A Simplified Version of Gödel's Theorem

Michael Redhead

"Isn't it a bad thing to be deceived about the truth and a good thing to know what the truth is?" Plato

Abstract

The Lucas - Penrose argument is considered. As a special case we consider sorites arithmetic and explain how the argument actually works. A comparison with Gödel's own treatment is made.

Truth is valuable. But certifiable truth is hard to come by!

The Case of Mathematics

Does truth equate with proof?

No: Gödel showed famously that truth outruns proof, even in the case of the natural numbers 0, 1, 2, 3, …

The Lucas-Penrose argument

The argument that ascertainable truth outruns proof in mathematics has been used by John Lucas and Roger Penrose to claim that human minds can do things which computers can't do, and hence that minds can't be (digital) machines.

Quote from Lucas and myself :

We are dealing with a different style of reasoning… an informal semantics concerned with truth rather than proof. Sceptics deny that we have such a concept of truth. We cannot prove to them that we have… (but) we have many intimations of truth, not all of which can be articulated as proofs.

Sorites arithmetic

This is the sort of arithmetic you learnt at school. The basic operation is moving from one number to the next along the number line and repeating the operation a finite number of times.

Addition and multiplication are defined recursively in the usual way.

So, addition is effectively reduced to adding one and one and one… repeatedly, and multiplication is repeated addition. We now have the machinery to check out some arithmetical facts like 279 x 681= 681 x 279.

Sorites axioms

1. Zero is a number
2. Every number has a unique successor
3. If two numbers have equal successors, then the number are equal
4. Every number other than zero is the successor of some number
5. Recursive definitions of addition and multiplication

Indeed,

(1) For any pair of number (m, n), we can prove m x n = n x m

But what we can't prove in Sorites arithmetic is the Commutative law of Multiplication (CM) in the form

(2) For all pairs of numbers (m, n), m x n = n x m

Why is this?

Because as the number get bigger, the proof increases in length without limit, so there is no finite proof that will work for all pairs of number. And this can't be avoided. In other words, we can't make the transition any → all.

Compare Any plum in this plum pudding is delicious → all the plums in the pudding are delicious.

This is a perfectly valid argument because we are claiming that a typical plum is delicious, not in virtue of being that particular plum, but just in virtue of being a plum.

But in the arithmetical example, there is no such thing as a typical pair of numbers.

The detailed proof for any particular pair depends crucially on which pair you choose.

But although we cannot prove CM, we can argue that it is true!

From (1) it follows that

(3) For any pair of numbers (m, n), it is true that m x n = n x m But (3) is strictly equivalent to:

(4) It is true that for all pairs of numbers (m, n), m x n = n x m And (4) is just CM!

So, in Sorites arithmetic CM can't be proved. (Notice that the argument depended crucially on the fact that the sequence of numbers is infinite). Note that we don't assume consistency. For the Gödel proofs of true but unprovable assertions we have to assume consistency. Without this nothing follows.

Response

Why not introduce a new axiom to strengthen Sorites arithmetic in such a way that CM can be proved? This step was taken by Peano in 1889, and in the resulting system known as Peano arithmetic, CM is both true and provable. But in 1931, Gödel showed that examples could be given of unprovable statements in Peano arithmetic and indeed in any strengthening of Peano arithmetic.

The Peano axioms

1. Zero is a number
2. Every number has a unique successor
3. If two number have equal successors, then the numbers are equal
4. Every number other than zero is the successor of some number
5. Induction axiom:

For any admissible predicate F, if F is true of zero, and if, given that F is true of n, then it is true of the successor of n, then it is true for all n.

Poincare (1952) famously claimed that this was not an analytic truth, but forced itself on us with such conviction, that it was a candidate for the elusive synthetic a priori. The induction axiom certainly cannot be proved from the first four axioms, so what is the source of the conviction that Poincare talks of?

My claim is that the essence of the Lucas-Penrose argument can be framed in the context of Sorites arithmetic and simple examples like CM, thus avoiding the formidable complexity of the Gödel construction.

We notice that the syntactic proof can be formalised, the semantic proof cannot. This is because in order to formalise it, we would have to express the notion of truth. But this is ruled out by Tarski’s theorem on the undefinability of truth, whose proof is closely connected to Godel's theorem. The definable truths are at most denumerable, whereas there are non-denumerably many sets of numbers.

Notice that the Gödel theorems say that if a sufficiently strong version of arithmetic is consistent, then the result follows. But we cannot prove consistency (by the second theorem). This is the basic conundrum. We have to assume consistency, but this moves outside the formalized theorems.

