PHILOSOPHICAL TRANSACTIONS A

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Research

Cite this article: Jaeger G. 2016 Grounding the randomness of quantum measurement. *Phil. Trans. R. Soc. A* **374**: 20150238. http://dx.doi.org/10.1098/rsta.2015.0238

Accepted: 6 January 2016

One contribution of 14 to a theme issue 'Quantum foundations: information approach'.

Subject Areas:
quantum physics, mathematical physics, statistical physics

Keywords:
quantum measurement, foundations of quantum theory

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Grounding the randomness of quantum measurement

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Julian Schwinger provided to physics a mathematical reconstruction of quantum mechanics on the basis of the characteristics of sequences of measurements occurring at the atomic level of physical structure. The central component of this reconstruction is an algebra of symbols corresponding to quantum measurements, conceived of as discrete processes, which serve to relate experience to theory; collections of outcomes of identically circumscribed such measurements are attributed expectation values, which constitute the predictive content of the theory. The outcomes correspond to certain phase parameters appearing in the corresponding symbols, which are complex numbers, the algebra of which he finds by a process he refers to as 'induction'. Schwinger assumed these (individually unpredictable) phase parameters to take random, uniformly distributed definite values within a natural range. I have previously suggested that the 'principle of plenitude' may serve as a basis in principle for the occurrence of the definite measured values that are those members of the collections of measurement outcomes from which the corresponding observed statistics derive (Jaeger 2015 *Found. Phys.* **45**, 806–819. (doi:10.1007/s10701-015-9893-6)). Here, I evaluate Schwinger's assumption in the context of recent critiques of the notion of randomness and explicitly relate the randomness of these phases with the principle of plenitude and, in this way, provide a fundamental grounding for the objective, physically irreducible probabilities, conceived of as graded possibilities, that are attributed to measurement outcomes by quantum mechanics.

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1. Introduction

The values of outcomes of quantum measurements, which are attributed expectation rather than definite values by quantum mechanics, have been widely considered to be intrinsically random. The apparent randomness of these outcomes of measurements on systems as described by the theory has often been taken to be a consequence of indeterminism, which itself has been connected by some with unpredictability. For example, it has been argued that the individual outcomes obtained through the measurement of quantum observables could not be predetermined by the quantum mechanical law of motion, because the observables are, in turn, generally value-indefinite (cf. [1]). In that case, the logical priority of such indefiniteness would need to be established. More generally, recently a number of investigators have emphasized the objective character of quantum probability (cf. [2], Sects. 3.2–3.3) and the derivation of the complex Hilbert space structure of quantum mechanics from objective principles [3,4], in reaction to suggestions that quantum probability is best understood as subjective in character, by some who purport to achieve such a derivation on the basis of radical Bayesian conception of probability in physics (see [2], Sect. 3.7, for a conceptual analysis of the approach). Here, an objective measurement-based reconstruction of basic elements of quantum mechanics first carried out by Julian Schwinger is considered with respect to its use of randomness and related notions, and a grounding for these in a logically prior principle, 'plenitude', is provided.

The stage is set for this grounding by Schwinger's explicit identification of the intended senses of several of the more philosophical of the terms and notions of which he made use, with the exception of randomness itself. Here, the question of the coherence of that set of notions is first addressed and a principled basis for the 'random' behaviour assumed by his scheme is found wanting. Such a basis is then offered. In particular, the plenitude principle, a metaphysical principle wielded by philosophers of the past, including Leibniz, and more recently in one form (in particle physics) by Paul Dirac and Murray Gell-Mann, is used to further ground the moves of Schwinger in his measurement-based reconstruction of quantum mechanics.

2. Chance and randomness

Let us first briefly compare randomness with chance and show the propriety of the sense of randomness used in our considerations here, namely, that identified with unpredictability. Randomness is primarily an abstract notion. The related notion of chance is more common. However, although the sense of chance intended by those using the term is usually less clearly specified than is that of randomness, there is some agreement that it can be grounded in the notion of probability. The notion of chance itself is also connected in some way with the notion of possibility since, for example, if there is no chance of an event taking place then that event is impossible, and if there is a *non*-zero chance of it taking place then the event *is* possible. It is thus perfectly natural that, for example, Leibniz understood probability as possibility given in degrees, i.e. *probabilitas est gradus possibilitas* [5]. In physics, probability has been a fundamental element of the theory of quantum mechanics, in which it has a clear mathematical prescription that supplies the expectation values of quantum measurement outcomes. Accordingly, the expressions yielding these expectation values are some of the results of Schwinger's investigation. Possibility has also been invoked by some in the interpretation of quantum mechanics within the differing, modal approach.

There is a range of understandings of the notion of randomness, beyond the simple and unproductive idea that the set of the so-called chance events is identical to the set of random events, which would render randomness and chance identical and the terms 'random' and 'chancy' synonymous [6]. Indeed, there may be useful sorts of randomness that do not involve objective chance and situations that involve such chance but are not, at least obviously, random. It is, therefore, important to distinguish the sense of randomness under consideration here from senses more directly related to the term 'chance'. In particular, one should note that the randomness under consideration here is not one related to *epistemic* probability. (Subjective probability is not clearly relatable to chance and possibility—nonetheless, it is valuable when considering related alternatives (cf. [7]).) This distinction is of significance here because the probabilities that emerge in the standard formulation of quantum mechanics are not standard in the sense that, if their application is not restricted, they fail to satisfy all the basic requirements of probability as supplied by Kolmogorov.

