Ancient Greek geometry of motion and its further development by Galileo and Newton

Mario Bacelar Valente

mar.bacelar@gmail.com

Abstract: in this paper we return to Marshall Clagett's view about the existence of an ancient Greek geometry of motion. It can be read in two ways. As a basic presentation of ancient Greek geometry of motion, followed by some aspects of its further development in landmark works by Galileo and Newton. Conversely, it can be read as a basic presentation of aspects of Galileo's and Newton's mathematics that can be considered as developments of a geometry of motion that was first conceived by ancient Greek mathematicians.

1. Introduction

In his study of kinematics – the space and time description of motion – in Europe's medieval period, in the book *The science of mechanics in the middle ages*, Marshall Clagett focus initially "in what way kinematics was nourished in antiquity" (Clagett 1961, 163). In Clagett's view:

Kinematics was fostered in antiquity by three distinguished currents of scientific activity: (1) the geometrization of astronomy, (2) the emergence of a geometry of movement, or generative geometry, and (3) the development of physical and mathematical treatises whose theoretical parts had a geometrical character. (Clagett 1961, 164)

Regarding point (1), Clagett considers Autolycus' work on the rotating sphere in the treatise *On the moving sphere*. In Clagett's view the function of the first three propositions of the treatise is as follows:

[To introduce] the basic concepts connected with the sphere in motion. All of these [propositions] describe the effect of a uniform rotation of the sphere on the movements of the points on the surface of [the] sphere. (Clagett 1961, 164)

To Clagett, in Autolycus' treatise, the "approach to movement is entirely geometrical" (Clagett 1961, 164). This is an important point to which we will return soon.

Another approach to kinematics is, according to Clagett, the emergence of a geometry of movement, or generative geometry, which we will call geometry of motion. Clagett addresses in some detail what he calls "Archimedes' strictly kinematical ideas as found in his [*On spirals*]" (Clagett 1961, 171).[1] Clagett presents Archimedes' definition of

1 We will not go into details regarding the third "current" in this work (see Clagett 1961, 175-184). Clagett focuses, in particular, on two passages in Aristotle's Physics regarding kinematics. These, even if very important in the Middle Ages, had in fact been superseded by Archimedes' work: "it should be pointed out that while there is evidence of the geometrical approach in these passages they are much less precise than those later mathematical writers like Archimedes" (Clagett 1961, 178).

spiral, based on the combination of two uniform motions, and also presents in detail the first two propositions of this treatise.

We departure from Clagett in his distinction between approach (1) and (2). Evidently, there is the difference that (1) refers to works on astronomy, while (2) to works on mathematics. In this sense, we have two currents. But, regarding the geometry developed in these different works it is a geometry of motion since, in Clagett's own words, we have an entirely geometrical approach to motion in Autolycus' treatise.

The present paper is organized as follows. Section 2 presents our account of basic elements of ancient Greek geometry of motion as found in the treatise of Autolycus, in a related treatise of Euclid, and in the treatise of Archimedes. Section 3 addresses the further development of a geometry of motion in Galileo's work on motion, in particular, his treatment of uniformly accelerated motion. Section 4 addresses Newton's geometry of motion that includes a geometrical treatment of continuously varying magnitudes.

2. The ancient Greek pure geometry of motion

While the pure geometry of Euclid's *Elements* might seem to be far from any geometrical treatment of motion, in book XI some of the solid geometrical objects seem to be instantiated by the rotation of planar geometrical objects. This is the case with the sphere, the cone, and the cylinder. We will just consider the definition of sphere:

When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere. The axis of the sphere is the straight line which remains fixed and about which the semicircle is turned. (Euclid 1908, 261)

That this does not have to be seen as implying a geometrical notion of rotation has been mentioned, e.g., by Funkenstein (1986) or Netz (2004). We could simply be facing an imaginary motion, not a motion-in-time, as a way of conceiving the object (Netz 2004, 120). Drawing from Aristotle's views, regarding the sphere, the cone, or the cylinder, Funkenstein claims the following:

Their motion means displacement only, of the same character involved in demonstrating the congruency of discrete figures; it could, for that matter, be thought of as instantaneous. (Funkenstein 1986, 306)

We have no quarrel with this position. However, we agree with it not due to Aristotle's views but due to the geometry of motion displayed in Euclid's own work and in related works. In the instantiation of a sphere by the rotation of a semicircle, there is no reference to time, or the type of motion attributed to the semicircle. But this is not the case of Euclid's treatment of a rotating sphere in his *Phaenomena*. We will just mention proposition 2. The first part of this proposition states that "in one revolution of the cosmos, the circle through the poles of the sphere will be perpendicular to the horizon twice" (Berggren and Thomas 1996, 55). In this proposition, while in the enunciation there is a reference to the revolution (rotation) of the cosmos, the cosmos is also identified as a sphere, and, in the proof, we consider a rotating sphere. It is the sphere qua geometrical object which is in a rotation. In this way, all the points on the sphere share this circular motion. That is, the motion is an attribute of geometrical objects. Time is also explicitly mentioned, even if it remains unclear its mathematical treatment (Berggren and Thomas 1996, 55-60).

Some aspects of the geometry of motion present in the *Phaenomena* are clearer in a related work by Autolycus, the treatise *On the moving sphere*.[2] Proposition 1 states the following:

If a sphere rotates uniformly about its axis, all the points on the surface of the sphere which are not on the axis will trace parallel circles that have the same poles as the sphere, and that are perpendicular to the axis. (Autolycos 2002, 41-3; see also Evans 1998, 87)

Here, it is made explicit that the sphere has a uniform rotation.[3] Also, in the treatise, time is explicitly taken into account in relation to the uniform rotation. Proposition 2 states the following:

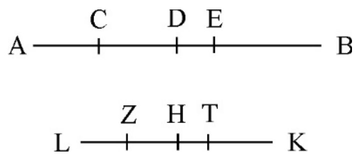
If a sphere rotates uniformly about its axis, all the points on the surface of the sphere describe in equal times similar arcs on the parallel circles on which they are moving. (Autolycos 2002, 44)

The proof of this proposition is geometrical. There is nothing outside mathematics taken into account. The sphere as a geometrical object is taken to be in a uniform rotation. Motion is intrinsic to the geometrical object.[4] During this motion, two chosen points describe similar arcs "in the same time" (Autolycos 2002, 46), which is proven mathematically in the demonstration. "Time" is not a notion external to geometry. However, like in the case of Euclid's *Phaenomena*, it remains unclear how it is addressed mathematically.

