Deductive Pluralism [1]

Abstract

This paper proposes an approach to the philosophy of mathematics, deductive pluralism, that satisfies the criterion of inclusiveness of mathematical practice. The basic argument is as follows: there are existing varieties of mathematics with incompatible foundations; given these varieties, a philosophy of mathematics should be inclusive; the criterion of inclusiveness and the fact of incompatible varieties implies a pluralistic philosophy; this in turn requires a deductivist approach within each variety; thus deductive pluralism. The advantages of deductive pluralism include the elimination of ontological problems, epistemological clarity, and objectivity.

1 Introduction

The basic argument of this paper is: there are existing varieties of mathematics with incompatible foundations, mathematical or logical; given these varieties, a philosophy of mathematics should be inclusive, i.e., not reject a variety as false in some absolute sense, and include existing and future varieties; the criterion of inclusiveness and the fact of incompatible varieties requires a pluralistic philosophy; this in turn requires within each variety a deductivist approach in which mathematical results explicitly or implicitly take the form of implications that the assumptions imply the conclusions after a long development from the foundations of definitions, theorems, and examples; thus deductive pluralism. This section will introduce the concept of a variety of mathematics and related concepts used in the rest of the paper. Section 2 considers factors that have motivated deductive pluralism. Section 3 discusses a few varieties of mathematics that are distinguished by incompatible mathematical or logical foundations, discussing their motivations, formalized mathematical and logical foundations, and highlighting some of their incompatibilities. The most widely used of these is *standard mathematics*, which has Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as the mathematical foundation and first order predicate calculus (FOPC) as the logical foundation. Section 4 discusses in more detail the criterion of inclusiveness and the components of deductive pluralism in light of the examples. Section 5 places deductive pluralism in the philosophy of mathematics by discussing its ontology, epistemology, and objectivity. Section 6 considers and answers some possible criticisms of deductive pluralism, including its consistency with mathematical practice and with the attitudes of philosophers and mathematicians who consider foundational questions. Section 7 considers other forms of pluralism, and finally section 8 considers deductive pluralism as it relates to some other philosophies of mathematics. Since logical assumptions are part of a foundation for a variety there is an appendix on relevant logical concepts which may be referred to as needed. There is also an

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appendix giving some examples of the historical development of mathematics towards an axiomatic (thus deductive) viewpoint.

As used in this paper, a variety of mathematics consists of the logical and mathematical foundations and the results (which in this paper include definitions, examples, and theorems) developed from the foundations. A variety will usually be considered as containing the maximal set of results (reflecting "the unity of mathematics"), although sometimes it may be useful to consider an axiomatized theory such as Dedekind-Peano arithmetic (PA) as a stand-alone variety as well as a theory within standard mathematics.

Varieties of interest are eventually formalized, and so it is useful to consider two aspects of the foundations of, and theories within, such a variety: its formal, axiomatized, and uninterpreted symbols (the syntax) and its interpretations (the semantics). The symbols can be divided into variable, logical, and relational symbols (constants, etc.), which are used to express the axioms. Associated with a formal system is a formal proof which is one that can be verified mechanically, such as by a computer proof assistant, examples of which will be introduced below. A deduction within a variety is seldom fully formalized but is usually rigorous, or sometimes still less formal. The concept of rigor is necessarily imprecise since it will vary between mathematicians, between areas of mathematics, and in different historical periods; however, in this paper a rigorous proof will be one that can in principle be formalized.

An interpretation assigns the symbols of a formal system to elements of another system used as a reference, and if the axioms of a theory hold in an interpretation then the interpretation is called a model. As a simple example with no non-logical axioms, consider propositional logic as the formal system with the interpretation in a set U. The variables correspond to subsets of U and the logical operations of conjunction, disjunction, and negation correspond respectively to the set operations of intersection, union, and complementation in U. Since the axioms of propositional calculus hold in this interpretation, it is a model. A formalism will usually have a particular intended interpretation reflecting its motivation, as is seen in the examples of section 3. There are advantages to viewing the foundations and theories as formal uninterpreted symbols with formal proofs. One advantage is the reduction of error since historically a common source of error has been the use of non-formal, imprecisely defined concepts, or proofs that rely on implicit assumptions that do not follow from the explicit assumptions. The desire to avoid such errors was a major motivation for univalent foundations discussed below. Another advantage is the possibility of multiple interpretations and thus multiple applications of the formalism since a formal proof applies to all interpretations. The formalism reflects the development over time of a variety, including abstraction and generalization from motivating examples, consideration of possible axioms, and new examples sometimes leading to new or modified axioms.

The varieties of mathematics may have different and incompatible logics used to derive conclusions from the foundational mathematical assumptions. Thus, an inclusive criterion for varieties of mathematics also requires an inclusive view of logics, at least for those used in mathematics. There are many varieties of logic, with different attitudes towards them. A type of pluralistic attitude has been advocated by Shapiro, who wrote: “Pluralism about a given subject, such as truth, logic, ethics, or etiquette, is the view that different accounts of the subject are equally correct, or equally good, or equally legitimate, or perhaps even true”(Shapiro, 2014, p. 13). This paper also has a pluralistic attitude; however whether two logics are “equally good” depends on the objectives and motivations. For example intuitionistic logic has been considered as part and parcel of constructive mathematics (as discussed in Section 3.4), thus FOPC is not as “good” as intuitionistic logic for constructivism. Other examples and their motivations are quantum logic, which denies the distributivity of conjunction over disjunction, and reflects Heisenberg uncertainty for incompatible observables such as position and momentum, and linear logic which reflects a resource-consciousness in which formulas represent resources which cannot be used without limit, as in computer logic gates. Thus, it is possible to view logics as tools whose suitability depends on the task, with First Order Predicate Calculus viewed as a general purpose (but not necessarily all-purpose) tool reflecting commonly accepted reasoning. The focus in this paper is on the implications for the philosophy of mathematics given the criterion of inclusiveness and the existing varieties of mathematics with incompatible foundations, exemplified in Section 3. The logics (and possible future logics) vary greatly so there is no useful overarching logic that can be applied within all the varieties. Thus, in this paper when considering varieties of mathematics, the mathematical and logical assumptions of varieties will be compared and then results within those varieties obtained by applying the logics of those varieties will be compared. Since the objective of considering the varieties is to show the incompatibilities, little beyond comparison will be needed. If deductions are needed outside a variety, during the discussion, then the general purpose logical tool of FOPC will be used.

2 Motivating Considerations

The first motivation concerns the status of consistency. A system is inconsistent if a contradiction can be derived. Inconsistency can be proven by exhibiting such a derivation, as Russell did with Frege’s system. In principle if a system is inconsistent then this can demonstrated by systematically listing all possible derivations; however, such a process will not terminate if the system is consistent. Thus, the problem lies in demonstrating that there can be no derivation of a contradiction. But this has become problematic for systems containing arithmetic, beginning with Gödel’s second incompleteness theorem which implies that a consistent first order formal system using FOPC as the logic and containing PA, such as standard mathematics, cannot prove its own consistency. Thus, for such systems, although consistency is not relative, depending only on the mathematical and logical assumptions, results asserting consistency are relative: such results state that if one system is consistent then so is another. A philosophy of mathematics should take onboard the relative nature of results asserting consistency.

The second motivation comes from the problematic nature, pointed out in philosophical discussions, of the existence of abstract objects (those not existing in spacetime) such as mathematical objects. There has been no philosophical agreement on even the meaningfulness of the question; for example, Balaguer (1998, 22) considered the question as essentially meaningless when he wrote that the existence of abstract objects "does not have any truth condition." One of the clearest approaches to abstract objects within mathematics is that of Hilbert (1902, 9–10) who equated the existence of such objects with consistency in his 1900 address introducing the Hilbert Problems. This 1900 suggestion has become problematic due to discoveries since then. First, as noted in the preceding paragraph, results asserting consistency have become relative. Thus, consistency (or existence using Hilbert's criterion) cannot be proven, so it must be assumed. This is essentially a deductivist position: statements assert what follows from the assumptions. A related problem is the multiplicity of contradictory but relatively consistent axiom systems, so that if existence is identified with consistency then there are contradictory assertions, including some based on standard mathematics. For example, consider the Continuum Hypothesis (CH), which states that any infinite subset of the reals must have the same cardinality as (be equinumerous with) either the reals or the natural numbers. If ZFC (using FOPC) is assumed to be a consistent basis for mathematics, then both ZFC+CH and ZFC+¬CH are also consistent (by the work of Gödel and Cohen). So, using consistency as the criterion for existence, if the existential assertions of ZFC hold then so do the contradictory assertions of CH and ¬CH. (Note the difference between this contradiction within one theory—the real numbers—and two distinct theories with incompatible assumptions that have examples within standard mathematics—e.g., Euclidean and non-Euclidean geometry.) Thus, the initially attractive idea of identifying mathematical existence with consistency runs aground on the results since the 1900 formulation, and so assertions of the absolute existence of mathematical abstracta are problematic and should be avoided.

The third motivation comes from noting that there are incompatible varieties of mathematics, with some examples given in the next section. Although some variety may be preferred for reasons such as philosophy, tradition, simplicity, or applicability, the question is: can the others be viewed as false in some absolute sense? If so, on what basis, particularly in light of the relative nature of results asserting consistency in mathematics which removes absolute consistency as a criterion?

3 Varieties of Mathematics

This section gives examples of varieties of mathematics focusing on factors needed in later sections, primarily incompatible assumptions or results, but also their motivations and formalizations. Only a few varieties are discussed below due to space limitations, with the selection based on considerations such as their comparability to standard mathematics, their prominence within mathematics, recent interest, or foundations based on something other than set theory. While working within a variety, the logic of that variety is applied to the previous results, ultimately to the foundational assumptions, to deduce theorems, to make definitions, or to construct examples.