Although there are a number of competing demands on scientifically useful senses of randomness that have been considered (cf. [6,8]), not all of these senses and demands are capable of being related to Schwinger's reconstruction of quantum mechanics; the notion of randomness used here, in particular, must at least imply the uniformity of the distribution of quantum measurement outcomes he requires, discussed below. However, randomness has been considered by some genuinely to arise in applications that are statistically uniform only in the limit of an infinite number of steps of a deterministic procedure because, for any realistic data collector (i.e. one with a finite dataset), only then would this data appear to be produced by a uniform distribution [9]. Moreover, it has been argued that the notion of randomness as an objective notion of probability of a uniform nature fails to capture entirely the absence of patterns in data. Such patterns might, alternatively, be captured by the so-called algorithmic randomness. This sense of randomness provides a probabilistic guarantee of such an absence of patterns, since almost all sequences are (computationally, Martin-Löf) random, yet is unable to guarantee absolutely the non-presence of computable patterns.

Specifically, algorithmic randomness has been considered to be characterized by an absence of patterns and defined (in algorithmic information theory) via the property of the incompressibility of finite strings of symbols, and the latter suffers first from the fact that the complexity of any string is only fixed up to a constant which depends on the universal Turing machine considered. In order to develop a more rigorous notion of algorithmic randomness, it would thus appear necessary to consider the limit case of infinite sequences of bits. One then finds that there are no maximally random sequences (cf. [10]). Thus, it would appear that the non-existence of maximally random sequences demonstrates that quantum (or any other form of) randomness cannot serve as an 'absolute' or 'true' sort of randomness [11]. The best one can achieve in this way, given its failure to identify an absolute sense of randomness, is to consider a series of increasingly strong senses of randomness. Moreover, classical systems may involve intrinsic unpredictability (even if necessarily essentially epistemic) due to sensitivity of behaviour on initial conditions. In the current context, particular care should be taken in regard to this point because measurement involves more than a single atomic system—at a minimum, measurement involves two systems, one of which might and often is considered describable by classical mechanics. However, when one recognizes that algorithmic randomness focuses on the randomness of *output* of a process, which in Schwinger's construction is irreducible within quantum mechanics, it appears inapplicable in our context here.

The different idea of basing the randomness arising from *unpredictability* has a number of other virtues which capture basic intuitions and practice surrounding randomness [6]. Moreover, it is invoked by Schwinger in his analysis of quantum measurement and his reconstruction of the formalism, as is shown in the following section. Thus, a version of this notion of randomness is that relevant here, the form of which is described in detail below.

3. The algebra of quantum measurements

Let us now review in some detail the prominent elements of Schwinger's 'phenomenological' reconstruction of the theory of quantum measurement and identify the formal locus of randomness within it. Schwinger based quantum mechanics on a measurement-process foundation by introducing the basic measurement symbol $M(a')$, which indicates the *selective* measurement process, a mathematical representative of the most basic integral physical operation of quantum measurement. Specifically, the effect of a selective measurement is the effect of a physical system capable of measurement of an ensemble of physical systems with the capacity to do two things: (i) to separate out subensembles each distinguished by a distinct discrete value

$\{a', a'', \ldots, a''^{\cdots'}\}$ of a property taken in measurement and (ii) to accept one of the finite number of resulting subensembles while rejecting the others. This selective process is exemplified for Schwinger by the operation of the Stern–Gerlach apparatus, which operates so as to filter z-spin values, that is separates portions of a beam by the spin-angular momentum component along a specific spatial direction z and then blocks all resulting beams but one, which it allows to pass.

The sum of two selective measurement symbols, e.g. $M(a') + M(a'')$, is taken to signify a less precise selective measurement, one that produces a subensemble associated with *any* of the values in the sum, neither of them being distinguished by the measurement. In accordance with the physical significance of these operations, the addition of symbols is both commutative and associative, whereas multiplication is *only* associative. The multiplicative numbers 0 and 1 are present with their standard arithmetic characteristics and multiplication is taken to be distributive over addition. Again in accordance with their physical significance, for different values a' and a'' one must have $M(a')M(a'') = \delta(a', a'')M(a')$ where the delta symbol has the standard meaning, that is, the measurement symbolized by $M(a')$ accepts every system produced by $M(a')$ and rejects every system produced by $M(a'')$ when $a'' \neq a'$. Thus, a measurement that does not distinguish *any* of the possible values of a' is the measurement that accepts all systems, that is, the *non-selective* measurement is symbolized as

$$\sum_i M(a') = 1 \tag{3.1}$$

and one that rejects *all* systems is symbolized by 0; these two symbols behave arithmetically in a way consistent with this physical interpretation.

The multiplication of these symbols, e.g. $M(a')M(a'')$, is then taken to symbolize a *succession* of measurements. With respect to the representation of physical properties, Schwinger considers two physical quantities, A_1 and A_2, to be *compatible* when the measurement of one does not destroy the (validity of the available) knowledge (of the value) of the result of the other gained by its measurement previously, that is, the second measurement to be non-disturbing of the first in that relative sense. For two compatible quantities, the selective measurements $M(a_1')$ and $M(a_2')$ produce an ensemble of systems for which quantities can be simultaneously assigned the values a_1 and a_2 for A_1 and A_2, respectively, regardless of the order in which they are performed: for this compound measurement $M(a_1')M(a_2') = M(a_2')M(a_1')$, which Schwinger symbolizes as $M(a_1'a_2')$, can be considered a discrete, integral process with distinct beginning and end [12].

A complete set of compatible physical quantities $A_1, A_2, \ldots, A_k$ is taken as one in which there is mutual compatibility for *each* pairing of these quantities and there is no quantity for which this is true that is *not* in this set, which is indicated simply by A. In that context, the measurement symbol

$$M(a') = \prod_{i=1}^{k} M(a_i') \tag{3.2}$$

and the systems selected are attributable specific values for the maximum possible number of attributes and are said to be *in state* a', a notion discussed in more detail not far below.

To encompass more general measurement situations, such as those in which the system is subjected to successive selective measurements that may include those of incompatible quantities, Schwinger considers the selective measurement of one complete set of properties, represented by B, and then another, represented by (a possibly different, possibly incompatible set of properties) A. The operation that rejects all entering systems except those in state b' and permits only systems in state a' to emerge from the apparatus is written, in a right-to-left convention, as $M(a', b')$. A succession of two such measurements is written as a product $M(a', b')M(c', d')$, with the convention, without loss of generality, as to priority that, again, the symbol to the *right* is prior to that on the left in the sequence of operations. The passage of all systems or no systems between the two stages is taken to be represented by the multiplicative numbers 1 and 0.