This is evidently different from the rotation of a semicircle in the definition of a sphere in Euclid's *Elements*. We agree that the geometry of the *Elements* is a static geometry, but we accept this not because of Aristotle's views but because of Euclid's geometry of motion.

The ancient Greek extant work where the geometry of motion is more clearly developed is in Archimedes' *On spirals*, in particular in what refers to the mathematical treatment of time. The Proposition 1 of this treatise states a result regarding the uniform motion of a point. In its demonstration reference is made to a definition of uniform motion, which is the same as the one at the beginning of Autolycus' *On the moving sphere* (Clagett 1961, 174). Proposition 1 states the following:

If some point is displaced with a uniform velocity along a certain line, and if upon this latter line we take two lines, they will have the same ratio between them as the times during which the point has traversed these lines. (Clagett 1961, 173; see also Netz 2017, 36)



2 Autolycus of Pitate was, according to different authors, an elder contemporary of Euclid (see, e.g., Heath 1981, 332). His treatises are considered the earliest extant mathematical works of ancient Greece (Autolycos 2002, 7). In Euclid's Phaenomena several results are assumed, part of which can be found in Autolycus' On the moving sphere (Berggren and Thomas 1996, 19).

3 This is also mentioned in the introduction of the *Phaenomena*. However, there are doubts regarding the authenticity of the introduction. In Berggren and Thomas's view "the introduction did not belong to the treatise originally as written by Euclid" (Berggren and Thomas 1996, 12).

4 This point was made by Clagett, taking into account the definition of uniform motion made at the beginning of Autolycus' treatise. Accordingly, "the approach to movement is entirely geometrical. We are not dealing with a gross body in movement but with a geometrical point" (Clagett 1961, 165). Here, we do not repeat Clagett's point exactly. We make the same point by considering propositions 1 and 2 of the treatise. This is because there are doubts regarding the authenticity of the definition (Autolycos 2002, 42, footnote 1).

In the proof of this proposition Archimedes takes into account that equal spaces are traversed in equal times; i.e., he considers the mathematical definition of uniform motion:

Since it is supposed that the point is carried in a uniform velocity along line AB, it is evident that it traverses each of the lines equal to CD in a time equal to that in which it traverses CD. (Clagett 1961, 173)

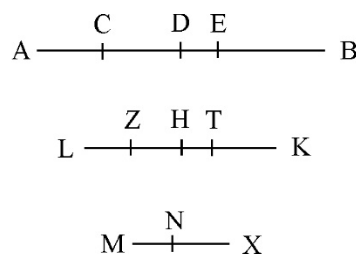
We speak of the mathematical definition of uniform motion and not simply the definition of uniform motion because both the space traversed and the time it takes to realize this motion are both treated mathematically.

At the beginning of the proof, it is stated the following: "let ZH be the time during which the point has traversed the line CD and let HT be the time in which the point has traversed line DE" (Clagett 1961, 173). In the accompanying diagram, the total time of motion LK is illustrated by a segment (a straight line), being ZH and HT illustrated as parts of this segment. We must be careful not to consider that Archimedes adopts a geometrical representation of time as a segment, in which case we might presume that time is a "physical" notion, not a mathematical one. Like in the case of book 5 of Euclid's *Elements*, the segment is adopted as an illustration of a mathematical magnitude (Euclid 1908, 138). Time is not represented by a geometrical object – a segment – time is a mathematical magnitude.

It is the treatment of time as a mathematical entity – a magnitude – that makes possible to consider the ratio between the times ZH and HT, in which, the notion of ratio is defined in book 5 of Euclid's *Elements*. Starting from the previous adoption of time as magnitude, proposition 1 establishes that in the case of a uniform motion of a geometrical point, the ratio between the spaces traversed, CD and DE, is proportional to the ratio between the times ZH and HT (during which the point traverses these spaces).

Regarding the mathematical treatment of velocity, this is made, in a way indirectly, in proposition 2, by considering the distances traversed by two points in the same time and with different uniform velocities. Velocity is not yet directly addressed mathematically, e.g., as a magnitude on its own. The only mathematized notions are the magnitude – the length – of segments and time as a magnitude. Proposition 2 states the following:

If two points are displaced with uniform velocity, each along a different line, and if on each of these lines one takes two lines such that the first line segment on the one line is traversed by its point in the same time as the first segment on the second line is traversed by the other point, and similarly the second segments on the lines are traversed in equal times, then the line segments on each line respectively are in the same proportion. (Clagett 1961, 173; see also Netz 2017, 39-40)



The comparison between the uniform velocities of each point is made indirectly by considering the distances traveled by each point in two equal times. Let CD be the distance traveled, along the line AB, by the first point in the time MN, and DE be the distance traveled in the time NX. Equivalently, let ZH be the distance traveled by the second point (with a possibly different uniform velocity), along the line LK, in the time MN, and HT be the distance traveled in the time NX. Archimedes shows that, for the first point, CD is to DE in the same ratio as MN to NX. In the same way, for the second point, the ratio of ZH to HT is proportional to the ratio of MN to NX. In this way, CD has the same ratio to DE as ZH to HT (Clagett 1961, 173-4; Netz 2017, 40).

Another important feature as a geometry of motion regarding Archimedes' approach is the composition of motions. This can be found in the definition of spiral:

If a straight line one of whose extremities is fixed turns with uniform speed in a plane, reassuming the position from which it started, and at the same time a point of that rotating line is moved uniformly fast on that line, starting from its fixed extremity, the point will describe a spiral in the plane. (Clagett 1961, 171; see also Netz 2017, 93)

We find here, like in the previous works by Autolycus and Euclid a uniform rotation, in this case not of a sphere but of a segment around a fixed extremity. The curve – the spiral – is not instantiated in the plane in terms of a make-believe motion. Using Funkenstein's terminology, we have a motion-in-time. It is described by a point undergoing two combined motions. The rotation of an "underlying" segment and the uniform motion of the point along the segment.

