Classical Electrodynamics of Moving Bodies - Unifying Faraday's Law of Induction with the Principle of Relativity

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Einsteins aim in his famous article "Zur Elektrodynamik bewegter Körper" was to unify Faraday's law of induction with the principle of relativity. But allthought his ideas lead to a satisfying solution, it turns out that in solving this problem, you do not need his reflexions on kinematics, which have become the special theory of relativity, at all. And even more: A mathematical analysis of Maxwells equations and a closer look on the phenomena of induction show that (absolute) time and space are in fact *necessary conditions of expierience.*

1 Do We Still Need the Theory of Relativity?

Within the new SI unit system, which will be endorsed on 20 May 2019, all basic units, especially the kilogram, will be put down to constants of nature. For the second and the meter this is already done since 1983. A second is defined through the *undisturbed* transition frequency $\Delta\nu_{Cs}$ between the two hyperfine levels of the groundstate of the caesium - 133 atom:

$$1s = \frac{9192631770}{\Delta\nu_{Cs}}.$$

The meter is determined by the second and the speed of light c in vacuum:

$$1m = \frac{c}{299792458} 1s = \frac{9192631770}{299792458} \frac{c}{\Delta\nu_{Cs}}.$$

What is considered whithin these definitions to be a disturbation and what not is strangely choosen. If there is a fraction of a light ray because of its transition from air to water, than this fraction is considered to be a disturbation from the propagation of light along straight lines. This light ray cannot be used to define a distance any more. Therefore the definition of the meter is just valid whithin the vacuum. If the light ray diverts from its straight line because of a gravitational field, than it is said that this line is still straight, but a straight line in a curved spacetime! Similar things apply to the definition of a second. If $\Delta\nu_{Cs}$ changes due to a fast movement of the atomic clock, one says that the clock shows the time dilation, if $\Delta\nu_{Cs}$ deviates from its ideal behaviour because of gravitation, this deviation shall come frome the curvature of spacetime. All other disturbations are considered to be disturbations and are excluded from the definitions of the second and the meter. This destinction is absolutely arbitrary. Einstein was able to make important progresses with the idea of spacetime curvature. His ability to make unusal choices made him a genius. But his ideas haven't been uncontroversial at his time.

> Why is the equivalence of the practically-rigid body and the body of geometry–which suggests itself so readily–denied by Poincaré and other investigators? Simply because under closer inspection the real solid bodies in nature are not rigid, because their geometrical behaviour, that is, their possibilities of relative disposition, depend upon temperature, external forces, etc. Thus the original, immediate relation between geometry and physical reality appears destroyed, and we feel impelled toward the following more general view, which characterizes Poincaré's standpoint. Geometry (G) predicates nothing about the relations of real things, but only geometry together with the purport (P) of physical laws can do so. Using symbols, we may say that only the sum of (G) + (P) is subject to the control of experience. Thus (G) may be chosen arbitrarily, and also parts of (P); all these laws are conventions. All that is necessary to avoid contradictions is to choose the remainder of (P) so that (G) and the whole of (P) are together in accord with experience. Envisaged in this way, axiomatic geometry and the part of natural law which has been given a conventional status appear as epistemologically equivalent. Sub specie aeterni Poincaré, in my opinion, is right. The idea of the measuring-rod and the idea of the clock co-ordinated with it in the theory of relativity do not find their exact correspondence in the real world. It is also clear that the solid body and the clock do not in the conceptual edifice of physics play the part of irreducible elements, but that of composite structures, which may not play any independent part in theoretical physics. But it is my conviction that in the present stage of development of theoretical physics these ideas must still be employed as independent ideas; for we are still far from possessing such certain knowledge of theoretical principles as to be able to give exact theoretical constructions of solid bodies and clocks.[1]

Nowadys our understanding has grown so far that the engineers and physicists of the global positioning system remove from the atomic clocks all discrepancies to the

[1]Einstein, A. *Geometry and Expierience*

ideal Euclidean behaviour regardless of the origin of these discrepancies. And witout noticing that in doing so they disagree with the special and the general theory of relativity and return to Poincaré's point of view.[2] Not everyone is pleased by this.