3.1 Standard Mathematics

The first example of a variety of mathematics will be standard mathematics (ZFC with FOPC as the logic). Most mathematics as practiced, both pure and applied, is standard mathematics. Since it is so dominant and extensive most other varieties of mathematics, including those discussed below, are careful to include many of the same or similar theories and results as standard mathematics (e.g., PA as interpreted within the variety). The motivations for the ZFC formalization include: the desire to answer the set-theoretic paradoxes while providing a foundation for mathematical practice; to fortify the position of the axiom of choice; and to support Hilbert's program of axiomatization (see section 2.9.1 "Motivations" in Ebbinghaus and Peckhaus (2007)).

In standard mathematics model theory studies these interpretations within set theory. However, the interpretation need not be in set theory, and need not be unique. For example, consider ZF–Inf: the Zermelo-Fraenkel axioms with the axiom of infinity negated. The intended interpretation is the collection of hereditarily finite sets, which is defined inductively by starting with the empty set and then taking only finite collections of other hereditarily finite sets: if $S_1, S_2, ..., S_k$ are hereditarily finite then so is the set $\{S_1, S_2, ..., S_k\}$. Ackermann (1937) gave a non-standard interpretation of ZF–Inf in the natural numbers by interpreting the variables as natural numbers and $m \in n$ satisfies the binary relation symbol "$\in$" (where same "$\in$" symbol is used in both the syntactic and semantic systems when there is no danger of confusion) if and only if the mth digit in the binary representation of n is 1, counting the digits from the right and starting at the 0th. Thus, $m \in n$ if 2^m is in the binary expansion of n. For example $22 = 2^4 + 2^2 + 2^1$ so $1 \in 22$, $2 \in 22$ and $4 \in 22$. The Ackermann interpretation, which is consistent relative to PA, is an example of a common situation: the syntactic system does not uniquely determine the semantic system so that there are other interpretations in addition to the intended interpretation. Note that a fully formal proof of a theorem (possibly verified by a computer proof assistant as discussed below) does not use the interpretation, and thus such a theorem holds in all interpretations.

3.2 Nonstandard Mathematics

Our next example, nonstandard analysis, is an extension of standard mathematics that was motivated by the desire to put infinitesimals on a rigorous foundation, and was developed by Abraham Robinson. The logic is the same as in standard mathematics, and there are many approaches to developing the infinitesimals. Nonstandard analysis is a conservative extension of standard mathematics in that any proposition stated in the language of standard mathematics that can be proven using nonstandard analysis can also be proven using standard mathematics. An example of this is the nonstandard proof by Bernstein and Robinson (1966) that every polynomially compact operator has a non-trivial invariant subspace, which appeared back-to-back with a standard proof. In their article they wrote that "[t]he proof is within the framework of Nonstandard Analysis" (Bernstein and Robinson, 1966, p 421), which illustrates that when a variety of mathematics other than standard mathematics is used the foundations are made explicit, especially if the work is in a journal containing standard mathematics in which standard foundations would otherwise be implicitly assumed.

3.3 Tarski-Grothendieck Set Theory

Tarski-Grothendieck set theory (TG), is an example of a non-conservative extension of ZFC with FOPC as the logic. TG set theory adds an axiom U to ZFC stating that every set is an element of a Grothendieck universe, where a Grothendieck universe is a set in TG that acts like a realization of the class of all ZFC sets, and so exists in TG but not in ZFC. A motivation is to provide a basis for the work of Grothendieck and others in algebraic geometry by clarifying and simplifying assumptions, although it is widely believed that most of the results using TG can also be obtained within ZFC alone, and thus avoid the risk of the additional assumption of U producing a contradiction. For example, the Stack Project (http://stacks.math.columbia.edu) has as one of its aims to avoid universes by using "partial universes", i.e., sufficiently large sets that vary with the context. Finally, TG has been formalized using the Mizar computer proof assistant (http://www.mizar.org).

3.4 Constructive Mathematics

Constructive mathematics was motivated by the philosophical criterion that mathematical objects should be constructible in some sense. The version considered here, sometimes referred to as BISH, was developed from the work of Errett Bishop. BISH is an example of a variety of mathematics in which the mathematical assertions and logic have both rules and intended interpretations different from standard mathematics. For example, in BISH if P and Q are propositions then $P \longrightarrow Q$ can be asserted only when a given construction for P can be constructively transformed into a construction for Q. Also, if P is a proposition then $P \vee \neg P$ can be asserted only when there is a constructive method of asserting P or a constructive method of asserting $\neg P$, which is not always possible, so the Law of Excluded Middle (LEM) does not hold in general. The logical foundation is intuitionistic logic, which is FOPC without LEM (also see intuitionistic logic in the appendix), and the mathematical foundations can be formalized as intuitionistic set theory. Standard mathematics is a proper extension of BISH, so that all theorems of BISH are also theorems of standard mathematics, but not conversely. Standard mathematics can also be viewed as an extension of PA. However, the situations differ: standard mathematics does not contradict the motivations or logic of PA, but does contradict the motivations of BISH since, for example, standard mathematics accepts LEM but BISH rejects LEM. Also, Bridges and Palmgren (2016) give as an example the comb (or Dirichlet), function (defined on the unit interval so that it is 1 on the rational numbers and 0 on the irrational numbers in the interval) which exists in standard mathematics but not in BISH.

3.5 Univalent Foundations

The *univalent foundations program* (http://homotopytypetheory.org), is a recent development with a motivation stated in the subtitle "A Personal Mission to Develop Computer Proof Verification to Avoid Mathematical Mistakes" of the paper by Voevodsky (2014, 8), and which is proposed as a "viable alternative to set theory." To do this the program is formalized using the computer proof assistant Coq (http://coq.inria.fr), which is used not only to check proofs but to develop them. Thus, univalent foundations explicitly addresses the question of the reliability of proofs discussed below in section 5. The intended interpretation is homotopy theory, and is an example of a variety of mathematics not based on set theory. Instead, it has as its basis an extension of the predicative, intuitionistic Martin-Löf type theory, with additional axioms such as univalence. The univalence axiom in implies that isomorphic structures can be identified. Identifying structures up to isomorphism is common in standard mathematics; however, isomorphic objects are not always identified. For example, the singleton sets $\{0\}$ and $\{1\}$ are isomorphic as sets (and by a unique isomorphism) but if they are identified then by extensionality the elements would be the same and so as a consequence $0 = 1$. Thus, univalent foundations are incompatible with standard set theory.

3.6 Category Theory

There have been proposals that some variety of category theory (CT) be a foundation for mathematics as an alternative to set theory. This approach is similar to univalent foundations in that the primary objective is usually a different foundation rather than a substantially different mathematics. It is also similar in that categorical foundations use topoi, which are a generalization of sets and whose logic is, in general, intuitionistic logic. Linnebo and Pettigrew (2011) survey some possibilities for using category theory as a foundation with some criteria, e.g., requiring independence from set theory and requiring some existential assertions (as ZFC asserts the existence of the empty set). Some theories are rejected: Synthetic Differential Geometry (SDG) as too narrow and the Category of Categories As Foundations (CCAF) as not independent of set theory. They then consider the Elementary Theory of the Category of Sets (ETCS) as a case study. ETCS is significantly different from set theory. In it everything is defined in terms of (category theoretic) arrows which presents problems for set membership since in this approach an element cannot be a member of more than one set, extensionality does not hold, and there are multiple (isomorphic) empty sets. In addition, although ETCS may be logically independent of set theory, it requires prior set theory for interpretations, for examples, and thus for comprehension.

3.7 Inconsistent Mathematics

Inconsistent mathematics is mathematics in which some contradictions are allowed (Mortensen, 1995). If a contradiction implies all statements then the system is trivial, thus the logic used cannot be standard logic. The most common alternative is some kind of relevant logic. Most of the work in this area has been in the logical foundations and their immediate consequences, although suggestions have been made for other possible applications including inconsistent databases, inconsistent pictures (such as those by Escher), earlier mathematics (such as infinitesimals), alternative accounts of the differentiability of delta functions, or solutions of inconsistent sets of equations. Inconsistent set theory is one of the most widely studied topics within inconsistent mathematics. The objective is often to have a set theory based on two assumptions: unrestricted comprehension (for any predicate P, $\exists z \forall x(x \in z \leftrightarrow P(x))$) and extensionality ($y = z \leftrightarrow \forall x(x \in y \leftrightarrow x \in z)$). As is well known the former leads to Russell's paradox by setting $P(x) = (x \notin x)$, and so to avoid triviality, in which all predicates hold, a non-explosive logic must be used. Also see the section on paraconsistent logic.

In this section we have briefly examined several varieties of mathematics. The list is not meant to be exhaustive: some varieties not discussed are various versions of finitism. However, the above varieties should be enough for the following discussion. If a philosophy of mathematics is to be inclusive of mathematical practice then it must accommodate these varieties, which have different logical assumptions (e.g., FOPC or intuitionistic), different set theoretic foundations (e.g., ZFC or ZFCU) or foundations not using set theory (e.g., univalent foundations). As a consequence objects, such as the Dirichlet comb function of section 3.4, may exist in one variety of mathematics but not in another variety. Thus, the discussions of the above varieties show that no single logical or mathematical foundation is feasible. Given this, the simplest and most natural approach within a variety is deductivism, as proposed in the introduction. The following sections expand on deductive pluralism, discuss the philosophical advantages of deductive pluralism, consider and answer possible arguments against deductive pluralism, and finally consider some related philosophies of mathematics.

4 The Components of Deductive Pluralism

The components of deductive pluralism are the requirement of inclusiveness, its consequence of pluralism, and the necessity of deductivism (sometimes referred to as if-thenism) within a variety. This section will discuss and justify these components.

