LOGICAL FOUNDATIONS OF PHYSICS. PART I. ZENO'S DICHOTOMY PARADOX, SUPERTASKS AND THE FAILURE OF CLASSICAL INFINITARY REASONING

RADHAKRISHNAN SRINIVASAN*
C-113, Symphony CHS Ltd.
Jain Mandir Marg, Nahar's Amrit Shakti Township
Chandivali, Mumbai 400072, India
rk_srinivasan@yahoo.com

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Abstract

Several variants of Zeno's dichotomy paradox are considered, with the objective of exploring the logical foundations of physics. It is shown that Zeno's dichotomy paradox leads to contradictions at the metamathematical (as opposed to formal) level in the basic classical infinitary reasoning that is routinely used in theoretical physics. Both Newtonian mechanics and special relativity theory suffer from these metamathematical inconsistencies, which occur essentially because the classical refutation of the dichotomy paradox requires supertasks to be completed. In previous papers, Non-Aristotelian Finitary Logic (NAFL) was proposed as a logical foundation for some of the basic principles of quantum mechanics, such as, quantum superposition and entanglement. We outline how the finitistic and paraconsistent reasoning used in NAFL helps in resolving the metamathematical inconsistencies that arise from the dichotomy paradox.

1 Zeno's dichotomy paradox

Probably the first serious challenge to the use of classical infinitary reasoning in physics came from Zeno's paradoxes of motion, in particular, the dichotomy paradox, originally proposed by Zeno in the fifth century BC [1, 2, 3]. Zeno's dichotomy paradox has remained completely unresolved for over 2400 years till date, despite claims to the contrary. And contrary to conventional wisdom,

*Permanent address (for correspondence): 1102 Tejas Heights, Plot 245, Mumbai Tamil Sangam Marg, Sion East, Mumbai 400022, India.

we will demonstrate that variants of the dichotomy paradox are not merely philosophical paradoxes, but actually lead to logical contradictions in classical and other systems of logic that support the infinite divisibility of space and time. The source of these contradictions, which occur at the metamathematical (rather than formal) level, is the fact that the classical resolution of the dichotomy paradox requires the completion of infinite tasks or supertasks [4]. Here we outline how these contradictions are resolved in non-Aristotelian finitary logic (NAFL) [5, 6, 7], which is a paraconsistent, finitistic logic that, in our view, plays an important role in the logical foundations of physics. In Part II of this two-part series of papers [8], the logic NAFL will be explained in detail. For a historically accurate account of Zeno's paradoxes of motion, in the way that Zeno actually formulated them, consult the appropriate references [1]. A modern version of the dichotomy paradox is all we will need for our purposes and is as follows.

Homer, initially located at $(x, t) = (0, 0)$, runs at a constant unit velocity along the x-axis towards a stationary target, say, a finish line located at $x = 1$, so that Homer's path is described by $x(t) = t$. Here x is a Cartesian coordinate and t is a dimensionless time in the standard real number system. As confirmed by experiment, Homer will reach the finish line in finite time, at $t = 1$, a conclusion which Zeno ingeniously contradicted. According to Zeno's argument, Homer first has to cover half the distance to the finish line in order to reach the point $x = 1/2$. He then has to cover half the remaining distance to the finish line in order to reach the point $x = 3/4$, and so on, *ad infinitum*. Zeno concluded that the Homer will never reach the finish line, essentially for the following two reasons (paraphrased in modern language).

- Homer has to run through the following infinite sequence of points, one at a time and in a finite time interval:

$$x_n = 1 - (1/2)^n, \; n = 0, 1, 2, \ldots, . \tag{1}$$

  This is an infinite task (supertask) which, contrary to conventional wisdom, arguably cannot be completed in finite time. Note that Homer reaches each point x_n in (1) at the corresponding instant t_n, where

$$t_n = 1 - (1/2)^n, \; n = 0, 1, 2, \ldots, . \tag{2}$$

- Homer has to traverse the following infinite sequence of distances in a finite time interval:

$$d_n = x_n - x_{n-1} = (1/2)^n, \; n = 1, 2, \ldots, . \tag{3}$$

  This is another supertask with an additional paradox of the infinitely many finite, nonzero distances d_n summing to a finite distance. Our strong intuition here is that this sum must diverge, contrary to what modern mathematics tells us. Note that Homer covers each distance d_n in a corresponding duration τ_n, where

$$\tau_n = t_n - t_{n-1} = (1/2)^n, \ n = 1, 2, \ldots, . \tag{4}$$

Next we will discuss in detail both of Zeno's objections to Homer reaching his target at $x = 1$.

1.1 Zeno's first objection — Supertasks

Running through the infinite sequence of points in (1) in finite time is equivalent to the process of counting the infinite sequence of natural numbers $\{0, 1, 2, \ldots\}$, one at a time and in a finite time interval. It is certainly paradoxical to claim that such a supertask can be completed, as the following argument shows.

Nonclassical induction argument Consider the process of counting the natual numbers in sequence. At any instant t during the counting process, at most one natural number n is counted and an infinite number of natural numbers remains uncounted. Since this is true for arbitrary t and n, one can conclude by induction that infinitely many natural numbers will always remain uncounted during every moment of such a counting process, which can therefore never be completed. In fact such a counting process does not even get started, and any 'progress' made in the counting is purely illusory, because the infinitely many uncounted numbers will always map one-to-one and onto the full sequence of natural numbers.

In our opinion, this induction argument is correct, despite (as we will show later) there being a classically unacceptable interchange of universal and existential quantifiers that is inherent in its conclusion. Therefore, the heavy machinery of modern mathematics and classical logic notwithstanding, Zeno's dichotomy paradox has never been satisfactorily resolved till date. Note that the nonclassical induction argument holds regardless of the speed of the counting process, which can *never* be completed, *even in principle.*

Define a local counting rate κ_n at each (x_n, t_n) as the reciprocal of the time taken by Homer to reach the next point x_{n+1}. From (4), we conclude that

$$\kappa_n = 1/\tau_{n+1} = 2^{n+1}, n = 0, 1, 2, \ldots, .$$

Here κ_n has to be understood as locally defining the rate at which the points of the sequence (1) are covered per unit time by Homer at each (x_n, t_n). Thus the counting rate is two points per unit time at $(x_0, t_0) = (0, 0)$, four points per unit time at $(x_1, t_1) = (1/2, 1/2)$, and increases exponentially with n. Nevertheless, only one point x_n of (1) is covered at each t_n, and in accordance with the nonclassical induction argument, there are always infinitely many points of this sequence that remain uncovered during each moment of Homer's run on $t \in [0, 1)$. Yet, we find that at $t = 1$ Homer has mysteriously bridged this seemingly unbridgeable gap and has covered all the (infinitely many) points x_n to reach $x = 1$. Note that the counting rate at $(x, t) = (1, 1)$ is zero because nothing remains to be counted.

In summary, Zeno's first objection may be stated concisely as follows. An infinite counting process that, by our nonclassical induction argument, *did not even get started* at any $t \in [0, 1)$ is deemed to be *complete* at $t = 1$, at which time nothing gets counted! Note that Zeno's first objection, though stated here in terms of the infinite sequence of spatial points x_n covered by Homer, applies equally to the infinite sequence of instants t_n in (2). Zeno could have asked, with equal validity, how Homer managed to complete experiencing (not necessarily consciously) infinitely many instants t_n sequentially and in a finite time interval. Thus we may conclude that Zeno's first objection strikes at the infinite divisibility of space and time, which is a technicality that is routinely accepted today by modern mathematicians and physicists.

1.2 Zeno's second objection — convergence of infinite summations

The classical refutation of Zeno's dichotomy paradox depends crucially on the following result:

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \tau_n = \sum_{n=1}^{\infty} (1/2)^n = 1. \tag{5}$$

Here d_n and τ_n are as defined in (3) and (4) respectively. Note that each term in these infinite summations is finite and positive. Given that there are no nonzero infinitesimals in the standard real number system, the question is how these infinitely many positive terms can sum to a finite magnitude. Our strong intuition here is that these infinite summations must diverge, despite the justification of (5) by modern mathematics. This intuition arises because infinitely many times any positive magnitude, no matter how small, is always infinite. One might attempt to argue that this intuition is wrong in the context of (5) because while each term $(1/2)^n$ in these summations is positive, there is no smallest term and secondly, because of the following mathematical fact:

$$\bigcap_{n=1}^{\infty} [0, (1/2)^n] = \{0\}. \tag{6}$$

Evidently, the intersection of the infinitely many nested intervals in (6) consists of the single point zero. Therefore one might argue that there is no apparent contradiction in the infinitely many positive magnitudes $(1/2)^n$ summing to a finite magnitude because no positive magnitude gets added infinitely many times, as our intuition tells us. However, this explanation only shifts the burden to (6), namely, how did infinitely many nested intervals, each of which has a positive length, intersect to a single point? Again our strong intuition is that if *none* of these nested intervals has a zero length, their intersection must also be an interval of positive length.

Hence Zeno's second objection is essentially that the infinite summations in (5) must diverge and consequently Homer will never reach the finish line at $x = 1$. Note that Homer has to cover the distances d_n sequentially in time, which implies that Zeno's first objection is also embedded in this supertask.

1.3 The classical refutation of Zeno's dichotomy paradox

The classical refutation of Zeno's dichotomy paradox employs modern mathematics to essentially establish the following.

- Homer does experience the actual infinity of instants t_n in (2) during which he crosses, in sequence, the actual infinity of points x_n in (1). Note that Homer completes two simultaneous supertasks here, one of which is purely temporal and the other, spatio-temporal.
- The infinite summations in (5) do converge to a finite value as indicated.
- From the premise that Homer has completed the supertasks of sequentially traversing the infinitely many distances d_n in the infinitely many durations τ_n, one may conclude [2] that Homer has reached $x = 1$ at $t = 1$.

Note that the distances d_n, and likewise, the durations τ_n, are entirely contained in the half open real interval $[0, 1)$. Therefore nontrivial arguments are needed for a proof of the final of the above points. The first step is to observe that the convergence of the infinite summations in (5) implies that Homer, after traversing these infinitely many distances/durations sequentially, must necessarily have reached a point infinitesimally close to $x = 1$ at an instant infinitesimally close to $t = 1$. The next step is to use the fact that the standard real number system does not contain any nonzero infinitesimals in order to conclude, as noted above, that Homer must necessarily have reached $x = 1$ at $t = 1$. This classically acceptable inference requires the assumption that Homer must exist at $t = 1$ after traversing the unit distance, which we take for granted in this paper. Benacerraf [9] has attempted to refute this inference via his postulation of a shrinking genie in place of Homer in Zeno's dichotomy paradox. However, Benacerraf's attempt fails because his genie does not meet the stated requirement of existence at $t = 1$. In Sec. 3, we demonstrate that Benacerraf's shrinking genie, which covers all the points $x \in [0, 1)$ and then vanishes before reaching $x = 1$, is of interest in its own right because it is even more paradoxical than Zeno's original formulation of the dichotomy.

The mathematical machinery needed to prove these results initially emerged with the development of the calculus by Newton and Leibniz in the seventeenth century. Over the next two hundred years or so, further work by Euler, Bolzano, Cantor, Cauchy, Dedekind, Frege, Hilbert, Lebesgue, Peano, Russell, Weierstrass and Whitehead, among others, culminating in the development of Zermelo-Fraenkel set theory in the early twentieth century, finally provided the various concepts needed to develop standard real analysis without the notion of nonzero infinitesimals. Today the majority view in the mainstream research community is that Zermelo-Fraenkel set theory with the axiom of choice (ZFC) provides rigorous foundations for standard real analysis, in which Zeno's paradoxes, among others, stand successfully refuted. See, for example, Ref. [10], where the authors assert that "...our notions of infinity and continuity are now so well developed that supertasks have lost their power to force refinement of these notions." It must also be emphasized that there still exists a significant dissenting minority view. Alternative systems, such as, predicative analysis [11] or nonstandard analysis [12], have been proposed for the resolution of Zeno's paradoxes.