That we are facing an entirely mathematical description of motion can be noticed by considering that Archimedes' definition of spiral is equivalent to what Netz calls the locus definition of spiral:

The condition of the locus of the spiral is this: that, for any two points on the spiral taken arbitrarily such as P1, P2, the ratio between the two points P1, P2 is the same as the ratio between the two radii at points P1, P2. (Netz 2017, 34-5)

Accordingly, "such is the locus definition of the spiral, translating into the language of proportion what, in Archimedes' language, is the language of uniform speeds" (Netz 2017, 35). This is a clear indication that Archimedes' geometrical approach under consideration is a geometry of motion, and not some sort of geometry applied to motion in which, possibly, some notions might be taken to be external to mathematics.[5]

3. Galileo' geometry to motion

Galileo mature mathematical treatment of motion can be found in his book *Discorsi e dimostrazioni matematichi intorno à due nuove scienze*. In this book, Galileo adopts the literary artifice of a dialogue between 3 characters. While the dialogues are in Italian, must of the mathematical treatment of motion is made in Latin. This is so because Galileo also employs a further literary artifice. In chapters 3 and 4 ("days" 3 and 4) one of the characters reads to the others a treatise on motion – *De motu locali* – written in Latin. This book within the book consists of three chapters. The first on uniform motion; the second on free fall and related motions like falling on an inclined plane, mathematically described as uniformly accelerated motions; the third on the motion of projectiles, mathematically described in terms of the composition of a uniform motion and a uniformly accelerated motion (Drake 1989; Wisan 1974, 107). According to Wisan:

The Latin treatise is organized like the works of the Greek mathematicians, Euclid and Archimedes. From definitions, axioms, and a single explicit postulate, Galileo derives a body of mathematical propositions by use of geometrical theorems and methods of the Greeks. (Wisan 1974, 114)

5 This is, e.g., the case with Archimedes' statics, where he blends mathematical notions with "physical" ones (see, e.g., Dijksterhuis 1956, 286-313; Netz 2009, 136-8).

As we will see, even if remaining as close as possible to the Greek ancient mathematics, Galileo resorts to a heuristic mathematics (in the demonstration of proposition 1 of the chapter on accelerated motion) to derive his most important result on uniformly accelerated motion (the proposition 2).

Galileo starts his treatment of uniform motion by defining it: "Equal or uniform motion I understand to be that of which the parts run through by the moveable in any equal times whatever are equal to one another" (Drake 1989, 148). As we have seen, this definition is taken into account in Archimedes' proposition 1 of *On spirals*. From this definition, Galileo makes explicit four consequences of it that he calls axioms. Axioms 1 and 2 are used in the demonstration of proposition 1, while axioms 3 and 4 are used in the demonstration of proposition 2. The role of these axioms is to enable a rigorous deduction of the propositions considering the definition of same ratio in book 5 of Euclid's *Elements* (Clavelin 1983, 27-31). Proposition 1 is the same as proposition 1 in Archimedes' *On spirals*. It is stated as follows: "If a moveable equably carried with the same speed passes through two spaces, the times of motion will be to one another as the spaces passed through" (Drake 1989, 149). Mathematically, the only difference is that while "Galileo's proof follows that of Archimedes [it] is given in greater detail showing, in particular, the application of Euclid's definition of equal ratios" (Wisan 1974, 282).

However, for this to be so, we have to remove, what for the moment we might call an apparent ambiguity in Galileo's terminology (which in fact is related, as we will see, to Galileo's conception of physical science). Where we have "moveable" we must read (geometrical) point; and where we have "space" we must consider the length of a (geometrical) segment, or simply, a line. To Galileo, a moveable is what we might call a material body (Drake 1989, xli), in relation to which we can make experiments regarding its "physical" motion – the motion as occurs in nature and described through experimentation. So, while this is a proposition of geometry of motion exactly like Archimedes', it is presented as a proposition about the "real world".

Mathematically, proposition 2 is proved in exactly the same way as proposition 1 but taking into account axioms 3 and 4 instead of axioms 1 and 2. There is, however, a quite important feature of this proposition that corresponds to a further development of Archimedes' geometry of motion. As we have seen in proposition 2 of *On spirals*, the comparison between the uniform velocities of two points each with a uniform motion is made indirectly by considering the distances traveled by each point in two equal times. In proposition 2 of Galileo's treatment of uniform motion, velocity is taken to be a magnitude.[6] The proposition is as follows: "If a moveable passes through two spaces in equal times, these spaces will be to one another as the speeds. And if the spaces are as the speeds, the times will be equal" (Drake 1989, 150). The velocity (speed) is a magnitude; as such, following Euclid, we can consider a ratio between two velocities and compare it with other ratios.

In Euclid's mathematics, as Clavelin remarks, "there can be relations only between homogeneous magnitudes – between spaces or between times, but, in all cases, not between spaces and times" (Clavelin 1983, 30-1). This has as a consequence the following:

6 The earliest work where velocity is treated as a magnitude seems to be the treatise Liber de motu of Gerard of Brussels, written between the late twelfth and mid-thirteenth century (Clagett 1961,184-186). According to Postulate 8 in this treatise, "The proportion of the movements [i.e. speeds] of points is that of the lines described in the same time" (Clagett 1961, 187).

Speed, of which the expression would require a ratio of type s/t is, therefore, not mathematically constructible and can only appear directly, in the form of an unanalyzable magnitude, in the theory that Galileo is setting up. (Clavelin 1983, 31)

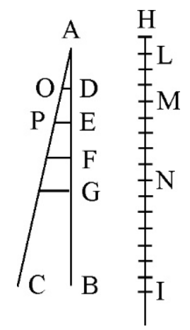
The following proposition – proposition 3 – states another relation of proportionality for the velocities of two bodies each with a uniform motion: "Of movements through the same space at unequal speeds, the times and speeds are inversely proportional" (Drake 1989, 150).

Besides these 3 propositions, Galileo's geometry of uniform motion consists of three more propositions. In these, we find what we might consider from the perspective of ancient Greek mathematics a mathematical innovation not fully justified. It regards the notion of compound ratio. We find the notion of compound ratio, e.g., in proposition 23 of book 6 of Euclid's *Elements* (Euclid 1908, 247-8), or in the proposition 31 of Archimedes' treatise *On conoids and spheroids* (Heath 1897, 148; Mugler 1970, 246). In both cases, one compounds two ratios between lengths – i.e., between homogeneous magnitudes. In Galileo's proposition 4, we find the ratio of speeds compounded with the ratio of times. In proposition 5, we have the ratio of spaces compounded with the inverse ratio of speeds. Finally, in proposition 6, we have the ratio of spaces compounded with the inverse ratio of times (Drake 1989, 151-2).

Even if in relation to the ancient Greek mathematics, Galileo compounded ratios are not fully justified, they are nevertheless fruitful in his development of a geometry of motion that goes beyond Archimedes'. In particular, proposition 4 is applied in the central proposition of Galileo's geometry of uniformly accelerated motion – proposition 2, known as the times-squared law or, simply, the law of fall.

