|\xi_x\rangle}{\langle \xi_x|\xi_x\rangle} \tag{12}$$

In the Weyl representation, a coherent state can be thought of as an isotropic state, centered on a point, x; the xs can be taken as their quantum numbers. Then (12) shows how the

[15]Specifically, $|\xi_x\rangle = \exp{(x_1 + ix_2)a^\dagger}|0\rangle$

point-values of the phenomenal field can also be understood as labelled by coherent states, taking the point in phenomenal space to be the corresponding quantum number.[16]

It is interesting to see how the phenomenal fields can be found in the theory, but unfortunately their significance is not particularly clear. In a field theory, the 'degrees of freedom' of the system – the physical magnitudes that characterize it – are the magnitudes of the field at each point: for each x we have a separate value $\phi(x)$, constrained just by the laws of motion. Quantum mechanically then we get a quantum operator for each point of the field, the quantum degrees of freedom. One could think of something like the collection of expectation values for the set of field degrees of freedom – a distinct operator $\phi(x)$ at each point x – for a given, particular state vector, as giving the classical field configuration. But in (12) the picture is reversed. The phenomenal fields are classical field configurations in phenomenal space, but they are understood as the expectation values for a single operator, for a set of state vectors. This must be the case because, at the root of the whole issue here, in the Weyl representation a field at a point has no physical significance, so there are only operators corresponding to full field configurations; all that could label phenomenal point-valued fields are states. But this situation undermines the usual understanding of a quantum system in which the statevector represents the state and operators yield the values of different magnitudes in that state; here the operator seems to represent the physical state, and a set of (coherent) states yield the values of the field at different points. If this identification is taken as revealing something deep about the ontology of the theory, instead of an interesting numerical correspondence, then some serious questions of interpretation of QFT arise.

More generally, Maudlin (2007) questions the feasibility of 'deriving' classical spacetime from some non-spacetime theory (he has in mind deriving 3-space from $3N$-configuration space, but the point generalizes). At the heart of his concern is that even if a formal derivation can be found, involving a mathematical correspondence between classical spacetime structures and structures defined in terms of a (more) fundamental non-spatiotemporal theory, it does not follow that the classical spacetime just *is* the more fundamental structure. Mathematical correspondences are too cheap: for instance, many systems are described by simple harmonic oscillator equations, but it would be a mistake to conclude that they were physically indistinguishable just because of this formal correspondence. According to Maudlin, for a reductive account, a formal derivation must also be 'physically salient'. I take this to mean that the derivation must veridically track the way in which fundamental structures 'combine' to physically constitute derivative ones. For instance, in ideal gas theory the *formal definitions* of 'temperature' as mean kinetic energy and 'pressure' as momentum transfer *track* the corresponding phenomenal thermodynamical quantities: kinetic energy is transferred between the molecules of the gas to liquid in a thermometer causing its expansion; and the pressure on the side of a vessel is due to the molecules contained colliding with it. The problem with a fundamental theory without spacetime is that our notions of what kinds of derivation track in this way are spatiotemporal notions, relying on

[16]Chaichian et al approach things from the other direction. They propose that the phenomenal fields should be labelled by the coherent states, and then conclude that they should be smeared according to (8).

colocation and dynamical interaction (of gas molecules with thermometers or vessel walls, say), for instance. But such familiar notions cannot apply if the physics involved is not, by supposition, itself spatiotemporal. So we face two problems: first, what new notions might apply? And second, even if we have a proposal, on what grounds can we conclude that we are correct?

If the analysis of this paper is correct, then non-commutative geometry is a nice example of this situation: the fundamental structure is algebraic, not a commutative geometry, and so concepts like 'spatial location' are not primitives of the theory. Rather, spatial structure is derived. In particular, we have discussed the proposal that it be recovered via Chaichian's ansatz, which I have also argued is not entailed by the theory, but an additional postulate. More precisely, it is an interpretational postulate, specifying how algebraic objects can 'combine' to physically constitute classical spatial structures – *a novel proposed non-spatiotemporal conception of which derivations are physically salient in the theory*. Thus the first problem can has been addressed in this case. (As in most cases, there is enough spatiotemporal structure in the underlying theory – which is after all a deformation of a commutative geometry – to find clues about how to reconstruct spacetime.)