The nonclassical induction argument given in Sec. 1.1 stands refuted in classical logic because of a classically unacceptable interchange of universal and existential quantifiers that is inherent in this argument. Thus to go from

$$\forall t \in [0,1)\, \exists n \in N\ (t_n > t) \tag{7}$$

to

$$\exists n \in N\, \forall t \in [0,1)\ (t_n > t) \tag{8}$$

is a classically unacceptable deduction. Here N is the class of all natural numbers, $\{0, 1, 2, \ldots\}$. Eq. (7) correctly asserts that Homer has experienced only finitely many instants t_n (as defined in (2)) up to each given time $t \in [0,1)$. Eq. (8) unsuccessfully attempts a seemingly straightforward generalization of (7) to the entire half open time interval $[0,1)$. In words, what the deduction of (8) from (7) expresses is that if only finitely many instants t_n have passed up to each given time $t \in [0,1)$, then only finitely many of the instants t_n can pass in the entire time interval $[0,1)$ and infinitely many of these instants will always remain inaccessible, which is precisely what the nonclassical induction argument asserts. In effect this deduction calls into question the very existence of the time interval $[0,1)$ by rejecting its infinite divisibility via the sequential passage of infinitely many instants t_n. In support of this classically banned deduction, it is natural to expect that what is true of each $t \in [0,1)$ must also be true of the entire time interval $[0,1)$ in the following sense. If *each* $t \in [0,1)$ is bounded above by some t_n (as asserted in (7)), then the collection of all such instants t that are bounded above by some t_n is precisely the time interval $[0,1)$, which essentially expresses that *all* the instants $t \in [0,1)$ have been exceeded by some t_n. In other words, *each* means *all* and this is exactly what (8) asserts.

1.3.1 Comments on the classical refutation

Eq. (8) requires the existence of an integer ρ such that Homer experiences only the $\rho + 1$ instants $(t_0, t_1, \ldots, t_\rho)$ in the time interval $[0,1)$. Clearly ρ cannot be any of the standard, finite integers and therefore such a t_ρ cannot exist within the standard real number system under consideration here. But in nonstandard analysis, a restricted form of the deduction of (8) from (7) is *required* for the existence of nonstandard numbers [13]. Therefore, in our view, the banning of the deduction of (8) from (7) within classical logic is somewhat arbitrary and does not have a clear logical basis.

What (7) establishes is that Homer can never consciously experience the sequential passage of infinitely many instants t_n, and that this is not just a practical limitation of the human mind, but a logical impossibility. What Homer actually experiences during the time interval $[0, 1)$ is an ever increasing, but always finite number of instants t_n. It is only at $t = 1$ that Homer retroactively concludes that all the instants t_n have indeed passed in the interval $[0, 1)$, but with no recollection even possible of how this transition from finite to infinite happened. One might attempt to argue that infinitely many instants t_n actually passed in an infinitesimal time interval as $t \to 1^-$ and that the human mind is incapable of grasping infinitesimals. However this explanation is refuted by the fact that in the standard real number system in which we are operating, the only infinitesimal magnitude is zero.

Nevertheless, the fact is that classical logic does block the deduction of (8) from (7), thus effectively allowing the existence of infinitely many instants t_n in the time interval $[0, 1)$ as a Platonic reality that is independent of the human mind. In the logic NAFL that we have proposed [5, 6, 7, 8], mathematical truths are formulated as axiomatic assertions of the human mind and NAFL rejects the philosophy of Platonism that is inherent in classical logic. From the point of view of NAFL, Homer's path is not 'pre-existing', but is actually constructed by him as he runs. NAFL accepts that the nonclassical induction argument given in Sec. 1.1 correctly establishes the logical impossibility of providing a construction for the completion of a supertask, such as, traversing infinitely many spatial / temporal points or distances / durations sequentially. In the NAFL version of real analysis [8, 7], the half open real interval $[0, 1)$ and the infinite sequences (1) and (2) do not exist. Therefore it is impossible to even formulate Zeno's dichotomy paradox in NAFL. The paradoxical blocking of the deduction of (8) from (7) is also avoided because it is impossible to legitimately state (7) in NAFL theories. Despite these seemingly severe restrictions, the NAFL version of real analysis does admit the infinite divisibility of space and time, albeit in a more restricted sense than in classical logic.

The arguments in support of the classical formulation are that it effectively blocks Zeno's dichotomy paradox, and that despite its nonconstructive and counterintuitive nature, no formal contradiction can be deduced. Further, the unrestricted infinite divisibility of space and time in the classical formulation has proven to be a highly useful idealization in modern scientific theories. In our opinion, what is mathematically or scientifically legitimate and rigorous must be determined purely by logical considerations. The perceived usefulness of classical theories does not justify the use of infinitary reasoning in these theories if there are logical objections to such use. Such logical objections must ideally be in the form of demonstrated contradictions, either formal or metamathematical, and we will pursue this in the ensuing sections. Here, a metamathematical contradiction is a contradiction outside of the formalism, which is often a formal theory in classical logic and is required to be a formal theory in NAFL. Therefore we will often use the words *metamathematical* and *metatheoretical* interchangeably.

1.4 Dichotomy paradox in an infinite relay run

Our goal is to demonstrate that the classical refutation of Zeno's dichotomy paradox does indeed lead to a logical contradiction. We first consider a scenario in which Homer's run on $x \in [0,1)$ is replaced by that of infinitely many men, say, Homer(n), $n \in N$, where N is the class of all natural numbers. This run, an infinite relay, is specified as follows.

- For each $n \in N$, Homer(n) starts his run at $(x,t) = (x_n, t_n)$ and ends it at $(x,t) = (x_{n+1}, t_{n+1})$. Here x_n and t_n are as defined in (1) and (2) respectively.
- It follows that for each $n \in N$, Homer(n) runs at average unit velocity in the specified spatial and temporal domains. Note that this requires Homer(n), who is at zero velocity at the start of his run, to accelerate to some finite velocity $v \geq 1$ and then decelerate to zero velocity at the end of his run.
- For each $n \in N$, Homer(n) carries a baton which he relays instantaneously to Homer($n+1$).
- Consider the special case where, for each $n \in N$, Homer(n) (with the baton) is imparted a positive impulse at $t = t_n$ and a negative impulse at $t = t_{n+1}$ that take him instantaneously to unit and zero velocities respectively. This case is interesting because the baton travels continuously at unit velocity on $t \in [0,1)$, as does Homer in the original formulation of the dichotomy paradox. We may also consider the specific example in which Homer(0), with the baton, is imparted a positive impulse at $t = 0$ that takes him to unit velocity and all subsequent impulses for $0 < t < 1$ are imparted due to elastic collisions, with transfer of baton, between Homer(n) and Homer($n+1$) for each $n \in N$. This case is very similar to the "beautiful supertask" of Laraudogoitia [14], with the only difference being that Laraudogoitia does not have a baton in his supertask. In particular, note the failure of the conservation of momentum, because all the Homer(n) are at rest at $t = 1$, while at $t = 0$, Homer(0) is at unit velocity.

At the outset, let us note that this is a highly idealized scenario. Each of the infinitely many Homer(n) and the baton are taken to be point masses. Further, as $n \to \infty$, Homer(n) will be subjected to unbounded accelerations and decelerations because of the need to maintain an average unit velocity over increasingly short distances. Nevertheless, the impulses imparted to the Homer(n) are finite and bounded as $n \to \infty$. Indeed, infinite accelerations are routinely handled in Newtonian mechanics, for example, during elastic collisions. Hence there are no fatal logical objections to this formulation.

It is clear that Homer's supertasks, discussed in Secs. 1.1 and 1.2, have been broken up into infinitely many finite tasks performed by the Homer(n). Therefore while each Homer(n) crosses only one of the intervals $[x_n, x_{n+1}]$, the totality of such intervals have been crossed collectively and in sequence. On the other hand, the baton does cross the infinite totality of these intervals sequentially and at an average unit velocity. Therefore the baton performs the said supertasks and plays the role of Homer in the original formulation of Zeno's dichotomy paradox. Let us consider this paradox in the present modified formulation.

- None of the Homer(n) reaches $x = 1$, in particular, at $t = 1$. This is already paradoxical, because the convergence of the infinite summations in (5) indicate that the Homer(n) collectively traveled a total unit distance and for a total unit duration. This paradox is explained by the fact that the Homer(n) have *collectively* gotten arbitrarily close to $x = 1$, in the sense that as $x \to 1^-$ there are infinitely many of the Homer(n) in every neighborhood of $x = 1$, without any of them actually reaching $x = 1$.

- The formulation of Zeno's dichotomy paradox in this setting is that the baton can never reach $x = 1$. Indeed, it seems almost obvious that the baton must always remain in the possession of one of the infinitely many Homer(n), none of whom reaches $x = 1$.

- The classical refutation of Zeno's dichotomy paradox now requires that the baton *does* reach $x = 1$ at $t = 1$, *despite* the fact that it would not be in the possession of any of the Homer(n) at $(x, t) = (1, 1)$! The convergence of the infinite summations in (5) indicate that the baton traveled a total unit distance and for a total unit duration. Therefore the baton does cross, in sequence, an actual infinity of distances d_n in an actual infinity of durations τ_n, after which it must *necessarily* reach $x = 1$ at $t = 1$. Indeed, at $t = 1$, it is clearly not possible for the baton to be at any location $x < 1$, because there would always exist some Homer(n) who would have relayed the baton beyond that point. It is important to note that there are no nonzero infinitesimals in the standard real number system, and so the baton cannot get arbitrarily close to $x = 1$ without actually reaching $x = 1$. Here we define the baton to be arbitrarily close to $x = 1$ if its location x satisfies $1 - \epsilon < x \leq 1$ for all $\epsilon > 0$. The reader should carefully contrast this situation with that of the infinitely many Homer(n), who *collectively* did manage to get arbitrarily close to $x = 1$, without any *individual* Homer(n) achieving that feat.

- It is obvious that the above classical refutation argument is highly problematic, since the baton's motion is only possible in the first place because one of the Homer(n) is powering it. Yet the baton reaches the finish line on its own steam, after the human beings fail to relay it there.

1.4.1 Contradiction from the infinite relay dichotomy paradox

Let us make the reasonable metamathematical stipulation that the baton cannot reach $x = 1$ on its own without being carried there by any of the Homer(n), and add this stipulation formally as an axiom. Then the infinite relay dichotomy paradox does result in a formal logical contradiction, namely, that the baton reaches $x = 1$ and the baton does not reach $x = 1$. If we choose not to add the above stipulation as an axiom, then the contradiction will remain metamathematical rather than formal.

The justification for this proposed axiom, and the consequent contradiction, is as follows. From the true premise that the baton has been *carried* past the locations $x = 1 - \epsilon$ for each $\epsilon > 0$, we had previously inferred that the baton has *reached* $x = 1$, and this is in accordance with the classical refutation of Zeno's dichotomy paradox. Now we would like to strengthen this inference by specifying that the baton can only *reach* $x = 1$ by virtue of its being *carrie*d there. This strengthening cannot be deduced formally within classical logic because it involves a classically illegal interchange of universal and existential quantifiers, as follows. From the premise that the baton has been carried past *each* location $x = 1-\epsilon$, one cannot automatically infer within classical logic that the baton has been carried past *all* such locations to $x = 1$. Nevertheless, one can see that this conclusion must follow from the stated premise. The baton's motion on $x \in [0, 1)$, as defined in the premise, is only due to its being carried by one of the infinitely many Homer(n) and the classically valid conclusion is that precisely such a motion leads to the baton reaching $x = 1$ (according to the classical refutation of Zeno's dichotomy paradox). The only possible conclusion from such a premise is that the baton has been carried to $x = 1$, because the premise does not define any other type of motion for the baton on $x \in [0, 1)$. But, due to the above-mentioned illegal interchange of universal and existential quantifiers, classical logic blocks this inference, which must nevertheless remain metamathematically valid. In Sec. 2, we will provide another justification for this inference, using the argument of limits. What we have established as true, but formally unprovable, is the axiom that there must exist some integer ρ such that Homer(ρ) has carried the baton to $x = 1$. Incidentally, such an axiom would solve the previously noted issue of the failure of the conservation of momentum. But no such integer ρ exists, at least not within the standard real number system that we have assumed, and we have the stated metamathematical contradiction, which would become formal if we choose to add this axiom to our formal system.

