Remarks on the applications of paraconsistent logic to physics*

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Dedicated to Michel Paty, scholar, philosopher, and friend.

Abstract

In this paper we make some general remarks on the use of non-classical logics, in particular paraconsistent logic, in the foundational analysis of physical theories. As a case-study, we present a reconstruction of P. -D. Février's 'logic of complementarity' as a strict three-valued logic and also a paraconsistent version of it. At the end, we sketch our own approach to complementarity, which is based on a paraconsistent logic termed 'para-classical logic'.

"He utters his opinions like one perpetually groping and never like one who believes he is in possession of definite truth."

Einstein on Bohr, *apud* A. Pais [37]

"Whoever writes a purely mathematical work can be considered as the author of a novel. He has the same privileges that are conceded *pictoribus atque poetis*. I can, v.g., invent a new curve and prove various of its properties. I can write a treatise on optics, where I take as an hypothesis that the light does not propagate according to a straight line, but according to a circular curve, or according to any curve whatsoever. And if my theorems and my solutions of the problems were legitimately derived from the principles I have proposed, no one can attribute to me any mistake."

J. Anastácio da Cunha (Portuguese mathematician, 1744-1787) [14]

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1 Introduction

Why paraconsistent logic? (henceforth simply 'PL'). Generally, there is no precise criterion for deciding whether a given abstract system can be regarded as a well defined logic or not. This is usually determined by the scientific community. Since nowadays paraconsistent logic is included among the best known systems of logic (just see the 03B43 Mathematics Subject Classification) [2], we can say that it has been accepted by the scientific community as *a logic*. But, are there other reasons which justify the important role this kind of logic has been assumed in present day scientific activity? Of course the study of this particular case can help us in the understanding the role played by non-classical logics in general.

The development of logic since the end of the XIX century has ascribed to this discipline a fundamental role in practically every field of contemporary knowledge, for instance in philosophy of science, metaphysics, ethics, mathematics, economy, computing sciences and even technology. Nowadays logic is 'mathematical' by nature, having rendered a field of knowledge which uses mathematical techniques. So, as in mathematics, we can say that a system of logic can be developed basically from two points of view: as a *pure* one or as an *applied* one. 'Pure' logic, as pure mathematics, can in principle be developed *in abstracto*, independently of possible applications. So, we can study paraconsistent logic or intuicionistic logic by themselves, basically with the aim of exploring their abstract mathematical properties. From this point of view, in developing a logical system, the logician can proceed as Hilbert has suggested, when he said that "[t]he mathematician [or the logician] will have to take account not only of those theories that come near to reality, but also, as in geometry, of all logically possible theories" [28] (see also our second motto above).

So, following these guidelines, we might develop abstract (pure) systems where some principle of classical logic is violated, as for instance that principle which entails that from contradictory premises any formula can be derived, in symbols, $\alpha \wedge \neg\alpha \vdash \beta$ (Duns Scotus Law, which is valid not only in classical logic, but in almost all the known logical systems, like intuicionistic logic). This is the way taken by the Russian philosopher N. A. Vasiliev, who perceived that the rejection of the law of non-contradiction could lead to a 'non-Aristotelian' logic in the same way as the violation of the parallel postulate of Euclidean geometry had conduced to non-Euclidean geometry. Vasiliev, as we shall recall below, is one of the forerunners of paraconsistent logic. In fact, we can sum up the meaning of paraconsistent logic in this way: it is devoted to the study of logical systems which can base inconsistent theories (that is, theories which have contradictory theses, like α and $\neg\alpha$) but which are not trivial (in the sense that not every well formed formula of their languages are also theses).

But in developing logic we could also proceed from the *applied* point of view, looking at some domain of knowledge where our intuition feels that some logic (in particular a paraconsistent one) could be useful for describing abstract structures that reflect the way certain deductive inferences are made within such a domain. One of the best known examples is provided by Birkhoff and von Neumann's approach to quantum logic, in saying that quantum mechanics would demand a logic distinct from the classical one [4]. As examples involving PL, we can recall that these logics have also been applied in technology, as we shall mention in the next section. On the other side, it has been also claimed that certain primitive civilizations (like the Azande) reason according to paraconsistent rules [39] (other examples shall be mentioned below). Of course we are not sustaining that they (the Zande people) really reason this way, but independently either they do that or not, the very interesting philosophical and anthropological discussions on the Zande way of reasoning can make sense only if we have at our disposal well developed abstract logical systems which fit such a possibility, for, without the (essential) formal counterpart, all discussions turn out to be pure speculation.

In this paper, we shall consider PL more from the 'applied' perspective, but before to begin with, let us comment on a general misunderstanding about the nature of PL. Sometimes we find someone saying that non-classical logics like PL were proposed because we think classical logic is wrong and that it must be replaced by a suitable one (in accordance with some philosophical criterion) [26, p. 1]. This is the case, for example, of intuicionistic Brouwer-Heyting logic, if we consider it as a culmination of Brouwer's original philosophy of mathematics. Brouwer's stance implies that, in a certain sense, classical mathematics has basic shortcomings and that a constructive mathematics should take its place; the underlying logic of this constructive mathematics being a new one, different from the classical. Nowadays, there are also some philosophers, especially in Australia, who strongly believe that classical logic should be replaced by another one (most of them think that the good logic would be a relevant logic).