As for the second problem, Huggett and Wüthrich (2013) proposes that such postulates, concerning how more fundamental structures compose to constitute less fundamental ones, are justified a posteriori, not a priori. (The paper briefly discusses NCFT and with other examples of theories without spacetime, along the lines found in this paper: identifying what spatiotemporal features are lost in each case, and explaining how they may be derived.) That is, how the fundamental gives rise to the less fundamental is not a matter of metaphysical necessity, but of physical contingency, and so is something that can only be discovered empirically, with the theory itself. For instance, if a theory of non-commutative geometry was empirically successful (in the usual ways, especially in making novel predictions that cannot be accounted for in any other known way) then both the theory *and* Chaichian's interpretational ansatz would be confirmed. That is, ultimately we justified in believing a derivation to be physically salient in the same way that any other scientific belief is justified: through successful confrontation with the data. No more is possible, but then it never is.

Thus, in addition to introducing NCFT and raising some specific interpretational questions, this paper presents it specifically as an example of derived or 'emergent' space, in order to illustrate and address Maudlin's challenge. There is a gap between non-commutative and commutative geometries, which can be *formally* filled by Chaichian's ansatz; but if this strategy were empirically successful then we would have scientific grounds to further believe that the derivation is physically salient, that the ansatz is a veridical statement of physical composition. The hope is that working through the example makes that claim plausible, or at least intelligible.[17]

[17] Of course it is logically possible to deny it, but I would say (a) that the historical record contains examples of similar changes in the concept of physical salience, and (b) that such a denial would collapse into a general antirealism, which is not my target here – rather the issue is whether there are special reasons to dispute the physical salience of derivations of spacetime.

4. Appendix: Technical Details

This final section is an appendix to the preceding, filling out some of the technical notions in a little more detail, giving an overview of the key concepts. Of course, for a more complete introduction the reader should consult the various surveys that I have cited, which the following digests.

NCG can be understood via a two-stage abstraction and generalization of standard differential geometry (with which the reader is assumed to be familiar): first one formulates an algebraic version of differential geometry; then one generalizes from that commutative algebra to non-commutative algebras. (See Figure 1). We will look at a particular realization of such an algebra; because work at a high level of abstraction risks losing contact with concrete physics, we will also examine how this realization can be used to derive physical consequences.

The starting point for the discussion is a differential manifold (representing space, not time) and the collection of smooth scalar fields on it: let X be the space, and C^∞_X the smooth fields on X. C^∞_X forms an algebra with respect to pointwise multiplication, and a representation theorem shows that if any two spaces agree on their C^∞ algebras, then they are the same manifold up to diffeomorphism: $C^\infty_X \cong C^\infty_{X'} \Rightarrow X \sim X'$.[18] Thus we can abstractly characterize the space by the algebra – the question is how other elements of differential geometry can also be captured algebraically.

Next, in developing differential geometry, smooth maps between manifolds, $\gamma : X_1 \to X_2$, play a central role. It is straight-forward to understand their algebraic representation. Such a map defines a pull-back map between fields: $\gamma^*(f)(x) \equiv f(\gamma(x))$. If γ maps the manifold onto itself, such a pull-back is a homomorphism of C^∞_X, since it preserves the algebra: $\gamma^*(f \cdot g) = \gamma^*(f) \cdot \gamma^*(g)$. Diffeomorphisms are invertible smooth maps, so they are represented by invertible homomorphisms – automorphisms – of C^∞_X.

In a moment we will return to the algebraic representation of differential geometry, but first note that C^∞_X is an abelian algebra: for any two fields, $[\phi(x), \psi(x)] = 0$. NCFT arises when one generalizes to a non-abelian algebra of 'fields', $\mathcal{A}$. We can continue to develop a 'geometry' without reference to a manifold by importing the algebraic representations into the new context; non-commutative 'smooth maps' are homomorphisms of $\mathcal{A}$, while 'diffeomorphisms' are its automorphisms. Of particular interest are the 'internal' automorphisms associated with any element $g \in \mathcal{A}$: $a \to g^{-1}ag$, for all $a \in \mathcal{A}$.