Our view is that this logical contradiction, whether formal or metamathematical, arises because the baton and the Homer(n) (collectively) are required to perform supertasks, namely, traversing infinitely many spatial and temporal intervals sequentially. Such supertasks are infinitary in nature and therefore logically inadmissible according to the strictly finitistic reasoning employed in the logic NAFL, which accepts the nonclassical induction argument of Sec. 1.1. In Ref. [8], we will describe in detail how NAFL avoids Zeno's dichotomy paradox and the resulting contradictions. We should also mention here that there are other proposed resolutions of Zeno's dichotomy paradox which make use of nonstandard real analysis [12] or the relativistic notion of spacetime (see Sec. 3.7 of Ref. [15]). But these resolutions are also problematic and are in any case not acceptable in the NAFL version of finitism, by whose yardstick the infinitary reasoning employed in relativity theory and nonstandard analysis stands rejected.

What the infinite relay scenario illustrates is that there is a difference between infinitely many Homer(n) relaying the baton versus Homer carrying the baton. In the first case none of the Homer(n) reaches $x = 1$, whereas in the latter scenario Homer does reach $x = 1$ with the baton. If one thinks about this difference, it is inexplicable from a logical point of view. Replacing Homer by infinitely many Homer(n) should not matter to the task at hand, namely, that of transporting the baton from $x = 0$ to $x = 1$ at an average unit velocity. Indeed, in the special case that we have considered, where the transfer of momentum from Homer(n) to Homer($n + 1$) occurs due to elastic collisions, the baton's motion on $x \in [0, 1)$ is completely unaffected by the said replacement of Homer. Clearly, the problem with this replacement, which breaks up Homer's supertasks into infinitely many parts, is purely logical in nature. In our view, the infinite relay scenario provides the clearest example of the logical illegitimacy of supertasks, which are an essential part of classical infinitary reasoning.

1.5 Realistic versions of Zeno's dichotomy paradox

Here we present more realistic modifications of Zeno's dichotomy paradox in the framework of classical logic, from which logical contradictions can be deduced. We wish to consider a scenario in which Homer dies, either while in motion or even while stationary. Motion is not required for this scenario, although we will consider both cases.

Definition of consciousness. An individual is defined to be conscious (unconscious) if he / she is capable (incapable) of having thoughts. At a given instant, a person who is conscious has a thought (assumed to occur instantaneously) and a person who is unconscious has no thoughts.

Definition of death. For our purposes, death is defined as a permanent loss of consciousness. Thus an individual is dead when he / she has lost consciousness and never regains it. An individual who is not dead is defined to be alive.

Here we are considering a sort of 'brain death', rather than standard clinical death. It is clear that our definitions satisfy the following logical constraints.

- At each instant of time, a person is either conscious or unconscious.
- Loss of consciousness, when it occurs, must necessarily be instantaneous.
- At each instant of time, a person is either alive or dead.
- Death, when it occurs, must necessarily be instantaneous.

It should be emphasized that these constraints are classical requirements. From the constraint that death must be instantaneous, it follows that there must exist a sharp moment of death. This requirement leads to a contradiction, as will be shown below.

1.5.1 The spatial dichotomy paradox and contradiction

First consider Zeno's dichotomy paradox as defined in Sec. 1, with Homer in uniform motion on $x \in [0, 1]$ at unit velocity. To this formulation, we add the following requirement.

- Homer is conscious at all locations $x \in [0, 1)$ and dies (instantaneously) when he reaches $x = 1$.

It follows that Homer has crossed infinitely many spatial intervals $[x_n, x_{n+1}]$, as defined in (1), in a conscious state. The infinitely many distances d_n that Homer has traversed sequentially sum to a total unit distance (see (3) and (5)). Therefore Homer has reached a location arbitrarily close to $x = 1$ in a conscious state. In other words, upon traversing these infinitely many distances, Homer's location x necessarily satisfies $1 - \epsilon < x \leq 1$ for all $\epsilon > 0$. Since there are no nonzero infinitesimals in the standard real number system, we conclude that Homer has in fact reached $x = 1$ in a conscious state. But this contradicts the requirement that Homer dies when he reaches $x = 1$. Therefore we have a contradiction that Homer is both alive and dead at the location $x = 1$.

In deducing this contradiction, we have made the following inference:

$$\begin{aligned}&\text{(Homer has traversed infinitely many distances } d_n \text{ in a conscious state)} \Rightarrow \\ &\text{(Homer has reached } x = 1 \text{ in a conscious state).}\end{aligned} \tag{9}$$

Note that the conclusion of (9) is a strengthening of the classical refutation of Zeno's dichotomy paradox discussed in Sec. 1.3, according to which one can only conclude that 'Homer has reached $x = 1$' from the premise on the left hand side of (9). That Homer is conscious at $x = 1$ cannot be formally deduced because of a classically illegal interchange of universal and existential quantifiers involved in such a deduction, as follows. From the premise that Homer is conscious after crossing *each* location x_n, one cannot automatically conclude within classical logic that Homer is conscious at $x = 1$ after crossing *all* such locations x_n. Nevertheless, one can see that this conclusion is true for the following reason.

The premise on the left hand side of (9) uses only data from $x \in [0, 1)$, because the infinitely many distances d_n (spanning the infinitely many intervals $[x_n, x_{n+1}]$ as defined in (1)) are entirely contained in $[0, 1)$. Therefore this premise does not include the added requirement that Homer dies when he reaches $x = 1$. Hence logically, any conclusion regarding Homer drawn solely from such a premise must necessarily require him to be in a conscious state. In particular, the classical refutation of Zeno's dichotomy paradox, as discussed in Sec. 1.3, does permit the conclusion 'Homer has reached $x = 1$' from this premise. This classical inference can therefore be legitimately strengthened to that in (9).

The premise used in (9) essentially states that Homer has traversed every point $x \in [0, 1)$ in a conscious state. If nonzero infinitesimal magnitudes existed, Homer would be conscious at some stage of the fact that he is infinitesimally close to $x = 1$. But the only infinitesimal magnitude in the standard real number system is zero, which allows only the possibility of Homer being conscious of having reached $x = 1$. Indeed, it is *topologically impossible* for Homer to consciously traverse the half open real interval $[0, 1)$ without being conscious of having reached $x = 1$, because Homer cannot run in such a way that he has a point ($x = 1$) available that he can cover 'next'. However, in Sec. 1.4 we saw that it is topologically possible for infinitely many Homer(n) to collectively traverse $[0, 1)$ without any of the individual Homer(n) reaching the 'next' point $x = 1$.

We have shown that (9) is true, but formally unprovable, and hence satisfies the metamathematical requirement of an axiom. Therefore the deduced contradiction must also remain metamathematical rather than formal, unless we choose to add (9) as an axiom to our formal system.

1.5.2 The temporal dichotomy paradox and contradiction

As noted previously, there is a purely temporal version of Zeno's dichotomy paradox, in which Homer is not required to be in motion. The paradox is stated as follows.

- Homer completes the supertask of experiencing (not necessarily consciously) infinitely many instants t_n, as defined in (2), sequentially in a finite time interval $[0, 1]$.
- The infinitely many durations τ_n, which Homer traverses sequentially, sum to a total unit duration (see (4) and (5)). This is paradoxical not only because another supertask has been performed, but also because one would expect the sum of infinitely many finite, nonzero durations to diverge. Hence Homer should never have been able to experience the instant $t = 1$, certainly not in the *alive* state.

The classical refutation of the temporal dichotomy paradox proceeds as discussed in Sec. 1.3. To get a contradiction from the classical refutation, we add the following requirement.

$$\forall t \text{ (Homer is conscious if } t \in [0, 1) \text{ and is dead if } t \geq 1). \tag{10}$$

The infinitely many durations τ_n that Homer has traversed sequentially in a conscious state sum to a total unit duration. Therefore Homer is conscious at a time arbitrarily close to $t = 1$, in the sense that Homer, upon traversing these infinitely many durations, must necessarily have experienced an instant t that satisfies $1 - \epsilon < t \leq 1$ for all $\epsilon > 0$. Since there are no nonzero infinitesimals in the standard real number system, we conclude that Homer is conscious at $t = 1$. But this contradicts (10), which requires that Homer dies at $t = 1$.

In deducing this contradiction, we have made the following inference:

$$\begin{aligned}&\text{(Homer has traversed infinitely many durations } \tau_n \text{ in a conscious state)} \Rightarrow \\ &\text{(Homer is conscious at } t = 1).\end{aligned} \tag{11}$$

The discussion of (11) proceeds entirely in analogy with that for (9). It is topologically impossible for Homer to consciously traverse all the durations τ_n, or equivalently, experience all instants $t \in [0, 1)$, without consciously traversing the infinitesimal duration required to experience the 'next' instant $t = 1$. However, in Sec. 1.4 we saw that it is topologically possible for infinitely many Homer(n) to collectively and consciously traverse the time interval $[0, 1)$ without any of the individual Homer(n) experiencing the instant $t = 1$. Eq. (11) is a strengthening of the classical refutation of Zeno's dichotomy paradox discussed in Sec. 1.3, according to which the conclusion on the right hand side of (11) should be 'Homer has experienced the instant $t = 1$', without the commitment that Homer is conscious at this instant. This strengthening is not formally permissible within classical logic because of a classically unacceptable interchange of universal and existential quantifiers that is inherent in its deduction, as follows. The fact that Homer is conscious after *each* instant t_n does not automatically imply that Homer is conscious at $t = 1$, after *all* such instants t_n. Nevertheless we may conclude that (11) is true, even if formally unprovable, because the premise on the left hand side of (11) essentially states that Homer is conscious for all $t \in [0, 1)$, and it follows as a logical consequence that any conclusion regarding Homer drawn solely from such a premise must necessarily require him to be conscious. In particular, the classically valid conclusion drawn from this premise, namely, 'Homer has experienced the instant $t = 1$', can be legitimately strengthened to that in (11).

We have shown that (11) is true, but formally unprovable, and hence (11) satisfies the metamathematical requirement of an axiom. But if (11) is formally added as an axiom, one may deduce the contradiction, from (10) and (11), that Homer is both alive and dead at $t = 1$. This contradiction will remain metamathematical if we choose not to add (11) formally as an axiom.

1.5.3 Attempted classical resolution of the contradictions

The temporal dichotomy paradox is certainly a realistic scenario, given the constraints of classical physics and classical logic. The contradictions deduced from both the spatial and temporal versions of the dichotomy paradox can be seemingly resolved classically, by formulating Homer's death as follows.

$$\forall t \text{ (Homer is conscious if } t \in [0, 1] \text{ and is dead if } t > 1). \tag{12}$$

This formulation satisfies the requirements of (9) and (11), namely, that Homer is conscious at $t = 1$. Since Homer is dead in the limit $t \to 1^+$, his death can be thought of as being instantaneous. Nevertheless, there is no precise moment of death, because for each instant $t_1 > 1$ at which Homer is dead, there must also exist an instant t_2 satisfying $t_1 > t_2 > 1$ at which Homer is dead. It follows from (12) that Homer is dead at times $t > 1$ that are *arbitrarily* close to $t = 1$, in the following sense:

$$\forall \epsilon > 0 \, \exists t \, ((1 \leq t < 1 + \epsilon) \text{ \& Homer is dead at time } t). \tag{13}$$

But one cannot infer from (13) that Homer is dead at times $t \geq 1$ that are *infinitesimally* close to $t = 1$ (equivalently, that Homer is dead at $t = 1$) because such an inference would require a classically illegal interchange of universal and existential quantifiers in (13). Hence the contradiction that Homer is alive and dead at $t = 1$ is formally blocked and does not immediately follow from (12). However, if we require that there must exist a precise moment of death, the said interchange of universal and existential quantifiers in (13) would become legal and one would conclude that such a moment of death can only be at $t = 1$. The contradiction would follow. In Sec. 2, it is shown that that this contradiction is also deducible from the failure of time reversal invariance for (12), which is a classical requirement. Therefore the attempted classical resolution of the contradictions from the dichotomy paradox via (12) does not really succeed.