But this opinion regarding classical logic does not fit ours. We think that classical logic is a fantastic subject which has and will continue to have strong interest and applications. The only problem is that in some specific domains, as we shall show below, other logics, in particular paraconsistent logics, may be more adequate for expressing certain philosophical or even technical reasons so as to make explicit some of the underlying structures which (apparently) fit more adequately what is being assumed in these fields, since classical logic (apparently) can't do that in full. This does not show that classical logic is wrong, but that its field of application should be restricted. At least, the use of non-classical logics in some fields helps us to better understand important aspects of these domains.

A nice example is the discussion of the nature of negation, which has been better understood with the raise of PL (another example is the significance of Russell's set –see [11]). Furthermore, it should be recalled that PL (in our view) keeps classical logic valid in its particular domain of application. Really, in this sense PL can be viewed not only as an 'heterodox' logic (or 'rival' logic [26], that is, as a logic which deviate from classical logic in what respects some of its principles), but also as a *supplement* to classical logic, for it coincides with this one if we take into consideration just what are called 'well-behaved propositions' (roughly, those propositions that obey the principle of non-contradiction). In short, and we hope this can be put definitively, at least according to our point of view, we don't intend to pray according to PL rules. PL may be useful in some domains, as shown below, but we shall continue to use classical logic, or other logics, when we find they are convenient or necessary.

We could also make the claim, with Granger [25], that paraconsistent logic can and should be employed in some developments, but only as a preliminary tool; in future researches, classical logic could finally be a substitute for it, as the underlying logic of those developments. In addition, it is worthwhile to note that constructive mathematics may be investigated from the point of view of classical logic and that in this way we are able to get really significant results.

In synthesis, there are in principle various 'pure' logics whose potential applications depend not only on a priori and philosophical reasons, but, above all, on the nature of the applications one has in mind. Our aim in this paper is only to call attention to some aspects of certain paraconsistent logical systems originated by physical questions. Tough the subject didn't achieve its definitive form yet, it seems really stimulating, specially from the philosophical point of view.

2 Paraconsistent logics: elements of its history

The forerunners of PL were specially the Polish logician Jean Łukasiewicz and the Russian logician Nicolai I. Vasiliev (for further historical details, see [3], [21]). Both, in 1910, following a 'pure' point of view, have presented general ideas which have contributed for the development of PL; Łukasiewicz discussed the possibility of violating the ancient Aristotelian Principle of Contradiction, but he did not elaborate any logical system to cope with his intuitions. It was his disciple, Stanislaw Jaśkowski, who did that in 1948. Jaśkowski constructed a system of propositional PL (known as *discussive logic*, in a piece originally written in Polish; the English translation of his work appeared only in 1969 [30]), where he distinguished between contradictory –inconsistent– systems and trivial ones, with the aim of 1) systematizing theories which contain contradictions, as in dialectics, 2) studying those theories where contradictions are caused by vagueness, and 3) directly studying empirical theories whose postulates or basic assumptions are contradictory (for historical details, see [3], [21]). Vasiliev was inspired by motivations quite similar to those of Łukasiewicz, although independently developed, culminating by presenting in 1912 and 1913 his system of *imaginary logic*, which expresses his conception that contradictions do not exist in our 'real' world, but only in a possible world created by our mind (*ibid*).

Starting in 1953 and 1954, the first author of this paper began the development of his ideas on paraconsistency (without knowledge of the above mentioned authors) in seminars given at the Federal University of Paraná, Brazil. Reasoning from a 'pure' point of view, he was motivated by certain mathematical problems, apparently being the first logician to develop the idea of PL as a field of living research, containing infinitely many logical systems which turn out to have relevant applications, as we shall see below (da Costa's systems $\mathcal{C}_n$,

$1 \leq n \leq \omega$, are nowadays well known; see [8], [10]).

The philosophical and the technical developments of these logics was remarkable (the Third World Congress on Paraconsistency was held in Toulouse in July 2003 [41]). Today, paraconsistent logic touches various areas like ontology, the philosophy of science, applied science, and technology, for example robotics, expert systems, flexible computing and medicine (see for instance [1], [36]). Moreover, the theoretical and technical counterparts evolved very much, giving rise, for instance, to paraconsistent model theory, paraconsistent set theory and paraconsistent geometry.

3 Paraconsistency and physics

The study of the relationships between logic and physics is a difficult and wide topic which cannot be examined in just one short paper.

We may recall the relevance of the foundational analysis of physical theories, a topic which is related to their axiomatization and of making explicit their mathematical counterpart. Historically, the importance of such a study was pointed out especially by Hilbert in the sixth of his celebrated 23 Mathematical Problems [28]. This kind of analysis, as it is well known, was fundamental also for the development of some XXth century philosophies, like neo-positivism, and has echo also in the semantical view, so as in the 'structuralist' approach initiated by Sneed, Stegmüller and others. However all of this is made within the paradigm of classical logic.