Our main example in this chapter is the algebra $\mathcal{R}^d_\theta$, defined as follows: let there be an even number, d, of elements $\hat{x}^i$ which fail to commute,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{13}$$

where θ^{ij} is an (antisymmetric) real-valued matrix. $\mathcal{R}^d_\theta$ is the algebra of linear combinations

[18] With a little more care and clarity. We assume that the topology is fixed, which can be achieved by giving the algebra of bounded-continuous fields (under certain assumptions). Then C^∞-isomorphism implies that X and X' will have the same *differential structure* – though they will not in general agree on any additional structures they may happen to be endowed with.

of products of these elements – scalar 'fields' expressed as power series of the $\hat{x}^i$s. Consider, for example, the internal automorphism – 'diffeomorphism' of $\mathcal{R}^d_\theta$ – implemented by

$$g = 1 + i\theta_{ij}\epsilon^i\hat{x}^j; \tag{14}$$

θ_{ij} is the inverse of θ^{ij} ($\theta^{ij}\theta_{jk} = \delta^i_j$), and the ϵ^is are d (real) infinitesimal parameters (this time, ordinary commutative ones). A couple of lines of algebra shows that

$$\hat{x}^i \to \hat{x}^i + \epsilon^i. \tag{15}$$

Thus, if we identify the $\hat{x}^i$s as 'co-ordinates', then we have a non-commutative analogue of C^∞_X (at least those fields that can be expressed as power series), in which $1 + \theta_{ij}\epsilon^i\hat{x}^j$ implements 'translations' by a constant vector ϵ^i. That said, for the purposes of this presentation it is best if the reader thinks of the formalism being presented as algebraic: in the first place, so doing puts off tricky questions about how familiar space enters the picture (something addressed at length in the main text); in the second, I argue above for an algebraic interpretation, so it will help if the reader understands the algebraic formulation. Thinking about the theory algebraically is further complicated by the use of spatial language to describe algebraic notions: for instance, elements of $\mathcal{A}$ are called 'fields' without reference to any scalar functions on a manifold, and automorphisms are called 'diffeomorphisms' . The reader is encouraged at this point to keep focussed on the algebraic meaning of these terms – and others soon to be introduced – licensed by the abstraction described. Above all, let us postpone serious discussion of how anything like a differential manifold appears!

To continue our development of NCG, let us return to differential geometry; after scalar fields, the next item to consider is the set of vector fields of the manifold, $T(X)$. These can be understood as generators of diffeomorphisms (more exactly, of pull-backs on scalar fields associated with diffeomorphisms): $f \to f + \epsilon V(f)$ for $V \in T(X)$. (Intuitively, vectors map scalar fields into their derivatives in a given direction at each point.) Leibniz's law, $V(fg) = V(f)g + fV(g)$, quickly follows:

$$fg \to fg + \epsilon V(fg) \tag{16}$$

$$\text{also} \quad fg \to (f + \epsilon V(f))(g + \epsilon V(g)) \tag{17}$$

$$\text{thus} \quad fg + \epsilon V(fg) = fg + \epsilon(V(f)g + fV(g)) + O(\epsilon^2). \tag{18}$$

((17) follows because the transformation is a diffeomorphism, which preserves the algebra, as we discussed above.) Thus in the algebraic formulation the role of vector fields is played by operators satisfying the Leibniz law: derivations, $Der(\mathcal{A})$. A simple commutator identity tells us that $[ab, c] = a[b, c] + [a, c]b$, so any commutator with an element of the field algebra is an 'internal' derivation:

$$V_c(a) \equiv [a, c] \quad (a, c \in \mathcal{A}). \tag{19}$$

We therefore have the interesting result that (unlike commutative field theory) elements of the field algebra generate infinitesimal transformations. For instance, defining $a + \delta_{\hat{x}} a \equiv (1 - i\epsilon\hat{x})a(1 + i\epsilon\hat{x}) = a + i\epsilon[a, \hat{x}] + O(\epsilon^2)$, we obtain

$$\delta_{\hat{x}} a = i\epsilon[a, \hat{x}] \qquad (= i\epsilon V_{\hat{x}}(a)). \tag{20}$$