The classical formulation of Homer's death in (12) requires that there must exist a final moment of consciousness, but no first moment of death. This is in contrast to reality because the time of death is a parameter that is observable, in the sense that it can be experimentally measured. The time of death is also routinely calculated in forensic science. On the other hand, no such measurement or calculation of a final moment of consciousness is available or even possible. In Ref. [8]. we will demonstrate that the proposed logic NAFL does support a precise moment of death, as in the following modification of (10):

- Homer dies at $t = 1$.

By the definition of death, this automatically implies that Homer is dead when $t \geq 1$, but NAFL does not permit the classical conclusion (as stated in (10)) that Homer was alive when $t \in [0, 1)$, for the following reason. The half open real interval $[0, 1)$ does not exist in the NAFL version of real analysis [8, 7], which effectively blocks Zeno's dichotomy paradox and the consequent classical contradiction that Homer is both alive and dead at $t = 1$. The time-dependent and paraconsistent nature of NAFL truth is ideally equipped to handle the process of instantaneous death, which is modeled as a step discontinuity in real analysis, without implying the existence of open or half open intervals of real numbers [8]. Likewise, the formulation of Homer's death in (12) requires the existence of the real interval $(1, \infty)$, which is not supported in the NAFL version of real analysis.

2 Role of supertasks in the time irreversibility of physical processes

In this section we prove that a class of supertasks can be viewed as physically realizable limit processes, wherein a limiting task must necessarily get performed as the supertask get completed. This enables a general justification for the various contradictions deduced previously. We also conclude that there is an inviolable arrow of time in many discontinuous physical processes, which contradicts the classical requirement of time reversal invariance [16, 17, 18] that follows from the governing laws of nature.

Definition of physical process. For our purposes, a physical process is in a Euclidean setting of space and time and is governed by the laws of nature, such as, Newton's laws of motion or Maxwell's laws of electromagnetism. Further, the arrow of time exists in a physical process and is always in the forward direction, that is, from past to present to future.

Definition of supertask. We extend the definition of supertask to any infinite sequence of events (not necessarily performed by a human being) whose domain is a strictly increasing, convergent sequence of times within a finite time interval in a given physical process.

Definition of extended real number system. The affinely extended real number system $\bar{\mathbb{R}}$ is defined [19] as $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, where $\mathbb{R}$ is the standard real number system and the new elements $\{-\infty, +\infty\}$, though not real numbers, satisfy the intuitive definitions of arithmetical and other operations (for example, $a + \infty = +\infty + a = +\infty$, whenever $a \in \bar{\mathbb{R}}$ and $a \neq -\infty$). Note that the symbol $+\infty$ is often written as just ∞ whenever the meaning is clear from the context. For our purposes, we will need the extended definition of the limit operation, which is again intuitive. For example, $\lim_{x\to 0^+}(1/x) = +\infty$ and $\lim_{x\to 0^-}(1/x) = -\infty$.

Metatheorem 1. *Consider a physical process in which the time t passes through a strictly increasing, convergent sequence $\{t_n\}$, $n \in N$, within a finite time interval, and let $t_b = \lim_{n\to\infty} t_n$. Let S_1 be the supertask defined by t assuming, in sequence, each of the infinitely many values in $\{t_n\}$. Then the minimum value of the time at which S_1 gets completed is $t = t_b$.*

Proof. The proof consists of two simple steps which follow from the fact that t_b is the limit point of the strictly increasing sequence $\{t_n\}$.

- S_1 is incomplete if $t < t_b$, as seen from:

$$t < t_b \Leftrightarrow \exists n\ (t < t_n < t_b) \Leftrightarrow (S_1 \text{ is incomplete}).$$

- S_1 is complete if $t = t_b$, which is obvious because t_b exceeds every value in $\{t_n\}$.

□

Remark 1. *Metatheorem 1 is a classically valid result, because it is a mathematical fact that if t exceeds every value less than t_b (as required for the completion of S_1), then $t \geq t_b$. However, there is a nonclassical component to Metatheorem 1 because classically, a time of completion is deemed not to exist for supertasks (see Remark 3), which do not have a last step. Nevertheless, it is clear from the above proof that t_b does satisfy the criteria required to qualify it as a time of completion of S_1.*

Remark 2. *It follows from Metatheorem 1 that if the supertask S_1 is complete, that is, if t sequentially passes through all the infinitely many values in*

$\{t_n\}$, *then* t *must necessarily attain the limiting value* t_b. *Hence the completion of the supertask* S_1 *can be viewed as a physical realization of the limit process* $\lim_{n\to\infty} t_n$. *Note that in the definition of a classical limit process* $\lim_{n\to\infty} t_n$, *it is assumed that* n *increases through arbitrarily large values without any requirement that* n *increases through an actual infinity of values. Hence a classical limit process does not require the completion of a supertask, as opposed to a physical process within classical logic. Metatheorem 1 can be generalized to a wider range of supertasks, as will be seen from Metatheorem 2.*

Metatheorem 2 (Supertasks as limit processes)**.** *Suppose that, in a given physical process, an object has a time-dependent state* $S(t)$, *where* $t \in [t_a, t_c]$ *and* $t_c > t_a$. *Here* t *is defined on the standard real number system* $\mathbb{R}$, *while, for our purposes,* $S(t)$ *is defined on the extended real number system* $\bar{\mathbb{R}}$. *At a given time* t_b, *where* $t_a < t_b \leq t_c$, *suppose that* $\lim_{t\to t_b^-} S(t)$ *exists. Then* $S(t_b) = \lim_{t\to t_b^-} S(t)$.

Proof. Consider any strictly increasing Cauchy sequence of times $\{t_n\}, n \in N$, defined within the time interval $[t_a, t_b)$ such that $\lim_{n\to\infty} t_n = t_b$. For example, taking $(t_a, t_b, t_c) = (0, 1, 2)$, consider the sequence $\{t_n\}$ defined in (2). Given the arrow of time in a physical process, the object in question completes the supertask S_2 of attaining, in sequence, an actual infinity of states $S(t_n), n \in N$. Clearly, S_2 occurs simultaneously with the supertask S_1 defined in Metatheorem 1, in the sense that there is a one-to-one correspondence in time between the steps of S_1 and S_2. It follows that the minimum value of the time at which S_2 gets completed is $t = t_b$. We claim that as t attains its limiting value t_b upon completion of S_1, the completion of S_2 must also happen with $S(t)$ attaining its limiting value $\lim_{n\to\infty} S(t_n)$, which exists, because $\lim_{n\to\infty} S(t_n) = \lim_{t\to t_b^-} S(t)$. For a simple proof of this claim, consider, for illustrative purposes, the case when $S(t)$ has a convergent series expansion of the form

$$S(t) = \lim_{n\to\infty} S(t_n) + \sum_{n=1}^{\infty} C_n (t - t_b)^n, \quad t \in [t_d, t_b), \tag{14}$$

for some t_d, where $t_a \leq t_d < t_b$ and the C_n are constants. Clearly, as t passes through an actual infinity of values t_n, the following must hold:

$$\forall \epsilon > 0 \ (|S(t) - \lim_{n\to\infty} S(t_n)| < \epsilon). \tag{15}$$

Here $|S(t) - \lim_{n\to\infty} S(t_n)|$ can be viewed as a distance on the real line, which, upon the completion of the supertask S_2, must shrink to less than every positive real number ϵ. This requirement must hold by the very definition of $\lim_{n\to\infty} S(t_n)$, irrespective of whether a convergent series expansion for $S(t)$ exists as postulated in Eq. (14). It is a direct mathematical consequence of (15) that the completion of S_2 must happen with $S(t)$ jumping to the value $\lim_{n\to\infty} S(t_n)$, because the extended real number system does not have nonzero infinitesimals. This is so *despite* the fact that (14) does not define $S(t)$ at $t = t_b$

and is entirely analogous to the fact that Metatheorem 1 requires the completion of S_1 to happen with t jumping to the value t_b, which is again outside the domain of definition of S_1. What is being asserted here is that both the values $t = t_b$ and $S(t) = \lim_{n\to\infty} S(t_n)$ must be physically realized upon completion of the supertask S_2. For example, if $S(t)$ is a velocity, then the limiting value of the time (t_b) and the velocity ($\lim_{n\to\infty} S(t_n)$) will be physically attained when S_2 gets completed. It follows that $S(t_b) = \lim_{n\to\infty} S(t_n) = \lim_{t\to t_b^-} S(t)$. $\square$

Remark 3. *The completion of the supertask S_2 in this proof can be viewed as the physical realization of a limit process, as was the case for the supertask S_1. That it is not possible for an actual infinity of values in the sequence $\{S(t_n)\}$ to be physically attained one at a time without the limiting value of $\{S(t_n)\}$ being physically attained is a nonclassical result, which must hold despite the fact that by definition, $\{S(t_n)\}$ does not contain its limit point. The definition of $\{S(t_n)\}$ is a classically valid, abstract construct made without any appeal to the notions of physical process, arrow of time or supertask. In general, classical logic permits the completion of the supertask S_2 without the commitment that the limiting value of $\{S(t_n)\}$ has been attained. Classically, S_2 is deemed to be complete in $t \in [t_a, t_b)$, despite the fact that a time of completion does not exist in this interval [10]. This paradoxical conclusion, rejected in Metatheorem 2, is classically justified by the fact that the supertask does not have a last step. In the version of real analysis supported by the logic NAFL [8, 7], this classical abstraction does not hold and sequences of real numbers are constructively defined such that they must necessarily include the limit points of all convergent subsequences.*

Remark 4. *To get an intuitive feel for why Metatheorem 2 must be true, consider the creation of the non-negative real line $[0, \infty)$ via the following supertask. During the time interval $[t_n, t_{n+1}]$ as defined in (2), extend a line segment from $x = n$ to $x = n + 1$ at uniform velocity, for each $n \in N$. Thus at $t = t_1 = 1/2$, the line segment $[0, 1]$ is created, which is then extended to the line segment $[0, 2]$ at $t = t_2 = 3/4$, and so on. Define $S(t)$ as the total length of the line segment at time $t \in [0, 1]$. We may show that*

$$S(t) = n + 2^{n+1}(t - t_n), \quad t \in [t_n, t_{n+1}].$$

In particular, $S(t_n) = n$, $n \in N$, and an application of Metatheorem 2 with $(t_a, t_b, t_c) = (0, 1, 1)$ shows that

$$S(1) = \lim_{n\to\infty} S(t_n) = +\infty.$$

Thus Metatheorem 2 tells us that the length of the non-negative real line is $+\infty$, despite the fact that it was created by extension of a line segment through a sequence of all and only finite lengths n in the standard real number system $\mathbb{R}$ (which does not include the limit point of the real sequence $\{n\}$, namely, $+\infty$). We can see that Metatheorem 2 is presenting the correct picture here, for obviously one cannot extend a line segment through all finite lengths without *creating an infinite length. Note also that the absence of* $\pm\infty$ *in* $\mathbb{R}$ *is a lacuna, if one interprets the existence of the real line in this constructive sense. This is why we have chosen the extended real number system* $\bar{\mathbb{R}}$ *in the statement of Metatheorem 2. Even this does not fix the logical issue that Metatheorem 2 raises, namely, that* $\pm\infty$ *must necessarily be present in the real number system, which is the case in the NAFL version of real analysis [8, 7].*

Remark 5. *The arrow of time in a physical process is crucially important in the proof of Metatheorem 2, which requires the value of the state* $S(t_b)$ *to be the limiting value of* $S(t)$ *as* t *(physically) approaches* t_b *from below. To the extent that classical logic does not impose any such restriction on the value of* $S(t_b)$*, Metatheorem 2 is a nonclassical result. For example, classical logic allows the possibility that* $S(t_b) = \lim_{t\to t_b^+} S(t)$ *(assuming this limit exists) and in the case that* $\lim_{t\to t_b^-} S(t) \neq \lim_{t\to t_b^+} S(t)$*, such a result would contradict Metatheorem 2. Even though Metatheorem 2 is formally unprovable within classical logic, one can intuitively see it to be almost obviously true, given the arrow of time in a physical process and given that supertasks are legitimate in classical logic. Note that the metamathematics of classical infinitary reasoning permits the truth of Metatheorem 2 despite the nonclassical content in its proof, because the concept of truth is not formalizable within the languages of classical formal systems, due to a result of Tarski (see Tarski's undefinability theorem [20]; the precise result, which holds for a class of formal languages with sufficient expressive power, in the sense of being capable of expressing self-reference and diagonalization, is that a truth predicate which identifies the true sentences of a classical formal language is not definable within that language). Hence Metatheorem 2 satisfies the criterion for addition to a classical formal system as an axiom, namely, that it is true, but formally unprovable.*

Corollary 1. *In the physical process defined in Metatheorem 2, suppose*

$$\lim_{t\to t_b^-} S(t) \neq \lim_{t\to t_b^+} S(t).$$

Then such a discontinuous physical process is not time reversible.