Concerning the use of non classical logics in physics, there have been only a few insights, not conclusive as far as we know up today. For instance, Bressan's suggestion of using certain modal logics was not fully developed; Reichenbach's three valued logic, although interesting from the point of view of the insights and clarification of some basic assumptions underlying quantum mechanics, has been criticized for not providing a full logical basis for such a discipline, mainly due to the lack of a detailed discussion of quantification (the propositional level of his logic does not suffice for physical purposes – see [27]). By the way, when we hear something about the relationships between logic and physics, we usually associate the subject with the so called 'quantum logics', a field that has its 'official' birth in Birkhoff and von Neumann's well known paper from 1936 [4]. Originally proposed to cope with some problems which are originated from quantum mechanics (like the apparent violation of the distributive law $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$), their fundamental work, which we could say came from an 'applied' stance (since their system was motivated by an empirical science), caused the development of a wide field of research in logic. Today there are various 'quantum logical systems', which have been usually studied as pure mathematical systems, practically far from applications to the microphysical world and from the insights of the forerunners of quantum mechanics (for a general and updated account on this field, see [20]). Recently, Dalla Chiara and Giuntini presented a 'paraconsistent quantum logic' ([18]; see also [19], [20]), in which certain 'classical laws', like the non contradiction and the excluded middle laws are violated, but their system will not be recalled here, in spite of its importance.

3.1 Truth in Physics

Another kind of relationship between PL and physics came from M. L. Dalla Chiara and G. Toraldo di Francia's important concept of truth as applied to physical theories. Informally speaking ([16, Chap. 3], [42, Sec. 1.10]) a physical law does not make reference to a specific physical system, but says that a mathematical relation holds among the values of certain physical magnitudes, as Newton's second law $f = m.a$, which relates the magnitudes 'force', 'mass' and 'acceleration'. More generally, a physical law may be written as $P(x_1, \ldots, x_n)$, where P expresses the mathematical relation which holds among the magnitudes $x_1, \ldots, x_n$. However, when we ask for the meaning or the truth value of a certain physical proposition, we need to make reference to certain states of physical systems, which we do in the metalanguage of our physical theory. In this metalanguage, we also describe experiments and measurement processes, and in particular we introduce certain *operational definitions* for the relevant physical magnitudes.

Given an operationally defined concept like force, F, it is assumed that due to the imprecision of measurements the acceptable values lie within a certain interval of real numbers (with an 'error' ϵ, which represents the precision of the instruments). For instance, given an operationally defined concept F of force related to a physical system s, there is an interval $[f - \epsilon, f + \epsilon] \subseteq \mathbb{R}$ such that whatever value p in such an interval can be accepted as the value of F for the system s in a certain state; in this case, we may say that F is equal to f with precision ϵ.

Within this semantic schema, it is possible to introduce the concept of a 'physical model' as a structure $\mathcal{M} = \langle M, E, T \rangle$, where M stands for the mathematical counterpart of $\mathcal{M}$ (for instance, usual functional analysis in the case of quantum mechanics), E is the 'experimental' counterpart of the model, which is composed by a class S of physical systems whose states are determined by certain physical magnitudes 'operationally defined' (what means that any magnitude is associated to a class of procedures suitable to provide the calculation of the values of this magnitude with a certain characteristic precision ϵ), and T is a translation map, that gives a mathematical interpretation to the elements of E.

Then, the authors characterize the concept of a sentence α being true with respect to a physical system $s \in E$ as follows. Firstly, α must be *defined* with respect to s, which intuitively means that what α asserts "has a meaning with respect to s" [15, p. 165], that is, supposing that α is the proposition $P(x_1, \ldots, x_n)$, then each of the magnitudes occurring in α determines a physical quantity X_i ($i = 1, 2, \ldots, n$) which can be measured in s (usually, they are represented by operators). Then, the results of the measurements in s of these physical quantities must admit n values $g_1, \ldots, g_n$ that in the mathematical counterpart M of the model satisfy the relation P with a prescribed precision ϵ.

When this is the case, we say that α is *true* with respect to s, written $\vDash_s \alpha$, and that α is true in the model $\mathcal{M}$, in symbols, $\mathcal{M} \models \alpha$, if and only if α is defined for at least one $s \in E$ and, for any s for which α is defined, then $\vDash_s \alpha$ (*ibid.*).

For instance, let us take again Newton's second law $f = m.a$. The three physical variables appearing in this equation correspond to three physical quantities *force* (F), *mass* (M) and *acceleration* (A), whose acceptable range of values for a certain physical situation s lie respectively within three intervals $[f_1, f_2] \subseteq \mathbb{R}$, $[m_1, m_2] \subseteq \mathbb{R}$ $[a_1, a_2] \subseteq \mathbb{R}$, each one of them expressing a certain precision ϵ for the measurements. Then $\vDash_s f = m.a$ when there exist three real numbers $p_1 \in [f_1, f_2]$, $q_1 \in [m_1, m_2]$ and $r_1 \in [a_1, a_2]$ such that $p_1 = q_1.r_1$.

However, due to the imprecision ϵ, there may be also other three real numbers p_2, q_2 and r_2, each one in the respective interval, so that $p_2 \neq q_2.r_2$, and these numbers are also *acceptable values* for the measurements of the corresponding physical quantities (in the same way as we could accept that in measuring our table we can find 'something like' 1.20 m). So, $\vDash_s \neg(f = m.a)$ too, that is, the negation of Newton's law should also be true with respect to the same physical situation s. This way, we may have, for a sentence α and physical situation $s \in E$, both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$, but of course not $\vDash_s \alpha \wedge \neg\alpha$, for this last case would entail the existence of three real numbers p', q' and r' belonging to the respective intervals such that $\vDash_s p' = q'.r' \wedge p' \neq q'.r'$, which is impossible [15, p. 168].