The measurement $M(a')$ has the special property that *no* change of state occurs under repeated measurement, that is,

$$M(a') = M(a', a'). \tag{3.3}$$

In a general basic measurement situation, one considers a system in the state that is subjected to successive measurements. Any incompatibility of two measurements will be manifested by the (imprecisely accountable) *disturbance* of the system state by a measurement, which Schwinger views as *the* characteristic distinguishing quantum from classical measurement. Consider, for example, the following sequence of two two-stage measurements. The properties of successive measurements of the type $M(a',a'')$ are symbolized by

$$M(a',a'')M(a''',a'''') = \delta(a'',a''')M(a',a''''). \tag{3.4}$$

If the order of the measurements is reversed, one then finds instead that

$$M(a''',a'''')M(a',a'') = \delta(a',a'''')M(a''',a''), \tag{3.5}$$

which is generally different from the first expression. This shows that the multiplication of measurement symbols is *non-commutative*. This is the algebraic embodiment of this distinguishing characteristic of quantum mechanics. These properties imply that both $M(a',a') = M(a',a')^{\dagger}$ and $M(a',a')^2 = M(a',a')$ with the interpretation that it is possible to confirm the result of such a measurement by repeating it, where $\dagger$ is the operator adjoint, discussed below. One can see by direct calculation also that the multiplication is distributive over addition.

Then, taking a bold step which Schwinger himself calls a 'leap of the imagination' [12] in relating measurement to state, Schwinger introduces system state symbols similar to Dirac's—about which more further below—giving them meaning through the following:

$$M(a',a'') = |a'\rangle\langle a''|, \tag{3.6}$$

where $|a'\rangle$ symbolizes the creation of a system of *type* a', whereas $\langle a'|$ symbolizes its destruction. Schwinger points out that the physical property A is characterized by the results of its measurement (the numbers a') and by the symbols of creation and annihilation of a system in the states labelled by those results, a'. 'It is though the entering a''-system is destroyed and an a'-system is created in its place. This mental two-step process is indistinguishable from the actual one' [12]. Measuring A of an ensemble for the system created in *state* a' is then naturally symbolized by $A|a'\rangle = |a'\rangle a'$ and $|a'\rangle A = a'|a'\rangle$.

One then finds that these symbols behave as a set of vectors dual to each other and to the measurement symbols, because it is natural to their interpretation that both

$$\langle a' \mid a''\rangle = \delta(a',a'') \tag{3.7}$$

and

$$\sum_{a'} |a'\rangle\langle a'| = 1, \tag{3.8}$$

the first by virtue of the fact that a' and a'' are different and the latter because all possible measurements are considered. Thus, one has for them both orthonormality and completeness, providing the basis for a *geometry* of states. One finds an explicit expression for the property A in terms of states by multiplying by the corresponding destruction symbols $\langle a'|$ and summing over all values of a':

$$A = \sum_{a'} |a'\rangle a'\langle a'|. \tag{3.9}$$

Schwinger points out that the physical property A is characterized by the possible results of its measurement (the numbers a') and by the symbols of creation and annihilation of a system in the states labelled by those results, a', so that one also has for any function $f(A)$ of A,

$$f(A) = \sum_{a'} |a'\rangle f(a')\langle a'|. \tag{3.10}$$

Returning to the consideration of successive measurements, first the creation of a state b' and then its being measured for A in some way (symbolized by $M(A)$), will yield that $M(A)\,|\,b'\rangle$ has the effect of the destruction of the b' state:

$$\langle b'|M(A)|b'\rangle. \tag{3.11}$$

By starting with a measurement that accepts *no* systems, then one state, two states and so on through increasingly less selective measurements—recalling that this involves summing over selective measurement symbols—one sees that the resulting quantities, which can be written in the form $p(a', b')$, are candidates for *probabilities*. Moreover, one has that $p(a', b') = p(b', a')$. Noting that the sum of probabilities is conventionally normalized to 1, one has the natural restriction that $p(a', b') = \langle b' \mid a'\rangle\langle a' \mid b'\rangle \geq 0$. This condition can be satisfied with $\langle b' \mid a'\rangle$ real, with $\langle b' \mid a'\rangle = \langle a' \mid b'\rangle$, but this would involve $\langle b'|$ being interchangeable with $|b'\rangle$, which is in conflict with the difference in the physical interpretation of the two. Instead, taking $\langle b' \mid a'\rangle$ and $\langle a' \mid b'\rangle$ to be complex and taking $\langle b' \mid a'\rangle = \langle a' \mid b'\rangle^*$ avoids any such conflict (see below). In fact, the matrix formed by the $p(a', b')$ can be seen to be *doubly stochastic*.

With the algebra of measurement symbols established, a measurement-based statistical interpretation of the quantum formalism is possible, in particular, by considering a triple product of measurements formed from the symbols of two incompatible properties, namely,

$$M(b')M(a')M(b') = \langle a' \mid b'\rangle\langle b' \mid a'\rangle M(b') = p(a', b')M(b'), \tag{3.12}$$

where $p(a', b') = \langle a' \mid b'\rangle\langle b' \mid a'\rangle$ is the invariant through which $\langle a' \mid b'\rangle$ gains physical meaning. One naturally expects the result of an intervening measurement to produce a fraction of the ensemble introduced at the beginning of this composition of measurements.