Proof. Without loss of generality, assume $(t_a, t_b, t_c) = (0, 1, 2)$. Metatheorem 2, as applied in the physical time variable t, implies that $S(1) = \lim_{t\to 1^-} S(t)$. Make the change of variable $t' = 2 - t$ and let $S'(t') = S(t)$. To arrive at a contradiction, apply Metatheorem 2 in the variable t', with $(t'_a, t'_b, t'_c) = (0, 1, 2)$. We find that $S(1) = S'(1) = \lim_{t'\to 1^-} S'(t') = \lim_{t\to 1^+} S(t)$, which contradicts the previous application of Metatheorem 2 in the variable t. $\square$

Remark 6. *Corollary 1 refutes an important classical requirement, namely, time reversal invariance of many physical processes, such as, those that obey the laws of classical mechanics. The reason for this contradiction is the nonclassical requirement of Metatheorem 2 that limit processes are physically realized via supertasks. The limit process in the time reversed direction, namely,* $\lim_{t\to t_b^+}$*, is physically realized via a different supertask than that used in the physical* *time variable t for the proof of Metatheorem 2, which yields a different limiting value for $S(t_b)$. The key observation here is that supertasks are not really time reversible, in the following sense. A time reversed supertask has no first step and hence cannot even be considered a well-defined task, let alone a supertask. In particular, for a supertask that terminates at a limit point in time ($t = t_b$ in this case), any first step in the time reversed direction would necessarily cover infinitely many steps of the original (physical) supertask, which therefore cannot be considered a supertask in the time reversed direction. Clearly, the time reversed supertask can have at most finitely many steps, because infinitely many steps of the original supertask have been coalesced into a single first step. It is important to note that one particularly absurd consequence of classical time reversal invariance in the infinite relay dichotomy paradox of Sec. 1.4 or in the "beautiful supertask" of Laraudogoitia [14], namely, the spontaneous self-excitation of velocities in an infinite sequence of particles that are initially at rest, with the consequent failure of conservation of momentum, is ruled out by Corollary 1.*

Remark 7. *Consider the simplest possible example of a discontinuous physical process, namely, an elastic collision between two particles of equal mass moving in opposite directions. Let the collision happen at $(x, t) = (0, 1)$, with the initial positions and velocities of the particles specified as follows. Particle 1 (Particle 2) starts at $x = -1$ ($x = 1$) and has velocity $v_1 = 1$ ($v_2 = -1$), if $t \in [0, 1)$. After the collision the velocity of Particle 1 (Particle 2) changes to $v_1 = -1$ ($v_2 = 1$), if $t \in (1, 2]$. Taking $S(t)$ for each particle to be its velocity and with $(t_a, t_b, t_c) = (0, 1, 2)$, an application of Metatheorem 2 in the physical time variable t yields $(v_1, v_2) = (1, -1)$ at the time of collision, $t = 1$. This implies that the momentum transfer due to the collision at $t = 1$ does not happen at $t = 1$, but does so for arbitrarily close times $t > 1$. For the time reversed version of this collision, make the change of variables $v' = -v$ and $t' = 2 - t$, and let $(t'_a, t'_b, t'_c) = (0, 1, 2)$. An application of Metatheorem 2 now yields $(v'_1, v'_2) = (1, -1)$ at $t' = 1$, which translates to $(v_1, v_2) = (-1, 1)$ at $t = 1$. Thus the time reversed application of Metatheorem 2 results in the contradictory conclusion that the momentum transfer due to the collision did happen at $t = 1$, which illustrates the failure, demonstrated in Corollary 1, of time reversal invariance.*

Remark 8. *Elastic collisions are probably among the most fundamental of physical processes occurring at the microscopic level. In Ref. [8], we will see that the logic NAFL, with its time-dependent truth and paraconsistency, supports the conclusion that the momentum transfer must happen precisely at the time of collision. Further, Metatheorem 2 cannot be formulated in NAFL, which does not permit the existence of half open real intervals or of sequences of real numbers that exclude their limit points. These facts enable the NAFL version of Newtonian mechanics to uphold time reversal invariance for discontinuous physical processes, such as, elastic collisions.*

Remark 9. *Metatheorem 2 and Corollary 1 force us to consider the following options.*

- *The first option is to accept that the time reversal invariance of the classical laws of nature, such as, the laws of Newtonian mechanics, requires the time reversibility of physical processes governed by these laws. Acceptance of this requirement would in turn throw up two possibilities. The first possibility is to reject Metatheorem 2, thereby ignoring its obvious truth and effectively ignoring a metamathematical inconsistency in classical infinitary reasoning, while maintaining the claim of formal consistency. The second possibility is to accept Metatheorem 2 and Corollary 1 as true within classical logic and thereby reject classical infinitary reasoning as inconsistent. It is this latter possibility that will be pursued in Ref. [8], to make the case for the logic NAFL.*

- *The second option is to drop the classical requirement of time reversal invariance of physical processes and accept that there is an inviolable arrow of time. This option is not really viable as a means of salvaging classical infinitary reasoning, for it would eventually lead to further inconsistencies. Most importantly, an inviolable arrow of time imposed as a logical requirement (via Metatheorem 2 and Corollary 1) would contradict classical theories, such as, Newtonian mechanics and in particular, the theory of special relativity, which requires coordinate invariance for its formulation. The conventional wisdom is that an arrow of time is not a logical requirement, but could physically exist, for example, as a result of initial conditions and certain physical laws, such as, the second law of thermodynamics.*

Remark 10. *We may now revisit the previously considered examples of Zeno's dichotomy paradox, and also consider the famous Thomson's lamp experiment, from the point of view of Metatheorem 2.*

- *The spatial and temporal dichotomy paradoxes (Secs. 1.5.1and 1.5.2) are covered by the following choice of $S(t)$:*

$$S(t) = t \text{ if } t \in [0, 1) \text{ and } S(t) = -1 \text{ if } t > 1. \tag{16}$$

  Here "$S(t) = t$" is interpreted as "Homer is conscious at time t (and has made a mark at $x = t$)", where we add the latter part in the spatial version of the paradox, and "$S(t) = -1$" is interpreted as "Homer is dead at time t". Applying Metatheorem 2 in the physical time variable t with $(t_a, t_b, t_c) = (0, 1, 2)$ yields $S(1) = 1$, which confirms the contradictions deduced in Secs. 1.5.1 and 1.5.2. This application of Metatheorem 2 also confirms the classical resolution of the spatial and temporal dichotomy paradoxes in Sec. 1.5.3 (see eq. (12)), which, however, is contradicted by an application of Metatheorem 2 in the time reversed direction, yielding $S(1) = -1$. In the spatial version of the paradox, $S(1) = 1$ implies that the markings would constitute a line segment of unit length. In particular, a mark was made at $x = 1$, indicating that Homer was alive at this location.

- *The spatial dichotomy paradox can be viewed as an attempt by Homer to construct the half open real interval $[0, 1)$ by extending a line segment* *through all lengths less than one in the infinite series* $\{1/2, 3/4, 7/8, \ldots\}$, *starting from* $x = 0$. *Metatheorem 2 tells us that this is not possible without extending the line segment to* $x = 1$, *that is, to length one, in the same way that it is impossible to extend a line segment through all finite lengths without creating an infinite length (see Remark 4). Thus it is a mathematical fact that Homer will reach* $x = 1$ *upon completing this supertask, as is classically permissible. Metatheorem 2 provides us with the stronger (nonclassical) result that Homer must necessarily reach* $x = 1$ *in the 'alive' state upon completion of this supertask.*

- *With* $S(t)$ *defined as in (16), consider the infinite relay run example of Sec. 1.4. In this case "*$S(t) = t$*" is interpreted as either "The baton has reached* $x = t$*" or " There exists* $n \in N$ *such that Homer(n) has carried the baton to* $x = t$*", and "*$S(t) = -1$*" is interpreted as "The baton has reached* $x = 1$ *and is not in the possession of any of the Homer(n) at time* t*". An application of Metatheorem 2 with* $(t_a, t_b, t_c) = (0, 1, 2)$ *yields* $S(1) = 1$, *whose first interpretation noted above is inexplicable and the second interpretation is a contradiction (as observed in Sec. 1.4), because, by definition, none of the infinitely many Homer(n) has reached* $x = 1$. *This contradiction cannot be resolved and represents a failure of classical infinitary reasoning. Note that application of Metatheorem 2 in the time reversed direction yields the contradictory* $S(1) = -1$, *which is inexplicable and is effectively a rejection of the classical requirement that the infinite relay supertask can be completed.*

- *Still retaining* $S(t)$ *as in (16), consider the Thomson's lamp experiment [4, 9], wherein a lamp is switched on (off) at time* t_n *if* n *is even (odd). Here the sequence* $\{t_n\}$ *is defined by (2). Clearly, the infinite succession of states "on" and "off" does not represent a convergent sequence. However, we may now interpret "*$S(t) = t$*" as "There exists* $n \in N$ *such that the lamp will be switched to a different state at time* $t_n > t$*", and "*$S(t) = -1$*" as "The unspecified state of the lamp remains constant at time* t*". Application of Metatheorem 2 with* $(t_a, t_b, t_c) = (0, 1, 2)$ *yields* $S(1) = 1$, *the falsity of which is essentially a denial that the supertask can be completed (see Remark 11 below). In the time reversed direction, Metatheorem 2 yields* $S(1) = -1$, *which is the contradictory conclusion that the supertask is complete at* $t = 1$, *with the state of the lamp being unspecified. Interestingly, both of these mutually contradictory points of view are prevalent in the literature [4, 10] and we have deduced both, assuming the classical requirement of time reversibility. This clearly points to a metamathematical inconsistency, triggered by Metatheorem 2, within the framework of classical infinitary reasoning.*