There are two elements of inclusiveness as used in this paper. The first is that no variety should be ruled out of consideration by viewing it as false in some absolute sense, unless, of course, a fatal error is found. But what if a variety is shown to be inconsistent, i.e., a contradiction is derived? Is this a fatal error? FOPC allows all statements to be derived from a contradiction; thus a system that contains contradictions must use a logic that prevents this, such as relevant logic, if triviality (all propositions hold) is to be avoided. And indeed there is research into inconsistent mathematics with a continuing body of publications. So, deductive pluralism will not rule out varieties that allow some contradictions. From within a variety, i.e., relative to that variety, some statements in another variety may be viewed as false. For example, from the viewpoint of constructive mathematics LEM (as a general principle) is viewed as false. The primary problem with considering a variety as false in an absolute sense is the lack of generally acceptable and principled criteria for such a decision. This lack is shown by the support given by respected mathematicians such as Bishop to constructive mathematics, Grothendieck to TG, and Voevodsky to univalent foundations. Nevertheless, some mathematicians or philosophers may view a variety as false in an absolute sense (e.g., Hauser and Woodin (2014) discussed below), and such a rejection may be itself unfalsifiable. In particular, recall the earlier discussion in section 2 on the lack of generally accepted criteria for the existence of abstract objects, including Hilbert's 1900 proposal to identify existence with consistency. This lack of criteria has two effects: it makes it difficult to falsify claims, but it also argues in favor of an inclusive view to avoid rejecting a variety for insufficient reasons. Thus, the criterion of inclusiveness is well justified.

The second component of inclusiveness is that a philosophy should apply to existing (and future) varieties. For example, there may be a philosophy solely concerned with constructive mathematics. This would be fine as far as it goes, but would be a partial philosophy of mathematics and thus not satisfy this paper's criterion of inclusiveness. Based on these considerations an inclusive philosophy is one that does not view the foundations or results of a variety as false in an absolute sense, and that is not a partial philosophy.

A pluralistic philosophy of mathematics should be combined with a deductive approach within a variety. By this is meant that a mathematical assertion implicitly or explicitly states that a conclusion follows from assumptions, ultimately from the logical and mathematical foundations after a long development of intermediate results–or, as a motto, mathematics studies what follows from what within a variety. In deductivism the foundational assumptions are not considered true in some absolute sense, but as a basis for work within the variety. For example, foundational assumptions in set theory may include the existence of an empty set and the existence of an infinite set. Additional set theoretic assumptions will assert that if some sets exist then others exist. A deductivist approach will not assert that the empty set exists or that an infinite set exists in some absolute sense (a problematic question for abstract objects) but rather will ask what conclusions can be drawn from such assumptions.

If, contrary to deductivism, some foundational assumptions of a variety are considered as true in some absolute sense then contradictory assumptions would be viewed as false. A philosophy that viewed foundational assumptions of a variety as false would likely reject that variety, violating the inclusiveness criterion. Thus, results within a variety should be viewed as holding or not relative to the foundational assumptions of that variety and the question becomes whether the results are correctly deduced from the assumptions: a deductive stance.

We should also note what deductive pluralism does not assert. Deductive pluralism, as advocated here, does not imply that formal foundations are unmotivated—some kind of game with symbols. Although in principle a set of axioms could have little motivation or be essentially random, such a system would be of little interest. Historically, as with the examples of section 3, a variety and its formalization has been motivated by considerations such as historical traditions, questions internal to mathematics or logic, scientific problems, applications, or philosophical concerns. These motivations may also supply the intended interpretation. Deductive pluralism is not a version of fictionalism (which views mathematical assertions as false) since in deductive pluralism mathematical results, implicitly or explicitly, take the form of assertions that the assumptions imply the conclusions. With this approach the assertions (i.e., implications) are also objectively true in that mathematicians favoring different varieties of mathematics can agree that given the assumptions, the logic to be used, and a deduction correctly using this logic then the conclusion follows. Finally, deductive pluralism need not be viewed as a version of nominalism (which denies the existence of abstracta) since it regards the question of the existence of abstract (mathematical) objects as indeterminate in the absence of generally acceptable criteria for their existence, as discussed in section 2.

5 Deductive Pluralism as a Philosophy of Mathematics

This section will discuss considerations which are relevant to any philosophy of mathematics: ontology, epistemology, and objectivity. One of the advantages of any version of deductivism is the elimination of ontological problems since no variety is considered as true in some absolute sense, and the statements are assertions that the assumptions imply the conclusions. This is similar to the view expressed by Carnap (1950) that the "reality" of abstract entities can only be considered within a linguistic framework. In practice a result will appear in a context in which many of the assumptions are implicitly understood, and the deductions will be from previous results and not from the foundations.

Any attempt to go beyond deductivism requires confronting the problematic question of the existence of abstract objects. There are many views, some of which consider the question as essentially meaningless, such as Balaguer's cited in section 2. In the context of mathematics, an ontological question arises about the universe of sets. If a universe of sets exists in an absolute sense, is it the universe of ZFC?, or TG?, or the intuitionistic set theory of BISH?, or perhaps some other? How can this be determined? Also, as pointed out in section 2, the 1900 suggestion by Hilbert identifying existence in mathematics with consistency ran afoul of discoveries since his proposal.

The question of epistemology for deductive pluralism centers on the consistency of the foundations (except for inconsistent mathematics noted above) and on the reliability of the proofs of the theorems. Since standard mathematics cannot prove its own consistency, most mathematicians implicitly assume consistency; but some do question the foundations and seek inconsistencies. For example in 2011 a respected mathematician, Edward Nelson, posted a claim that he had found a contradiction within the Dedekind-Peano axioms, but an error in his reasoning was soon located and the claim was withdrawn (Nelson (2015)). The failures of these explicit efforts and the long development without finding contradictions give some support to the assumption that the standard foundations are consistent. If a contradiction were found then a likely result would be a modified set of axioms that avoids the contradiction and preserves (almost all) mathematics, as occurred with the discovery of Russell's paradox.

Are the theorems reliable given the foundations? The theorems are usually supported by rigorous, but not fully formalized, proofs. There can be considerable disagreement on when a published proof has sufficient detail, but, as discussed above, a common idea is that it should be possible to expand a published proof to obtain a formal proof, e.g., one which can be checked by a computer proof assistant. According to Mackenzie (2001, 323) mechanization of proofs in the mathematical literature has supported the belief that these rigorous proofs are reliable:

> Research for this book has been unable to find a case in which the application of mechanized proof threw doubt upon an established mathematical theorem, and only one case in which it showed the need significantly to modify an accepted rigorous-argument proof. This is testimony to the robustness of 'social processes' within mathematics.

Nothing is perfect and there are errors in published proofs which may lie undetected for many years, especially in those which are seldom examined. However, mechanical checking, as with Coq (used with univalent foundations) or Mizar (used with TG set theory with results in the *Journal of Formalized Mathematics*), reduces the chance for error and provides a robust check on mathematics.

In deductive pluralism mathematical results, implicitly or explicitly, take the form of assertions that the assumptions imply the conclusions. With this approach the assertions (i.e., implications) are also objectively true in that mathematicians favoring different varieties of mathematics can agree that given the mathematical assumptions, the logic to be used, and a deduction correctly using this logic, then the conclusion follows.

6 Possible Objections

This section will consider and answer some possible objections to deductive pluralism. The objections will be roughly organized into several categories: first objections that may apply to some versions of deductivism (or if-thenism) but not to deductive pluralism, then objections based on absolute views, on applications, and finally those based on mathematical practice. As examples of the first category, Maddy (1989, 1124) defines if-thenism as the "study of logical consequence." This is similar to deductivism as discussed in this paper, but there are possible differences: based on the remarks below, there seems to be an implicit assumption tying it to one variety of mathematics rather than the pluralist approach advocated in this paper. For example, Maddy considered it a "petty embarrassment" to if-thenism that an assertion about the integers, such as "2+2=4", could follow from several foundations. This concern does not apply to deductive pluralism which explicitly acknowledges and accepts pluralism, and as we wrote in section 3 most varieties of mathematics are careful to include many of the same or similar theories and results as standard mathematics, including arithmetic. Maddy (1989, 1124) then wrote in discussing the Continuum Hypothesis that

> if we move to the idea of second order consequence, the Continuum Hypothesis becomes a real question in its own right, in the sense that it either follows or doesn't follow from second order ZF. But CH is just the sort of question If-thenism hopes to count as meaningless.

Again, this objection seems to be based on an implicit assumption of a single variety of if-thenism. But this does not apply to deductive pluralism, which does not restrict the logical component to first order logic, and allows varieties using FOPC that decide CH. For example, let V=L be the axiom of constructability which states that all sets (V) are constructible sets (L). Then ZF+(V=L) implies CH (using FOPC), as well as implying ZFC. So, from the point of view of deductive pluralism CH is not meaningless, but is indeterminate in some varieties of mathematics but not in others. Thus, when considering an objection to some form of deductivism (if-thenism) it needs to be ascertained if the objection also applies to deductive pluralism as advocated here.

6.1 Objections Based on Absolute Views

There are assertions that some statements about the natural numbers (starting from 1 rather than 0) are unquestionably true in an absolute sense. Some examples of these are: "there is one prime between 15 and 18" (concerning small natural numbers); "2+3 = 5" (concerning arithmetic of small natural numbers); or even "there are infinitely many primes" (concerning the (potential) infinity of the natural numbers). Since these statements are made with little or no discussion of the implicit assumptions (and there are always assumptions—"nothing comes from nothing") about the meaning of the terms and about the nature of the natural numbers, the following comments are somewhat tentative. For writers with this view concepts about the natural numbers have an intuitive, or indeed "natural", and inevitable status. However, not all societies have such concepts. For example, according to Frank et al. (2008) the Pirahã of Amazonia have no number words but can do one-to-one matching tasks when these tasks do not require memory. This requires matching discrete and persistent objects, perhaps with canonical collections, such as fingers. For example, if disjoint collections of 2 and 3 discrete, persistent objects are matched with 2 and 3 fingers, then the total collection is matched with 5 fingers. Thus, natural numbers are not so natural, arising in some, but not all, societies possibly from experience in matching collections of discrete and persistent objects. Or, in brief, matching is natural, not numbers.