This definition of truth reflects a kind of *empirical truth*, having interesting consequences pointed out by the authors, like the non truth-functionality of the logical connectives, in the sense that the truth of a conjunction is not equivalent to the simultaneous truth of both conjuncts. To us here, it is interesting to note the 'paraconsistent aspect' of this definition of truth. It is clear from the above that this concept is related with paraconsistency, for we can have both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$. Truly, in [13] it was shown that the logic which describe this phenomenon is a particular paraconsistent logic, namely, a Jaśkowski's logic (one of the characteristics of this kind of logic is that from α and $\neg\alpha$ we don't necessarily derive the conjunction $\alpha \wedge \neg\alpha$).

3.2 Février's logic of complementarity

Among the suggestions for the use of non-classical logics in physics, we recall those which are related to three-valued logics. The best known cases are those of H. Reichenbach and P. Destouches-Février, although Février's logic isn't strictly a three-valued logic,[1] since the truth tables for the connectives are not functions from the cartesian product of truth-values to the set of truth-values. In what follows, after making some remarks on her 'logic of complementarity', we present a reconstruction of it as a strictly three-valued logic, and afterwards we introduce a paraconsistent version of it.

[1] Although frequently mentioned in the literature, Février's system $L_{c,3}$ has not been discussed in detail; for instance, S. Haack [26] only mentions Février's logic $L_{c,3}$, while Jammer [29] considers it, but without discussion on the logical system proper. We think that this is due to the fact that her system was never presented in a systematic way.

Let us first recall that the concept of 'complementarity' was introduced in quantum mechanics by Niels Bohr in his famous 'Como Lecture' in 1927.[2] The consequences of his ideas were fundamental for the development of the Copenhagen interpretation of quantum mechanics and constitute, as is largely recognized in the literature, one of the most fundamental contributions for the development of quantum theory. Notwithstanding their importance, Bohr's ideas on complementarity are controversial. Truly, it seems that there is no general agreement on the precise meaning of his *Principle of Complementarity.* Although some authors like C. von Weizsäcker and M. Strauss have tried to elucidate Bohr's principle from a logical point of view, their developments never found an acceptable agreement from the scientific community. For instance, it is well known that Bohr himself rejected von Weizsäcker's attempt of logically describing his principle (cf. [29, p. 90]). Strauss' proposal of a logic in which two propositions, say α and β (which stand for complementary propositions) may be both accepted, but not their conjunction $\alpha \wedge \beta$, was considered as 'inadvisable' by R. Carnap, although it seems to deserve attention from a present day point of view.[3]

In a series of papers, which culminated in the presentation of the book *La Structure des Théories Physiques* [22], P. Destouches-Février sketched a propositional logic to deal with complementary propositions, termed $L_{c,3}$.[4] Her main motivations for developing $L_{c,3}$ are Heisenberg's Uncertainty Relations, which according to her are to be taken as basic physical principles and not as consequences of the mathematical formalism of quantum theory [29, p. 362]. The central idea is that the composition (read: the conjunction) of complementary propositions like those referring to position and moment of a certain particle, are to be regarded as *propositions incomposables*, to which the logical conjunction should not be performed. But even so she writes $p\&q$ for such a conjunction, and says that it has to have the logical value *absolutely false*, A. In other words, even by taking for granted that there are "*propositions incomposables*, pour lesquelles on ne peut jamais affirmer le produit logique" [22, p. 33], she assumed that the conjunction of these propositions has a third true value, distinct from true and false, the *absolutely false*, which according to her would indicate that such a conjunction could not be performed in a strict sense (this shows that her system is not in conformity with standard many-valued logic).

Février uses a binary connective '&' for conjunction, but in reality it is taken to stand ambiguously for two distinct conjunctions: one for the 'composable' propositions and another for the 'incomposable' ones, and both conjunctions are characterized semantically by a kind of 'generalized matrices'.[5] Perhaps we can motivate the use of these connectives following Nagel (cf. [35]): if 'a_i' is defined as 'The coordinate q_i of a particle i has value q_{0_i}' and 'b_i' as 'The component p_i of the momentum of the particle i has the value p_{0_i}', then there are two

[2]All the references of this section, not explicitly referred to here, may be found in [12].

[3]Mainly if we take into account Jaśkowski's discussive logic [13].

[4]See the reviews of some of her works [33], [34] and [35].

[5]She does the same concerning disjunctions, also using '$\vee$' ambiguously for both composable and incomposable propositions.

matrices for characterizing respectively the conjugate product '$a_i \& b_i$' and the non-conjugate products '$a_i \& a_k$', '$b_i \& b_k$' and '$a_i \& b_k$' for $i \neq k$ (these matrices are presented below). So, at the bottom she distinguished (by some meta-rule, not made explicit) between propositions which can and which cannot be composed, by introducing appropriate connectives (conjunctions and disjunctions) to link them.