Inductive support is provided by the fact that no inconsistencies have so far been produced. ‘Syntactic’ means referring to grammar, instead of ‘semantic’, which refers to meaning as well as grammar. Gödel places great stress on the constructivist aspect of syntax, but the argument leads to a conundrum, as we have seen, which can only be solved in terms of the semantic approach, in the way we have described.

But does the argument show that minds are not machines? For any given unprovable statement, we can always work in a stronger system where it becomes provable. (This is used to deny the Lucas-Penrose argument). But no machine can deal with all unprovable statements. This is the basis of the Lucas-Penrose argument that we are following. Notice our argument for the truth of CM involves the semantic conception of truth, and moreover the truth of statements which are universally quantified over infinite domains.

In such a case do we really have a clear conception of truth? In a slogan: IS CANTOR'S PARADISE ANOTHER NAME FOR LALA LAND?

Of course, we are interested, not in transfinite entities, but we are dealing with infinite quantities. To be sure if we could assume a strict finitism, this might be a possible let-out for our own approach.

Notes and References

Gödel assumed the Peano axioms to be ω-consistent. This was later relaxed by Rosser to the weaker assumption of consistency. See Machover (1996). For an up-to-date version of Gödel, along the lines of Boolos and Jeffrey, see Smith (2007).

See Lucas (1961), (1970) and Penrose (1989), (1994). Penrose employs a simplified form of the Gödel argument due to Turing.

For complete references to the many different types of response to the Lucas-Penrose argument, see Lucas (2000) and Penrose (1994).

What predicates are admissible depends on whether we choose to employ first- or second- order logic.

In more formal terms, we have sought to demonstrate that sorites arithmetic is ω-complete in a semantic sense, while recognizing that it is w- incomplete in syntactic terms. From this perspective, Gödel's theorem establishes the ω-incompleteness of Peano arithmetic. The truth of the Gödel sentence then follows from an argument similar to the one given in the text showing that Peano arithmetic is ω-complete in a semantic sense.

It is important to notice that none of these theorems can be proved using the logical principle of Universal Generalization (UG). UG allows us to pass from the fact that a typical member of a collection possesses a property to the claim that all members of the collection possess that property. But the decisive point is that, for the purposes of these theorems, all the numbers are uniquely different- there is no such thing as a typical number. We stress again that we are talking of theorems which are true of the intended interpretation. There will, of course, be non-standard interpretations of sorites arithmetic (in first-order logic), but these will include an initial segment isomorphic to the natural numbers. It is with regard to this initial segment that we are claiming the theorems to be true. We owe this point to Richard Healey. An additional remark: these theorems could be proved if we employed an infinitary logic incorporating the so-called ω-rule. But such logics cannot of course be implemented on a machine with a finite number of operations. This is what Lucas has in mind when he talks of digital machines.

See Berto (2009) p. 154, following Smullyan (1992) p.112. Notice that the Gödel theorem says that if a sufficiently strong version of arithmetic is consistent then the result follows. But we cannot prove consistency (by the second theorem). This is the basic conundrum. We have to assume consistency, but this moves us outside the formalized theorems. Inductive support, as we have seen, is provided by the fact that no inconsistencies have so far been produced. Gödel makes great stress on the constructivist aspect of syntax, but the argument leads to a conundrum, as we have seen, which can only be solved in terms of the semantic approach in the way we have described.

This has been challenged by Raatikainen (2005) who claims correctly that Tarskian semantics invoke a failure of showing that truth outruns provability. But we do not of course have to believe in Tarskian semantics. See, in this respect the views of Kirkham (1995). Further points are made in the reply by Lucas and Redhead (2007).

I make some comments on the paper. True means agreement with fact or reality. Correspondence means agreement with statements of objective facts. Their own variations on this notion such as coherence theory, pragmatic theories, etc. We follow here the Ancient Greek usage!

Proof of my result. Since for Gödel true means that it is unprovable, i.e. it means what it says, and then goes to show that it is unprovable, the hard part. In my example unprovability is taken for granted so to speak, and we concentrate on playing about with truth!

For our example truth correlates with the universal quantifier, whereas provability does not. Since the theorems are analytically true, we can replace proof with truth, in the sense that they express defining properties of the numbers 0, 1, 2, etc. If the theorems were false, we would not be talking about numbers, i.e. if we are talking about numbers then the theorems are true.

Notice the step from (3) to (4) is an instance of universal generalisation in quantification theory.

Recursion moves from n to n-1 to n-2 etc until it stops at the so- called base case.

If we employ an infinitary logic incorporating the so-called ω-rule, then our argument fails.

But such logics cannot be implemented on a machine that has a finite number of operations. This is what Lucas has in mind when he talks of digital machines.

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