Or again, derivatives with respect to coordinates generate translations along the axes: $a \to a + \epsilon^i \partial_i a$. Comparing with the internal translation automorphism on $\mathcal{R}^d_\theta$ given above (14), we see:

$$\begin{aligned} a &\to a + \epsilon^i \partial_i a && (21)\\ &= (1 - i\epsilon^i \theta_{ij}\hat{x}^j)a(1 + i\epsilon^i \theta_{ij} x^j) && (22)\\ &= a + \epsilon^i [a, i\theta_{ij}\hat{x}^j] + O(\epsilon^2). && (23) \end{aligned}$$

So in $\mathcal{R}^d_\theta$

$$\partial_i = [\cdot, i\theta_{ij}\hat{x}^j].^{19} \tag{24}$$

It is easy to see that $\partial_i \hat{x}^j = \delta_i^j$. Then, as in familiar differential geometry, we can express general vector fields as 'column vectors' of coefficients. Moreover, if we assume that all spatial derivatives are of this form – i.e., that these are covariant derivatives – we effectively endow $\mathcal{R}^d_\theta$ with a 'flat' non-commutative geometry.

All the results that we have obtained so far (and others to be obtained) are collected in the following table, which is presented to summarize the overall scheme.

[19] Somewhat confusingly, authors often switch between this notation and writing the same expression as $[\cdot, \partial_i]$ In practice it does not matter for many purposes since $[[\cdot, A], [\cdot, B]] = [\cdot, [A, B]]$. The same applies to covariant derivatives.

Commutative		**Non-Commutative**	
Diff'tial Geometry $\leftrightarrow$	*Field Algebra*	$\rightarrow$ *Field Algebra*	e.g., $\mathcal{R}^d_\theta$
• Manifold, X	C^∞_X	Non-Abelian, $\mathcal{A}$	$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
• $\gamma : X_1 \rightarrow X_2$	$\gamma^*(f)(x) \equiv f(\gamma(x))$	$Hom(\mathcal{A})$	$Hom(\mathcal{R}^d_\theta)$
• Diffeomorphism	Invertible	$Aut(\mathcal{A})$ – esp. $a \rightarrow g^{-1}ag$	$\hat{x}^i \rightarrow \hat{x}^i + \epsilon^i \Rightarrow g = 1 + i\theta_{ij}\epsilon^i\hat{x}^j$
• Tangent vector fields, $V \in T(X)$	$f \rightarrow f + \epsilon V(f) \Rightarrow$ Leibniz law	$Der(\mathcal{A})$ – esp. $V_c(a) = [a, c]$	$\hat{\partial}_i a = [a, i\theta_{ij}\hat{x}^j]$
• Vector bundle, $F = R \times X$	$\Gamma(F)$, module over C^∞_X	M, module over $\mathcal{A}$	e.g. Fock space
• ∇_i, A_i, F_{ij}, $\mathcal{L}$, $\int \mathrm{d}^n\hat{x}$, S ...			

To do familiar physics we will need other geometric objects, especially 'internal', in addition to tangent, vector fields. These represent fields with gauge symmetries: for instance, imposing local U(1) symmetry on a complex scalar field, $\phi(x)$, produces an action in which $\phi(x)$ represents a charged field coupled to the electro-magnetic field. In general then we need vector bundles, $F = R \times X$, where R is a vector space (here we assume a trivial topology for the bundle).