Here we interpret her proposal by distinguishing between two conjunctions, namely, $\&_c$, which link 'composable propositions', and $\&_i$, to be used for the incomposable ones (similarly for disjunctions). So, using matrices like Février's, we can interpret $L_{c,3}$ as a standard three-valued logic (we shall use the same notation, $L_{c,3}$, to designate the resulting system). We remark that in our system we shall always perform conjunctions. To be more precise, we adapt Février's terminology, taking as basic the following connectives $\&_c$ (first conjunction), $\&_i$ (second conjunction), $\rhd$ (exclusive disjunction), $\vee_c$ (first disjunction), $\vee_i$ (second disjunction), $\equiv$ (first equivalence), $\simeq$ (second equivalence), $\rightarrow$ (implication), $\mathbf{N}$ (first negation) and $\sim$ (second negation), which are defined by the following matrices, whose designated truth-value is T ([22, pp. 34-39]):

$\&_c$	T	F	A
T	T	F	A
F	F	F	A
A	A	A	A

$\&_i$	T	F	A
T	A	A	A
F	A	A	A
A	A	A	A

$\rhd$	T	F	A
T	A	T	T
F	T	A	F
A	T	F	A

$\vee_c$	T	F	A
T	T	T	T
F	T	F	F
A	T	F	A

$\vee_i$	T	F	A
T	A	T	T
F	T	A	F
A	T	F	A

$\equiv$	T	F	A
T	T	F	F
F	F	T	F
A	F	F	T

$\simeq$	T	F	A
T	T	F	F
F	F	T	T
A	F	T	T

$\rightarrow$	T	F	A
T	T	F	F
F	T	T	F
A	T	T	T

p	T	F	A
$\mathbf{N}\ p$	F	T	A
$\sim p$	F	T	T

The problem (as already pointed out by Nagel in his review [35]) is that she doesn't provide any criterion for distinguishing between propositions which can and which cannot be composed. Some meta-rule is to be supposed. Anyway, by following her ideas, and accepting that such a distinction can be made (the way we shall present her system below, by accepting that conjugate or complementary propositions can always be joined by the connectives $\&_i$ and $\vee_i$, makes such a rule dispensable), the intuitive explanations of these symbols may be as follows. $\&_c$ is the conjunction of propositions which *can* be composed ('composable' propositions), while $\&_i$ is the conjunction of complementary propositions (whose conjunction receives the truth value A, the *absolutely false* –'faux absolu'). So, according to Février's interpretation, the consideration of two (complementary) propositions like

$p =_{\mathsf{df}}$ "the component p_x of y has a value between p_0 and $p_0 \pm \Delta p_0$ at time t"
$q =_{\mathsf{df}}$ "the coordinate x of y has a value between x_0 and $x_0 \pm \Delta x_0$ at time t"

are subjected to Heisenberg's Uncertainty Principle, hence their conjunction $p \&_i q$ has to have truth value A if Δp_0 and Δx_0 don't satisfy Heisenberg's inequality. The connective $\rhd$ stands for (as she says) a generalization of the classical exclusive disjunction; $p \rhd q$ is true when only p or q is true, absolutely false when the propositions (which can be composed) have the same true value and false otherwise.

The connectives $\vee_c$ and $\vee_i$ are the disjunctions which link propositions which can and which cannot be composed respectively. $\equiv$ has the same motivation it has in classical logic, so that $p \equiv q$ is true if and only if both p and q have the same true value. But $\simeq$ is a little bit different: $p \simeq q$ is true when either p and q are both true or not true, and it is false when one of the propositions is true and the other is not true, that is, F or A. The table of the implication $\rightarrow$ generalizes the classical material implication.

Now concerning the negations. Due to the semantic characterization of the connectives, it results (as noted by Février) that the Double Negation rule does not hold for $\sim$, so that $\mathbf{N}(p \&_x q)$ is not equivalent to $\mathbf{N}p \vee_x \mathbf{N}q$, where x is in both cases either c or i. The same can be said concerning $\sim (p \&_x q)$ and $\sim p \vee_x \sim q$. Other results are given by the theorems below (proven from the matrices above, taking T as designated).

We remark that our reconstruction of Février's logic is not in accordance with her views, although it surely reflects some of her intuitions regarding the logic of quantum mechanics. Taking validity and the semantic concepts defined as usual, we have the following theorems, where Latin letters will stand for formulas and capital Greek letters for sets of formulas:

Theorem 1

(i) We have in $L_{c,3}$:

$\models p \rightarrow \sim\sim p$

$\models p \&_i \sim p \rightarrow q$

$\models p \&_i \mathbf{N}p \rightarrow q$

$\models \sim (p \&_c \sim p)$

$\models \sim (p \&_i \mathbf{N}p)$

$\models \sim (p \&_c \mathbf{N}p)$

$\models (p \rightarrow q) \rightarrow ((p \rightarrow \sim q) \rightarrow p)$

$\models p \vee_i \sim p$

$\models p \rightarrow (q \rightarrow p)$

$\models p \&_c q \rightarrow p$

$\models p \,\&_i q \rightarrow p$

(ii) The Deduction Theorem holds in the following form: if $\Gamma, p \models q$, then $\Gamma \models p \rightarrow q$.