Remark 11. *Note that in the case of the Thomson's lamp experiment, Metatheorem 2 in the physical variable* t *assumes that the supertask can be completed, to arrive at a contradictory result. Hence this application of Metatheorem 2 can be viewed as a proof by contradiction that the supertask cannot be com-*

pleted. It is easy to see that this objection can be made in general for the entire class of supertasks considered in Metatheorem 2. For example, taking $\{t_n\}$ *and* $S(t)$ *to be as defined in (2) and (16) respectively, we interpret "*$S(t) = t$*" as "*$\exists n \in N(t_n > t)$*" and "*$S(t) = -1$*" as "*$\forall n \in N(t_n < t)$*". Application of Metatheorem 2 with* $(t_a, t_b, t_c) = (0, 1, 2)$ *in the physical time variable* t *yields* $S(1) = 1$*, which is clearly false. Classically, this false result is used to deny the validity of Metatheorem 2 with the argument that the properties of the interior points* $t \in [0, 1)$ *(including the sequence* $\{t_n\}$*), do not carry over to the end point (limit point)* $t = 1$ *as required by Metatheorem 2. However, we take the position that Metatheorem 2 is obviously true, given the validity of the classical assumption that supertasks can be completed. Thus the falsity of the result predicted by Metatheorem 2, namely,* $S(1) = 1$*, is actually a proof by contradiction that supertasks cannot be completed and can be viewed as a confirmation of the non-classical induction argument of Sec. 1.1, which the logic NAFL supports [8]. We can possibly ignore this objection when the sequence of states* $S(t_n)$ *converges, for example, in the spatial and temporal versions of Zeno's dichotomy paradox. But we would still run into another contradiction, namely, the failure of time reversal invariance, which is a classical requirement. Therefore, in our view, Metatheorem 2 ultimately establishes via contradictions that supertasks cannot be completed, which calls into question the validity of the classical refutation of Zeno's dichotomy paradox in Sec. 1.3, and indeed, the validity of all of classical infinitary reasoning.*

3 Benacerraf's shrinking genie

Benacerraf [9] proposed a version of Zeno's dichotomy paradox (see Sec. 1) in which Homer is replaced by a genie whose height $h(t)$ shrinks continuously in proportion to the distance covered on $x \in [0, 1]$, as follows.

$$h(t) = h_0(1 - t), \quad x(t) = t, \quad t \in [0, 1], \tag{17}$$

where we assume that the genie is a one-dimensional creature and $h_0 > 0$ is the initial height of the genie. Clearly, the genie, which runs at unit velocity, reaches every point $x \in [0, 1)$, but does not reach $x = 1$, where it vanishes. Here it is assumed that the genie no longer exists when its height has reduced to zero. An application of Metatheorem 2 with $S(t) = h(t)$ and $(t_a, t_b, t_c) = (0, 1, 1)$ confirms that $h(1) = 0$, *i.e.*, the genie must vanish at $x = 1$. Next choose $S(t)$ as in (16), where we interpret "$S(t) = t$" as "At time t, the genie is at nonzero height" and "$S(t) = -1$" as "At time t, the genie has vanished". An application of Metatheorem 2 with $(t_a, t_b, t_c) = (0, 1, 1)$ yields $S(1) = 1$, which is the contradiction that the genie must reach $x = 1$ at nonzero height. Benacerraf used this example in an attempt to illustrate his point that the genie can complete the supertask of sequentially traversing the infinitely many points x_n in (1) without entailing that the genie must reach the limit point $x = 1$, as required by Metatheorem 2, which is therefore false from Benacerraf's point of view. However, we again take the position that Metatheorem 2 is true subject to the underlying classical assumption that the genie can complete supertasks, which must stand falsified in Benacerraf's example.

To further elaborate on our position that Metatheorem 2 correctly predicts a contradiction in this scenario, consider a hypothetical universe in which the genie has no length scale available other than its own height. The only objects in this universe are shrinking genies, all of which are of the same height on $t \in [0, 1]$. As specified in (17), assume that a genie is traveling at unit velocity in empty space on $x \in [0, 1)$ towards a stationary target (another genie) at $x = 1$. When the genie is at $x = 1/2$, its height is $h_0/2$. However, the genie has no way of perceiving that it has either reached $x = 1/2$ or that its height has reduced. The genie can only judge its progress towards the target in terms of the number of units of its own height that remain to be covered. Thus if the genie is at some point $x \in [0, 1)$, the distance to the target is $(1 - x)$ and the height of the genie is $h_0(1-x)$. The ratio of these two lengths remains constant at $1/h_0$, which implies that the genie would not perceive any motion towards the target or any shrinking of its height. Intuitively, the genie would always have to take the same number of "steps" to reach the target and there is no other length scale available to it. Indeed, the genie can only perceive that its height has shrunk against a fixed standard length, like the meter scale, which it does not have. Hence, from the point of view of the genie, it remains stationary at fixed height h_0 and at a fixed unit distance from the target genie on $t \in [0, 1)$. In other words, the height of the genie is *set* at h_0 as its standard unit of length in place of the meter scale, and the genie judges motion based on this standard. At $t = 1$, the genie must cover the unit distance to the target at $x = 1$ and must also vanish instantaneously. Clearly, these two constraints can be satisfied if and only if the genie perceives an infinite velocity at $t = 1$. Note that both constraints are important here. For example, merely requiring that the genie vanishes at $t = 1$ would imply (in the genie's coordinate system) that the genie has not even started the supertask of covering the unit distance to the target before vanishing.

Thus in terms of the genie's coordinates, we are able to deduce the contradiction, without any appeal to Metatheorem 2, that the genie has reached $x = 1$ at height h_0 (and has collided with the target genie of the same height), and the genie has not reached $x = 1$, where it vanishes (along with the target genie) instantaneously. This contradiction can only be resolved by rejecting the classical assumption that the genie can complete supertasks. Note that there are two supertasks of interest here, namely, that the traveling genie catches up with the target genie and also that the heights of the genies reduce to zero at $t = 1$ (*i.e.*, Zeno's dichotomy paradox exists in two dimensions in this example). Let the traveling genie apply Metatheorem 2 to this scenario in its own coordinate system, taking $S(t)$ on $t \in [0, 1)$ to be the fixed height $h(t) = h_0$ of the genies and also the fixed distance $d(t) = 1$ between the genies respectively, and with $(t_a, t_b, t_c) = (0, 1, 1)$. Clearly, the traveling genie would conclude that $(h(1), d(1)) = (h_0, 1)$, implying that neither of these supertasks can be completed, as predicted by the nonclassical induction argument of Sec. 1.1 and in contradiction to the requirements of classical logic.

Let there be a stationary shrinking genie at $x = 0$ in the postulated hypothetical universe. The stationary genie would consider its height to be fixed at h_0 (assumed to be the only available length scale) and would therefore observe that both the traveling genie and the target genie are also at fixed height h_0, at a fixed unit distance from each other, and are both receding at an increasing velocity proportional to $(1-t)^{-2}$, as can be seen from the following. Recall that the traveling genie starts at $x = 0$ and travels at unit velocity, so that its path is described by $x(t) = t$, and the target genie is at $x = 1$. Let $(x_1(t), h_1(t))$ and $(x_2(t), h_2(t))$ be the coordinates of the traveling genie and the target genie respectively, from the viewpoint of the stationary genie, which fixes its height at h_0 as the standard unit of length. It is easy to see that

$$
\begin{aligned}
& x_1(t) = \frac{t}{1-t}, \quad x_2(t) = \frac{1}{1-t}, \quad t \in [0,1) \to (h_1(t) = h_2(t) = h_0), \\
& \frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{1}{(1-t)^2}, \quad t \in [0,1) \to (x_2(t) - x_1(t) = 1), \\
& x_1(1) = x_2(1) = \frac{dx_1}{dt}(1) = \frac{dx_2}{dt}(1) = +\infty, \qquad (18)
\end{aligned}
$$

where $x_1(t) \in \bar{\mathbb{R}}$, $x_2(t) \in \bar{\mathbb{R}}$, $t \in [0,1]$ and $\bar{\mathbb{R}}$ denotes the extended real number system, which has been used to remove the singularity at $t = 1$. From the point of view of an observer external to the postulated hypothetical universe, the infinities occur here because the length scale of the stationary genie has shrunk to zero at $t = 1$. Observe that $(x_2(1) - x_1(1))$, being of the form $(\infty - \infty)$, is undefined in the extended real number system. We also need to define $h_1(1)$ and $h_2(1)$. There are three possibilities here, all of which lead to contradictions, as follows. A natural definition, which follows from an application of Metatheorem 2, would be

$$x_2(1) - x_1(1) = 1, \quad h_1(1) = h_2(1) = h_0, \qquad (19)$$

which implies that the traveling genie does not complete the two supertasks of catching up with the target genie and also vanishing (along with the target genie) at $t = 1$. Alternatively, we could define, in partial violation of Metatheorem 2,

$$x_2(1) - x_1(1) = 0, \quad h_1(1) = h_2(1) = h_0. \qquad (20)$$

Here the viewpoint of the stationary genie would confirm the contradiction that the traveling genie catches up with the target genie at nonzero height h_0 and collides with it before both vanish instantaneously. Thirdly, we could set, in violation of Metatheorem 2,

$$h_1(1) = h_2(1) = 0. \qquad (21)$$

In this case, by assumption, neither the genies nor their coordinates $x_1(1)$ and $x_2(1)$ exist at $t = 1$ and therefore the traveling genie does not complete the supertask of catching up with the target genie before both vanish.

We have seen that the contradictions that exist in (18) at $t = 1$, which follow from (19)–(21), are also present in (17) if we accept Metatheorem 2. While the contradictions in (18) do not result from an explicit use of Metatheorem 2, (18) does require the use of the extended real number system in order to remove the singularities that would otherwise result at $t = 1$. The consequent infinities in (18) at $t = 1$ may also be thought of as resulting from a tacit use of Metatheorem 2 (see Remark 4). As noted previously, these contradictions must be attributed to the underlying classical assumption that supertasks can be completed.

3.1 Newtonian kinematics versus relativistic kinematics

It should be emphasized that from the point of view of Newtonian kinematics, (17) and (18) are entirely equivalent. Note that replacement of h_0 by $h_0(1-t)$ in (18) would imply that the length scale of the genie is frozen at the initial value (h_0) of its height, which is now shrinking by a factor of $(1-t)$. Therefore the distances $x_1(t)$ and $x_2(t)$ must also shrink by the same factor, which would imply

$$x_1(t) = t, \quad x_2(t) = 1,$$

and we recover (17), as we should. What (18) expresses is that if, at any point of time, we assign the number h_0 to the height of the genies as the standard of length, the stated distances and velocities would follow. Keeping in mind that the number we assign to a length standard is fixed and purely arbitrary and all other lengths are relative to this standard, this is an entirely consistent and kinematically equivalent description of the physical 'reality' that (17) expresses. An observer external to the postulated hypothetical universe would hold the view that the genies are shrinking as in (17), while the genies would maintain that their universe is expanding with respect to their fixed height h_0, as in (18). Purely from the point of view of kinematics, it *ought* to be impossible to say which of these two scenarios is the underlying 'reality', if one rejects Platonism as a philosophy of physics. However, this equivalence between (17) and (18) does not hold in the kinematics of special relativity theory (SR), wherein the Lorentz transformations apply and according to which (18) is illegal, essentially due to an illegitimate choice of the standard of length by the genies. Indeed, SR requires that the length standard be chosen such that the velocity of light in vacuum is a *defined* constant c, in order to conform with the light postulate of SR. Assuming that the standard of length in the Cartesian coordinate system of (17) is chosen to uphold the defined value c of the velocity of light, the time τ_g taken by light to traverse the height of the genies would be measured, by an observer external to the postulated hypothetical universe, as

$$\tau_g = \frac{h_0(1-t)}{c}, \quad t \in [0, 1). \tag{22}$$

Hence the external observer would conclude that the genies are shrinking, and that their height is $h_0(1-t)$. The genies would also measure exactly the same value of τ_g as in (22) for light to traverse their height, which they fix at the value h_0 as their standard of length. It follows that the velocity of light in vacuum (c_g) in the coordinate system of (18) would be *measured* by the genies as

$$c_g \approx \frac{c}{1-t}, \quad t \in [0, 1), \tag{23}$$

where the approximation is valid to leading order as $c \to \infty$ or as $h_0 \to 0$. Clearly, (23) is in violation of the light postulate of SR and it follows that (18) is illegitimate in SR. Note also that the velocities of the genies in (18) exceed the velocity of light c_g in (23) by an order of magnitude as $t \to 1^-$, which is also not allowed in SR. Thus relativistic kinematics picks out (17) as a Platonic reality and one may conclude that Platonism, rejected in the proposed finitistic logic NAFL, is inherent in SR. This issue is important and deserves further elaboration. When relativistic kinematics requires that the genies are shrinking, we have a right to ask, shrinking with respect to what? If one views the shrinking of the genies as the motion of one endpoint of the genie towards the other, a consistent theory of kinematics must hold that such a shrinking motion can only be relative to another material object of fixed length, which we may postulate as the length standard. But the only material objects in the postulated hypothetical universe are the genies themselves. Therefore the only alternative for the genies is to hold their own height as the fixed standard of length and conclude that their universe is expanding with respect to this standard, in violation of SR. A theory based solely on kinematics cannot reject this alternative if it establishes that motion is truly relative, as it should. One could assert that the genies are shrinking with respect to their initial height h_0. But if this is the case, at any given time $t > 0$ there is no material object in the postulated hypothetical universe that would correspond to the length h_0. Relativistic kinematics defines this standard length h_0 without any reference to material objects, as the distance traveled by light in vacuum in the time h_0/c, where c is a *defined* constant. In our view, this is essentially an assertion of the existence of absolute space, e.g., the pre-relativistic ether, which SR itself rejects as a completely unverifiable Platonic reality. In other words, the kinematics of SR freezes the initial height h_0 of the genies in the vacuum (ether) of absolute space and purports to measure the shrinking of the genies with respect to this preexisting, Platonic and unverifiable length standard. Note that an observer external to the postulated hypothetical universe, for whom (17) holds, is assumed to have access to a material object of precisely the length h_0, which can in principle be held as the length standard and with respect to which c can be recovered as a *measured* constant.