The expectation value of property A for systems in the state b' is the average of the possible values of A, weighted by the probabilities of occurrence that are characteristic of the state b'; one can write the probability formula as

$$p(a', b') = \operatorname{tr} M(a')M(b'). \tag{3.13}$$

The additivity of $p(a', b')$ and the fact that $\sum_{a'} p(a', b') = 1$, which follows from this, shows $p(a', b')$ to be stochastic, establishing the core of its interpretation as the probability that a' is observed in the measurement of A of a system just previously measured to be in state b', with first selective measurement of B playing the role of state preparation. In order to completely qualify the probability interpretation, the requirement mentioned above that $\langle b' \mid a'\rangle$ and $\langle a' \mid b'\rangle^*$ are complex conjugate numbers that establishes the non-negativity of $p(a', b')$ requires that $\lambda(a')^* = \lambda(a')^{-1}$ and so, significantly, requires them to take the form

$$\lambda(a') = \mathrm{e}^{\mathrm{i}\phi(a')}, \tag{3.14}$$

in which the values of the real numbers $\phi(a')$, i.e. phases, importantly are *unrestricted* and so *all possible*, a point taken up in detail in §5.

Thus, we see that correspondence between operators and physical quantities is such that a function $f(A)$ of the property A is assigned the operator $f(A)$, as seen in equation (3.10), and the operators associated with a complete set of compatible physical quantities form a complete set of commuting Hermitian operators. The function of A that exhibits the value unity in the state a' and zero otherwise is given by the operator $M(a')$. The operator adjoint (†) was introduced with the convention that the symbol for selective measurement equivalent to a pair of selective measurements performed in opposite order is taken to be *equal* to the adjoint of that for the original order, with the Hermitian self-adjoint operators being unchanged under the operation, because the choice to interpret symbols from the left or right in equations involving them is conventional. This use of complex numbers in the measurement algebra means that the dual algebra in which all numbers are replaced by their complex conjugate numbers produces no physical difference from the original. The formation of the adjoint within the complex conjugate algebra is called 'transposition': $X^{\mathrm{T}} = X^{*\dagger} = X^{\dagger *}$. Thus, the algebra generated by the selective measurement symbols is canonically isomorphic to the algebra of all operators in Hilbert space $\mathcal{H}$, with the observables identified with the self-adjoint elements and the involution with the adjoint.

Now, in order to find the symbolic counterpart of the *non*-selective measurement, sometimes called a 'pre-measurement', which Schwinger takes as the first stage of any measurement prior to that of selection, he considers a system in state c' that is subjected to the selective measurement $M(b')$ and then an A measurement with the final outcome a'. This occurs with probability

$$p(a', b', c') = p(a', b')p(b', c') = |\langle a' \mid b'\rangle\langle b' \mid c'\rangle|^2 = |\langle a' \mid M(b') \mid c'\rangle|^2, \tag{3.15}$$

whereas if the intermediate stage involves no discrimination of values at all, that is, no intermediate measurement occurs (of any observable B), one would have

$$p(a', 1, c') = |\langle a' \mid c'\rangle|^2 \tag{3.16}$$

$$= \left|\left\langle a' \left| \sum_{b'} M(b') \right| c' \right\rangle\right|^2 \tag{3.17}$$

$$= \left| \sum_{b'} \langle a' \mid M(b') \mid c'\rangle \right|^2 . \tag{3.18}$$

Finally, if a non-selective intermediate measurement, say of an observable B, occurs, one would have instead

$$p(a', b, c') = \sum_{b'} p(a', b')p(b', c') \tag{3.19}$$

$$= \sum_{b'} \big| \, |\langle a' \mid M(b') \mid c'\rangle| \big|^2 \tag{3.20}$$

$$= \big| \, |\langle a' \mid M_b \mid c'\rangle| \big|^2. \tag{3.21}$$

This differs from what is seen in equation (3.18) in that there is an absence of interference terms between states with differing values for b'.

Schwinger then argues from this that the appropriate decomposition of the symbol for *non-selective* B measurement is

$$M_b = \sum_{b'} e^{i\phi_{b'}} M(b'), \tag{3.22}$$

'where the real phases $\phi_{b'}$ are independently, randomly distributed quantities. The uncontrollable nature of the disturbance produced by a measurement thus finds its mathematical expression in these random phase factors' [13]. As for the overall process, Schwinger considers it quantum mechanically indivisible.

> The uncontrollable disturbance attendant upon a measurement implies that the act of measurement is indivisible. That is to say, any attempt to trace the history of a system during the measurement process usually changes the nature of the measurement that is being performed. Hence to conceive of a given selective measurement $M(a', b')$ as a compound measurement is without physical implication. It is only of significance that the first state selects systems in the state b', and the last one produces them in the state a'; the interposed states are without meaning for the measurement as a whole. Indeed, we can even invent a nonphysical state to serve as the intermediary. We shall call this mental construct the null state 0 [14, p. 258].

With the null state so defined, one can write

$$M(a', b') = M(a', 0)M(0, b'), \tag{3.23}$$

showing that it is consistent to consider the formal state 'during' measurement as a fiducial one. The arbitrariness that remains in these symbols is expressed in the relations

$$M(a', b') \rightarrow \mathrm{e}^{-\mathrm{i}\phi(a')} M(a', b')\, \mathrm{e}^{\mathrm{i}\phi(b')}, \tag{3.24}$$

which implies the accompanying substitutions

$$\langle a' | \rightarrow \mathrm{e}^{-\mathrm{i}\phi(a')} \langle a' | \quad \text{and} \quad | b' \rangle \rightarrow \mathrm{e}^{\mathrm{i}\phi(b')} | b' \rangle. \tag{3.25}$$

Note that, for those favouring state descriptions over process descriptions, the characteristics of the symbols $M(a', b')$ are also derivable from those of $\langle a'|$ and $|b'\rangle$. In addition, the substitution (3.25) transforms the measurement operators in accordance with (3.24) [14].