(ii) Modus Ponens is a semantical valid rule of inference: if both p and $p \rightarrow q$ are true, then q is true. Other semantically valid inference rules are:

$$p, \sim p \vee_c q \models q, \qquad \sim p, p \vee_i q \models q.$$

Theorem 2 *The following formulas express that p has the values T, F and A respectively:*

$p \equiv \sim (p \&_i p)$

$p \equiv \sim (p \equiv p)$

$p \equiv (p \&_i p)$

Theorem 3 *The following schemes are not valid in $L_{c,3}$:*

$\sim\sim p \rightarrow p$

$p \,\&_i \sim p$

$p \,\&_c \sim p$

$p \,\&_c \mathbf{N}p$

$p \,\&_i \mathbf{N}p$

$p \,\&_c \sim p \rightarrow q$

$p \,\&_c \mathbf{N}p \rightarrow q$

$p \rightarrow (\sim p \rightarrow q)$

$p \rightarrow (\mathbf{N}p \rightarrow q)$

$\mathbf{N}(p \,\&_i \mathbf{N}p)$

$\mathbf{N}(p \,\&_c \mathbf{N}p)$

$\mathbf{N}(p \,\&_i \sim p)$

$\mathbf{N}(p \,\&_c \sim p)$

$p \vee_i \mathbf{N}p$

$p \vee_c \sim p$

$p \vee_c \mathbf{N}p$

$(p \rightarrow \sim p) \rightarrow p$

$(\sim p \rightarrow p) \rightarrow p$

$(p \rightarrow q) \equiv (\sim p \vee_c q)$

$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$

$(p \rightarrow q) \equiv (\mathbf{N}q \rightarrow \mathbf{N}p)$

$(p \rightarrow q) \equiv (\sim q \rightarrow \mathbf{N}p)$

$(p \rightarrow q) \equiv (\mathbf{N}q \rightarrow \sim p)$

$(p \rightarrow q) \equiv (\sim p \vee_i q)$

$(p \rightarrow q) \equiv (\mathbf{N}p \vee_i q)$

$(p \rightarrow q) \equiv (\sim p \vee_c q)$

$(p \rightarrow q) \equiv (\mathbf{N}p \vee_c q)$.

Although Février did not develop her logic in detail, she made some interesting remarks on complementarity, based on her informal discussion. As she says, even if we mix up the values F and A, her logic does not reduce to classical logic. So, she concludes, "within a theory where Bohr's complementarity is introduced, it is impossible to use classical logic for the calculus of experimental propositions; it is necessary to use a logic of complementarity" (*op. cit.*, p. 40). This is expressed by means of 'Février's theorem': In a theory where the experimental propositions satisfy a complementary condition, it is necessary to use a logic of complementarity which is irreducible to classical logic (*loc. cit.*).

However, Février does not confine her logic to experimental propositions. Speaking of complementarity and wave mechanics, she concludes that complementarity logic is necessary also for dealing with theoretical propositions which are deduced within a schema which includes complementarity, hence her logic being also applicable to theoretical propositions. According to her,

> "... la logique de complémentarité ne joue plus seulement pour les énoncés expérimentaux, mais aussi pour les énoncés théoriques, ce qui va imposer une structure toute nouvelle à la mécanique ondulatoire.
>
> "Ceci fait apparaître un autre aspect de la complémentarité de Bohr, plus profond que celui s'est révélé par les raisonements de Heisenberg sur les mesures, et qui est la complémentarité entre l'aspect corpusculaire et l'aspect ondulatoire. Cette fois il ne s'agit plus d'une simple complémentarité de fait, mais une complémentarité de droit, qui doit faire partie du corps même de la théorie, et non plus être seulement une limitation aux possibilités expérimentales." (*ibid.*, pp. 44-45).

Her belief that the complementarity logic is the right logic for a theory involving complementary propositions makes her able to answer the three questions posed at the beginning of her book, which are close to F. Gonseth's ideas that logic has an empirical face, so that the principles of logic should express general physical laws; the first, asking whether logic is universal and unique, normative and a priori, expressing the laws of a pure reason and not expressing any contend of knowledge, is answered in the negative. The same holds for the second question, which asks whether logic is arbitrary under certain conditions of coherence (consistency) and hence being a kind of tautology, that is, a syntax independent of all knowledge. But the third question is answered positively; the question asks whether logic, in each of its applications, should be adapted to the domain of knowledge in which it will be applied. In this case, logic would not be neither a priori and independent of whatever application, nor an arbitrary syntax: as she says, "[n]ous voyons qu'il n'y a pas une logique unique et universelle, normative *a priori*, exprimant les lois d'une raison pure, et indépendante de toute connaissence (...) mais qu'elle est adapteé à chaque domaine de connaissance, en particulier à chaque théorie physique, et que, par conséquent, elle exprime un certain contenu de connaissance" [22, p. 41]. Classical logic, she says, due to the above 'Février's theorem', is suitable for classical physics, including relativity and "a certain theory of fields" (*ibid.*), complementarity logic being the logic adequate for quantum theories.

It is important to observe that if in $L_{c,3}$ we call a proposition *normal* if and only if it can assume only the truth-values T or F, then the propositional logic of normal propositions is the standard propositional calculus. This implies that, in a determinate sense, classical logic is contained in $L_{c,3}$ (of course, this way we may introduce normal predicates etc., and rebuild classical logic inside Février's system). In order to do that, one has to give a (metalogical) definition of normal proposition, predicate, etc., as well as one needs to characterize Février's incomposable propositions.

Before sketching a way of looking to $L_{c,3}$ from a 'paraconsistent point of view', we shall turn to some criticisms that were presented to her system and philosophical position.