> Globally, the current situation in the Global Navigation Satellite System (GNSS) is almost analogous to the following one: imagine that a century after Kepler, the astronomers were still using Kepler's laws as algorithms to correct epicycles by means of "Keplerian effects". Similarly, a century after Einstein, one still uses the Newtonian theory and corrects it by "relativistic or Einsteinian effects" instead of starting with Einstein's gravitational theory from the beginning.[3]

There is something strange about this criticism. Nobody would say that the engineers and physcists of the global positioning system would disobey laws of nature! The fact that the GPS endorses an Euclidean frame of reference proves that geometry is in fact conventional. So one should rather change the physical theory than the GPS. But the theory of relativity is something the physicists are very proud of. Poincaré thought about the problem of synchronising clocks and the usage of non-eucledian geometry even before Einstein. But he derived very different conclusions:

> ... we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the laws of optics, and suppose that light is not rigorously propagated in a straight line. It is needless to add that every one would look upon this solution as the more advantageous. Euclidean geometry, therefore, has nothing to fear from fresh experiments.[4]

Seemingly Poincaré was very wrong about his fellow scientists but he was right about what he said on geometry. Furthermore we are nowadays already in possesion of the modified laws of optics he spoke of, so there is no need to hold on to the theory of relativity.

2 Basic Definitions

A body $\mathcal{B}$ is a subset of a four dimensional pseudo-riemannian manifold $(\mathcal{M}, \mathbf{g})$. We assume global hypebolicity[5] for $\mathcal{M}$, which means that topologically $\mathcal{M} = \mathbb{R} \times \Sigma$. The elements of $\mathcal{B} \cap \Sigma$, denoted χ, are called particles. Whithin $\mathcal{B}$ they follow an orbit $\Phi_t(\chi) \subset \mathcal{B}$ of a *one parameter group of diffeomorphisms* Φ_t. A one parametergroup of diffeomorphisms is a map $\mathbb{R} \times \Sigma \to \Sigma$ such that for all $t, s \in \mathbb{R}$, we have $\Phi_t \circ \Phi_s = \Phi_{t+s}$. The equation

$$\frac{d\Phi_t(\chi)}{dt} = v(\chi)$$

[2] Ashby, N., Weiss, M. (1999): "Global Positioning System Receivers and Relativity", [1]

[3] Pascual-Sanchez, J.-F. ,San Miguel, A. and Vicente, F., "Introducing relativity in global navigation satellite systems", p.1, [10]

[4] Poincare, H., *Science and Hypothesis*, p. 74 [11]

[5] Wald, R. M., *General Relativity*, [12] pp. 200

defines a vectorfield $v(\chi)$ on Σ, which is called the generator of the one parameter group Φ_t. We introduce on $\mathcal{B}$ two 2- forms $\mathbf{F}$ and $\mathbf{G}$ and a 3-form

$$\mathbf{j} = \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho} j^{\rho} dx^{\mu} \wedge dx^{\lambda} \wedge dx^{\rho}, \tag{1}$$

the current density of the electric charge. The following relations[6]

$$d\mathbf{F} = 0, \tag{2}$$
$$d\mathbf{G} = \mathbf{j} \tag{3}$$

should hold good between those forms. What we are interested in is the time evolution of the fields $\mathbf{F}, \mathbf{G}$

$$\begin{aligned} \mathcal{L}_v \mathbf{F} &= d i_v \mathbf{F} + i_v d\mathbf{F} \\ \mathcal{L}_v \mathbf{G} &= d i_v \mathbf{G} + i_v \mathbf{dG}. \end{aligned}$$

The *Lie derivative* $\mathcal{L}_v$ of a form $\mathbf{W}$ is by definition[7]

$$\begin{aligned} \mathcal{L}_v \mathbf{W} &= \lim_{t\to 0} \frac{1}{t} \left(\Phi_t^* \mathbf{W} - \mathbf{W} \right) \\ &= \frac{d}{dt} \Phi_t^* \mathbf{W} \mid_{t=0}, \end{aligned}$$

d is the operator of *exterior derivative* and i_v is the operator of the *inner product* of a form with the vector field v. The equations above can be proven mathematically [8] and do not contain any information about the body $\mathcal{B}$. But we can insert the informations given in the equations (2) and (3) and get:

$$\mathcal{L}_v \mathbf{F} = d i_v \mathbf{F} \tag{4}$$
$$\mathcal{L}_v \mathbf{G} = d i_v \mathbf{G} + i_v \mathbf{j}. \tag{5}$$

The electric and magnetic fieldstrenghts are given by $\mathbf{E} = i_v \mathbf{F}$ and $\mathbf{H} = i_v \mathbf{G}$. The electric and magnetic flux densities are defined by $\mathbf{B} = \Phi^* \mathbf{F}$ and $\mathbf{D} = \Phi^* \mathbf{G}$. The material relations between those quantities are given by

$$\mathbf{G} = *\mathbf{F}. \tag{6}$$

As discussed in the introduction, we chose not to interpret the tensor g, which is needed to define the operator $*$, to be the metric of a curved spacetime. The tensor g and the operator $*$ describe the possible deviation of the propagation of light from straight lines. Herewith our theory is complete. But we still have to figure out how to evaluate its content.