The introduction of the null states allows the following interpretation. The measurement process that selects a system in the state b' and produces it in the null state,

$$M(0, b') = \Phi(b'), \tag{3.26}$$

can be considered as the annihilation of a system in the state b'; and the production of a system in the state a' following its selection from the null state,

$$M(a', 0) = \Phi(a'), \tag{3.27}$$

can be considered as the creation of a system in the state a', if the null state is understood as a sort of 'vacuum' state. Thus, $M(a', b')$ is indiscernible from the compound process of the annihilation of a system in the state b' followed by the creation of a system in the state a':

$$M(a', b') = \Psi(a')\Phi(b'). \tag{3.28}$$

The extension of the measurement algebra to include the null state supplies the properties of the Ψ and Φ symbols:

$$\Psi(a')^{\dagger} = \Phi(a') \tag{3.29}$$

and

$$\Phi(b')^{\dagger} = \Phi(b'). \tag{3.30}$$

Furthermore,

$$\Psi(a')\Psi(b') = \Phi(a')\Phi(b') = 0, \quad M(a', b')\Phi(c') = \Psi(a')M(b', c'), \tag{3.31}$$

whereas

$$M(a', b')\Psi(c') = \langle b' \mid c' \rangle \Psi(a'), \quad \Psi(a')M(b', c') = \langle a' \mid b' \rangle \Phi(c') \tag{3.32}$$

and

$$\Phi(a')\Psi(b') = \langle a' \mid b' \rangle M(0). \tag{3.33}$$

Schwinger views quantum state disturbance in measurement as objectively significant 'because at a higher level of classical measurement, disturbances produced by a measurement can be admitted if they are *known* and therefore could be compensated for' [15]. He viewed the difference in the quantum case as due to the fact that

> the measurement act involves a strong interaction [at the atomic level]. I repeat: on the microscopic scale it is necessarily strong because we cannot cut the strengths of the charges in half; we cannot change the properties of these fundamental particles; we must accept them as they are—and so the measurement unavoidably produces a large disturbance, which we cannot correct for in each individual instance, for we cannot control what happens in each individual event in any detail... the program of computing what the effect of the disturbance was and correcting for it is, in general, impossible. Accordingly, the two basic tenets of the theory of macroscopic measurement are both violated. Either the interactions cannot be made arbitrarily weak because of the phenomenon of atomicity, or if we wish to accept this and correct for it, we cannot do so because we do not have a detailed, deterministic theory of each individual event [15, p. 12].

The senses of determinism and causality invoked by Schwinger are discussed in §5. Before considering these, let us first discuss the key move of taking the phase values in the long run to take all their allowed values in a randomly distributed manner in detail in the following section.

4. The randomness assumed

First, let us note, following Schwinger, that the selective measurements can be mathematically related to measurements where no absorption occurs, and instead 'nature makes a choice' of phase for each possible measurement outcome, by the substitution of infinite imaginary numbers for the phases—so that one has $e^{i(i\infty)} = e^{-\infty} = 0$—of those terms of the sum corresponding to values that are *not* obtained, with the physical interpretation that selection is the rejection of all but the corresponding subensemble, and so is 'absorptive', resulting in each event belonging to the subensemble corresponding to shared outcome value(s). Finally, note that since non-selective measurements reject *no* systems, one must have

$$\sum_{a'} p(a', b, c') = 1, \tag{4.1}$$

which is equivalent to the unitarity of the operators $M(b)$, that is,

$$M_b^\dagger M_b = M_b M_b^\dagger = 1. \tag{4.2}$$

The randomness assumed in the reconstruction by Schwinger of quantum mechanics is that which he invokes when considering the mathematical forms of the symbols he induces for the non-measurement and the non-selective measurement in (3.15)–(3.21). (These can be seen as taking place relative to what is often called a measurement basis.) Schwinger reconstructed these general forms through the comparison of their corresponding different but related physical circumstances. (Note that the comparison is similar to that carried out by Heisenberg [16] which involves more specific examples and suggests that the randomness arises in a more mechanistic way.)

Given these forms (cf. [13]), what is first required is the averaging of phase values (which in the non-measurement case, by contrast, is only one value rather than a range of values) in the formation of subensembles of the ensemble of measured systems that is a necessary characteristic of any measurement, often taken to define 'pre-measurement'. To accomplish this averaging, Schwinger takes the real phases $\phi(b')$ of equation (3.22) to be independent, randomly distributed quantities, something which he justifies by appeal to the idea—found in previous writings of Bohr and Heisenberg, although for the latter, the operative verb is German *kontrollieren*[1] rather than 'to control' more pertinent to the issue of accounting than to control—that there is an uncontrollable disturbance of the state of a quantum system if measured. The effect of this random behaviour is the elimination of any interference to which these phases would otherwise give rise; in the terminology of contemporary foundations of quantum theory, there is intrinsic decoherence. These moves present a number of issues, addressed successively below, after the following short résumé of the elements of Schwinger's justification [13].

In Schwinger's view, the law of motion of quantum mechanics applies directly only to ensembles, so that one has no theory of the individual systems. The effect on each element of an identically prepared ensemble of the set of interactions involved in any measurement is, therefore, not specified via this law—individual behaviour need only be such that it would not lead to contradictions of its implications. Schwinger explains that with interactions taking place at the atomic scale, which are by necessity relevant in any measurement involving the atomic level, unpredictability will be present. Again,

[1] I thank John Stachel for recently reminding me of this point, though it seems that Schwinger himself was not so conscious of this shift due to transposition between the related languages. See also [17] on a related point.

> Either the interactions cannot be made arbitrarily weak because of the phenomenon of atomicity, or if we wish to accept this and correct for it, we cannot do so because we do not have a detailed, deterministic theory of each individual event... [15, p. 12]

Such a correction is something which could be accomplished *were there* precise predictability understood as determinism, but it is unavailable. Thus, Schwinger views quantum measurement to be a locus of indeterminism in relation to individual events—possibly in addition to other differences of quantum from classical physics—and that interdeterminism is to be expressed (at least) in the behaviour of the phases $\phi(b')$.