To represent F algebraically we consider $\Gamma(F)$, the set of sections of F (i.e., smooth maps from the points of X to R). Now, if I multiply a section by a scalar field I get another section. In algebraic terms, $\Gamma(F)$ is a 'module' over C^∞_X: it is just like a vector space over C^∞_X *except* that C^∞_X is a commutative ring not a field. (C^∞_X is not a field like the real numbers, because functions with zeroes don't have inverses.) This notion can be immediately generalized to a non-abelian algebra: in NCG a vector bundle is any module over $\mathcal{A}$ (which is certainly a ring because it is non-commutative). As an example: a change of coordinates in $\mathcal{R}^d_\theta$ turns θ^{ij} into a canonical form in which all elements are zero except for 2×2 antisymmetric matrices along the diagonal – the d non-commuting coordinates are transformed into $d/2$ pairs of mutually non-commuting coordinates, which commute with all other coordinates. (Hence there have to be an even number of non-commuting dimensions.) Formally, the only non-trivial commutation relations are:

$$[\hat{y}^{2r}, \hat{y}^{2r+1}] = i\theta_r, \quad r = 1, \ldots, d/2. \tag{25}$$

Thus the new coordinates form pairs satisfying a raising-lowering operator algebra, and so have a representation on a Fock space. Other elements of $\mathcal{R}^d_\theta$ are sums of products of the coordinates, so they all act on the Fock space – which is thus a module over $\mathcal{R}^d_\theta$. So one kind of matter field in NCFT is represented by the given Fock space (other fields can be represented by other modules over $\mathcal{R}^d_\theta$).

In both the abelian and non-abelian cases we take vector bundles to be modules over the field algebras. But that similarity masks a crucial difference that helps explain the power and significance of non-commutation. Because C^∞_X is commutative, it can only have 1-dimensional representations; it simply multiplies sections (pointwise) for instance. If we want to consider non-trivial transformations of sections – as we do when we consider gauge transformations – we have to introduce new internal operations on $\Gamma(F)$. But because $\mathcal{A}$ is non-commutative, it has modules with representations that are not 1-dimensional – it can already have the desired gauge group algebra! For instance, in $\mathcal{R}^d_\theta$, the translations discussed above are operators on the Fock module – in general, the distinction between spatial and internal transformations and symmetries is erased.

This point is central to the framework, and explains a number of its features, though we will not dwell on it. I would however direct the attention of those who work on the interpretation of gauge theory to this feature of NCG. The fact that the distinction can be erased in this way suggests that – in some ways at least – there is a real distinction to draw. Those who see argue for little or no distinction in the familiar case of commutative spacetime would do well to consider NCG (for instance, those who view general covariance as a gauge symmetry) – though I offer no conclusions here.

In the remainder of this appendix I will sketch, even more briefly than the foregoing, how QFT can be developed within NCG; naturally the references cover the material more fully. To develop physically realistic QFT we need to give the non-commutative analogue of geometry on a vector bundle – on the module already introduced. First there is the connection, a map from section-vector field pairs into sections, satisfying a version of the Leibniz Law:

$$\nabla : \Gamma(F) \times T(X) \to \Gamma(F) \tag{26}$$

such that

$$\nabla_V(f\gamma) = V(f)\gamma + f\nabla_V(\gamma) \text{ for all } V \in T(X), f \in C^\infty_X, \gamma \in \Gamma(F). \tag{27}$$ [20]

The connection can always be expressed in terms of coordinate derivatives and a vector potential, A (globally if the bundle's topology is trivial, but at least locally):

$$\nabla_V(\gamma) = V^i(\partial_i + A_i)\gamma \equiv V^i\nabla_i\gamma. \tag{28}$$

Finally, the curvature tensor is the commutator of a pair of derivatives: $F_{ij} = [\nabla_i, \nabla_j]$. If

[20] I emphasize here that we are discussing a connection on an 'internal' vector bundle, not on the tangent bundle. We continue to focus on flat space, in which the covariant derivative components are just coordinate derivatives.

the vector potential has no internal indices, so its components are numbers, then $[A_i, A_j] = 0$ (assuming the numbers are commuting), and

$$F_{ij} = [\partial_i + A_i, \partial_j + A_j] = [\partial_i, \partial_j] + [\partial_i, A_j] + [A_i, \partial_i] + [A_i, A_j] = \partial_i A_j - \partial_j A_i. \tag{29}$$

Physically, the sections of F represent matter fields, while the vector potential and curvature tensor represent a gauge field: for instance, the electromagnetic vector potential and field. In electromagnetism the components are indeed numbers, and (29) indeed gives the field. Thus we have the ingredients needed to write down a lagrangian density, $\mathcal{L}$, for the free gauge field and its coupling to other fields, in particular the elements of $\Gamma(F)$.