The inability of relativistic kinematics to support (18) in the postulated hypothetical universe suggests a possible metamathematical inconsistency in SR, because our contention is that (18) can only be rejected, if at all, by invoking alternative theories, such as, dynamics. Indeed, in Sec. 4, we will demonstrate that Metatheorem 2 also contradicts the key requirement of SR that invalidates (18), namely, that c is an unattainable upper bound for the velocities of massive objects. Hence, in our opinion, the attempt to invoke relativity theory as a solution to Zeno's dichotomy paradox, e.g. in Sec. 3.7 of Ref. [15], fails.

3.2 The NAFL fix for the contradictions in (17)-(18)

As noted previously, the main objection to (18) is that the use of the extended real number system leads to the contradictions which follow from (19)–(21). We have pointed out that the contradictions also exist in (17) if one accepts Metatheorem 2 and must be attributed to the underlying assumption of classical logic that supertasks can be completed. Here we will briefly outline how these contradictions can be resolved in the logic NAFL; the details will be available in Ref. [8]. By eliminating these contradictions, the NAFL version of Newtonian kinematics upholds the equivalence between (17) and (18), as any legitimate theory of kinematics should.

In the logic NAFL, it is not even possible to formulate Benacerraf's example of the shrinking genie, which is postulated to exist only when $t \in [0, 1)$, because the half open interval $[0, 1)$ does not exist in the NAFL version of real analysis [8, 7]. Therefore the first modification required by NAFL to Benacerraf's formulation is that the genie must continue to exist even when its height has shrunk to zero, *i.e.*, the genie must be treated as a point mass at $t = 1$ in (17) and (18). Of course, such a formulation, which satisfies conservation of mass, would invalidate the basic premise in Benacerraf's argument. The next step needed in NAFL to uphold the equivalence between (17) and (18) is to modify (21) as follows:

$$h_1(1) = h_2(1) = 0, \quad x_2(1) - x_1(1) = 0. \tag{24}$$

Note that the coordinates $x_1(1)$ and $x_2(1)$ must now exist, because the genies do exist at $t = 1$. With this formulation, (17) and (18) (along with (24)) are equivalent in the NAFL version of Newtonian kinematics *without* the contradictions that result in classical logic from an application of Metatheorem 2. This is so because neither Zeno's dichotomy paradox nor Metatheorem 2 can be formulated in NAFL, which does not permit the existence of half open real intervals or of sequences of real numbers that exclude their limit points. Further, the paraconsistent and time-dependent nature of NAFL truth correctly handles the step discontinuities at $t = 1$ in $h_1(t)$, $h_2(t)$ and $(x_2(t) - x_1(t))$ that result from (24), without implying the existence of open or half open intervals of real numbers, as elaborated in Ref. [8].

4 Metamathematical inconsistencies in special relativity theory and Newtonian mechanics

We present another argument for the inconsistency of special relativity theory (SR) that uses Metatheorem 2 without any appeal to the failure of time reversal invariance, which, as noted in Sec. 2, is already a fatal contradiction. Both of these logical inconsistencies in SR, which occur at the metamathematical (rather than formal) level, are also present in classical Newtonian mechanics. We will briefly outline how the finitistic reasoning of the logic NAFL can be used to fix these inconsistencies for Newtonian mechanics, but not for SR, whose reasoning is infinitary by the NAFL yardstick.

In a region where SR is applicable, let a particle of rest mass $m_0 > 0$ have a linear velocity $v \geq 0$ with respect to an inertial frame of reference. It is well known that v must satisfy the constraint $v < c$, where c is the velocity of light in vacuum. Thus SR allows the particle to attain every velocity less than c, but not $v = c$, which is a singular point at which the theory becomes ill defined. In particular, consider the following velocity profile:

$$\frac{v(t)}{c} = t, \quad \text{if } t \in [0,1) \quad \text{and} \quad \frac{v(t)}{c} = 0.99, \quad \text{if } t \in [1,2]. \tag{25}$$

Here t is a dimensionless time as measured by a clock at rest in the inertial frame of reference. Note that $v \to c$ as $t \to 1^-$, and hence there is a discontinuity in the velocity at $t = 1$, which could happen, for example, due to a collision. As the particle accelerates on $t \in [0,1)$, its mass m and energy E increase without bound:

$$m(t) = \frac{m_0}{\sqrt{1-t^2}}, \quad E(t) = m(t)c^2, \quad t \in [0,1). \tag{26}$$

Nevertheless, the energy and mass of the particle remain finite and $v < c$ at each instant $t \in [0,2]$, which makes (25) a legitimate definition in SR. It should be emphasized that we are concerned here with only the *logical* feasibility of (25) and not its *practical* feasibility, which is obviously highly problematic. Indeed, the primary logical requirement is consistency. Even if a logical inconsistency is demonstrated in an impractical scenario, it has serious implications for SR as a deductive system.

Apply Metatheorem 2 to (25) and (26) with $(t_a, t_b, t_c) = (0,1,2)$, and with $S(t)$ taken in turn as $v(t)/c$, $m(t)$ and $E(t)$. This yields

$$v(1) = c, \quad m(1) = E(1) = +\infty, \tag{27}$$

which contradicts (25). Note that the extended real number system $\bar{\mathbb{R}}$ has been used to define $m(1)$ and $E(1)$. From (27) we conclude that SR must necessarily hit the singularity at $t = 1$, where the theory becomes ill defined and inconsistent, despite the fact that the velocity profile defined in (25) is strictly within the range of validity of SR. This logical inconsistency in SR is at the metamathematical level and will become formal if we add Metatheorem 2 as an axiom to SR.

To understand why Metatheorem 2 is presenting the correct picture in this case, namely, one of inconsistency of SR, let us temporarily ignore (27) and assume that (25) is legitimate, as required by SR (and as also supported by an application of Metatheorem 2 in the time reversed direction). Consider the energy lost by the particle due to the discontinuous drop in velocity at $t = 1$. Intuitively, one feels that the particle lost an infinite amount of energy at $t = 1$ because, from (26), $E(t) \to \infty$ as $t \to 1^-$. But this intuitive conclusion is false, because, as noted, $E(t)$ is finite at each instant $t \in [0,2]$. There is no question of the particle losing an infinite amount of energy if in the first place its energy did not become infinite at any point of time, in particular, "just before" $t = 1$, as our intuition falsely tells us. The only remaining possibility is that the energy loss at $t = 1$ is finite, but this is also clearly false because the energy of the particle has exceeded every finite magnitude as $t \to 1^-$. Hence loyalty to SR leads us to the untenable conclusion that the energy lost by the particle due to the velocity discontinuity at $t = 1$ can be neither finite nor infinite.

The key to resolving this contradiction is in the assertion made above that $E(t)$ has exceeded every finite magnitude. This is the same as accepting that the velocity of the particle has exceeded every magnitude less than c, which implies that the particle completed the supertask of attaining infinitely many velocities in the sequence $\{v(t_n)\}$, where t_n is as defined in (2). The proof of Metatheorem 2 uses the completion of precisely such supertasks to conclude the inconsistency of SR via (27). Clearly, the velocity of the particle cannot be increased through every magnitude less than c without the particle attaining $v = c$, and this is a basic mathematical fact (in the same way that it is impossible to extend a line segment through every finite length without creating an infinite length, as noted in Remark 4). Note that even with (27) in place, we *cannot* conclude that an infinite amount of energy is acquired by the particle at $t = 1$. We may only conclude that SR is inconsistent and one cannot believe the predictions of an inconsistent theory formulated in classical logic, which allows an arbitrary proposition to be deduced in inconsistent theories (via the principle of explosion, also known as *ex falso quodlibet* (EFQ)).

4.1 Classical objection and rebuttal

A possible classical objection to the above claim of inconsistency might be that the supertasks used in the proof of Metatheorem 2 cannot be completed as assumed because they require an infinite amount of energy. The substance of this objection is that the supertasks in question, if completable, must get completed in the half open time interval $[0, 1)$ without the existence of a time of completion in $[0, 1)$ because supertasks do not have a last step. This would result in the particle acquiring an infinite amount of energy in the time interval $[0, 1)$ without the commitment that SR has hit the singularity at $t = 1$. According to this classical argument, the fact that the supertasks remain incomplete for each $t \in [0, 1)$ does not imply that the supertasks are incomplete in the entire half open time interval $[0, 1)$, for such a conclusion would require a classically illegal interchange of universal and existential quantifiers. Thus the classical objection amounts to asserting that (25) is illegitimate in the sense of being unphysical, without the consequence that SR is inconsistent, that is, the limit processes ($v \to c^-$, $m(t) \to \infty$ and $E(t) \to \infty$) as $t \to 1^-$ are not actually realized at $t = 1$. This argument is rejected in Metatheorem 2, wherein the completion of the said supertasks is nonclassically equated with the physical realization of these limit processes at $t = 1$. Each step of these supertasks adds only finite amounts of energy and mass to the particle and is therefore physically realizable *in principle*. It follows by induction that there is no upper limit to the number of steps of these supertasks that can be completed. It is a requirement of classical logic that such a step by step infinite process must result in completion, which does not happen at any point of time in the interval $[0, 1)$.

Let us further consider the specific example of a particle P attaining each velocity in the sequence $\{v(t_n)\}$, where v and t_n are as defined as in (25) and (2) respectively. This supertask may be broken up into infinitely many finite parts, as was done in Sec. 1.4 for the infinite relay dichotomy paradox. Let $\{P_n\}$ represent an infinite sequence of particles such that for each $n \in N$, the particle P_n is accelerated to the velocity $v(t_n)$ at time t_n (say, via instantaneous, finite impulses). Clearly, there is no particle in $\{P_n\}$ that has reached the velocity c or has acquired infinite energy and mass, and hence each velocity in $\{v(t_n)\}$ is in principle attainable from the standpoint of SR. It should follow as a logical consequence that for each $n \in N$, a single particle P may be accelerated to velocity $v(t_n)$ at time t_n, because logically, if the infnite-particle supertask is completable, the same holds for the single-particle supertask. This is so because there is a one-to-one correspondence in time between each of the infinitely many particles achieving velocity v_n at time t_n and a single particle achieving each velocity v_n at time t_n. Hence SR should permit the single-particle supertask to be completed without P reaching the velocity c (and acquiring infinite energy and mass), as required by (25). Yet this classical conclusion is contradicted by Metatheorem 2, as demonstrated in (27). As in Sec. 1.4, we may conclude that the reason for this metamathematical inconsistency in SR is the fact that classical logic requires that supertasks should be completable.