Some criticisms are based on the view that a particular foundation is true or false in an absolute sense rather than in a relative sense, in which a sentence is true within a variety if it follows from the assumptions of that variety using that variety's logic. Mathematicians are almost always working within the context of a variety of mathematics, usually standard mathematics. Within such a context if asked whether a mathematical object exists or a result holds, they will answer in the affirmative. However, if the context is broader and more philosophical, then the attitudes may be more nuanced. Thus, in discussing such attitudes it is necessary to try to determine the context. A similar point was made by the mathematician Hersh (1997, 39) who wrote: "Writers agree: The working mathematician is a Platonist on weekdays and a formalist on Sundays." This can be interpreted as stating that when doing mathematics (on weekdays) within the context and implicit assumptions of a variety a mathematician can assert existence, e.g., of the empty set, but when reflecting more broadly on mathematics or considering foundational questions (on Sundays) a more deductivist view may be adopted. The philosopher Papineau (1990, 161) made a similar distinction between contexts when he wrote "mathematical practice only establishes *hypothetical* facts about what follows from what, whereas mathematical judgements standardly make *categorical* claims of existence." Mathematicians do make implicit or explicit judgements when deciding what foundational assumptions to make, but these judgements may not be intended to assert absolute claims, even by those advocating a new foundation. For example, Mumford (2000, 198) suggested that statistical random variables should be a primitive concept with stochastic set theory as a foundation for mathematics, which required either the axiom of choice or the power set axiom to be dropped. The motivation for this proposal was relevance to possible applications rather than absolute claims. Similarly, the advocates of univalent foundations view it as a viable alternative to set theory with reduced chances of errors in proofs rather than as a correction of absolutely false foundations.

Philosophers (who may or may not support deductivism) have commented on the attitudes of mathematicians towards foundations and deductivism. Maddy (1989, 1223–224) generalized about the attitude of mathematicians when she wrote:

> What you hear from the mathematician intent on avoiding philoso-

> phy often sounds more like this: "All I'm doing is showing that this follows from that. Truth has nothing to do with it. Mathematics is just a study of what follows from what."

Of course, from the point of view of deductive pluralism the characterization of mathematics as studying "what follows from what" is not an avoidance of philosophy but an assertion of a deductivist philosophy—recall the motto in section 4: mathematics studies "what follows from what" within a variety. In a similar vein Clarke-Doane (2013, 470) wrote that "[m]athematicians are overwhelmingly concerned with questions of logic—questions of what follows from what." Thus, according to these philosophers mathematicians generally (when considering philosophy) do not assert absolute views. Now, it is true that some mathematicians do have attitudes that assert absolute truth claims about abstract objects, especially in set theory since those studying large cardinals ("the higher infinite") are likely to reject or even consider as absolutely false axioms that exclude their objects of study. For example, the axiom of constructability, V=L, together with ZF resolves a variety of problems as noted above. However, ZF+(V=L) is inconsistent with many of the large cardinal axioms (although it is consistent with Grothendieck universes). Thus, Hauser and Woodin (2014, 13) wrote: "In fact the assertion V=L itself is almost certainly false because among other things it rules out the existence of measurable cardinals." More generally, Hamkins (2014, 25) wrote that this is a common view: "Set theorists often argue against the axiom of constructability V=L on the basis that it is restrictive." But he also wrote that this view is based on an absolute set concept. However, an absolute set concept has limits for those who accept standard mathematics since ZFC is incompatible with Reinhardt cardinals, so ZFC itself rules out some large cardinals. Of course, from the point of view of deductive pluralism absolute attitudes about sets are rejected since they would rule out those with other views, such as those who would accept ZF+(V=L). Absolute beliefs about mathematical objects may provide motivation for research, but do not affect mathematical assertions since they state that within a variety, correctly using the mathematical and logical assumptions of that variety, an implication holds: an assumption implies a conclusion. In such a case, the implication holds whether or not the assumption is viewed as an absolute truth. For example, if ZFC is extended with the axiom of measurable cardinals, then the rigorous proof that the existence of a measurable cardinal implies that V $\neq$ L holds whether or not one believes in the absolute existence of measurable cardinals.

Some philosophers assert the need for absolute consistency, as when Resnik (1997, 142) wrote that "[deductivism] is an unsatisfactory doctrine. Mathematicians want to know that their systems have models; and they want to know this absolutely, and not just relative to a metaphysical theory." Recall that a model is an interpretation of a deductive system in which the axioms hold. However, Gödel's completeness theorem and the Gödel-Mal'cev theorem imply that a first order theory using FOPC is consistent if and only if it has a model. Thus, for such systems Resnik's assertion is also about consistency. Certainly, mathematicians would like to know their system is consistent, but by Gödel's second incompleteness theorem this cannot be proven within the system. So, in practice mathematicians implicitly assume consistency (except for a few such as Nelson, mentioned above). In addition, adherents of different varieties of mathematics have contradictory views about what should be consistent: users of TG set theory want it to be consistent, while strict constructivists may not believe that ZFC, much less TG, is consistent. Even Resnik in the pages preceding his assertion quoted above, in a discussion of mathematical practice, wrote that "[t]he real issue concerns what is true if [the axioms] are true, and in the course of proving theorems one provides conclusive evidence for such conditional truths" (Resnik, 1997, 140).

A final example of an objection relying on an absolute concept of truth Hellman (1989, 26) wrote that a "decisive objection" to if-thenism is to suppose that an arithmetic sentence is implied by some assumptions but that the antecedent is false, e.g., that there is no natural number sequence. Then using FOPC, in which a false sentence implies all sentences, the assumptions would imply all sentences. The first problem is that the assignment of absolute truth or falsity to assumptions about abstracta brings us back to the problematic nature of truth conditions for abstracta discussed in section 1. Putting this aside, inconsistent assumptions would lead to problems for any philosophy. In fact, deductivism can reduce the problems by putting more stress on the foundational assumptions and by treating them as assumptions rather than as absolute truth. Deductive pluralism further improves the situation by promoting the awareness that there is a plurality of varieties with incompatible assumptions. Thus, when considering absolute views we note that when mathematicians consider philosophical questions or a new foundation they often do not endorse absolute views; when mathematicians do have absolute views, these do not affect the mathematics but reflect personal preferences; and when absolute views are endorsed there is no agreement (e.g., on large cardinals).

6.2 Objections Based on Applications

There are objections against inclusiveness based on applications. Some of these view applications as determining the validity of foundations: the existence of applications of mathematics is used not only to justify mathematics but to allow attribution of absolute truth or falsity to mathematical assertions. The question for this paper is not if some areas of mathematics might be considered as justified by those interested in particular applications, but whether objections relying on applications require the rejection of other areas of mathematics, which would contradict the inclusive criterion of this paper. Some such objections view mathematics as subordinate to other disciplines, thus regarding mathematics as a means rather than as an end in itself. Such an attitude is a personal preference and as such not binding on others. But views subordinating mathematics to applications also have internal problems. First some examples will be given with comments specific to them, and then more general remarks. Resnik (1997, 99) wrote: "On my account, *ultimately* our evidence for mathematics and mathematical objects is their usefulness in science and practical life." Similarly, Azzouni (1994, 84) wrote: "In particular, the truth or falsity of a particular branch of mathematics or logic turns rather directly on whether it is applied to the empirical sciences." However, applicable mathematics can have contradictory assumptions. For example, both standard mathematics and BISH can be applied to the empirical sciences, but the former asserts the law of excluded middle and the existence of the comb function while the latter deny both. Which then is "true"?

A somewhat broader view was taken by Quine (1986, 400) who allows for "rounding out" theory beyond applications, such as to uncountable infinities. He suggests that the axiom V=L affords a convenient cut-off, disallowing inaccessible cardinals. However, inaccessible cardinals exist in TG set theory as Grothendieck universes. Also, as mentioned above, the suggestion of V=L as a cut-off may be in conflict with many set theorists who take absolute views and consider the axiom V=L as false, or at least too restrictive. So even this expansive version conflicts with some of the practices of contemporary mathematicians, and illustrates the problem of attempting to restrict mathematics based on applications or even on "rounding out."

Thus, the advocates of subordinating mathematics to empirical science do not agree among themselves as to the criteria to be used. Some, as in the quotes from Resnik and Azzouni, would seem to have rather restrictive criteria while Quine was more expansive. For all such advocates there is a problem in that the minimum amount and kind of mathematics needed for an application will generally be uncertain, both at a particular stage of mathematical development and over time.

There are arguments against if-thenism based on the idea that the applicability of mathematics argues against if-thenism. This paper's interest is whether these arguments hold against deductive pluralism. Maddy (1989, 1124–125) wrote:

> But for all this, the argument that seems to have clinched the case against If-thenism for Russell and Putnam is a version of Frege's problem, a problem about applications. Reformulated for the If-thenist, it becomes: how can the fact that one mathematical sentence follows from another be correctly used to derive true physical conclusions from true physical premises?