[6] Göckler, M., Schücker, T. *Differential geometry, gauge theories, and gravity*, [6] p. 44

[7] Choquet-Bruhat, Y. and DeWitt-Morette, C. and Dillard-Bleick, M., *Analysis,Manifolds and Physics*, [2] p. 147

[8] ibid., p. 207

3 The principle of relativity

Most and maybe all experimental results are given to us within time and space. Therefore we define a time function

$$\begin{aligned} f:\mathcal{B} &\longrightarrow \mathbb{R} \\ \Phi_t(\chi) &\longmapsto t \end{aligned}$$

and mappings from $\mathcal{B}$ into space $\mathbb{R}^3$. The reference configuration κ_0 is given by

$$\begin{aligned} \kappa_0 : f^{-1}(0) &\longrightarrow \mathbb{R}^3 \\ \chi &\longmapsto (X,Y,Z) \end{aligned}$$

and the momentary configuration κ_t is similarly defined[9]

$$\begin{aligned} \kappa_t : f^{-1}(t) &\longrightarrow \mathbb{R}^3 \\ \Phi_t(\chi) &\longmapsto (x,y,z). \end{aligned}$$

The time function f together with the reference configuration κ_0 or the momentary configuration κ_t introduce coordinates on $\mathcal{B}$. If we use the reference configuration $\mathbf{W} = \mathbf{W}(t,X,Y,Z)$ we say: "$\mathbf{W}$ is given in the Lagrangian representation." If we use the momentary configuration $\mathbf{W} = \mathbf{W}(t,x,y,z)$ we say: "$\mathbf{W}$ is given in the Eulerian representation."[10] The *movement* of a particle is the mapping[11]

$$\begin{aligned} \gamma_t := \kappa_t \circ \Phi_t \circ \kappa_0^{-1} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (X,Y,Z) &\longmapsto (x,y,z). \end{aligned}$$

It can be used to calculate the Lagrangian representation of a field from the Eulerian representation and vice versa. Let $U_i \subset \mathbb{R}$ be an open Intervall, then

$$\begin{aligned} \phi_i : f^{-1}(U_j) &\longrightarrow U_j \times \mathbb{R}^3 \\ \Phi_t(\chi) &\longmapsto (f(\Phi_t(\chi)), \kappa_{it}(\Phi_t(\chi))) = (t,x,y,z) \end{aligned}$$

defines a *local trivialization* on $\mathcal{B}$. Given two local trivializations ϕ_i and ϕ_k then

$$\kappa_{it} \circ \kappa_{kt}^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$$

shall be an element of a *structural group* G. The mappings

$$\begin{aligned} g_{ik} : U_i \cap U_k &\longrightarrow G \\ t &\longmapsto g_{ik}(t) = \kappa_{it} \circ \kappa_{kt}^{-1} \end{aligned} \tag{7}$$

are called transition functions.[12] The choice of a group G defines the geometry of the spaces $f^{-1}(t)$. Before Einstein one assumed that a passive transformation $g_{ik}(t) :=$

[9] Greve, R., *Kontinuumsmechanik*, [7] p.1
[10] ibid., p.3
[11] ibid., p.2
[12] Choquet-Bruhat, Y. and DeWitt-Morette, C. and Dillard-Bleick, M., *Analysis, Manifolds and Physics*, [2] pp. 125

$\kappa_{it} \circ \kappa_{kt}^{-1}$ corresponds to an active transformation $\gamma_t := \kappa_t \circ \Phi_t \circ \kappa_r^{-1}$, the movement of a *body of reference.* This body should be a rigid body in the sense that there is a metric $\mathbf{h}$ for which

$$\mathcal{L}_v \mathbf{h} = 0. \tag{8}$$

If the motion is not a rigid body motion, then $\mathcal{L}_v \mathbf{h}$ is the strain tensor. There are three possible geometries for which equation (8) can be fullfilled: the Euclidean, the hyperbolic and the spherical geometry. The point of view taken here was given by Henri Poincare in [11]. Note that the metric $\mathbf{h}$ is different from the metric $\mathbf{g}$ we introduced in our basic definitions and haven't used yet. The **principle of relativity** says:

> The laws of physics must have the same form in any frame of reference.