All these features are straight-forward to translate into NCFT. Since the vector bundle is replaced by a module M, and vector fields by derivations, the 'connection' is a map

$$\hat{\nabla} : M \times Der(\mathcal{A}) \to M \tag{30}$$

such that

$$\hat{\nabla}_V(am) = V(a)m + a\hat{\nabla}_V(m) \text{ for all } V \in Der(\mathcal{A}), a \in \mathcal{A}, m \in M. \tag{31}$$ [21]

The connection can again be expressed in 'components'

$$\hat{\nabla}_V(m) = V^i(\partial_i + A_i)m \equiv V^i \hat{\nabla}_i m, \tag{32}$$

where the A_i take on values in $\mathcal{A}$. The 'curvature tensor' is defined the same way, but now since the algebra is non-Abelian, even if the 'vector potential' has no internal indices, $[A_i, A_j] \neq 0$. That is, unlike commutative field theory, in NCFT the gauge fields are automatically non-commutative. Thus

$$F_{ij} = [\partial_i + A_i, \partial_j + A_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j], \tag{33}$$

and the field has terms quadratic in the potential. Since the field appears in the 'lagrangian', it will automatically have additional terms compared to the commutative case; more specifically there can be no free field, because there will necessarily be self-interactions.

To proceed further, we need to say a little more about the Weyl transform, introduced in the main text. This is a bijection between the elements of $\mathcal{R}^d_\theta$ and C^∞ functions that fall off rapidly at infinity, which allows standard analysis to be used to perform computations in NCFT (and more, as I discuss in §3.2). Naturally, to respect non-commutivity, field multiplication is no longer represented by ordinary multiplication but by the 'Moyal star', defined in (2). To preserve the algebra, the coordinate derivatives we found above (24) map

[21] Because $\mathcal{A}$ is non-commutative, elements of the algebra can act either to the left or right of the module. To keep the presentation minimal I just give expressions for a left module, though they are easily modified to obtain the corresponding expressions for a right module.

to ordinary partial derivatives. Since we consider fields that vanish at infinity $\int d^n x \partial_i \phi = 0$, so integration in NCFT can be represented by a trace, Tr:

$$\text{Tr } \partial_i a = \text{Tr}[a, i\theta_{ij}\hat{x}^j] = i\text{Tr}(a\theta_{ij}\hat{x}^j - \theta_{ij}\hat{x}^j a) = 0, \tag{34}$$

since the trace is cyclic.

With this 'integral' in hand, an 'action', S, can be derived from a lagrangian density. The value of this observation – what makes this framework useful for physics – is that the usual framework for least action physics applies in NCFT *because* the derivation of the Euler-Lagrange equations and Noether theorem and the like do not depend on commutivity![22]

Moreover, the Weyl transform gives a way to turn commutative field theories into meaningful non-commutative ones: take a commutative action and replace all multiplication with Moyal star multiplication – the Weyl transformation maps it into a NCFT action, minimizing it leads to an equation of motion of the Weyl transforms of physical NCFT solutions.[23] (Following Szabo (2003), 17-9) consider, for instance, a (commutative) massive scalar field with a $\phi^4(x)$ interaction, whose lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_i\phi(x))^2 + \frac{m^2}{2}(\phi(x))^2 + \frac{\lambda}{4!}(\phi(x))^4 \tag{35}$$

$$\rightarrow \frac{1}{2}\partial_i\phi(x) \star \partial_i\phi(x) + \frac{m^2}{2}\phi(x) \star \phi(x) + \frac{\lambda}{4!}\phi(x) \star \phi(x) \star \phi(x) \star \phi(x). \tag{36}$$