Proposition 2 of the "chapter" on uniformly accelerated motion from *De motu locali* – the book within the book – states the following:

> If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratio of their times; that is, are as the squares of those times. (Drake 1989, 166)



Like in Archimedes' geometry of motion, Galileo's geometry of motion treats time as a magnitude. This is not only the case with uniform motion but also with uniformly accelerated motion. The proof of proposition 2 starts by stating: "Let the flow of time from some first instant A be represented by the line AB, in which let there be taken any to times, AD and AE" (Drake 1989, 166). This wording shows the (apparent) ambiguity, above-mentioned, regarding mathematical propositions (and their demonstrations) and their relation to experience, e.g., as experimental statements: "flow of time" seems as a reference to an intuitive perception of time as measured in experimentation. "Representation" might seem to point to a vagueness in the way the measured time is addressed. This wording seems to refer to the motion in nature – the naturally accelerated motion – but in fact, here, we are proving a mathematical proposition. Time is a magnitude. AD and AE are two homogeneous magnitudes. What Galileo proves is a mathematical relation – in the form of a relation between ratios. Galileo takes HI to be "the line in which the uniformly accelerated moveable descends from point H as the first beginning of motion" (Drake 1989, 166). Again, like in the proposition on uniform motion, moveable is a geometrical point in the demonstration of the proposition but refers to the physical counterpart – the material body with which experiments are made. We are told that in the time AD the point moves through a space (geometrical segment) HL, and in time EA the point moves through a geometrical segment MH. Using proposition 1 (of the chapter on uniformly accelerated motion) and the above-mentioned proposition of the chapter on uniform motion, Galileo proves that "MH and HL have the same ratio as do the squares of EA and AD" (Drake 1989, 166). The ratio of the lengths of two segments (i.e. the ratio of two homogeneous magnitudes) is proportional to the ratio of the squares of two times (again, the ratio of two homogeneous magnitudes).

So, why the ambiguity in the wording of the propositions? Galileo proposes to develop a new science of motion. Inspired by the work of Ptolemy (Drake 1978, 52), Galileo combines a rigorous as possible mathematics presented in a Euclidean way (definitions, postulate, axioms, deductive sequence between propositions, Euclidean structure of proofs) with observation, which in Galileo's case means experimentation, in a way that, e.g., the law of fall as a mathematical proposition corresponds to this law as an experimental law. In fact, the experimental law of fall, relating measured distances of fall to measured times, was established by Galileo around 1604 (Drake 1989, xv-xxix), while the mathematical proposition 2 in its last published form seems to be a late development, from late 1635, made just previous to the publication of Galileo's book, and motivated by Galileo's intention in providing a more consistent deduction of it than in his previous demonstrations of his experimental law as a mathematical proposition (Drake 1989, 370-1; Wisan 1974, 286-295).

Regarding the mathematics of his new science, it is a geometry of motion. According to Galileo himself, he follows Archimedes. It is a fact that contrasting his approach, e.g., with that o Archimedes' in *On Spirals*, according to Galileo, contrary to Archimedes, he is not "inventing as pleasure some kind of motion and theorizing about its consequent properties" (Drake 1989, 153). For Galileo, "since nature employs a certain kind of acceleration for descending heavy things, [he] decided to look into their properties" (Drake 1989, 153). But this refers to his decision of treating natural fall, not to how he sees the mathematics employed in his new science. In relation to this in a letter from 1639 Galileo states the following:

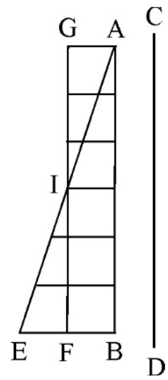
I assume nothing except the definition of the motion I wish to treat of and whose properties I wish to demonstrate, imitating in this Archimedes in his *Spiral lines* [the treatise *On spirals*], where he, having explained what he means by motion made in the spiral that is compounded from two uniform motions, one straight and the other circular, goes on immediately to demonstrate its properties. I declare that I want to examine what symptoms occur in the motion of a moveable which, leaving from the state of rest, goes moving with speed growing always in the same way […] And putting in nothing more, I come to the first demonstration, in which I prove the distance passed by such a moveable to be in the squared ratio of the times, and then I go on to demonstrate a large number of other properties […] I argue *ex supposition* concerning the motion defined in the above way, so that even if the consequences did not correspond to the events of natural motion of descending heavy things, it would matter little to me, just as it in no way derogates from the demonstrations of Archimedes that there is found in nature no moveable that is moved through spiral line. (Drake 1978, 395-6)

The adoption of a pure geometry – in this case, a pure geometry of motion – is one of the elements of Galileo's conception of physical science, the other, as mentioned, is establishing a relation to experience by experimentation.

The proof of proposition 2 corresponds in Clavelin words to the "Euclidean orthodoxy" (Clavelin 1983, 42). To be more precise, this is just the case if we take

Galileo’s extension of the notion of compound ratios to that of compounding two non-homogeneous ratios as a sound extension of the theory of proportion as set on book 5 of Euclid’ *Elements*. However, the demonstration of the crucial proposition 1 is a quite different matter. As it is, in relation to ancient Greek mathematics, this demonstration comes out of the blue.[7] Proposition 1 states the following:

> The time in which a certain space is traversed by a moveable in uniformly accelerated movement from rest is equal to the time in which the same space would be traversed by the same moveable carried in uniform motion whose degree of speed is one-half the maximum and final degree of speed of the previous, uniformly accelerated motion. (Drake 1989, 165)



The time AB of motion is a magnitude, which is illustrated (represented) by a line. Galileo considers the notion of an instant of time, which due to the adopted representation, is treated as a point. Here, we have a subtle move from a line (a segment) simply as an illustration, like, e.g., in Archimedes’ work, to a much more “active” role. A geometrical segment represents another mathematical entity – time as magnitude. This endues time with properties of segments as determined in Euclid’s *Elements*. In this case, it becomes meaningful to consider a “point of time”. Another novelty is the notion of degree of speed. This corresponds somewhat to our notion of instantaneous velocity. To each of the points in the time line AB there is an associated degree of speed, which is represented by a line drawn perpendicularly to the segment AB. They are a sort of indivisibles, the aggregate of which is represented by a geometrical figure (Clavelin 1983, 41; Jullien 2015, 97-103).