In our context this principle means that all physical laws must be invariant under the action of the structure group G. Since we wrote down all equations in an coordinate independent way, we already achieved that. But Maxwells electrodynamics is not *generally covariant.* In special relativity two different reference configurations κ_{i0} and κ_{k0} belong to two different fibrations f_i and f_k. This is called the "relativity" or "conventionality of simultaneity". Because of this there are no transition functions in special relativity although the exact same Lorentz transformations belong to the hyperbolic geometry with the fibration $f(t, x, y, z) = \sqrt{c^2t^2 - x^2 - y^2 - z^2}$. Einsteins fibration within a choosen coordinate system is simply $f(t, x, y, z) = t$. This construction cannot be generalized to arbitrary moving frames of reference, which could otherwise easily be achieved by equation (7). How to describe a rotating frame of reference was something that caused Einstein a lot of headache. In the end Einstein gave up the principle of relativity towards the **principle of general covariance**.

> *The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).*
> It is clear that a physical theory which satisfies this postulate will also be suitable for the general postulate of relativity.[13]

A modern formulation of the principle of general covariance can be found in [12]:

> An important principle - which goes under the name of *general covariance* - applies to the form of the laws of physics in the prerelativity describtion of space as well as in special relativity and general relativity. The principle of general covariance in this context states that the metric of space is the only quantity pertaining to space that can appear in physics. Specially, there are no prefered vector fields or preferred bases of vector fields pertaining only to the structure of space which appear in any law of physics.[14]

The theory presented here is not generally covariant since there is a preferred vector field: the generator $v(\chi)$ of the one parameter group Φ_t. Due to this preferred vector

[13] Einstein, A., *The Foundation of the General Theory of Relativity*, [4] p. [10]
[14] Wald, R. M., *General Relativity*, [12] p.57

field coordinate-systems, in which the time coordinate is given by the foliation $f : \mathcal{M} \to \mathbb{R}$, i.e. the momentary and reference configuration, are more important than others.

4 Faraday's Law of Induction

The reason that there are almost no applications of $\mathbf{F} = 0$ or $d\mathbf{G} = \mathbf{j}$ to be found in the literature might be the following difficulty:
Consider the equation

$$d\mathbf{F} = 0. \tag{9}$$

By setting $\mathbf{F} = -E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz + B_z dx \wedge dy - B_y dx \wedge dz + B_x dy \wedge dz$ we obtain from $d\mathbf{F} = 0$

$$\begin{aligned} & (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t}) dt \wedge dx \wedge dy - (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t}) dt \wedge dx \wedge dz \\ + & (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t}) dt \wedge dy \wedge dz + (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) dx \wedge dy \wedge dz \\ = & 0. \end{aligned}$$

Although the components of this equation are equal to the components of the Maxwell equations

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{10}$$

$$\nabla \vec{B} = 0 \tag{11}$$

only the solutions of the older Maxwell equations (10) and (11) have a direct physical meaning. If you write equation (10) in the language of differential forms

$$d(E_x dx + E_y dy + E_z dz) + \frac{\partial}{\partial t}(B_z dx \wedge dy + B_y dz \wedge dx + B_x dy \wedge dz) = 0. \tag{12}$$

the electric field is a 1-form and in $d\mathbf{F} = 0$ the electric field is a 2-form. Note that the equation (12) is defined on three-dimensional space rather than four-dimensional spacetime. The integration of (9) does not lead to the same results as the integration of (12).
Equation (4) ($\mathcal{L}_v \mathbf{F} = d i_v \mathbf{F}$) solves this problem, because $i_v \mathbf{F}$ is indeed a 1-form. Let $S \in \mathcal{B}$ be a surface

$$S : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow f^{-1}(0).$$

We derive Faraday's law of induction from

$$\int_S \mathcal{L}_v \mathbf{F} = \int_S d i_v \mathbf{F}.$$

The left-hand side is

$$\begin{aligned}\int_S \mathcal{L}_v \mathbf{F} &= \int_S \frac{d}{dt}\Phi_t^* \mathbf{F}\ |_{t=0} \\ &= \frac{d}{dt}\int_S \Phi_t^* \mathbf{F}\ |_{t=0}\ .\end{aligned}$$

