Large Cardinals and the Iterative Conception of Set

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Abstract

The independence phenomenon in set theory, while pervasive, can be partially addressed through the use of large cardinal axioms. A commonly assumed idea is that large cardinal axioms are species of *maximality principles* for the iterative conception, and assert that the length of the iterative stages is as long as possible. In this paper, we argue that whether or not large cardinal principles count as maximality principles depends on prior commitments concerning the richness of the subset forming operation. In particular we argue that there is a conception of maximality through *absoluteness*, that when given certain technical formulations, supports the idea that large cardinals are consistent, but false. On this picture, large cardinals are instead true in *inner models* and serve to *restrict* the subsets formed at successor stages.

Introduction

Large cardinal axioms are widely viewed as some of the best candidates for new axioms of set theory. They are (apparently) linearly ordered by consistency strength, have substantial mathematical consequences for independence results (such as consistency statements and Projective Determinacy[1]), and often appear natural to the working set theorist, providing fine-grained information about different properties of transfinite sets. They are considered mathematically interesting and central for the study of set theory and its philosophy.

In this paper, we do not deny any of the above views. We will, however, argue that the status of large cardinal axioms as *maximality* principles is questionable. In particular, we will argue that there are conceptions of maximality in set theory on which large cardinal axioms are viewed as *restrictive* principles that serve to leave out the consideration of certain subsets formed under the iterative conception of set.

Our strategy is as follows: We first (§1) explain how large cardinals have been seen to be related to the iterative conception of set, and how they might be viewed as maximality principles. Specifically, we will canvass the idea that large cardinal axioms assert that the stages in the iterative conception go as far as very large ordinals. We then (§2) present a different conception of maximality under the iterative

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[1]See [Schindler, 2014] for a textbook treatment of large cardinals and determinacy.

conception, the *absoluteness* idea that 'possible' sets should be witnessed already in the universe. We link this idea to subset formation, and point out that under the iterative conception there is a priority of width considerations rather than height considerations. Next (§3) we present three troublesome cases for large cardinals based on ideas of maximality through absoluteness (concerning Reinhardt Cardinals, inaccessible cardinals, and the existence of ω_1). Given this picture, we argue, large cardinals serve as *restrictive* principles rather than maximising principles; when they are asserted, they serve to *leave out* subsets. We then (§4) argue that this intuition can be made formally precise using Maddy's notion of *restrictiveness*. However, we also (§5) argue that the roles played by large cardinals in contemporary set theory are left relatively untouched. In particular, we contend that even in the anti-large cardinal frameworks proposed, large cardinal axioms are still fruitful objects of study and can be used to play their usual foundational roles of indexing consistency strength, constructing models, and justifying axioms of definable determinacy. Finally (§6) we make some concluding remarks and identify some open questions.

1 Large cardinals, the iterative conception of set, and maximality

In this section, we provide some required background on large cardinals and the iterative concept of set.[2] We then explain how one might think that the iterative conception legislates in favour of large cardinals on the basis of their status as maximality principles.

1.1 Large cardinals

Given a set theory capable of axiomatising a reasonable fragment of arithmetic (i.e. able to support the coding of the relevant syntactic notions), we start our discussion with the following celebrated theorem:

Theorem 1. [Gödel, 1931] (Second Incompleteness Theorem). No consistent[3] recursive theory $\mathbf{T}$ capable of axiomatising primitive recursive arithmetic can prove its own consistency sentence[4] (often denoted by '$Con(\mathbf{T})$').