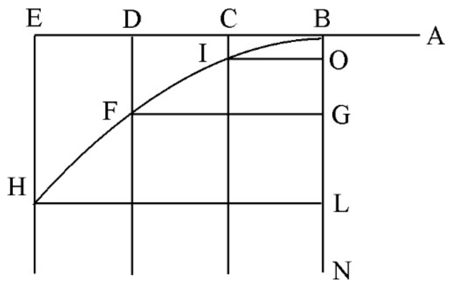
Galileo establishes a one-to-one correspondence between the degrees of speed represented by a triangle to those represented by a parallelogram. From this correspondence, Galileo concludes that “equal spaces will be run through in the same time” (Drake 1989, 166), in the uniformly accelerated motion or the corresponding uniform motion in which the speed is one-half the final degree of speed of the accelerated motion. The justification for this step has been given differently by different authors (see, e.g., Damerow et al. 2004, 241; Clavelin 1974, 301; Wisan 1974, 292). The important point for us is that it is not made explicit in the demonstration. The reason we think is that Galileo, even if trying to achieve the Euclidean ideal of rigor in his geometry of motion, is developing a heuristic mathematics to deal with the uniformly accelerated motion. In particular, a degree of speed is not a fully developed mathematical entity.

While his mathematics is heuristic, as a demonstration of proposition 1 it is applied indirectly in the demonstration of proposition 2. This proposition is not somewhat ad hoc because of this. It can be seen as a pure geometry counterpart of the experimental law of fall – that is, the law stated in proposition 2 but taking space to correspond to the measured distance gone by a material body, undergoing a naturally accelerated motion, and time to correspond to the measured time. It might seem that Galileo failed to give a mathematical foundation to proposition 2; on the contrary, he pointed – with his heuristic mathematics – to the needed further development of the geometry of motion.

7 This is not to say that the historical development of the approach applied by Galileo is not well-known. That is not the case. The main elements of this historical development are the work of some scholars, associated to the Merton College, on the intension and remission of forms, in particular, their “Merton rule”, and Oresme’s geometrical representation of qualities and his work on the “Merton rule”. (see, e.g., chapters 5 and 6 of Clagett 1961).

With his mathematical description of the uniformly accelerated motion, Galileo extended the realm of ancient Greek geometry of motion. His further work on the motion of projectiles enabled Galileo to apply Archimedes' approach to the composition of motions to the case in which one of the motions is a uniformly accelerated motion. To see the continuity with Archimedes' approach it will suffice to consider proposition 1 of the chapter on the motion of projectiles. It is as follows:

> When a projectile is carried in motion compounded from equable horizontal and from naturally accelerated downward [motion], it describes a semiparabolic line in its movement. (Drake 1989, 217)



This proposition is restated further on as follows: "The line described by a heavy moveable, when it descends with a motion compounded from equable horizontal and natural falling [motion], is a semiparabola" (Drake 1989, 221).

In both cases, like in previous ones, the proposition is stated using non-mathematical terms, making reference to motion in nature: "projectile", "heavy moveable", "naturally accelerated downward", "descends", "natural falling". Even if this is so we are dealing with a proposition of pure geometry of motion. The terminology points towards the relation between the mathematically described motion – the motion of a point, to its counterpart in experience – the motion of a material body; it serves as a "bridge" between the geometrical and the experimental.

When we come into the demonstration it's all geometry. For the proof of this proposition, it is necessary to consider a proposition regarding parabolas demonstrated by Apollonius (Drake 1989, 219). Here, we are well within the orthodoxy of ancient Greek geometry. In the proof, we consider a geometrical point (whose counterpart in experience is a moveable and called as that) having two motions: (a) a uniform motion, initially along a line BE which we consider as horizontal (again to relate with experience); (b) a uniformly accelerated motion, initially along a line BL which we consider vertical. From a mathematical point of view, like in the case of the composition of motions in Archimedes' *On spirals*, this combination of motions is instantiated on a geometrical plane. To the points C, D, and E, in the "horizontal" line correspond the points O, G, and L, in the "vertical" line BE. These are determined by "dropping" perpendiculars from points C, D, and E. The length of these perpendiculars CI, DF, and EH are determined by the times-squared law. Using parallels to the line BE this determines the points O, G, and L in BE.

Galileo demonstrates that "the square of HL will be to the square of FG as line LB is to BG, while the square of FG [will be] to the square of IO as GB is to BO" (Drake 1989, 222). In this way, by taking into account Apollonius' proposition, Galileo concludes that "points I, F, and H lie in one and the same parabolic line" (Drake 1989, 222; see also Wisan 1974, 258-9). Like in the case of Archimedes' spiral, a curve in the geometrical plane – a semiparabola – is instantiated in the plane by considering a geometrical point undergoing two combined motions. Like in the case of the spiral, we can see Galileo's approach as a "definition" of a semiparabola adopting the geometry of motion. This is equivalent to the locus definition of the parabola made adopting static geometry. Like Archimedes' geometry of motion, its further development by Galileo corresponds to a pure mathematics – a pure geometry of motion.

4 Newton's geometry of motion

In his *Philosophiae naturalis principia mathematica* (usually referred to as the *Principia*), Newton makes use of a new kind of geometry that while encompassing Euclid's geometry goes beyond it. It is, in fact, a geometry of motion superseding the one developed by Archimedes and the heuristic superseding of this by Galileo.

While the *Principia* is a very long treatise, its core can be found in the first three sections of book 1. According to Newton, to be able to read book 3 it suffices "to read with care the definitions, the laws of motion, and the first three sections of book 1" (Newton 1999, 439).

The *Principia*, published in 1687, arose from a vast expansion of a small treatise written by Newton in the autumn of 1684 entitled *De motu corporum*, which consisted just in three definitions, four hypotheses, four theorems, and seven problems (Newton 1974, 30-74). For our purpose, we will need to consider the core of the core of *De motu corporum*. This consists in Newton's determination of a geometrical expression that characterizes the motion of a point, around a "central" point, along a closed curve, like a circumference or an ellipse, on a geometrical plane. This is made in theorems 1 and 3 of *De motu corporum*. These theorems correspond in the *Principia* to the propositions 1 and 6 of book 1, which are in different aspects more elaborated versions of the initial theorems. One crucial difference between the treatments made in the two works is that while important features of the geometry applied by Newton in the first treatise remain implicit (for example in hypothesis 4 that corresponds to lemma 10 of the *Principia*), in the *Principia* we have an initial section with eleven lemmas where this is made explicit (Newton 1999, 79-89).[8]

There is a further work to consider that is relevant for our purpose. In it, Newton makes clearer general aspects of his geometry of motion underlying the propositions of *De motu corporum* and the *Principia*. It is a treatise not published or even completed, named *Geometria curvilinea*, written around 1680 (Newton 1971, 409-413).