In the last step we were allowed to interchange derivation and integration since the image of S lies in $f^{-1}(0)$. For the right-hand side we use Stokes' theorem

$$\int_S di_v \mathbf{F} = \int_{\partial S} i_v \mathbf{F}.$$

By defining $i(v)\mathbf{F} := \mathbf{E}$ and $\Phi_t^*\mathbf{F} := \mathbf{B}$ we finally get

$$\frac{d}{dt}\int_S \mathbf{B} = \int_{\partial S} \mathbf{E}. \tag{13}$$

In the Lagrangian description we have

$$\begin{aligned}\mathbf{E} &= E_i dX^i \\ &= E_x dX + E_y dY + E_z dZ\end{aligned}$$

and we set[15]

$$\begin{aligned}\mathbf{B} &= \frac{1}{2}\epsilon_{\mu\nu\lambda} B^\mu dX^\nu \wedge dX^\lambda \\ &= B_z dX \wedge dY + B_y dZ \wedge dX + B_x dY \wedge dZ.\end{aligned}$$

Therefore equation (13) is in agreement with equation (12).

Example 1 ***Rotating Conductor in a Constant Magnetic Field***
Consider a square rotating around the y-axis in a constant magnetic field . The square is given by

$$\begin{aligned} S : [-5cm; 5cm] \times [-5cm; 5cm] &\longrightarrow \kappa_r^{-1}([-5cm; 5cm] \times [-5cm; 5cm] \times 0cm) \\ (X, Y) &\longmapsto \chi(X, Y)\end{aligned}$$

and its boundary by

$$\begin{aligned} \partial S_1 : [-5cm; 5cm] &\longrightarrow \kappa_r^{-1}([-5cm; 5cm] \times -5cm \times 0cm) \\ (X, -5cm) &\longmapsto \chi(X, -5cm), \\ \partial S_2 : [-5cm; 5cm] &\longrightarrow \kappa_r^{-1}(5cm \times [-5cm; 5cm] \times 0cm) \\ (5cm, Y) &\longmapsto \chi(5cm, Y), \\ \partial S_3 : [-5cm; 5cm] &\longrightarrow \kappa_r^{-1}([5cm; -5cm] \times 5cm \times 0cm) \\ (X, 5cm) &\longmapsto \chi(X, 5cm), \\ \partial S_4 : [-5cm; 5cm] &\longrightarrow \kappa_r^{-1}(-5cm \times [5cm; -5cm] \times 0) \\ (-5cm, Y) &\longmapsto \chi(-5cm, Y).\end{aligned}$$

[15] Göckler, M., Schücker, T., *Differential geometry, gauge theories, and gravity*, [6] p. 44

The movement shall be

$$\begin{array}{ccc} \gamma_t : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} Xcos(2\pi s^{-1}t) \\ Y \\ Xsin(2\pi s^{-1}t) \end{pmatrix}. \end{array}$$

and the electromagnetic field is given by

$$\mathbf{F} = 0.1T dx \wedge dy.$$

Although this electromagnetic field is constant in the Eulerian representation it is not constant in the Lagrangian representation, the rest frame of the rotating conductor

$$\begin{aligned} \mathbf{F} &= 0.1T dx \wedge dy \\ &= -0.2\pi X sin(2\pi s^{-1}t) V m^{-2} dt \wedge dY + 0.1T cos(2\pi s^{-1}t) dX \wedge dY. \end{aligned}$$

The electric fieldstrenght is $\mathbf{E} = -0.2\pi X sin(2\pi s^{-1}t) V m^{-2} dY$ *and the magnetic flux density is* $\mathbf{B} = 0.1T cos(2\pi s^{-1}t) dX \wedge dY$. *We now compute the lefthand side*

$$\begin{aligned} \frac{d}{dt}\int_S \mathbf{B} &= \frac{d}{dt}\int_{-5cm}^{5cm}\int_{-5cm}^{5cm} 0.1T cos(2\pi s^{-1}t) dX dY \\ &= \frac{\partial}{\partial t} cos(2\pi s^{-1}t) \cdot 10^{-3} T m^2 \\ &= -2\pi sin(2\pi s^{-1}t)) \cdot 10^{-3} T m^2 s^{-1} \\ &\approx -6.283 \cdot 10^{-3} sin(2\pi s^{-1}t) V \end{aligned}$$

and the righthand side of equation (13)