Given then some appropriately strong set theory $\mathbf{T}$, we can then obtain a *strictly stronger* theory by adding $Con(\mathbf{T})$ to $\mathbf{T}$. So, if we accept the standard axioms of Zermelo-Fraenkel set theory with Choice (henceforth '$\mathbf{ZFC}$') then, $\mathbf{ZFC}+Con(\mathbf{ZFC})$ is a strictly stronger theory, and $\mathbf{ZFC}+Con(\mathbf{ZFC}+Con(\mathbf{ZFC}))$ is strictly stronger still. More generally:

Definition 2. A theory $\mathbf{T}$ has *greater consistency strength* than $\mathbf{S}$ if we can prove

[2]One might feel that this section covers well-known ground. We include it simply for clarity and because our main point is rather philosophical in nature: The place of large cardinals in the iterative conception requires further sharpening of how sets are formed in the hierarchy. For this reason, we hope that the philosophical claims of the paper will be readable and open to scrutiny by a relatively wide audience, even if some of the technical details are somewhat tricky in places. Time-pressed readers are invited to proceed directly to §1.3.

[3]Strictly speaking, this is Rosser's strengthening, but we suppress the usual discussion of ω-inconsistency for clarity.

[4]The consistency sentence for a theory $\mathbf{T}$ is a sentence in the language of $\mathbf{T}$ that states that there is no code of a proof of $0=1$ (or some other suitable contradiction) in $\mathbf{T}$.

$Con(\mathbf{S})$ from $Con(\mathbf{T})$, but cannot prove $Con(\mathbf{T})$ from $Con(\mathbf{S})$. They are called *equiconsistent* iff we can both prove $Con(\mathbf{T})$ from $Con(\mathbf{S})$ and $Con(\mathbf{S})$ from $Con(\mathbf{T})$.[5]

The interesting fact for current purposes is that in set theory we are not limited to increasing consistency strength solely through adding Gödel-style diagonal sentences. The axiom which asserts the existence of a transitive model of $\mathbf{ZFC}$ is stronger still (such an axiom implies the consistency of theories with transfinite iterations of the consistency sentence for $\mathbf{ZFC}$). As it turns out, by postulating the existence of certain kinds of models, embeddings, and varieties of sets, we discover theories with greater consistency strength. For example:

Definition 3. A cardinal κ is *strongly inaccessible* iff it is uncountable, regular (i.e. there is no function from a smaller cardinal unbounded in κ), and a strong limit cardinal (i.e. if $|x| < \kappa$ then $|\mathcal{P}(x)| < \kappa$).

Such an axiom provides a model for *second-order* $\mathbf{ZFC}_2$ [namely $(V_\kappa, \in, V_{\kappa+1})$]. These cardinals represent the first steps on an enormous hierarchy of logically and combinatorially characterised objects.[6] More generally, we have the following rough idea: A large cardinal axiom is a principle that serves as a natural stepping stone in the indexing of consistency strength.

In the case of inaccessibles, many of the logical properties attaching to the cardinal appear to derive from its brute size. For example, it is because of the fact that such a κ cannot be reached 'from below' by either of the axioms of Replacement or Powerset that $(V_\kappa, \in, V_{\kappa+1})$ satisfies $\mathbf{ZFC}_2$. In addition, this is often the case for other kinds of cardinal and consistency implications. A *Mahlo cardinal*, for example, is a strongly inaccessible cardinal κ beneath which there is a stationary set (i.e. an $S \subseteq \kappa$ such that S intersects every closed and unbounded subset of κ) of inaccessible cardinals. The fact that such a cardinal has higher consistency strength than that of strong inaccessibles (and mild strengthenings thereof) is simply because it contains many models of these axioms below it.

It is not the case, however, that consistency strength is inextricably tied to size. For example, the notion of a *strong*[7] cardinal has lower consistency strength than that of *superstrong*[8] cardinal, but the least strong cardinal is larger than the least superstrong cardinal.[9] The key point is that despite the fact that the least superstrong is not as *big* as the least strong cardinal, one can always *build* a model of a strong cardinal from the existence of a superstrong cardinal (but not vice versa). Thus, despite the fact that a superstrong cardinal can be 'smaller', it still validates the consistency of the existence of a strong cardinal.

Before we move on to our discussion of the iterative conception, we note two phenomena concerning large cardinals that make them especially attractive objects of study:

[5] A subtlety here concerns what base theory we should use to prove these equiconsistency claims. Number theory will do (since consistency statements are number-theoretic facts), but we will keep discussion mostly at the level of a suitable set theory (e.g. $\mathbf{ZFC}$).