Finding Euclid's *Elements* "scarcely adequate" for his purposes, Newton developed a geometry extending Euclid's one, whose "elements" he presented in the book 1 of his treatise (Newton 1971, 423-5). Newton's definition of a curve (line) is made in terms of the motion of a point, like in the case of Archimedes' *On spirals*: "The locus of a moving point is the line, straight or curved, which that point describes in its movement" (Newton 1971, 427). Newton considers two postulates that, according to Whiteside, are "an addition to the 'static' set given by Euclid in his *Elements*" (Newton 1971, 428). The first postulate regards the motions of lines that are admissible. The second postulate regards the lines that are taken to be given in Newton's extended geometry. Postulate 1 is as follows:

> That any line may move in any geometrical fashion whatever. By 'geometrical fashion' I understand such a fashion of moving that any position of a line moved in it can be geometrically designated. (Newton 1971, 427)

A simple example of this postulate at work is found, e.g., in proposition 26. In this proposition, it is considered the partial rotation of a line in relation to a point: "P will be the pole of angular motion of the straight line BD" (Newton 1971, 467). This is similar to the case of Archimedes' *On spirals*, where, as we have seen, Archimedes considers a straight line "one of whose extremities is fixed [that] turns with uniform speed in a plane" (Clagett 1961, 171).

---

8 This had already, basically, been incorporated in the second revision of the earlier treatise (see Newton 1974, 107-123).

Considering now postulate 2 it is as follows: "That there are given the lines described by points or the intersections of lines moved in a geometrical fashion" (Newton 1971, 429). In simple terms this postulate says what kind of curve (line) we can consider in Newton's geometry: (a) curves generated by the motion of points; (b) curves generated by the point of intersection of curves in motion.

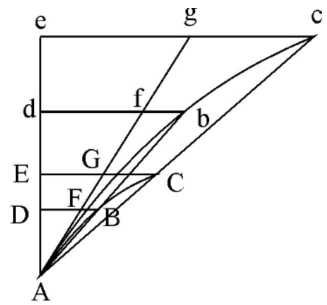
By adopting explicit definitions and his two postulates, Newton is presenting his geometry of motion adopting a well-framed system like that of Euclid's *Elements*; in fact, it is an addition to it. What we have just seen should show beyond any doubt that Newton is developing a pure geometry of motion, as "formal" as Euclid's geometry is.

Another aspect of Newton's geometry is his adoption of a notion of geometrical limit. This comes about by Newton consideration of changing magnitudes in his geometry. This was not part of Archimedes' geometry of motion. As we have seen, Galileo made a heuristic approach to the change of velocity by adopting the notion of degree of velocity and a representation of it as a straight line. However, Galileo was unable to give a geometrical description of changing magnitudes, like velocity in the case of a non-uniform motion. Magnitudes, for Newton, are generated by continuous motion, or at least their change depends on the underlying continuity of motion or time (Guicciardini 2009, 171 & 180). We have, in Guicciardini's words, "geometrical magnitudes varying by continuous flow" (Guicciardini 2009, 218). This enables to consider "nascent" or "evanescent" magnitudes (like the distance covered at the beginning of an accelerated motion), and the limit of the ratio of two magnitudes when these are "nascent" or "vanishing" magnitudes – the "first" or "ultimate" ratio. This mathematical approach is made explicit in the axiom 6 of *Geometria curvilinea* (Newton 1971, 427), and made use of throughout the treatise. According to Guicciardini:

The demonstrations in '*Geometria curvilinea*' […] depend on the determination of the limits of ratios and sums of vanishing magnitudes. Typically, Newton needed to evaluate the limit to which the ratio between two geometrical magnitudes tends when they vanish simultaneously. (Guicciardini 2009, 218)

Here, we will focus on the limit approach as adopted and deployed in the *Principia*. The lemma 9 of section 1 of book 1 is as follows:

If the straight line AE and the curve ABC, both given in position, intersect each other at a given angle A, and if BD and CE are drawn as ordinates to the straight line AE at another given angle and meet the curve in B and C, and if then points B and C simultaneously approach point A, I say that the areas of the triangles ABD and ACE will ultimately be to each other as the squares of the sides. (Newton 1999, 83)



In the proof of this lemma, Newton takes the points B and C (in the curve ABC) each to have a motion so that they come to coincide with point A at the same time. "Ultimately", when the points are meeting, the (finite) curvilinear figures Abd and Ace coincide with the triangles Afd and Age (since the chords come to coincide with the tangent Ag). From a previous lemma, it follows that the ratio of the areas Afd and Age is as the squares of their sides Ad and Ae. According to Newton, "areas ABD and ACE are always proportional to these areas, and sides AD and AE to these sides" (Newton 1999, 83).[9]

9 Notice that implicit in the demonstration is the idea that the length of Ae is fixed, while Ad and Ae are constantly proportional to AD and AE. This enables to "follow in the finite" the variations of the curvilinear triangles ABD and ACE (see De Gandt 1995, 231-2).

From this it follows that "areas ABD and ACE also are ultimately in the squared ratio of the sides AD and AE" (Newton 1999, 83).

Let us now, like it was made before by Galileo, take the time as a mathematical magnitude to be represented by another mathematical entity – a straight line. In this case, let the time gone by point B in its motion starting at A be AD, and the time gone by point C in its motion starting from A be AE. Instead of considering a motion of B and C towards A and an ultimate ratio, we are considering the nascent motion of points B and C starting at A. Like with Galileo, velocity is a magnitude. However, now it is a geometrical magnitude varying by continuous flow. Newton takes the velocity generated during the continuous motion, at the times AD and AE, as represented by DB and EC (again following Galileo). According to Newton, "the spaces described by these velocities will be as the areas ABD and ACE" (Newton 1999, 84). By a direct application of lemma 9 Newton demonstrates lemma 10: "The spaces which a body describes [when having a centripetal acceleration] are at the very beginning of the motion in the squared ratio of the times" (Newton 1999, 83-4).[10] Here, Newton obtains Galileo's proposition 2 (for uniformly accelerated motion) – Galileo's law of fall – as a limiting result: at the beginning of an accelerated motion, the spaces run through are to each other as the squares of the times. Newton result enables to apply Galileo's law of fall in cases where the acceleration is not constant or always in the same direction. It is, in this sense, a generalization of Galileo's law. Also, it is not a heuristic result like it was the case with Galileo. Newton's geometry is not just a development of an Archimedean geometry of motion like Galileo's; it is also a geometry of continuously flowing magnitudes. As such, Newton can address changes in magnitudes – in this case, velocity – within his geometry.