Since this objection is considered important, it will be discussed in some detail, beginning with the views of Russell and Putnam. Russell (1937, v) wrote in the preface to the second edition of *Principles of Mathematics*: "The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to modify." This logicism is Russell's version of if-thenism: "PURE Mathematics is the class of all propositions of the form 'p implies q'." What Russell did criticize in the second edition is a formalism in which the symbols are left uninterpreted. However, as pointed out above, in deductive pluralism a formalism is accompanied by interpretations, which provide motivation for the formalism. Thus, Russell's objections do not apply to deductive pluralism. Now we will consider Putnam's views. Maddy cites Putnam (1975) in which he wrote (p. 60): "In this paper I argue that mathematics should be interpreted realistically ... ." This raises the question of what Putnam meant by realism in mathematics. Later in the article he wrote (p. 69):

> A realist (with respect to a given theory or discourse) holds that (1) the sentences of that theory or discourse are true or false; and (2) that what makes them true or false is something *external*—that is to say, it is not (in general) our sense data, actual or potential, or the structure of our minds, or our language, etc. Notice that, on this formulation, it is possible to be a realist with respect to mathematical discourse without committing oneself to the existence of 'mathematical objects.' The question of realism, as Kreisel long ago put it, is the question of the objectivity of mathematics and not the question of the existence of mathematical objects.

Thus, Putnam discussed two separate issues: the primary concern of realism which was identified with objectivity, and the existence of mathematical objects. If-thenism was mentioned later (p 74): "I believe that the position most people find intuitive...*is* realism with respect to the physical world and some kind of nominalism or if-thenism with respect to mathematics". This, depending on his concept of if-thenism, may be at odds with his definition of realism in mathematics quoted above. In any case, deductive pluralism is objective, as was noted at the end of section 5, and so realistic in Putnam's formulation, and it regards the question of the existence of abstract (mathematical) objects as indeterminate (as discussed at the end of section 4). Thus, the positions of Russell and Putnam are at most partially in conflict with deductive pluralism.

Now we will consider Maddy's question above: "how can the fact that one mathematical sentence follows from another be correctly used to derive true physical conclusions from true physical premises?" The fact of one statement following from another shows a correct deduction. Whether this leads to an (approximately) correct conclusion about a natural system depends on the quality of the observations, on the quality of the model of the natural system, and on the mathematics and logic applied to the natural system model. (The term "natural system model" is used to distinguish a model of a natural system from models interpreting formal systems.) So, deductive pluralism needs to consider why there are mathematical theories, either part of or an entire variety, that can be successfully used in a natural system model.

To sketch the process, given a natural system the researcher picks out certain features of interest (e.g., the number of items, relations between items, properties of items, etc.), which may be idealized. These are described in a natural system model which is developed based on factors such as scientific traditions and the possibility of using mathematical tools to deduce conclusions. The natural system model may be initially vague, with interactions between the intuitions and mathematical theory (existing or specially developed) leading to a better natural system model. Once this is developed, the observational data from the natural system is used to deduce new results, which are then compared back to the natural system.

Now, how does deductive pluralism fit into this sketch? Pluralism is a natural fit, supplying many possible mathematical tools that might be used, not only from standard mathematics but also from other existing varieties of mathematics (e.g., an algorithmic approach using BISH, or nonstandard mathematics such as Albeverio et al. (1985)), and the possibility of developing new varieties, such as those using fuzzy logic or some version of quantum logic. Multiple theories within standard mathematics have been used for models of physical reality with incompatible assumptions such as Newtonian physics (flat spacetime) or General Relativity (curved spacetime), quantum physics (noncommuting observables) or classical physics (commuting observables), and continuous or discrete systems. Thus, there is a wide range of mathematical theories, within standard mathematics and also other varieties, that can be used in developing the natural system model.

With respect to deductivism, Resnik (1980, 118) wrote: "Moreover, [deductivism] appears to account nicely for the applicability of mathematics, both potential and actual; for when one finds a physical structure satisfying the axioms of a mathematical theory, the application of that theory is immediate." However, there is more of an interaction than Resnik suggests between the natural system and mathematical theory. Often an application starts with the natural system and then the search for appropriate mathematics. For example, General Relativity was formulated over several years in the early twentieth century interweaving physical intuitions and the mathematical tools of Riemannian geometry developed in the mid-nineteenth century, and extending these tools to semi-Riemannian geometry.

Finally, it should also be noted that mathematical deductions sometimes give conclusions applicable to natural systems because they are designed to do so, since the mathematical theory and applications to natural systems are developed together by an individual or by a research community. Some examples are the development of Newtonian gravitational theory and of the calculus, the interaction between the development of quantum mechanics and operator theory, and string theory which has had major interactions with new mathematics such as Calabi-Yau manifolds. As an example of the conjoined development of natural system models and theory in biology and statistics, Ronald Fisher has been called a founder of modern statistics and the greatest biologist since Darwin by Dawkins (2011): "Not only was he the most original and constructive of the architects of the neo-Darwinian synthesis. Fisher also was the father of modern statistics and experimental design."

In this section we have considered objections to deductive pluralism based on applications. Some objections subordinate mathematics to other disciplines and consider it as a means rather than an end in itself. But as discussed above such objections to inclusiveness are personal choices and also have internal problems. Other objections seem to implicitly assume that there can be only one variety of deductivism, or that the assumptions of a variety are unmotivated, neither of which is the case for deductive pluralism. We have also considered some ways mathematics, viewed deductively, can be successfully applied: conjoined development or selection from a large collection of mathematical theories.

6.3 Objections Based on Mathematical Practice

Although some philosophers such as Papineau consider mathematical practice as deductivist, there are objections to deductivism based on mathematical practice, i.e., concerns internal to mathematics. Maddy (1989, 1124) wrote that "we need to ask what mathematicians were doing before arithmetic was axiomatized. Was it not mathematics?" It *was* mathematics, which does not require explicit axiomatization for a deductivist approach. Historically mathematics had implicit or "evidently true" assumptions about arithmetic or geometry leading to deductions of new results. Over the course of time these assumptions have been made more explicit (such as in Euclid's work) and eventually become formal axiomatic systems (such as in PA), thus clarifying that mathematics studies "what follows from what." The need for pluralism, in addition to the clarification of assumptions provided by axiomatization, has become clear more recently with the development of incompatible foundations. These developments in mathematics should be reflected in the philosophy of mathematics, as they are in deductive pluralism.

Resnik (1980, 133–36) wrote that "deductivism is a powerful and appealing philosophy of mathematics," but he expressed concerns about "loose ends" related to mathematical practice. The first concern was that the deductivist "would need to explain why realism is acceptable in nuclear physics but not in mathematics." The basic answer to this objection is that physics is about objects that exist in spacetime, but mathematics is about abstract objects that do not exist in spacetime. Also, as seen above, Putnam identified realism in mathematics with objectivity, so in this sense deductive pluralism is realistic. Another of Resnik's concerns was that "deductivism may be unable to present a satisfactory epistemology for deductive reasoning itself." As has been noted, different varieties of mathematics have different views about the rules for deductive reasoning (e.g., the acceptance of LEM), so in deductive pluralism the logic is part of the foundational assumptions.

A final objection along this line is that deductivism is incomplete. Hellman (1989, 9) wrote that "a straightforward formalist or deductive approach is ruled out by the Gödel incompleteness theorems: no consistent formal system can generate all sentences standardly interpreted as truths 'about the intended type of structures(s)'." We will first expand on this. Consider a first order theory T using FOPC and containing PA, such as ZFC or PA itself. Since it uses FOPC, if T is inconsistent then every sentence can be proven. In the rest of this paragraph we assume that T is consistent and consider interpretations in which the axioms of the theory hold (i.e., models), including an intended interpretation. By the first incompleteness theorem there is a sentence G that holds in the intended interpretation but cannot be proven. However, by the completeness theorem a sentence can be proven if and only if it holds in all interpretations, so there must be other interpretations in which G does not hold. Thus, G does not hold in general, but only in some interpretations. This is an inherent property of first order theories using FOPC, independent of any philosophy of mathematics. A philosophy of mathematics might try to avoid this situation is by ruling out interpretations other than the preferred one, such as by claiming that truth in mathematics stands outside formal systems and is otherwise accessible—considered in section 6.1 on absolute views. However, such an attitude has sometimes been problematic. Consider an historical analogue. Let Eu be the axioms of Euclidean geometry without the parallel postulate P. P cannot be proven from Eu, it holds in the intended interpretation of Eu (the Euclidean plane), but does not hold in other interpretations, such as hyperbolic geometry, which have become a focus of modern mathematics. The broader perspective of deductive pluralism considers varieties with different mathematical and logical foundations, and does not require a particular interpretation.

7 Other Forms of Pluralism

Various forms of pluralism have been advocated. Rudolf Carnap in *The Logical Syntax of Language* (Carnap, 1937, p. xv) wrote:

> Let any postulates and any rules of inference be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols. By this method, also, the conflict between the divergent points of view on the problem of the foundations of mathematics disappears ... . The standpoint which we have suggested—we will call it the *Principle of Tolerance* ... [thus] before us lies the boundless ocean of unlimited possibilities.

Koellner (2009, p. 98) considered Carnap's position as too radical and that "[t]he trouble with Carnap's entire approach (as I see it) is that the question of pluralism has been detached from actual developments in mathematics." Koellner then went on to consider pluralism with respect to additional axioms for ZFC with the general view that the choices are not arbitrary. (His paper used the last lyrical phrase of the quotation from Carnap as an epigraph and coda.) Since both postulates and rules of inference are included in Carnap's position it can be viewed as a generalization of deductive pluralism. However, since deductive pluralism is based on actual mathematical practice, it avoids Koellner's criticism. In addition, the position advocated here, deductive pluralism, is somewhat stronger than mere tolerance may suggest, since these varieties of mathematics should be included in a philosophy of mathematics, not only tolerated as views external to the philosophy.