3.3 Some criticism

In 1954, McKinsey and Suppes made a review of Février's book in which they presented several criticisms to the idea of employing her logic in physics and in particular to her 'theorem' mentioned above, which says that modern physics demands a logic with more than two truth-values [34].[6] McKinsey and Suppes' review is important not only for their discussion of Février's theses, but for presenting to the reader interesting insights related to a general discussion on the relation between logic and physics.

Let us begin by recalling in short some of the main arguments against Février's ideas. The first objection is concerning with Février's 'theorem' above,

[6]In the same volume, McKinsey published a critical review of another work of her where similar views are defended [33].

in conformity with which quantum theory would demand a three valued logic. McKinsey and Suppes (in short, McK-S) don't see why this would be necessary, and according to them Février's arguments are not conclusive. We don't want to discuss this topic now, for it will be treated below within a more general context. But McK-S also say that in order to base a physical theory on a non-classical logic, it would be necessary to present such a field as a formal system, and Février doesn't do that. They still suggest that before completing such an "herculean labor", perhaps it would be better to get an axiomatization of quantum theory in the 'ordinary mathematical sense', through a set-theoretical predicate formulated in standard set theory (hence, using classical logic), as they themselves (with Sugar) made with classical particle mechanics (see [38]). Of course, the use of classical logic, although keeps physics with an adequate mathematical apparatus for expressing physical laws and results, opens the door for formidable philosophical problems that arise in connection with the use of standard set theory in quantum mechanics, as shown for instance in [9] (see also [10]) and [17]).

McKinsey and Suppes' criticism that Février restricted her system to the sentential calculus, while quantifiers should be used in whatever logic suitable for quantum mechanics, is of a deep nature. Interesting to remark that the same criticisms were made by Hempel concerning Reichenbach's three valued logic [27] and by Church in what respects Birkhoff and von Neumann's non-distributive logic [7]. We think that they are right in this point. If Février had restricted the idea of complementarity, and hence the application of the connectives $\&_i$ and $\vee_i$, to 'experimental propositions' only, the situation would be more satisfactory. In this case, we could say that her sentential calculus concerns experimental propositions only, which could obey the laws of her three-valued logic, and so keeping classical logic to cope with the metamathematical discourse, including the use of quantifiers, and maybe set theory. But this is not what she does. As already recalled above, Février intends that her system can be extended also to wave mechanics (cf. [22], pp. 43ff).

Keeping in mind the possibility of reconstructing classical logic inside our $L_{c,3}$ (as remarked at the end of the last section when we talked about 'normal' propositions), criticisms like those of McKinsey and Suppes can be surmounted, and so the door is open to sustain the thesis that $L_{c,3}$, even in the form proposed by Février, is the underlying logic of a possible axiomatization of quantum mechanics: the truth-value A only appears in connection with experimental propositions, the remaining ones being two-valued. Therefore, as we have indicated, one can make a synthesis of $L_{c,3}$ and classical mathematics.

This kind of discussion shows that the axiomatization of a given empirical theory is not something which is always totally determined. It depends on the several aspects of the theory that we consider, explicitly or implicitly, appropriate to take account of it. So, for example, Ludwig [31] studies an axiomatization of quantum mechanics based on classical logic. Both stances, that of Février (according to our interpretation) and that of Ludwig, are in principle right, since they treat different perspectives of the same empirical domain, and only the future of physics will perhaps decide which is the better solution, what involves pragmatic factors.

3.4 A paraconsistent look on Février's logic

Let us define a logic $L^p_{c,3}$ with the same language as that of $L_{c,3}$, and giving to the connectives the same semantic characterization above. But now, we take T and A as designated truth-values. That is, we take the matrix (with the same tables as above)

$$\mathcal{M} = \langle \{T, F, A\}, \{T, A\}, \&_c, \&_i, \rhd, \vee_c, \vee_i, \equiv, \simeq, \rightarrow, \mathbf{N}, \sim \rangle$$

to characterize what we will call the 'paraconsistent complementarity three valued logic'. The notion of semantic consequence of a set of formulas is introduced in the standard way: we say that a proposition p is a consequence of a set Γ of propositions, in symbols, $\Gamma \models p$, if and only if for all valuations in which the propositions of Γ have a designated value, p has also a designated value.

It is easy to see that $p \,\&_i \sim p$ is a tautology of $L^p_{c,3}$ (that is, it has always a designated value, namely, A), but not every proposition of its language is a tautology, for instance, $p \,\&_c \sim p$ is not. Hence, $L^p_{c,3}$ is inconsistent (perhaps we should say '$\sim$-inconsistent', for this inconsistency is related to the second negation $\sim$) though it is not trivial and, as it is easy to see, can be the underlying logic of inconsistent but non trivial theories; that is, it is paraconsistent.

In our logic every proposition receives a truth value (T, F or A), so do $p\&_i q$ and $p\&_c q$. But when p and q are complementary, the first conjunction always receives the designate value A, while the second one obeys the table for $\&_c$. Then, the two conjunctions of such propositions *can* be meaningfully performed. We remark that this approach resembles Reichenbach's logic in that quantum propositions have always truth-values. This is in accordance with physics, where we can meaningfully speak of position *and* momentum of a certain particle, although the problem whether the position and the momentum can be both precisely measured at the same time is subjected to Heisenberg's principle. In $L^p_{c,3}$, the formulas assuming only T and A are those which are acceptable.