The only way to correctly infer that supertasks cannot be completed is via the nonclassical induction argument of Sec. 1.1, and if we accept this argument (as does the logic NAFL), all of classical infinitary reasoning, including SR, would become inconsistent. Let us also note that from a classical point of view, it is not logically inconsistent to suppose the existence of energy sources that can supply an unlimited, but finite amount of energy. Such sources could be distributed along the entire path of the particle (say, as a force field) and supply the required energy for the supertasks, step by step.

4.2 The Newtonian case and the NAFL resolution

Essentially the same arguments as in Sec. 4 apply for the Newtonian case, with the only difference being that Newtonian mechanics becomes inconsistent at $v = +\infty$ rather than at $v = c$. Consider the following velocity profile, with respect to an inertial frame of reference, of a particle of mass m:

$$v(t) = \frac{1}{\sqrt{1-t}}, \quad \text{if } t \in [0, 1) \quad \text{and} \quad v(t) = 1, \quad \text{if } t \in [1, 2]. \tag{28}$$

The velocity $v(t)$ of the particle and its kinetic energy, $E(t) = mv(t)^2/2$, become unbounded as $t \to 1^-$. An application of Metatheorem 2 with $(t_a, t_b, t_c) = (0, 1, 2)$ yields $v(1) = E(1) = +\infty$, which contradicts (28) and implies the metamathematical inconsistency of classical Newtonian mechanics. Again the supertasks used in the proof of Metatheorem 2 require only a finite amount of energy at each step and should be classically completable because they are defined entirely within $t \in [0, 1)$ and v is finite everywhere in (28).

This metamathematical inconsistency in classical Newtonian mechanics is fixable in the logic NAFL, as will be elaborated upon in Ref. [8]. The NAFL version of real analysis [8, 7] does not allow quantification over real numbers. Superclasses of real numbers can be defined without quantification in NAFL, but such superclasses must necessarily be closed, that is, include their limit points. In particular, only closed intervals of real numbers are allowed in NAFL and the half open time interval $[0, 1)$ does not exist. The superclass of all real numbers (*i.e.*, the real line) must necessarily include the limit points $\pm\infty$ in the NAFL version of real analysis. It follows that (28) is not a legitimate definition in the NAFL version of Newtonian mechanics, which does not allow the paradoxical classical assertion that a particle can attain (and exceed) every finite positive velocity when $t \in [0, 1)$ without attaining $v = +\infty$ at $t = 1$. The fact that a massive object can attain an infinite velocity if an infinite amount of energy is put into it is *not* an inconsistency in Newtonian mechanics; it only represents a practically unrealizable scenario. Every finite value of the energy / velocity is *logically* (but not *practically*) attainable for a massive object in Newtonian mechanics and hence NAFL, unlike classical logic, mandates that an infinite value of the energy / velocity is also a logically attainable limit. Secondly, as noted in Remark 8, time reversal invariance does hold in the NAFL version of Newtonian mechanics.

Unfortunately, these NAFL fixes for Newtonian mechanics are not applicable to SR, which *requires* open (and half open) intervals of real numbers, as well as time reversal invariance, for its definition. That a massive object is allowed to attain every velocity less than c, but not $v = c$, is an integral fact of SR which requires the existence of the half open real interval $[0, c)$, not permitted in NAFL. In other words, SR requires that $v = c$ for a massive object is a *logically* unattainable limit at which SR becomes inconsistent. Hence SR is an infinitary theory as per the strict definition of finitism in NAFL, which also does not permit non-Euclidean geometries, such as, Minkowski spacetime. Nevertheless, SR is a highly successful theory and it remains to be seen if some of its formulas are recoverable in NAFL via a suitable modification of Newtonian mechanics in an Euclidean framework of space and time.

5 The sentence "I committed suicide"

There has been considerable interest among philosophers and logicians in self-referential 'liar' sentences, such as, "I am a liar" or "This sentence is not true", which have attracted a voluminous literature [21, 22, 23]. While there is a consensus among the mainstream thinkers that these liar sentences are illegitimate, there is no clarity on why this should be the case from a logical point of view. Part of the problem is that the dominant philosophy of classical logic is Platonism, wherein truths are eternal and independent of the human mind. From a Platonic standpoint, the sentence "I am a liar" is logically equivalent to "X is a liar", where X is the individual to which "I" in the original sentence refers. Note that the latter sentence, unlike the original, is merely a statement of a matter of fact that is independent of the human being (other than X) who utters it. Thus the Platonic equivalent of "I am a liar" seems to be perfectly legitimate and does not appear to be tied in any way to infinitary reasoning. Classical logic does permit certain self-referential sentences. For example, the Gödel sentence [24], equivalent to "This sentence is not provable", can be formalized in finitistic classical theories.

Here we consider the sentence "I committed suicide" (henceforth abbreviated as ICS), which, as we will demonstrate, has a significance for finitary reasoning that has remained unexplored so far. To study ICS from a classical point of view, we set up the temporal dichotomy paradox (see Secs. 1.5.2 and 1.5.3) as follows.

- Homer shoots himself at time $t = 0$.
- Homer is alive if $t \in [0, 1]$.
- At $t = 1$, Homer has the thought "I committed suicide". Here we assume, for convenience, that thoughts can occur instantaneously.
- Homer is dead if $t > 1$.
- The autopsy establishes the cause of death as Homer's self-inflicted bullet injury and the time of death as $t = 1$.

Note that the sentence ICS occurs as Homer's final thought before he dies. At first sight the truth of ICS in this scenario seems almost obvious. The 'I' in "I committed suicide" refers to the individual Homer in the present context, which therefore dictates that

$$\text{I committed suicide} \Leftrightarrow \text{Homer committed suicide.} \tag{29}$$

The sentence on the left hand side of (29) is one that can only be asserted by a human being, while that on the right hand side is a statement of a matter of fact, with no apparent connection to the human mind. Nevertheless this equivalence is classically valid because of the prevalent philosophy of classical logic, namely, Platonism. It follows that ICS is true from the point of view of classical logic because Homer did commit suicide, as confirmed by the autopsy.

One might object to the above classical interpretation of ICS as follows. On the left hand side of (29) is an assertion by Homer of his suicide as a *completed* act, for the sentence ICS essentially means "I have killed myself". There is a seeming contradiction here, for Homer was alive at $t = 1$ when he made this mental assertion of his own death, and indeed, *had* to be alive, for dead men can have no thoughts. The counter to this objection is that classical truths in the Platonic universe are eternal, that is, time-independent. From the point of view of classical logic, the time at which Homer asserted ICS is immaterial to its truth because it is a fact that Homer's suicide was eventually proven. This time-independence is precisely what the equivalence in (29) expresses.

There is a second, more interesting semi-classical interpretation of the truth of ICS in the present context that overrules the above objection, albeit controversially. Although a time of death does not formally exist in the given formulation of Homer's suicide, the idea is to nevertheless impose the time of death as $t = 1$ (as confirmed by the autopsy) in a *metamathematical* sense, via an application of Metatheorem 2 in the time reversed direction. The resulting contradiction that Homer is both alive and dead at $t = 1$ gets pushed to the metatheory, while the formal classical theory remains consistent. From this point of view, Homer did witness his own death at $t = 1$, although this contradiction is not formally provable, given that Metatheorem 2 is a nonclassical, metamathematical result. Therefore Homer's assertion of ICS at $t = 1$ can be argued to be true even when the time-dependence of truth is taken into account, without any need to invoke the equivalence in (29). This interpretation is interesting because it clearly ties the truth of ICS to classical infinitary reasoning, which, in the process of refuting Zeno's temporal dichotomy paradox (see Secs. 1.5.2 and 1.5.3), enables Homer to witness his own death by permitting him to complete the supertask of sequentially traversing infinitely many durations in finite time. Whereas, according to the purely classical, Platonic interpretation of ICS, as symbolized by (29), the truth of ICS does not have any obvious connection to infinitary reasoning. In fact, according to conventional wisdom, the sentence ICS can be formalized in finitistic classical theories.

We briefly touch upon the interpretation of ICS in the logic NAFL [8]. We have already noted in Sec. 1.5.3 that a precise moment of death must exist in NAFL, which does not support (12). The NAFL formulation of Homer's suicide would be as follows.

$$\text{Homer shoots himself at time } t = 0 \text{ and dies at } t = 1. \tag{30}$$

It is clearly impossible for a dead Homer to assert ICS at $t = 1$. Therefore the contradiction that Homer witnessed his own death is not permitted in NAFL. Homer can only assert ICS when he is alive, e.g., at some time $\bar{t}$, where $0 \leq \bar{t} < 1$. NAFL truths, being axiomatic assertions of the human mind, are time-dependent [5, 6, 7, 8]. It follows that ICS is unambiguously false in NAFL, *despite* the fact that Homer did commit suicide, because Homer was alive *when* he made the assertion ICS. Clearly, the equivalence in (29) does not hold in NAFL, which rejects the classically acceptable philosophy of Platonism. By the NAFL yardstick, a different equivalence must hold in the present context:

$$\text{ICS}(t) \Leftrightarrow \text{At time } t\text{, Homer committed suicide and witnessed his own death.} \tag{31}$$

Here, ICS(t) denotes the sentence ICS asserted at time t, which, in the present context, refers to Homer's suicide and can only be asserted by Homer himself. Clearly, the NAFL restrictions imply that Homer can assert ICS(t) truthfully if and only if Homer did commit suicide at time t and is able to witness his own death at that time, as asserted in (31). Note that "committed suicide" in (31) refers to the *completion* of the act of suicide, that is, to Homer's death; the *initiation* of Homer's suicide, namely, the act of shooting himself, has no relevance in this context. Evidently, the right hand side of (31) (and consequently, ICS(t)) is a metamathematical contradiction which, as we have seen from the NAFL formulation of Homer's suicide in (30), is unambiguously false, and indeed, impossible. It is worth noting that this impossibility occurs because, unlike the case in classical logic, the nonclassical induction argument of Sec. 1.1 holds in NAFL and does not permit Homer to complete supertasks. In contrast, we have seen that according to the semi-classical interpretation, Homer has to complete supertasks (as required by Metatheorem 2) in order to be able to witness his own death, and hence the truth of ICS(t) in (31) is permitted by classical infinitary reasoning, unlike the finitary reasoning of NAFL.

The logical impossibility of completing supertasks and in particular, the inability of the human mind to complete them, is precisely the reason why infinite sets cannot exist within NAFL theories [8]. Hence there is a close parallel between the falsity of ICS(t) in (31) and the nonexistence of infinite sets in NAFL theories, both of which are metatheoretical results upheld by the nonclassical induction argument of Sec. 1.1. It should be emphasized that there is no formal refutation of sentences like "I committed suicide" or "Infinite sets exist" within NAFL theories, whose syntax does not even permit such sentences.

In summary, NAFL provides new criteria for finitary reasoning by refuting classical infinitary reasoning at the metamathematical, rather than formal, level. While choosing between classical logic and NAFL, the reader should consider whether it is strictly rigorous to accept Platonism and / or the infinitary, nonconstructive reasoning that allows sentences like ICS to be true within classical logic. We believe that the interests of rigor are better served by accepting the finitism of NAFL, which makes ICS unambiguously false and indeed, absurd.

6 Concluding Remarks

In this paper, we have reexamined Zeno's dichotomy paradox in some detail and found that its variants lead to metamathematical inconsistencies in classical infiinitary reasoning, including Newtonian mechanics and the theory of special relativity. The main reason for these metamathematical (as opposed to formal) inconsistencies is that the classical refutation of Zeno's dichotomy paradox (see Sec. 1.3) requires supertasks to be completed. In Part II of this two-part series of papers [8], we will outline the basic principles of the finitistic logic NAFL, including real analysis, and show how these and other paradoxes (including a few arising from quantum mechanics) are resolved.

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