Another form of pluralism was advocated by Pedeferri and Friend (2011). Their proposal was a form of methodological pluralism, allowing "deviant" proofs "where mathematicians use steps which deviate from the rigorous set of rules methodologies and axioms agreed to in advance." Rigorous proofs were not required to be fully formal: there can be missing steps that in principle can be filled by relatively routine work in to produce a fully formal proof, which is consistent with the usage of this paper. They claimed that there are many deviant proofs and gave as the central case study the classification of finite simple groups. The basis for the claim that a portion of the classification was deviant was an interview with Serre by Raussen and Skau (2004) in which, according to Pedeferri and Friend, Serre found that deviant methods were used to overcome an impasse. This does not correctly represent the issue, which was the classification of "quasi-thin" groups and which at one point relied on an unpublished manuscript. Those who considered that the classification was complete at that time viewed the quasi-thin case as having been satisfactorily dealt with by the manuscript. Serre considered it as a substantial gap. The question was not one of "deviant" methodology: all the classification was carried out with standard mathematics and methods. The question was whether the manuscript was sufficient. As it turned out Serre was correct and the quasi-thin case was completed at about the time of the Serre interview. Methodological pluralism was considered as part of a larger program of pluralism in Friend (2013). In this work Friend advocated pluralism with respect to mathematics, including inconsistent mathematics. She did not consider foundations containing both mathematical and logical components. Instead, she suggested the use of some paraconsistent logic when the varieties of mathematics are compared. No specific version of the many types of paraconsistent logic was advocated, and no example of its use was given. As discussed in the introduction there is the problem that any overarching background logic that is compatible with all the logics that are and may in the future be used in varieties of mathematics must be so weak that it is difficult to imagine a useful version. For example, it must not satisfy the law of excluded middle (rejected by constructive analysis), not satisfy distribution of conjunction over disjunction (rejected by quantum logic), etc. When the mathematical and logical foundations are considered together, as in deductive pluralism, the attempt to find an overarching logic is unnecessary. As also pointed out in the introduction, with deductive pluralism the foundational assumptions can be compared, then within a variety the logic of that variety can be applied to the mathematical assumptions of that variety, then the results can be compared. Thus, deduction is primarily within a variety and only comparison is used between varieties. If deduction is needed outside the varieties then the general purpose logical tool of FOPC can be used.

There are also advocates for pluralism of two varieties of mathematics or for pluralistic extensions of an existing variety. Davies (2005) discussed standard (called "classical" in the paper) and constructive mathematics, with an emphasis on the justification of constructive mathematics. The paper viewed each of these two varieties as valid within its own context. Davies (2005, p. 272) wrote that "[o]ne should simply accept each mathematical theory on its merits, and judge it according to the non-triviality and interest of the results proved within it." This is pluralism with respect to two varieties and the phrase "proved within it" contains a suggestion of deductivism. Thus, deductive pluralism is compatible with this view, extending it to general varieties of mathematics and grounding them in an explicitly deductivist format. An example of pluralism within a particular area is the approach to set theory developed by Hamkins (2014), which he calls the set-theoretic multiverse, in which there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe.

This section has considered some related work in the philosophy of mathematics and has shown that some approaches are consistent with pluralism or deductivism. Thus, deductive pluralism as advocated in this paper provides a systematic approach that encompasses much of this other work.

8 RELATED PHILOSOPHIES

This section considers the relationship between deductive pluralism and some other philosophies of mathematics. The objective is not to analyze or criticize them in any detail, which space considerations do not allow, but to only indicate some of the major points where they intersect with deductive pluralism. One problem of discussing these is that there are often multiple versions of each philosophy. Thus, only some features of other philosophies most relevant to deductive pluralism are considered.

8.1 Fictionalism

Fictionalism is a variety of nominalism since it asserts the non-existence of abstracta. Balaguer (2013) wrote that the basic tenets of fictionalism are that (1) mathematical theorems and theories assert the existence of abstracta, (2) abstracta do not exist, (3) and thus mathematical theorems and theories are false. Deductive pluralism denies this syllogism since (1) is not accepted as discussed at the end of section 5: in deductive pluralism what is asserted is not the absolute existence of abstracta but only statements that a conclusion follows from assumptions, ultimately from the logical and mathematical foundations; in brief that mathematics is concerned with "what follows from what."

This paper also maintains that contrary to (1) a deductivist approach is consistent with the views of many mathematicians. It is difficult in general to determine the views of mathematicians about foundational ontological questions since they almost always write and speak within the context of standard mathematics and within that context their assertions are true: the conclusions follow from the implicit assumptions of that variety. Section 6.1 provided examples of statements by philosophers and mathematicians (e.g., Mumford, Clarke-Doane, Maddy, and univalent foundations) that support the contention that mathematicians have a deductivist view when their broader attitudes are sought, not an absolutist view as asserted in the first assumption of the above syllogism.

Balaguer also discussed another fictionalist slogan that asserts mathematical statements are "true in the story of mathematics." This use of the word "story" asserts an analogy to fiction, and adds unnecessary baggage to nominalism. Literary fictions deal with events in imaginary spacetimes, e.g., Sherlock Holmes in London, which is not the case for mathematical objects such as numbers. As Burgess (2004, p. 35) wrote in his conclusion to a discussion of fictionalism: "I think that in view of this radical difference between mathematics and novels, fables, or other literary genres, the slogan 'mathematics is a fiction' not very appropriate, and the comparison of mathematics to fiction not very apt." In any case, the slogan "true in the story of mathematics" can be given an interpretation consistent with deductive pluralism. To do this we consider a "story" to be a variety of mathematics and the assertion that "a statement is true in a story of mathematics" becomes "a statement is implied within a variety of mathematics."

8.2 Realism

Some philosophies of mathematics have a realistic view of mathematical concepts or entities. Platonism is a strong realism since the entities and concepts are viewed as eternal, acausal, objectively true, and mind independent. Such views usually contradict deductive pluralism since they reject incompatible varieties. However, there are many versions of realism, including the one given by Putnam which could classify deductive pluralism as realistic since the implications are objective.

9 Summary

The thesis of this paper is that the criterion of inclusiveness of mathematics plus the fact of incompatible varieties of mathematics require deductive pluralism. Section 4 argued in favor of the inclusiveness criterion and then showed that, given the varieties of incompatible mathematics discussed in section 3, deductivism is needed within each variety. Section 5 discussed the ontological and epistemological advantages of deductive pluralism while section 6 discussed and answered possible objections to deductive pluralism.

Now recall some of the advantages of deductive pluralism. Deductive pluralism is inclusive of mathematical practice since it allows various logical and mathematical foundations, and is flexible enough to allow for future developments. It is also consistent with attitudes expressed by many mathematicians and philosophers of mathematics as is shown in several ways, e.g., by the expressed view of mathematicians who consider altering the standard foundations (such as Mumford and those working in univalent foundations) and by the statements of philosophers of mathematics that mathematicians, especially when pressed, assert that they are concerned with "what follows from what."

Deductive pluralism also has significant philosophical advantages. Mathematical assertions take the form of implications, ultimately from the foundations. As a consequence the ontological problem of the existence of abstract objects is eliminated and the problem of epistemology is reduced to the correctness of proofs. Also, mathematical assertions are objectively true in the sense that mathematicians supporting any variety of mathematics would agree that within another variety a proof correctly following the logic of that other variety establishes that the conclusion follows from the logical and mathematical assumptions.

10 APPENDIX: LOGIC

This appendix will present in more detail some logical assumptions that differ between the varieties of mathematics and will discuss some logical results used in the discussion of these varieties. There is a distinction between syntax (primarily form) and semantics (related to meaning or truth). Thus, when a statement is considered as true, it is implicitly meant as true in some interpretation. As an introduction to interpretations of formal systems some examples of interpretations of logics in terms of sets will also be given.

10.1 Classical Sentential Logic

Most of mathematics uses classical sentential logic and its extension to First Order Predicate Calculus (FOPC). Propositions are combined using conjunction $\wedge$, disjunction $\vee$, negation $\neg$, and other connectives into new propositions. If a formula has a free variable, e.g., $P(x)$, the universal quantifier $\forall$ or existential quantifier $\exists$ can be used to bind the free variables, e.g., $\forall x P(x)$, producing a sentence, which by definition has no free variables. The main deductive rule is *modus ponens*: if P holds and if $P \to Q$ holds then Q holds. In classical logic implication is defined as "material implication": $P \to Q$ is equivalent to (or defined as) $\neg P$ holds or Q holds, i.e., $\neg P \vee Q$. In this logic a false sentence implies every sentence, since if P is false, $\neg P$ is true, $\neg P \vee Q$ holds, and so $P \to Q$ ("explosion" is when a false statement implies every statement). Non-classical logics often retain *modus ponens* but do not use material implication. A second element of classical sentential logic that varies is the Law of Excluded Middle (LEM): for any sentence P either P holds or $\neg P$ holds and so $P \vee \neg P$ always holds.

An interpretation of sentential logic can be given in which a sentence corresponds to a set in the Boolean algebra of all subsets of a fixed set U (the universe). In this interpretation $\vee$ corresponds to set union $\cup$, $\wedge$ corresponds to set intersection $\cap$, and negation $\neg$ corresponds to set complement within U. When discussing interpretations the same letter will be used for a sentence and its interpretation to simplify notation if there is no danger of confusion.

10.2 Intuitionistic Logic

Intuitionistic logic is used in several varieties of mathematics, including constructive mathematics. This logic rejects LEM and consequently rejects the general form of proof by contradiction $\neg\neg P \to P$. However some particular proofs by contradiction still go through since by a theorem of Brouwer $\neg\neg\neg P \to \neg P$ holds in intuitionistic logic.

An interpretation of intuitionistic logic can be given in which a sentence corresponds to an open set in a fixed topological space U where $\vee$ and $\wedge$ are as in the Boolean set interpretation of classical sentential logic (since the union and intersection of two open sets are both open), but negation corresponds to the interior of the set complement $\mathrm{int}(A^c)$ (since the complement of an open set is not generally open) and instead of material implication, where $A \to B$ is defined as $\neg A \vee B$, the intuitionistic interpretation takes the interior: $A \to B$ corresponds to $\mathrm{int}(A^c \cup B)$. Since *false* corresponds to the empty set and *true* corresponds to its complement, U, LEM corresponds to $A \cup \mathrm{int}(A^c) = U$, which need not hold for all A. Thus LEM fails as desired in this interpretation of intuitionistic logic.