$L^p_{c,3}$ constitutes a propositional logic with the help of which we are also able to talk about the foundations of quantum mechanics, similarly to what happens with $L_{c,3}$. In addition, it is possible to extend $L^p_{c,3}$ to a strong paraconsistent logic to cope with theoretical contradictions and to make it strong enough to encompass a great part of extant mathematics. Thus, we have another alternative road to found some parts of quantum mechanics, in addition to Février's and Ludwig's systems.

3.5 Using paraclassical logic

In [12], we followed another way to surmount some of the problems of quantum mechanics. To give an idea of this alternative, firstly let us introduce a new kind of physical theory, whose language is supposed to contain complementary propositions; such theories are called $\mathcal{C}$-theories or *complementary theories*.

A $\mathcal{C}$-theory has a language, possesses a set of axioms, and its notion of consequence, also called *paraclassical* consequence, is introduced as follows: if $\Gamma \cup \{p\}$ is a set of formulas of T, then we say that p is a paraclassical consequence of Γ, and we write $\Gamma \vdash_{\mathsf{P}} p$, if and only if

(i) $p \in \Gamma$, or

(ii) p is a classical valid formula, or

(ii) there exists a consistent (according to classical logic) subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash p$ (in classical logic).

Let us suppose that a $\mathcal{C}$-theory T be such that there are formulas p, q and r of its language satisfying the conditions (a) $T, \vdash_{\mathsf{P}} p$ and $T, \vdash_{\mathsf{P}} q$; (b) $T, p \vdash_{\mathsf{P}} r$ and $T, q \vdash_{\mathsf{P}} \neg r$. In this case, we say that p and q are complementary theorems of T; despite (b) above, it happens that in general $r \wedge \neg r$ is not a theorem of T. So, in a $\mathcal{C}$-theory, we may deal with propositions like 'x is a particle' and 'x is a wave', each of which entailing the negation of the other, without the consequence that they will conduce to a strict contradiction, that is, to a formula of the form $r \wedge \neg r$.

It is important to remark that, according to our approach, complementary propositions are not necessarily such that one of them is the negation of the other (this is a particular case) but, more generally, propositions each of which having *consequences* which may contradict some consequence of the other. This is, we believe, in accordance with some of the ideas raised out by Bohr himself, as we have emphasized in [12], although we shall not develop this topic here. Anyway, our definition is quite general and of course is not restricted to physics, being useful in other situations as well. Let us give an example, which could be more elaborated.

Suppose a judge who is confronted with 'complementary normative propositions' that *should* hold simultaneously, like to preserve the free will of the citizens and to preserve also the obligations imposed by the State. In doing so, sometimes a contradiction may arise. For instance, take the case of some prisoners who are going hungry to conquest some advantages, say more time for visit. The judge, under the 'complementary' situation of having to take into account both the free will of the prisoners in going hungry to sustain their position and the role of the State in preserving one's life (since such a strike may conduce some prisoners to serious illness), is confronted with a situation involving complementary propositions (normative complementary propositions). This kind of situation exemplify very well that ideas involving complementary situations are not restricted to physics, as Bohr himself had already suggested [6]. Anyway, interesting it may be, this topic shall be left to be analyzed in another time.

4 General remarks

We are inclined to agree with some of Février's remarks (which, as mentioned above, follow a tendency derived from Gonseth [24, Chap. 8]), mainly upholding that applied logic has an empirical counterpart. Nevertheless, we have some observations to make. First, taking into consideration our distinction between pure and applied logic, it is not necessary to eliminate the a priori traces of logic. Of course we are not simply endorsing the position that there is just one logic and that it is independent of whatever field of knowledge. What we say is that logic *can* be studied independently of any application, as a pure mathematical system, hence being a priori in certain sense. Any applied logical system possesses an a priori dimension and an a posteriori one. For instance, we could begin by studying Février's system, which is supposed to be motivated by the empirical science, verifying whether it can be axiomatized, afterwards proving a completeness theorem and so on. From another point of view, logic deals with the underlying structures of inference of particular domains or theories, and in this sense a field like the quantum world may suggests that a different logic (that is, other than classical logic) could be useful to cope with certain features which cannot be dealt with by means of classical logic. As an example, if we accept the view (advanced by E. Schrödinger, M. Born and others) that quantum objects are *non-individuals*, having no individuality in the sense that one is always indistinguishable of any other of a similar species, then it seems that in looking at the quantum world as constituted by entities of this kind, classical logic (with its Leibniz's Principle of the Identity of Indiscernibles) and classical mathematics (founded on the very notion of set, that is, collections of *distinguishable* objects), should be revised (concerning these points, see [23]). So, different 'perspectives' of a domain of science may demand for distinct logical apparatuses, which put us on a philosophical point of view very different from the classical.

The possibility of using non-standard systems does not necessarily entail that classical logic is wrong, or that (in particular) quantum theory *needs* at the moment another logic. Physicists probably will continue to use classical (informal) logic in the near future. But we should realize that other forms of logic may help us in the better understanding of certain features of the quantum world as well, not easily treated by classical devices, as the concepts of complementarity and of non-individuality show.

To summarize, we think that there is not just one 'true logic', for distinct logical (so as mathematical and perhaps even physical) systems can be useful to approach different aspects of a wide field like quantum theory. If we push this view a little bit, although we shall not develop this philosophical point here, we could say that our philosophical position may be called pluralist (but not relativist).[7]

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