First, note an important property of star-product integration. From (2): $\int d^d x f(x) \star g(x) = \int d^d x f(x) \cdot g(x)$ + terms of the form:

$$\theta^{ij}\theta^{i_2 j_2} \ldots \theta^{i_n j_n} \int d^d x\ \partial_i \partial_{i_2} \ldots \partial_{i_n} f(x) \cdot \partial_j \partial_{j_2} \ldots \partial_{j_n} g(x). \tag{37}$$

Focusing on the i and j contributions, we integrate by parts twice (remembering surface terms vanish),

$$= \theta^{ij} \ldots \int d^d x\ \partial_j \ldots f(x) \cdot \partial_i \ldots g(x) \tag{38}$$

$$= \theta^{ji} \ldots \int d^d x\ \partial_i \ldots f(x) \cdot \partial_j \ldots g(x) \tag{39}$$

$$= -\theta^{ij} \ldots \int d^d x\ \partial_i \ldots f(x) \cdot \partial_j \ldots g(x), \tag{40}$$

where we have relabeled $i \leftrightarrow j$, and then used the antisymmetry of ϕ^{ij}. Thus (37) = (40) = −(37), so all expressions of that form vanish. Hence

$$\int d^d x\ f(x) \star g(x) = \int d^d x\ f(x) \cdot g(x). \tag{41}$$

That is, in an integral over products it makes no difference whether Moyal or ordinary multiplication is used. Hence the free terms in the lagrangian density make no difference

[22]We won't show this in general, but the reader might like minimize the following action to obtain the equations of motion: $S = \text{Tr}\{g^{ij}/2 \cdot \partial_i\phi \cdot \partial_j\phi + V(\phi)\}$, with $\phi \in \mathcal{R}^d_\theta$. (Hint: first show that the product rule, $\text{Tr}(f \cdot \partial_i g) = -\text{Tr}(g \cdot \partial_i f)$ follows from (19) and (34).)

[23]Note that the infinitesimal line elements of the integral are to be multiplied in the usual way, since the integral symbol, including them, is Weyl transformed to the trace, as we discussed above.

to action – since it involves an integral over space. In particular, in perturbative non-commutative QFT the free field is unchanged: the propagator is as for the corresponding commutative theory. The difference comes in the interacting terms, so in the form of the vertices of the theory. To see how a little more clearly it is helpful to work in terms of Fourier transforms of the fields. In that case we can rewrite the Moyal star (2) as:

$$f(x) \star g(x) = f(x)\ e^{(\frac{i}{2})\theta^{ij}\overleftarrow{\partial_i}\cdot\overrightarrow{\partial_j}}\ g(x) \tag{42}$$

$$= \int \frac{\mathrm{d}^{\mathrm{d}}k_1}{(2\pi)^2}\frac{\mathrm{d}^{\mathrm{d}}k_2}{(2\pi)^2}\ \tilde{f}(k_1)\tilde{g}(k_2 - k_1)\ e^{-\frac{i}{2}k_1\theta k_2}e^{ik_2\cdot x} \tag{43}$$

(where $k_1\theta k_2 \equiv \theta^{ij}k_{1i}k_{2j}$). This form of the product can be used, with a little work, to show that

$$\phi(x) \star \phi(x) \star \phi(x) \star \phi(x) = (\prod_{a=1}^{4}\int \frac{\mathrm{d}^{\mathrm{d}}k_a}{(2\pi)^D})\ e^{(\sum_{a=1}^{4}k_a)\cdot x}\ e^{-\frac{i}{2}\sum_{a<b}k_a\theta k_b}, \tag{44}$$

where all terms are understood to be inside the four momentum integrals. In this form we can read off that in perturbative QFT vertex terms will pick up a momentum-dependent phase. (The second exponent: the first produces a momentum conserving delta-function on spacetime integration.) This produces the effects of the non-commutative space.