10.3 Paraconsistent Logic

A paraconsistent logic is one that does not allow the derivation of all sentences in the case that some sentence and its negative have both been derived. In classical logic if both P and $\neg P$ are asserted, then any sentence Q can be asserted: from a contradiction everything follows—*ex contradictione quodlibet* (ECQ). Thus, a paraconsistent logic must change classical logic to prevent this explosion and thus triviality (in which all statements can be derived). Various proposals have been made for paraconsistent logic; one of the most common is *relevant logic* in which the conclusion of a deduction must be relevant to the assumption. A way of attempting to do this is to require both A and B to have a common term as a precondition for the assertion of $A \to B$. In ECQ the conclusion need not be relevant to the assumption, so relevant logic blocks explosion.

An interpretation of paraconsistent logic is closed set logic, a dual to the interpretation of intuitionistic logic. In this approach a sentence corresponds to a closed set in a fixed topological space. As with the interpretation of intuitionistic logic, $\vee$ corresponds to union and $\wedge$ corresponds to intersection. The interesting case is again negation. Since in general the complement of a closed set is not closed, negation corresponds to the closure of the complement $\overline{A^c}$. In parallel with the intuitionistic case $A \wedge \neg A$ corresponds to $A \cap \overline{A^c}$, which need not be empty (i.e., *false*).

10.4 Model Theory

A few results are used from FOPC (in which there is only one type of variable), model theory, and Gödel's theorems.

Let L_0 be a logic, such as FOPC. A first order language L is an extension of L_0 obtained by adding relation, function, and constant symbols. (These can all be considered relation symbols, e.g., a constant symbol is a 0-ary relation symbol.) One of these relation symbols will be the binary equivalence relation of equality, if it is not considered to be part of the logic. A first order L-theory T is L together with a collection of sentences, which can be viewed as axioms, in the language L. (Sometimes the term "theory" is used for both the axioms and all sentences that can be deduced from them.) If S is a collection of sentences and a sentence ϕ can be deduced from S by a finite number of applications of the rules of deduction (such as *modus ponens*) then ϕ is a syntactic (or deductive) consequence of S, which is written symbolically as $S \vdash \phi$. A collection S of sentences is inconsistent if there is some sentence ϕ such that both ϕ and $\neg\phi$ can be deduced, i.e., $S \vdash \phi$ and $S \vdash \neg\phi$.

Standard model theory uses sets, often not in the context of a specific set theory. In this approach an interpretation of L is in an L-structure: a set (or domain) over which the variables range together with assignments sending constant, relation and function symbols to constants, functions, and relations on the domain. Thus, we have four elements: a logic, a language, a theory (all three formal and generally uninterpreted), and an interpretation of the language. The L-structure interpreting T is assumed to have a consistent way of determining if a relation is satisfied. The logic, language, and theory are together referred to as a first order (deductive) system An L-structure M is said to be a model of an L-theory T, or M satisfies T, if all the sentences of T interpreted in M are satisfied in M. Symbolically this is written $M \models T$, read as M models T. A sentence ϕ in the language L is defined to be true or semantically valid (or model-theoretically valid) if it is satisfied in all interpretations, i.e., $M \models \phi$ for all interpretations M. Thus "true" in model theory (and more generally in mathematics) means true in all models. The models symbol is also used in the slightly different form $S \models \phi$ where S is a collection of sentences in L, ϕ is a sentence in L, and $S \models \phi$ means that every model of S is also a model of ϕ. When $S \models \phi$ holds we say that ϕ is a semantic consequence of S. Thus there are two versions of consequence: syntactic consequence $S \vdash \phi$ and semantic consequence $S \models \phi$.

The following results from logic and model theory are used:

- Gödel's completeness theorem for first order systems implies that the two notions of consequence agree: $S \models \phi$ if and only if $S \vdash \phi$.

- Gödel's completeness theorem and the Gödel-Mal'cev theorem imply that a first order theory is consistent if and only if it has a model. Thus, an interpretation should not be referred to as a model unless consistency is proven (or assumed).

- Gödel's (first) incompleteness theorem and its extensions imply that in any consistent formal system containing arithmetic there are statements in the language of the system such that neither the statement nor its negative can be proven in that system.

- Gödel's second incompleteness theorem implies that any consistent first order system containing arithmetic cannot prove its own consistency. Thus, most results are about relative consistency rather than consistency. Note that if a system is inconsistent then in FOPC any statement can be proven, including the statement that the system is consistent.

- The compactness theorem implies that if every finite subset of a first order system with countably many variables has a model, then the system as a whole has a model.

- The Löwenheim-Skolem theorem implies that a first order system has a model with a countably infinite domain if and only if it has a model with an uncountably infinite domain.

As an example of these concepts we will consider the first order Dedekind-Peano axiomatization of the natural numbers (with intended interpretation $\mathbb{N} = \{0, 1, ...\}$) and its extension to Dedekind-Peano arithmetic (PA). The formal language L_N of the natural numbers is $(S, 0, =)$ where S is a function symbol (interpreted as successor), 0 is a constant symbol, and $=$ is the equivalence relation of equality. The theory PN of the natural numbers adds to the language L_N the Dedekind-Peano axioms:

i. $\forall x \neg (S(x) = 0)$

ii. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$

iii. $(\phi(0) \text{ and } \forall x(\phi(x) \rightarrow \phi(S(x)))) \rightarrow \forall x \phi(x)$

Axiom (iii) is the axiom schema of induction where, for simplicity, ϕ is assumed to be any unary predicate formula. (In general n-ary predicate formulas are used.) The arithmetic operations can be defined using these three axioms to give the full set of axioms for the formal first order theory of Dedekind-Peano arithmetic, PA.

The formal theory PA has the intended interpretation $(\mathbb{N}, S, 0, =)$ (where for simplicity the relations in this interpretation are again given the same names as the formal relation symbols). By the Löwenheim-Skolem theorem if there is a countable model for a first order theory, then there are models of all infinite cardinalities. This is an example of the inability of first order theories to distinguish orders of infinity. By the second incompleteness theorem if PA is consistent it cannot prove its own consistency, and thus by the completeness theorem the intended interpretation $(\mathbb{N}, S, 0, =)$ cannot be proven to be a model of PA (without additional assumptions).

Assume that PA is consistent and so has a model M. Then a nonstandard model of PA can be constructed from it by adding a new natural number constant symbol c to L_N giving L'_N with symbols $(S, 0, =, c)$. (The constant c can be interpreted as an infinite number.) The theory T'_N is defined to have the same sentences as PA with the addition of the countable set of sentences $\neg(c = 0), \neg(c = S(0)), \neg(c = S(S(0))), ...$. Let F be a finite subtheory of T'_N. Then F has a model with c interpreted as a suitable element of the domain of M not corresponding to any element of F. So by the compactness theorem for first order logic there is a model for the infinite theory T'_N, and thus for PA. This model is a nonstandard model that is not isomorphic to M.

Since proofs in standard mathematics apply FOPC to the axioms of ZFC, a (fully formalized) proof holds in all interpretations. This can cause some seeming contradictions. For example the Löwenheim-Skolem theorem implies that a first order system such as ZFC has a model (i.e., is consistent) with a countably infinite domain if and only if it has a model with an uncountably infinite domain. So, assuming consistency, the real numbers can be defined and proven to be uncountable in any interpretation. This appears to be a contradiction to the Löwenheim-Skolem theorem, but it is resolved by recalling that a set is countable if and only if there is a one-to-one function from the natural numbers onto the set. Thus, from the (internal) perspective of an interpretation there may not exist enough such one-to-one functions so that a set is uncountable, while from the (external) perspective of another interpretation such a one-to-one function exists. Thus, every interpretation "thinks" that it is the intended interpretation. From a deductive perspective this does not matter since a deduction from the axioms of ZFC applies to all interpretations.

10.5 Second Order Logic

Some considerations concerning second order logic are needed in this paper. In second order logic there are two types of variables, first order variables ranging over the elements of the domain and second order variables ranging over sets of elements. The second order variables are sometimes considered as properties, but we will take an extensional approach in which a set corresponds to all elements having that property. The standard (or canonical) interpretation of second order logic is to use "all" subsets of a domain, although there is a problem in deciding what "all" means. The model-theoretic results listed above do not generally hold for second order logic: second order logic is not complete, since $S \models \phi$ may hold but not $S \vdash \phi$; the compactness theorem does not hold; and the Löwenheim-Skolem theorem does not hold.

Quine famously referred to second order logic as "set theory in sheep's clothing" (Quine, 1986, p. 66), and Shapiro wrote that "second-order logic, as understood through standard semantics, is intimately bound up with set theory", (Shapiro, 2012, p. 305). Considering the problems of second order logic such as incompleteness, its close relation to set theory, its use of sets in its model-theoretic semantics, its relative lack of development compared with FOPC, and no clear mathematical advantages (e.g., see Vaananen (2012)) mathematicians have generally stuck with the traditional approach of standard set theory with FOPC rather than use second order logic.

11 Appendix: Historical Examples

Mathematics has been practiced for thousands of years. Over this period mathematicians have abstracted, generalized, reinterpreted, axiomatized, and formalized past work. This section gives two examples of the trend towards formalization: the natural numbers as formalized by the Dedekind-Peano axioms, and group theory.