A related sort of random event is also introduced by Schwinger, namely, the appearance of the value attributed to any physical property on the basis of measurement of an incompatible property. In such measurement situations,

> the system that is being measured is disturbed in an uncontrollable way in such a manner that if we now went back and asked for the value of the first physical property, checking to see that it still had the same value as before, we would now find not at all the same value but a *random assortment* [emphasis mine] of all the possible values that it could assume, with various probabilities that depend in detail upon precisely what we have done. This is so because the second measurement has introduced a new physical situation... [15, p. 13].

The new physical situation here is captured by the mathematical description given the ensemble after the measurement has taken place, again expressed via the behaviour of the phases of measurement symbols.

Thus, Schwinger sees two novel aspects in quantum mechanics related to the disturbance involved in measurement at the scales of the fundamental particles, both of which involve randomness: (i) a loss of coherence between available alternative quantum states that occur upon measurement and (ii) the assortment of these values over the members of the resulting (initially pure) ensemble. Schwinger's theoretical goal in performing the reconstruction discussed here was to induce the 'whole new scheme of mathematics,' namely, that of quantum mechanics 'developed by Heisenberg, Born, Schrödinger and others, essentially in the years 1925 to 1929', something necessary because fundamental classical mechanical assumptions about the relationship between measured values and physical properties no longer holds in the quantum realm.

> I wish to point out that the failure of these fundamental assumptions means equally well a failure of the ability to represent physical phenomena in the microscopic realm by numbers which change in time as we do in the macroscopic or classical domain. Something of an entirely different mathematical nature is needed, such that it represents, or mimics, the basic facts of microscopic measurement [15, p. 13].

The result of his effort is 'a whole new theory of microscopic measurement', on which a 'phenomenological' explication of the quantum kinematics is to be founded.

Schwinger used the terms 'determinism' and 'causality' involved here with the following meanings.

> By *causal*, one means essentially that when the state of the system is given at a particular time—and we must return to precisely what we mean by 'state'—then the state is completely determined at any other time; this is what we mean by causality. Causality is inference in time: given the state of affairs at one time, the state of affairs is uniquely determined at another time. What makes it *deterministic* is that the knowledge of the state also determines all possible physical phenomena precisely [15, p. 2].

One sees that, for Schwinger, quantum mechanics is indeterministic and, only when measurement is involved, generally acausal for individual measurements; one cannot, in general, determine future states of affairs from complete (as is possible) knowledge of the current state of affairs.

Thus, for him, quantum mechanics 'is a causal, statistically deterministic theory... You will have perfectly determinate, statistical predictions but no longer individual predictions' [15, p. 15], that is, individual measurement outcomes are *unpredictable*.

Although, mathematically, all that Schwinger requires of *randomness* is the uniformity of the distribution of the phases $\phi(b')$ and of the quantum measurement outcomes in selective measurements, his notion of determinism allows his scheme to be consistently understood as one involving randomness in the sense of unpredictability at the individual system level. The main question left unanswered so far is that of the ultimate grounding for this random behaviour of these elements of the state upon measurement.

5. Grounding quantum randomness beyond complementarity

We see from the foregoing that Schwinger's reconstruction of quantum mechanics can be consistently understood as one in which there is physical randomness in the sense of unpredictability of physical states of affairs, albeit one that was left otherwise ungrounded. He held that the statistical nature of quantum mechanics renders it silent regarding individual outcomes. On this basis, he also held that the so-called measurement problem is not a genuine problem for his approach. Nonetheless, because the novel aspects of quantum mechanics are related to the disturbance involved in measurement and the loci of randomness in his approach are precisely those associated with the quantum measurement problem as typically formalized, it is important not to overlook this context: the question of the ground in principle for the occurrence of the definite, individual outcomes of measurements which constitute the members of these ensembles has remained unanswered by Schwinger's measurement-based approach to quantum mechanics. Although the random behaviour under measurement is in need of a grounding that is on a par with that provided by natural law, quantum mechanics is silent as to the values appearing in these outcomes beyond a formal prescription allowing them to be naturally interpreted as random variables.

A physical explanation of what takes place during measurement at the level of the atomic and subatomic realms by a successor physical theory is not something coming into play in Schwinger's induction of quantum mechanics—ultimately, he viewed quantum field theory, and later his source theory, as successor theories which might supplant basic quantum mechanics for a number of reasons beyond our scope here. Within traditional quantum mechanics, Schwinger simply took Bohr's principle of complementarity, which he described as 'perhaps the widest philosophical principle that has emerged from these studies of microscopic physics', as his ground for understanding the quantum mechanical world via the quantum mechanical description [15].

> Bohr's principle of complementarity is the statement that we have in microscopic physics first of all a new world (that's the important thing to recognize), in which classical analogies fail... Two distinct classical pictures can hold under different physical situations, never simultaneously, and the applicability of one picture prevents the applicability of the other—the two classical analogies are mutually exclusive. But both pictures are on the same footing... This is, in essence, the entirely new situation which has no counterpart in any of the classical philosophical modes of thought. It is something that simply must be accepted [15, p. 17].

Therefore, for those demanding a firmer ground for physical state change, until a superior explanation—if one becomes available at all—of what takes place during measurement at the level of the atomic and subatomic realms is provided in a single physical picture given by some (successor) theory, the adequacy of the physical picture provided by *quantum mechanics* in the face of the nature of quantum measurement would be in question.

Historically, this concern has been couched in the form of worries about the 'acausal' nature of quantum mechanics, because before the great success of quantum mechanics in providing explanations of atomic-level phenomena, causal explanations had been generally required and are still preferred for all physical phenomena. Indeed, Schwinger considers quantum mechanics a (statistically) causal theory, even though one lacking underlying 'hidden variables', because it is a theory of ensembles not of individuals. No grounding for the physical state change accompanying definite measurement outcomes, beyond its being necessitated by their existence as brute matters of fact which remain problematic in the absence of such a successor theory or a special assumption absolving the theory of responsibility for providing a detailed dynamics of measurement, is offered by Schwinger.