Or, to give another example, a commutative field with a $U(1)$ symmetry has a lagrangian

$$\frac{-1}{4g^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_\mu - iA_\mu)\psi - m\bar{\psi}\psi \tag{45}$$

which becomes on 'Weyl quantization'

$$\frac{-1}{4g^2}F_{\mu\nu} * F^{\mu\nu} + i\bar{\psi} * \gamma^{\mu}(\partial_\mu\psi - iA_\mu * \psi) - m\bar{\psi} * \psi. \tag{46}$$

(Remembering that $F_{\mu\nu}$ has an extra term, given in (33).) For a discussion of the Feynman rules that follow from this lagrangian, and their consequences, see (Hayakawa (1999)).

References

Alexey Anisimov, Tom Banks, Michael Dine, and Michael Graesser. Comments on non-commutative phenomenology. 2001. URL `http://arxiv.org/abs/hep-ph/0106356v4`.

Jonathon Bain. Algebraic substantivalism and the hole argument. *Preprint*, Unpublished.

Harvey R Brown. *Physical relativity: space-time structure from a dynamical perspective*. Clarendon Press, Oxford, 2005. ISBN 0199275831 (acid-free paper). URL `http://www.loc.gov/catdir/toc/ecip0517/2005023506.html`.

Jeremy Butterfield. The hole truth. *The British Journal for the Philosophy of Science*, 40 (1):1–28, March 1989.

Sean M. Carroll, Jeffrey A. Harvey, V. Alan Kostelecky, Charles D. Lane, and Takemi Okamoto. Noncommutative field theory and lorentz violation. 2001. URL `http://arxiv.org/abs/hep-th/0105082v1`.

M. Chaichian, A. Demichev, and P. Presnajder. Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences. *Nucl.Phys.*, B567:360–390, 2000. URL `http://arxiv.org/abs/hep-th/9812180`.

M. Chaichian, P. Prešnajder, and A. Tureanu. New concept of relativistic invariance in nc space-time: twisted poincaré symmetry and its implications. 2004. URL `http://arxiv.org/abs/hep-th/0409096v1`.

Henk W. de Regt. Spacetime visualisation and the intelligibility of physical theories. *Stud. Hist. Phil. Mod. Phys*, 32(2):243–65, 2001.

Dennis Dieks. Space and time in particle and field physics. *Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics*, 32(2): 217 – 241, 2001. ISSN 1355-2198. doi: DOI: 10.1016/S1355-2198(01)00004-1.

Michael R. Douglas and Nikita A. Nekrasov. Noncommutative field theory. *Reviews of Modern Physics*, 73:977–1029, 2001.

John Earman. *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time*. MIT Press, Cambridge, MA, 1989.

John Earman and John Norton. What price spacetime substantivalism? the hole story. *The British Journal for the Philosophy of Science*, 38(4):515–25, December 1987.

Robert Geroch. Einstein algebras. *Communications in Mathematical Physics*, 26:271–275, 1972.

M. Hayakawa. Perturbative analysis on infrared and ultraviolet aspects of noncommutative qed on r^4. 1999. URL `http://arxiv.org/abs/hep-th/9912167v1`.

Nick Huggett. Essay review: Harvey r. brown: Physical relativity: Spacetime structure from a dynamical perspectiverobert disalle: Understanding spacetime: The philosophical developments of physics from newton to einstein. *Philosophy of Science*, 76(3):404–22, July 2009.

Nick Huggett and Christian Wüthrich. Emergent spacetime and empirical (in) coherence. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 44(3):276–285, 2013.

Nick Huggett, Tiziana Vistarini, and Christian Wüthrich. Time in quantum gravity. In Adrian Bardon and Heather Dyke, editors, *The Blackwell Companion to the Philosophy of Time*. Blackwell, 2012.

Fedele Lizzi. Noncommutative spaces, 2009.

Tim Maudlin. Completeness, supervenience, and ontology. *Journal of Physics A: Mathematical and Theoretical*, 40:3151–3171, 2007.

Robert Rynasiewicz. Rings, holes and substantivalism: On the program of Leibniz algebras. *Philosophy of Science*, 59:572–589, 1992.

Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *Journal of High Energy Physics*, 09:032, 1999. URL `http://arxiv.org/abs/hep-th/9908142v3`.

Richard J. Szabo. Quantum field theory on noncommutative spaces. *Physics Reports*, 378: 207–299, 2003.