To see the importance of lemma 10, let us consider theorems 1 and 3 of *De motu corporum* (corresponding to the propositions 1 and 6 of book 1 of the *Principia*). Theorem 1, that is necessary for the demonstration of theorem 3, enables to relate the areas spanned by the radius of a curve to the time the end-point of the radius takes to move through the corresponding arcs of the curve. Implicit in the demonstration is the possibility of taking a curve, which can be considered as corresponding to the curve of a point with a uniform motion combined with an acceleration towards a center – like a circumference or an ellipse –, as the limit of a polygonal figure (Pourciau 2003, 267-311). Newton shows, for the polygonal figure, that the triangular areas, spanned by the radius during the motion of the point in equal times, are equal. By considering the limit in which the polygonal figure coincides with the curve, Newton obtains the same result for the curve: to equal times correspond equal areas (Newton 1974, 35-7).

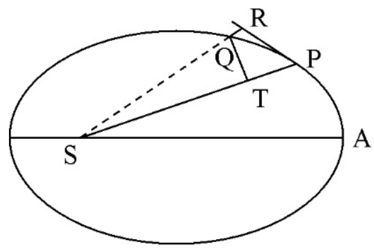
As just mentioned, Newton considers curves that can be seen as resulting from a point moving by a combination of two motions. One is a uniform motion; the other an accelerated motion. The case Newton addresses in theorem 3 is similar to that of the motion of projectiles addressed by Galileo. It is, however, a more general case. In the case of the projectile, a point has a "horizontal" uniform motion combined with a "vertical" uniformly accelerated motion. In the case dealt with by Newton, the situation is as follows:

The fall or quasi-fall of the [geometrical point] is not vertical with constant direction but is instead directed toward a point, the fixed center S. Moreover, the intensity of the [acceleration] varies from point to point in [the plane] […] the moving [point] is subjected

10 Newton presents his lemma 10 in terms of force. However, as mentioned by Guicciardini, "force is often equated with acceleration" (Guicciardini 1999, 14). In the theorems 1 and 3 that we will be considering we deal with centripetal forces. Force is proportional to acceleration, where the constant of proportionality is the mass. In the propositions, this constant is not made explicit. In fact, if we make a dimensional analysis of the result of proposition 6 we are obtaining a magnitude with the dimensions of m/s^2; i.e., the dimensions of an acceleration. We adopt "centripetal acceleration" instead of force for this reason.

to [an acceleration] that varies, though perhaps only slightly, along its path. In [Newton's] case, then, Galileo's law is applicable only in the infinitely small […] is thus valid only if [the arc under consideration] is very small or 'nascent'. (De Gandt 1995, 12)

Let us see in detail the role of lemma 10, in the form of hypothesis 4, in theorem 3. The enunciation of the theorem is as follows:



> If a body P is orbiting round the centre S shall describe any curved line APQ, and if the straight line PR touches that curve at any point P and to this tangent from any other point Q of the curve there be drawn QR parallel to the distance SP, and if QT be let fall perpendicular to this distance SP: I assert that the centripetal [acceleration] is reciprocally as the 'solid' SP^2 x QT^2/QR, provided that the ultimate quantity of that solid when the points P and Q come to coincide is always taken. (Newton 1974, 41)

Lemma 10 is at work at the beginning of the demonstration of theorem 3. According to Newton:

For in the indefinitely small configuration QRPT the line-element QR is, given the time, as the centripetal [acceleration] and, given the [acceleration], as the square of the time, and hence, when neither is given, as the centripetal [acceleration] and the square of the time jointly. (Newton 1974, 41)

The proportionality of QR, for a given time, to the acceleration follows from the second law of motion, which in *De motu corporum*, is still a tacit assumption (De Gandt 1995, 32). This does not depend on taking the limit when the points P and Q come to coincide. In this case, however, we are in a situation where lemma 10 applies. In relation to point P, we are considering the "commencement" of an accelerated motion covering a distance QR (corresponding to the nascent arc PQ). According to lemma 10, the distance covered is proportional to the square of the time. According to Newton, since neither the time or the acceleration are given, the distance covered due to the accelerated motion is proportional to the centripetal acceleration and the square of the time conjointly (Brackenridge 1995, 91-2). From theorem 1, it follows that "the area SQP [is] proportional to the time (or its double, SP x QT) taken twice" (Newton 1974, 41-3). From this result, it follows that the centripetal acceleration is inversely as SP^2 x QT^2/QR when the points P and Q come to coincide. This result shows that the nascent arc PQ results from the combination of a uniform motion along the tangent PR and, in this nascent arc, a constant acceleration (proportional to QR/ SP^2 x QT^2) towards S leading to covering the distance QR. We are in the same situation as that described by Galileo related to the motion of projectiles. The nascent arc PQ is the nascent arc of a semi-parabola. We find in Newton's demonstration of theorem 3 all the elements of Galileo's extension of ancient Greek geometry of motion, improved by Newton's incorporation of changes in magnitudes into his more fully developed geometry of motion.

5. Conclusion

Newton's geometry as developed, or applied, in *Geometria curvilinea*, *De motu corporum*, and in the *Principia*, can be seen as a geometry of motion as envisaged by

Marshall Clagett in relation to part of the ancient Greek geometry. Mathematicians like Autolycus, Euclid, and Archimedes made the first steps into going beyond a "static" geometry as that of the *Elements*. With Autolycus' *On the moving sphere* and Euclid's *Phaenomena*, we find the geometrical treatment of a uniformly rotating sphere. Time is explicitly taken into account even if it remains unclear its mathematical treatment. In Archimedes' *On spirals*, we find a geometrical treatment of (rectilinear) uniform motion in which like the distances, time is treated as a magnitude, and the uniform velocity is characterized in terms of the proportionality between the ratio of distances and the ratio of times. A more mathematical treatment of velocity is made by considering the distances traversed by two points in the same time and with different uniform velocities. However, velocity is not yet treated as a magnitude. The comparison between the uniform velocities of each point is made indirectly by considering the distances traveled by each point in two equal times. Another feature of *On spirals* is the composition of motions with which is defined the spiral. This curve is described by a point undergoing two combined motions: the uniform rotation of a segment and the uniform motion of the point along the segment.