Grounding the existence of these only statistically governed individual outcomes in an independent fundamental principle, rather than merely assuming them to be randomly occurring in the statistical sense, would be a significant development in justifying its otherwise obvious explanatory power. This is the primary motivation here for looking to the plenitude principle (suitable formulated) as such a ground that differs from those offered by Bohr (complementarity) and Heisenberg (actualization of potentiality) which might be considered, given that it would not be a stretch to consider Schwinger otherwise an adherent of the so-called orthodox interpretation associated with the Copenhagen school, as well as from the invocation of the mechanism of decoherence sometimes associated with the 'new orthodoxy' in the interpretation of quantum mechanics [18]. The principle of plenitude has the advantages—as Schwinger's approach itself has over those previous manners of handling the nature of quantum measurement viewed as distinct from classical measurement—of logical elegance and simplicity.

The sort of explanations for the evolution of the physical states of individual systems, since the introduction of Newtonian mechanics (except, until Einstein, those given by Newton's theory of gravitation), were almost always causal. The associated demand can be traced back in large part to principles of the sort exemplified by the principle of sufficient reason, advocated by Leibniz and Spinoza, namely:

Sufficient reason. 'For every fact F, there must be an explanation why F is the case'.

In Hume's view, the relation of causation is necessary for allowing us to make predictions, that is, to infer from observations (matters of fact) to (presently) unobserved matters of fact [19]. He offered a minimalist set of properties a cause should possess in relation to an effect, namely: (i) 'Contiguity in time and place is a requisite circumstance to the operation of all causes'. (ii) 'Priority in time is another requisite circumstance in every case'. (iii) 'A third circumstance [is] constant conjunction betwixt the cause and the effect...' In addition, he noted that 'Beyond these three circumstances, I can discover nothing in this cause' [19]. Thus, for Hume, they are *equivalent to* causation.

Most famously, Laplace set the following standard of causation as applied in *physics*, historically accepted as satisfied by classical mechanics:

> We ought...to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain, and the future, as well as the past, would be present to its eyes [20, p. 4].

Thus, causation has traditionally been understood in either strong and weak forms, for example, the following.

Causation (strong). 'Every effect is produced through lawful necessity by a cause. One can enforce asymmetry by stipulating that causes precede their effects'.

Causation (weak). 'An effect cannot temporally precede its cause'.

Even for those who do not accept the requirement that causation be understood via a natural law or accept a strong form of causation itself, the consideration of causal factors can be of value, in particular, in providing explanations of the sort 'F happened because C happened beforehand'; for example, 'This photodetector clicked because a photon impinged upon its front surface' is a valid explanation of a particular event by appeal to physical theory.

Before quantum theory, whenever theories did not offer precise predictions of the observed system states, the corresponding imprecision was expected to be understandable as both statistical *and* in terms of an underlying, more precise set of laws for individuals having, in principle, state (or process) specifications as precise as required. Thus, the former states would be statistical states reducible, in principle, to the latter individual states, the evolution of which could ideally be causally described. It is, however, broadly accepted that the behaviour of the ensembles described by quantum mechanics *cannot be* understood in this way. This is another sense in which quantum mechanics presents us with 'an entirely new situation', as Schwinger called it.

Nonetheless, while still invoking the principle of sufficient reason, one can consider a different sort of explanation from a strictly causal one, even within mechanics, which has historically apparently been most amenable to such explanations. Indeed, the principle of sufficient reason, stated as above, leaves open the question of what constitutes a sufficient reason under the peculiar new circumstances of the realm of quantum physics, allowing for the provision of a principled grounding for the definite outcomes we observe there.

6. Plenitude and possibility

I will now argue that the principle of plenitude, suitably applied, provides a fundamental grounding for the definite outcomes of individual quantum measurements in the context of Schwinger's reconstruction of the theory. The principle of plenitude is perhaps most famous in its use by Leibniz in the realm of logic (cf. [21]). A distinction between logical and physical possibility has emerged since his use of it, as has its application in relation to them both. The goal here is to show how the principle of plenitude applied to physical descriptions, in particular, to the phases associated with the possible states of an individual quantum system initially belonging to a given pure ensemble, helps explain the observed phenomenon of the appearance of definite outcomes as results of measurements of quantum systems.

First, it should be noted that definite measurement outcomes are matters of fact, just as in classical mechanics, despite differences of the physics of the interactions involved when they take place. In that sense, the existence of these outcomes is not in question as an aspect of the physical world. The open issues are, rather, (i) that of their definiteness and (ii) that of their relation to the laws of quantum mechanics, not simply the mutual consistency of the two that has been brought into question in the form of the measurement problem, which Schwinger dismisses because he views the theory as inherently one of ensembles rather than ensemble *members* directly, but rather of the relationship between the behaviour of the individual members of the quantum ensemble that is subject to the law of motion which describes them only indirectly.

A traditional 'explanation' of the appearance of definite measurement outcomes has been the invocation of the so-called projection postulates, the most widely accepted being the Lüders rule, another being von Neumann's prescription for the post-measurements state, which in some cases does not prescribe a pure state but rather a mixed state, distancing itself from definiteness in such cases. These rules are merely prescriptions for the re-assignment of the quantum state description upon measurement conditionally on the outcome of measurement, not *mechanisms* for state change (or process). By contrast, the elements of Schwinger's algebra of measurement describe the *changes* of state during measurement which come about randomly with certain probabilities with measurements corresponding to the random occurrence of specific phase parameters in each measurement, being in that way a more adequate, albeit, as he calls it, a phenomenological, description. However, on their own, both of these treatments appear *ad hoc*.

The consideration of measurement processes within the context of possible worlds theory allows for the consideration of different modalities, that is, ways in which what takes place could be the case or must be the case for the events they comprise. Recall that, in that context, a proposition is *necessary* just in the case it is true in all possible worlds, a proposition is *possible* just in the case it is true in some possible worlds, and it is *contingent* just in the case it is true in some but not all possible worlds.[2] Given that physical possibilities are a fundamental aspect of quantum mechanics, in that ostensibly irreducible probabilities are attributed to future measurement outcomes, this context is clearly pertinent. We thus consider now the principle of plenitude as the ground for the actual appearance of definite measurement outcomes and measurement symbol phase-parameter values upon measurement when a number of values are possible. I wish first to note, however, as Jaako Hintikka pointed out, that the principle of plenitude does not assert that there is a plenitude of contemporaneous actual realizations, but only a sort of equation between possibilities and their realizations, which is something essential to understand in the context of our application of the principle here.