$$\begin{aligned} \partial S_1^* \mathbf{E} &= 0 \\ \partial S_2^* \mathbf{E} &= -10^{-2}\pi sin(2\pi s^{-1}) V m^{-1} dY \\ \partial S_3^* \mathbf{E} &= 0 \\ \partial S_4^* \mathbf{E} &= 10^{-2}\pi sin(2\pi s^{-1}) V m^{-1} dY \end{aligned}$$

therefore

$$\begin{aligned} \int_{\partial S} \mathbf{E} &= \int_{-5cm}^{5cm} -10^{-2}\pi sin(2\pi s^{-1}) V m^{-1} dY \\ &+ \int_{5cm}^{-5cm} 10^{-2}\pi sin(2\pi s^{-1}) V m^{-1} dY \\ &= -2 \cdot 10^{-3}\pi sin(2\pi s^{-1}t)) V \\ &\approx -6.283 \cdot 10^{-3} sin(2\pi s^{-1}t) V. \end{aligned}$$

Now we compute the same situation but this time the conductor shall be static and the electromagnetic field is time-dependent.

Example 2 ***Fixed Conductor in a Rotating Magnetic Field***
We consider the same square S as in example 1. Its movement is

$$\begin{array}{rcl}\gamma_t : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix}. \end{array}$$

The electromagnetic field is

$$\mathbf{F} \;=\; -0.2\pi x sin(2\pi s^{-1}t)Vm^{-2}dt \wedge dy + 0.1Tcos(2\pi s^{-1}t)dx \wedge dy.$$

The field $\mathbf{F} = 0.1Tcos(2\pi s^{-1}t)dx \wedge dy$ alone would not satisfy $d\mathbf{F} = 0$, but $\mathbf{F} = -0.2\pi x sin(2\pi s^{-1}t)Vm^{-2}dt \wedge dy + 0.1Tcos(2\pi s^{-1}t)dx \wedge dy$ does. Because the conductor does not move the field is the same in the Eulerian and Lagrangian describtion.

$$\begin{aligned}\mathbf{F} &= -0.2\pi x sin(2\pi s^{-1}t)Vm^{-2}dt \wedge dy + 0.1Tcos(2\pi s^{-1}t)dx \wedge dy \\ &= -0.2\pi X sin(2\pi s^{-1}t)Vm^{-2}dt \wedge dY + 0.1Tcos(2\pi s^{-1}t)dX \wedge dY.\end{aligned}$$

The rest can be read of example 1.

Both examples are identical in the Lagrangian describtion and give therefore identical results. The next example will be taken from *The Feynman Lectures of Physics.* In order to stress the differences of our approach to Feynmans and to make clear how we are solving the problem Einstein had with classical electrodynamics, I will give some quotes of Feynman and Einstein.

> So the "flux rule" - that the emf in a circuit is equal to the rate of change of the magnetic flux through the circuit - applies wether the flux changes because the field changes or because the circuit moves (or both). The two possibilities - "circuit moves" or "field changes" - are not distinguished in the statement of the rule. Yet in our explanation of the rule we have used two completely distinct laws for the two cases - $\mathbf{v} \times \mathbf{B}$ for "circuit moves" and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t}$ for "field changes."[16]

Using two different laws for the same phenomenon obviously violates the principle of relativity as Einstein pointed out.

> It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there

[16] R.P. Feynman, *The Feynman Lectures of Physics*, [5] p.17-2

arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case. Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, ...[17]

Regardless of Einsteins work, Feynman insists that the phenomena described in example (1) and (2) are two different phenomena.

> We know of no other place in physics where such a simple and accurate general principle requires for its understanding an analysis in terms of *two different phenomena*. Usually such a beautiful generalization is found to sterm from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the "rule" as combined effects of two quite seperate phenomena. We must look at the "flux rule" in the following way. In general, the force per unit charge is $\mathbf{F}/q = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. In moving wires there is the force from the second term. Also, there is an $\mathbf{E}$-field if there is somewhere a changing magnetic field. They are independent effects, but the emf around the loop of wire is always equal to the rate of change of magnetic flux through it.[18]

Consider a disc rotating in a constant magnetic field along its axis of rotation. We can expect that there is no change in the magnetic flux through the disc. Nevertheless there is an emf. Feynman calls this an "exeption to the flux rule." He says that this phenomenon must be explained by the force $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$. [19] But we will use, as we have done in example (1) and (2), the transformation of the electromagnetic field into the restframe of the disc instead.