The "naturalness" of the natural numbers was discussed at the beginning of section 6.1. Explicit definitions of the natural numbers have been given since early times. For example, Euclid (1908), Book VII, definition 1 states that "a unit is that by virtue of which each of the things that exist is called one" and definition 2 states that "[a] number is a multitude composed of units." The definition of unit is unclear or circular, and multitude is not defined. Of course, not all concepts can be defined if infinite regress is to be avoided. Euclid also uses implicit assumptions, and there have been various proposals on how to fill in the gaps. When it comes to proof Euclid interprets numbers as geometrical line segments. For example, proposition 1, in which a condition is given for two numbers to be prime to one another, begins "[f]or, the lesser of two unequal numbers AB, CD . . .", where these are line segments. Thus, Euclid is an early example of the use of definitions, interpretations (as line segments), and implicit assumptions. Newton (1769, p. 2) defined numbers, including rationals and irrationals, by abstracting from ratios: "By number we understand not so much a multitude of unities, as the abstracted ratio of any quantity, to another quantity of the same kind, which we take for unity." By the end of the nineteenth century the widely used properties of the natural numbers were axiomatized by the Dedekind-Peano axioms, and by their extension to Dedekind-Peano Arithmetic, PA. The applicability of the natural numbers is thus to be expected since PA is based on the natural practice of cultures with discrete, stable, finite collections of objects. The finiteness property is a notable difference between many applied uses of numbers and the axioms of PA which might lead to inconsistency: the inductive axiom produces an infinity, potential or actual, of natural numbers. As noted in the above discussion of standard mathematics, some mathematicians have believed that PA is inconsistent due to the inductive axiom.

As another example of the growth of mathematical concepts consider the group concept. As discussed in Kleiner (1986) the concept developed from a variety of sources: in the eighteenth century Euler studied modular arithmetic and Lagrange studied permutations of solutions to algebraic equations; in the nineteenth century Jordan defined isomorphisms of permutation groups and Cayley extended the study of groups beyond permutations to other examples, such as matrices. Although Cayley was ahead of his time in abstracting the concepts to sets of symbols, group elements were usually considered as transformations until the twentieth century. The first study of groups without assuming them to be finite, without making any assumptions as to the nature of their elements, and formulated as an independent branch of mathematics may have been the book "Abstract Group Theory" by O. Shmidt in 1916. Thus, analogous to the axiomatization of the natural numbers the axiomatization of group theory occurred as the result of a long period of development.

In these and other examples history shows that basic mathematical concepts can arise over a long period of gradual development, abstraction, generalization, and eventual axiomatization and formalization. These concepts develop naturally and are not arbitrarily selected variations on existing concepts.

References

Ackermann, W. (1937). Die Widerspruchsfreiheit der allgemeinen Mengenlehre. *Mathematische Annalen*, *114*, 305–315.

Albeverio, R., R. Høegh-Krohn, J. Fenstad, and T. Lindstrøm(1986). *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*. San Diego, Ca: Academic Press.

Azzouni, J. (1994). *Metaphysical Myths, Mathematical Practice*. Cambridge: Cambridge University Press.

Balaguer, M. (1998). *Platonism and anti-Platonism in Mathematics*. Oxford: Oxford University Press.

Balaguer, M. (2013). Fictionalism in the Philosophy of Mathematics. In E. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy (Fall 2013 Edition)*. Retrieved from http://plato.stanford.edu/archives/fall2013/entries/fictionalism-mathematics.

Bernstein, A. and A. Robinson (1966). Solution of an Invariant Subspace Problem of K. T. Smith and P. R. Halmos. *Pacific J. Math*, *16*, 421–431.

Bishop, E. (1967). *Foundations of Constructive Analysis*, New York: McGraw-Hill.

Bridges, D. and E. Palmgren (2016). Constructive Mathematics. *The Stanford Encyclopedia of Philosophy*. Retrieved from http://plato.stanford.edu/archives/win2016/entries/mathematics-constructive/.

Burgess, J. (2004). Mathematics and *Bleak House*. *Philosophia Mathematica (3) 12*, 18–36.

Carnap, R. (1937). *The Logical Syntax of Language*. English translation by A. Smeaton, London: Kegan Paul.

Carnap, R. (1950). Empiricism, Semantics, and Ontology. *Revue Internationale de Philosophie*, *4*, 20–40.

Clarke-Doane, J. (2013). What is Absolute Undecidability?. *Nôus*, *47*, 467–481.

Davies, E. (2005). A Defense of Mathematical Pluralism. *Philosophia Mathematica (3) 13*, 252–276.

Dawkins, R. (2011). Retrieved from http://www.edge.org/conversation/who-is-the-greatest-biologist-of-all-time.

Ebbinghaus, H-D and Peckhaus, V. (2007). *Ernst Zermelo: An Approach to His Life and Work*. Berlin: Springer.

Euclid (1908). *The Thirteen Books of Euclid's Elements*. English translation by J.L. Heiberg, with introduction and commentary by T.L. Heath, Cambridge: Cambridge University Press.

Frank, M.C. et al. (2008). Number as a Cognitive Technology: Evidence from Pirahã Language and Cognition. *Cognition 108*, 819-–824.

Friend, M. (2013). Pluralism and "Bad" Mathematical Theories: Challenging our Prejudices. In K. Tanaka et al. (Eds.), *Paraconsistency: Logic and Applications* (pp. 277–307). Berlin: Springer.

Hamkins, J. (2014). A Multiverse Perspective on the Axiom of Constructiblity. In Chitat Chong et al. (Eds.), *Infinity and Truth* (pp. 25–45). New Jersey: World Sci. Publ.

Hauser, K. and W.H. Woodin (2014). Strong Axioms of Infinity and the Debate about Realism. *The Journal of Philosophy, 111*, 397–419.

Hellman, G. (1989). *Mathematics without Numbers*. Oxford: Clarendon.

Hellman, G. and J. Bell (2006). Pluralism and the Foundations of Mathematics. In S. Kellert, H. Longino, and C. Waters (Eds.), *Scientific Pluralism* (pp. 64–79). Minneapolis: Univ. Minnesota Press.

Hersh, R. (1997). *What is Mathematics, Really?*. Oxford: Oxford University Press.

Hilbert, D. (1976). Mathematical Problems. In F. Browder (Ed), *Mathematical Developments Arising from Hilbert Problems* (pp. 1–34). Providence, R.I.: American Mathematical Society.

Jané, I. (2005). Higher Order Logic Reconsidered. In S. Shapiro (Ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic* (pp. 781– 810). Oxford: Oxford University Press.

Kleiner, I. (1986). The Evolution of Group Theory: a Brief Survey. *Mathematics Magazine 59*, 195–215.

Koellner, P. (2009). Truth in Mathematics: the Question of Pluralism. In O. Bueno and Ø. Linnebo (Eds.), *New Waves in Philosophy of Mathematics* (pp. 80–116). Houndmills, England: Palgrave Macmillan.

Linnebo, Ø. and R. Pettigrew (2011). Category Theory as an Autonomous Foundation. *Philosophia Mathematica (3) 19*, 227–254.

Mackenzie, D. (2001). *Mechanizing Proof: Computing, Risk, and Trust*. Cambridge, MA.: The MIT Press.

Maddy, P. (1989). The Roots of Contemporary Platonism. *The Journal of Symbolic Logic, 54*, 1121–1144.

Mortensen, C. (1995). *Inconsistent Mathematics*. Boston: Kluwer Academic Publishers.

Mumford, D. (2000). The Dawning of the Age of Stochasticity. In V. Arnold et al, (Eds.), *Mathematics: Frontiers and Perspectives* (pp. 197–218). Providence, R.I.: American Mathematical Society.

Nelson, E. (2015). Elements. https:arXiv.org/pdf/1510.00369v1.

Newton, I. (1769). *Universal Algebra*, translated by T. Wilder, London: W. Johnson.

Papineau, D. (1990). Knowledge of Mathematical Objects. In A.D. Irvine (Ed), *Physicalism in Mathematics* (pp155–182). Dordrecht: Kluwer Academic Publishers.

Pedeferri, A. and M. Friend (2011). Are Mathematicians Better Described as Formalists or Pluralists? *Logic and Philosophy of Science IX*, 173–180.

Penrose, R. (2005). *The Road to Reality*. New York: Alfred A. Knopf.

Putnam, H. [1975]: What is Mathematical Truth? In *Mathematics, Matter and Method. Philosophical Papers, vol. 1*, Cambridge: Cambridge University Press.

Quine, W.V. (1986). Reply to Charles Parsons. In L. Hahn and P. Schilpp (Eds.), *The Philosophy of W.V. Quine* (pp. 396–403). La Salle, Ill: Open Court.

Raussen, M. and C. Skau (2004). Interview with Jean-Pierre Serre. *Notices of the American Mathematical Society, 51*, 210–214.

Resnik, M. (1980). *Frege and the Philosophy of Mathematics*. Ithaca, New York: Cornell University Press.

Resnik, M. (1997). *Mathematics as a Science of Patterns*. Oxford: Clarendon Press.

Russell, B. (1937). *Principles of Mathematics, 2nd ed.* New York: W.W. Norton.

Shapiro, S. (2012). Higher-Order Logic or Set Theory: A False Dilemma. *Philosophia Mathematica (3) 20*, 305–323.

Shapiro, S. (2014). *Varieties of Logic*. Oxford: Oxford University Press.

The Univalent Foundations Program Authors (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. The Univalent Foundations Program, Princeton, N.J.: Institute for Advanced Study. Retrieved from http://homotopytypetheory.org/book.

Väänänen, J. (2012). Second order logic or set theory? *Bulletin of Symbolic Logic 18(1)*, 91-121.

Voevodsky, V(2014). The Origins and Motivations of Univalent Foundations. *The Institute Letter, Summer 2014* (8–9). Princeton, NJ: Institute for Advanced Study. https://www.ias.edu/ideas/2014/voevodsky-origins.