In Galileo's *Discorsi e dimostrazioni matematichi intorno à due nuove scienze*, we find a further development of the ancient Greek geometry of motion. His treatment of uniform motion is pretty much that of Archimedes, however, with an important addition: velocity is taken to be a magnitude. Where we find a crucial contribution by Galileo is in his geometrical treatment of the uniformly accelerated motion. Even if adopting a heuristic approach, Galileo demonstrates that the spaces run through at any times whatever are to each other as the squares of those times. Galileo also applied Archimedes' approach to the composition of motions to the case in which one of the motions is a uniformly accelerated motion and the other a (rectilinear) uniform motion. In this case, Galileo found that a point describes a semi-parabolic line in its movement.

With Newton, we find two new features, one of them of the utmost importance. He explicitly mentions that he's developing a geometry that goes beyond that of the *Elements*, and by adopting explicit definitions and postulates related to geometrical motion (i.e. the motion of geometrical objects), Newton presents a geometry of motion adopting a well-framed system like that of the *Elements*. Besides this, Newton addresses within his geometry continuously flowing magnitudes; i.e., Newton is able to address changes in magnitudes. Both these features can be found in *Geometria curvilinea*. However, the importance of Newton's geometrical treatment of continuously varying magnitudes is at full display in *De motu corporum* and the *Principia* (where the mathematics of changing magnitudes is more fully developed). Newton adopts in *De motu corporum*, as hypothesis 4, a generalization of Galileo's law of fall, which is demonstrated in the *Principia* as lemma 10 by using his approach in terms of the geometrical limit that enables to determine the ratio of changing magnitudes (specifically, the first or ultimate ratio of nascent or vanishing magnitudes). Newton considers curves that can be seen as resulting from a point moving by a combination of two motions. One is a uniform motion; another an accelerated motion due to a centripetal acceleration. The direction and magnitude of the centripetal acceleration change along the curve; the acceleration is always directed towards a center and its magnitude is, for a nascent arc beginning at a point P, proportional to $QR/ SP^2 \times QT^2$. This result is obtained by considering the validity of Galileo's law of fall in the limit; i.e., for a nascent arc. In this case, we have the same combination of motions dealt with by Galileo: the nascent arc is the nascent arc of a semi-parabola. With Newton, Galileo's heuristic treatment of a changing velocity is substituted by a geometrical treatment of continuously flowing magnitudes that is part of a well-framed geometry of motion.

References

Autolycos de pitate, *La sphere en movement. Levers et couchers héliaques. Testimonia*. Paris: Les Belles Lettres.

Berggren, Len, and Thomas, Robert (1996). *Euclid's Phaenomena*. Providence: American Mathematical Society.

Brackenridge, J. Bruce (1995). *The key to Newton's dynamics: the Kepler problem and the Principia*. Berkeley: University of California Press.

Clagett, Marshall (1961). *The science of mechanics in the middle ages*. Madison: The University of Wisconsin Press.

Clavelin, Maurice (1974). *The natural philosophy of Galileo*. Cambridge: MIT Press.

Clavelin, Maurice (1983). Conceptual and technical aspects of the Galilean geometrization of the motion of heavy bodies. In W. R. Shea (ed.), *Nature mathematized, vol. 1*. Dordrecht: D. Reidel Publishing Company.

Damerow, Peter, Freudenthal, Gideon, McLaughlin, Peter, and Renn, Jürgen (2004). *Exploring the limits of preclassical mechanics, second edition*. New York: Springer.

De Gandt, François (1995). *Force and geometry in Newton's Principia*. Princeton: Princeton University Press.

Dijksterhuis, Eduard (1956). *Archimedes*. Copenhagen: Ejnar Munksgaard.

Drake, Stillman (1978). *Galileo at work. His scientific biography*. Chicago: The University of Chicago Press.

Drake, Stillman (1989). *Two new sciences, second edition*. Toronto: Wall & Emerson.

Euclid, (1908). *The thirteen Books of the Elements,* second edition. Translated with introduction and commentary by Sir Thomas L. Heath, from the critical edition of Heiberg. Cambridge: Cambridge University Press.

Evans, James (1998). *The history and practice of ancient astronomy*. Oxford: Oxford University Press.

Funkenstein, Amos (2004). *Theology and the scientific imagination*. Princeton: Princeton University Press.

Guicciardini, Niccolò (1999). *Reading the principle: the debate on Newton's mathematical methods for natural philosophy from 1687 to 1736*. Cambridge: Cambridge University Press.

Guicciardini, Niccolò (2009). *Isaak Newton on mathematical certainty and method*. Cambridge: MIT Press.

Heath, Thomas (1897). *The works of Archimedes*. London: C. J. Clay and Sons.

Heath, Thomas (1981). *A history of Greek mathematics*, vol. 1. New York: Dover Publications.

Jullien, Vincent (2015). Indivisibles in the work of Galileo. In V. Jullien (ed.), *Seventeenth-century indivisibles revisited*. Heidelberg: Birkhäusen.

Mugler, Charles (1970). *Archimède*, vol. 1. Paris: Les Belles Lettres.

Netz, Reviel (2004). *The works of Archimedes*, vol. 1. Cambridge: Cambridge University Press.

Netz, Reviel (2009). *Ludic proof: Greek mathematics and the Alexandrian aesthetic*. Cambridge: Cambridge University Press.

Netz, Reviel (2017). *The works of Archimedes, vol. 2*. Cambridge: Cambridge University Press.

Newton, Isaac (1971). *The mathematical papers of Isaak Newton*, vol. 4. Edited by D. T. Whiteside. Cambridge: Cambridge University Press.

Newton, Isaac (1974). *The mathematical papers of Isaak Newton*, vol. 6. Edited by D. T. Whiteside. Cambridge: Cambridge University Press.

Newton, Isaac (1999). *The principia: mathematical principles of natural philosophy*. The authoritative translation by I. B. Cohen and A. Whitman. Oakland: University of California Press, 1999.

Pourciau, Bruce (2003). Newton's argument for proposition 1 of the Principia. *Archive for History of Exact Sciences* 57, 267-311.

Wisan, Winifred (1974). The new science of motion: A study of Galileo's *De Motu Locali*. *Archive for History of Exact Sciences* 13, 103-306.