[17] Einstein, A., *Electrodynamics of Moving Bodies*, [3] pp.37
[18] R.P. Feynman, *The Feynman Lectures of Physics*, [5] p.17-2
[19] ibid., pp.17-2

Example 3 ***Exception to the "flux rule": Rotating disc in a constant magnetic field***
The surface S of the disc is given in cylindric coordinates (R, Φ, Z) by

$$\begin{array}{rcl} S : [0; 5cm] \times [0; 2\pi] & \longrightarrow & \kappa_r^{-1}([0; 5cm] \times [0; 2\pi] \times 0) \\ (R, \Phi) & \longmapsto & \chi(R, \Phi), \end{array}$$

its boundary by

$$\begin{array}{rcl} \partial S : [0; 2\pi] & \longrightarrow & \kappa_r^{-1}(5cm \times [0; 2\pi] \times 0) \\ (\Phi) & \longmapsto & \chi(\Phi), \end{array}$$

its motion by

$$\begin{array}{rcl} \gamma_t : \mathbb{R} \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} R \\ \Phi \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} r \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} R \\ \Phi + 2\pi s^{-1} t \\ 0 \end{pmatrix} \end{array}$$

and the electromagnetic field is given by

$$\begin{array}{rcl} \mathbf{F} & = & 0.1T dr \wedge r d\phi \\ & = & 0.2Vm^{-2}\pi R dt \wedge dR + 0.1TR dR \wedge d\Phi. \end{array}$$

The electric fieldstrength is $\mathbf{E} = 0.2Vm^{-2}\pi R dR$ *and the magnetic flux density is* $\mathbf{B} = 0.1TR dR \wedge d\Phi$. *The electric flux through the surface is constant*

$$\begin{array}{rcl} \int_S \mathbf{B} & = & \int_0^{5cm} \int_0^{2\pi} 0.1TR dR d\Phi \\ & = & 2.5\pi \cdot 10^{-4} Tm^2, \end{array}$$

therefore

$$\frac{\partial}{\partial t} \int_S \mathbf{B} = 0.$$

Because the component $\partial S^* \mathbf{E}$ *of the electric fieldstrenght along* ∂S *is zero*

$$\int_{\partial S} \mathbf{E} = 0$$

holds as well. If you measure a potential difference between the center of the disc and its boundary, as the picture in the Feynman Lectures indicates, you get a result which is not 0.[20] *Let* Γ *be the path from the center to the boundary*

$$\begin{array}{rcl} \Gamma : [0; 5cm] & \longrightarrow & \kappa_r^{-1}([0; 5cm] \times 0 \times 0) \\ R & \longmapsto & \chi(R). \end{array}$$

[20] R.P. Feynman, *The Feynman Lectures of Physics*, [5] p.17-3

The emf is

$$\int_\Gamma \mathbf{E} = \int_0^{5cm} 0.2\pi V m^{-2} R dR$$
$$= 2.5\pi \cdot 10^{-4} V.$$

This result is nevertheless no exception to the "flux rule".

Now we can compare with Einsteins words Feynmans approach - the "old manner of expression"- with the point of view taken here:

> 1. If a unit electric point charge is in motion in an electromagnetic field, there acts upon it, in addition to the electric force, an "electromotive force" which, if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector-product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old manner of expression.)
> 2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force which is present at the locality of the charge, and which we ascertain by transformation of the field to a system of co-ordinates at rest relatively to the electrical charge. (New manner of expression.)
> The analogy holds with "magnetomotive forces." We see that electromotive force plays in the developed theory merely the part of an auxiliary concept, which owes its introduction to the circumstance that electric and magnetic forces do not exist independently of the state of motion of the system of co-ordinates. Furthermore it is clear that the asymmetry mentioned in the introduction as arising when we consider the currents produced by the relative motion of a magnet and a conductor, now disappears.[21]

Allthough Einsteins physical intuition was correct in this case, his mathematical treatment of time and space was not. I believe no one ever really used electrodynamics in its four vector formalism or in its differential form formalism to describe an experiment without adding further assumptions concerning time and space, which are in fact not in agreement with what Einstein said about time and space. I think that Kant got it right when he said time and space were pure forms of our intuition (Anschauung). Faraday's law of induction is percived in experiments as if the change of a magnetic flux through a surface induces an electromotive force in its boundary. The word "induces" indicates that Faraday's law is an experimental fact, taken from the law of cause and effect. But Hume recognized that you never can tell out of empirism that one phenomenon is the cause of another. You just can tell that - up to now - both phenomena appeared together. Kant recognized that this is not a criticism towards the law of cause and effect. That law still is necessary to turn a mere perception into an experience based on experiments. The law of cause and effect is just not taken from the phenomena, it was put into the phenomena by our understanding. Kant argued that this is not the only contribution of our understanding.