> It is as much or as little a Principle of the Paucity of Possibilities as a Principle of Plenitude of their Realizations. No plenitude of any sort can be extracted from the Principle except in conjunction with a sufficiently strong assumption concerning the richness of the range of possibilities whose eventual realization is asserted by the Principle. Hence Lovejoy's term 'the Principle of Plenitude' is a misnomer... [23, p. 6].

One of the primary reasons philosophers have used the principle in a way that pertains to how it will be used here was the assumption that one could characterize modalities in statistical terms [23]. This again supports the pertinence of the principle to our case. Indeed, an entire class of interpretations of quantum mechanics, the *modal interpretations*, is founded upon the legitimacy of such an application. Such interpretations, however, differ greatly from the specific issue and approach we are dealing with here. Among other aspects, those interpretations are the so-called no-collapse interpretations, in which the quantum state is taken only to describe the possible, and *never the actual properties* of the system (understood as dispersion-free expectation values), which is not the case in Schwinger's approach to quantum mechanics.

Before proceeding, let us also briefly note two specific uses of the principle of plenitude in the realm of quantum theory, specifically processes within particle physics, the latter more direct than the former. In relation to electron–positron annihilation, which was predicted by his quantum electrodynamics, Dirac commented that 'there appears to be no reason why such processes should not actually occur somewhere in the world. They would be consistent with all the general laws of Nature...' [24]. This case could be captured by the general statement 'That which is not forbidden is allowed', that is, in some (here, physical) sense possible. Gell-Mann later suggested a stronger proposition along such lines for use in theoretical particle physics, which he called the 'totalitarian principle', namely, 'Everything not forbidden is compulsory', that is, in some sense (here, physical) necessary [25]. Both of these examples can be considered specific instances of the application of the principle.

A general form of the principle of plenitude in relation to the natural world can be formulated as follows.

> *Plenitude*. 'A state of affairs or process which is not precluded by law or by the actual current state of affairs or processes will occur'.

This might be viewed as implying the following proposition, suited to the context of the reconstruction of quantum mechanics as carried out by Schwinger.

> *Physical plenitude principle* (PPP). 'If no physical law or current state of affairs or processes precludes a set of possible states of affairs or processes, then the members of this allowed set will occur with some likelihood'.

[2]Thus, a sentence is considered necessary (possible, contingent) just in the case it expresses a necessary (possible, contingent) proposition [22].

It is this proposition that relates to the situation arising in the quantum measurement, considered in the previous section, and to the randomness assumed by Schwinger.

As a rule, when the plenitude principle, in one form or another, comes into play, it does so in the face of some form of impotence of principle or law. In the current case, the impotence involved is the inability of the law of motion to determine what happens over the full course of the measurement process—and so, in the context of observational practice, the inability of individual measurements to occur without disturbance coupled together with an inability of experimenters to precisely account or compensate for an inevitable disturbance due to this failure to predict the effect of this disturbance in the individual case, something expressed by the Heisenberg uncertainty principle (HUP), for example, for a spectrum of discrete eigenvalues [16,26]. (The HUP expresses how measurements of non-commuting properties are mutually limiting: they do not preclude one another from being carried out, but they limit precision of joint measurement, and hence preclude the preparation of joint 'elements of reality' due to the corresponding disturbance.)

The plenitude principle provides an appropriate answer to the question of how a formal description of the measurement process, such as that provided by Schwinger, involving as it does indeterminate events, can be grounded in principle on a level of logical priority comparable with that of the quantum mechanical equations of motion which operate at all times other than the critical stage of interaction between measurement apparatus and measured systems. It provides justification for the attribution of particular values of each phase parameter and one finite amplitude, providing just one among the set of possible outcomes that a measuring apparatus allows as alternatives: (i) By the PPP, because the phases within the quantum state of a subatomic system, while under measurement, are beyond the purview of the quantum mechanical equation of motion, all phase values will arise at the subatomic level with equal likelihood and because they are mutually exclusive and no one has priority over any other in the context of any single event or process, they will randomly occur in the corresponding ensemble. (ii) Similarly, the phases of the quantum state amplitudes of subatomic systems are beyond the purview of the quantum mechanical equation of motion during measurement, so that the individual outcomes will occur in such a way that a finite value is provided to only one of those possible values in any measurement as described in Schwinger's formulation (with likelihoods constrained only *at the ensemble level* by the statistics corresponding to the values provided by the Born rule), most clearly so in the limit of measurements made with perfect precision, where the physical law makes *no prediction whatsoever* for the value of quantities incompatible with the observable measured. Notably, these are the two elements of measurement typically considered in need of explanation by an adequate solution of the problem of measurement in quantum mechanics.

7. Conclusion

Schwinger provided physics with a mathematical reconstruction of quantum mechanics on the basis of the characteristics of sequences of measurements occurring at the atomic level of physical structure, the central component of which is an algebra of symbols corresponding to quantum measurements, conceived of as discrete processes [27]. The outcomes of measurements are determined by the values of the phase parameters appearing in the corresponding symbols. These phases were taken by Schwinger to take a range of random, uniformly distributed definite values. Here, this assumption was considered in the context of recent critiques of the notion of randomness and explicitly related to the principle of plenitude, which provides a fundamental grounding for the objective, physically irreducible probabilities, conceived of as graded possibilities that are attributed to measurement outcomes by quantum mechanics. In this way, new light is shed on the nature of measurement on which, Schwinger showed, the theory could be based.

Competing interests. The author declares that he has no competing interests.
Funding. I received no funding for this study.

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