[21] Einstein, A., *On the Electrodynamics of Moving Bodies*, [3] pp.54

> So I tried first whether *Hume's* objection might not be presented in a general manner, and I soon found that the concept of the connection of cause and effect is far from being the only concept through which the understanding priori thinks connections of things *a priori*; rather, metaphysics consists wholly of such concepts.[22]

Time and space as well are contribution of our understanding and do not belong to the things in themselves.

> Geometry bases itselfes on the pure intuition of space. Even arithmetics forms its concepts of number through successive addition of units in time, but above all pure mechanics can form its concepts of motion only by means of the representation of time. Both representations are, however, merely intuitions; for if one eliminates from the empirical intuitions of bodies and their alterations (motion) everything empirical, that is what belongs to sensation, then space and time still remain, which are therefore pure intuitions that underlie *a priori* the empirical intuitions, and for that reason can never themselves be eliminated; but, by the very fact that they are pure intuitions *a priori*, they prove that they are mere forms of our sensibility that must precede all empirical intuition (i.e., the perception of actual objects), and in acordance with which objects can be recognized *a priori*, though of course only as they appear to us.[23]

If we eliminate from the theory we discussed here everything empirical, we are still left with the manifold $\mathcal{M}$ and its topology, which was described in section 2. The reason why we do not work with $d\mathbf{F} = 0$ and $d * \mathbf{F} = \mathbf{j}$ directly - as it is thought to be possible and necessary in the context of general relativity - is that our mind - the way we construct experiments and percieve their results - works with time and space and not with a four dimensional spacetime. It is said that in absence of strong gravitational fields and velocities small compared to the speed of light, it is not necessary to use electrodynamics in its general and coordinate independent form, since electrodynamics in its non-relativistic formulation is already able to give good results. I think this is just an excuse for not beeing able to use $d\mathbf{F} = 0$ and $d * \mathbf{F} = \mathbf{j}$ without returning to a prerelativistc point of view. Similar things happen in astronomy. The cosmological models have got the same topology we gave in section 2, including an absolute (cosmological) time and furthermore Newton's theory of Gravitation is used for calculating galaxy rotations instead of Einsteins equation; with the justification that you can do that in absence of strong gravitational fields and velocities small compared to the speed of light. But you get horribly wrong results.

[22] Kant, I., *Prolegomena to Any Future Metaphysics*, [9] p. 4:260
[23] ibid., [9] p. 4:283

References

[1] Ashby, N., Weiss, M. "Global Positioning System Receivers and Relativity“ in *NIST Technical Note 1385* 1999.

[2] Choquet-Bruhat, Y., De Witt-Morette, Dillard-Bleick, M., *Analysis, Manifolds and Physics*. Elsevier Science B.V., Amsterdam, 1982.

[3] Einstein, A. "Electrodynamics of Moving Bodies“ in *The Principle of Relativity* Dover Publications, New York, 1952.

[4] Einstein, A., "The Foundation of the General Theory of Relativity“ in *The Collected Papers of Albert Einstein*, vol.6, The Berlin Years: Writings 1914 - 1917, Princeton University Press, 1997.

[5] Feynman, R. P.,*The Feynman Lectures on Physics*. vol. I, Addison - Wesley, Reading, 1966.

[6] Göckler, M., Schücker, T., *Differential geometry, gauge theories, and gravity*, Cambridge University Press, Cambridge, 1987.

[7] Greve, R., *Kontinuumsmechanik - Ein Grundkurs für Ingenieure und Physiker*. Springer-Verlag, Berlin, Heidelberg, 2003.

[8] Hawking, S.W. und Ellis, G.F.R., *The Large Scale Structure of Space- Time*. Cambridge University Press, Cambridge 1973.

[9] Kant, I., *Prolegomena to Any Future Metaphysics*. Cambridge University Press, 1997.

[10] Pascual-Sanchez, J.-F. ,San Miguel, A. and Vicente, F., "Introducing relativity in global navigation satellite systems“,*Annalen der Physik*, Volume 16, Issue 4, pages 258–273, 2007

[11] Poincare, H., *Science and Hypothesis*. The Walter Scott Publishing CO., LTD, New York, 1905.

[12] Wald, R. M., *General Relativity*, The University of Chicago Press, Chicago and London, 1984.