[6] Often, combinatorial and logical characterisations go hand in hand, such as in the case of measurable cardinals. However, sometimes it is not clear how to get one characterisation from another. Recently, cardinals often thought of as having only combinatorial characterisations have been found to have embedding characterisations. See [Holy et al., S] for details.

[7] A cardinal κ is *strong* iff for all ordinals λ, there is a non-trivial elementary embedding (to be discussed later) $j : V \longrightarrow \mathfrak{M}$, with critical point κ, and in which $V_\lambda \subseteq \mathfrak{M}$.

[8] A cardinal κ is *superstrong* iff it is the critical point of a non-trivial elementary embedding $j : V \longrightarrow \mathfrak{M}$ such that $V_{j(\kappa)} \subseteq \mathfrak{M}$.

[9] See [Kanamori, 2009], p. 360.

Fact 4. The 'natural' large cardinal principles appear to be linearly ordered by consistency strength.

One can gerrymander principles (via metamathematical coding) that would produce only a partial-order of consistency strengths[10], however it is an empirical fact that the large cardinal axioms that set theorists have naturally come up with and view as interesting *are* linearly ordered.[11] This has resulted in the following:

Fact 5. Large cardinals serve as the the natural indices of consistency strength in mathematics.

In particular, if consistency concerns are raised about a new branch of mathematics, the usual way to assess our confidence in the consistency of the practice is to provide a model for the relevant theory with sets, possibly using large cardinals.[12] For example, worries of consistency were raised during the emergence of category theory, and were assuaged by providing a set-theoretic interpretation, which then freed mathematicians to use the category-theoretic language with security. For instance, Grothendieck postulated the existence of universes (equivalent to the existence of inaccessible cardinals), and Mac Lane is very careful to use universes in his expository textbook for the working mathematician.[13] These later found application in interpreting some of the cohomological notions used in the original Wiles-Taylor proof of Fermat's Last Theorem (see [McLarty, 2010]). Of course now category theory is a well-established discipline in its own right, and quite possibly stands free of set-theoretic foundations. Nonetheless, set theory was useful providing an upper bound for the consistency strength of the emerging mathematical field. More recently, several category-theoretic principles (even some studied in the 1960s) have been calibrated to have substantial large cardinal strength.[14]

This observation concerning the role of large cardinals in contemporary mathematics point to a central desideratum for their use:

> **Interpretative Power.** Large cardinals are required to *maximise interpretative power*: We want our theory of sets to facilitate a unified foundational theory in which all mathematics can be developed.[15]

Maximising interpretative power entails maximising consistency strength; we want a theory that is able to incorporate as much consistent mathematics as is possible whilst preserving a sense of intended interpretation, and hence (assuming the actual consistency of the relevant cardinals) require the consistency strength of our framework theory to be very high.

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[10]See [Koellner, 2011] for discussion.

[11]There are some open questions to be tied up, for example around strongly compact cardinals and around Jónsson cardinals.

[12]See here, for example, Steel:

> "The central role of the theories axiomatized by large cardinal hypotheses argues for adding such hypotheses to our framework. The goal of our framework theory is to maximize interpretative power, to provide a language and theory in which all mathematics, of today, and of the future so far as we can anticipate it today, can be developed." ([Steel, 2014], p. 11)

[13]See [Mac Lane, 1971], Ch.1, §6. Also interesting here is [McLarty, 1992], Ch. 12.

[14]See [Bagaria and Brooke-Taylor, 2013] for details. The consistency strength is really quite high; many category-theoretic statements turn out to be equivalent to Vopěnka's Principle.

[15]This idea is strongly emphasised in [Steel, 2014] and has a strong affinity with Penelope Maddy's principles UNIFY and MAXIMIZE (see [Maddy, 1997] and [Maddy, 1998]). We will discuss the latter in due course.

1.2 The iterative conception of set

It seems then that large cardinals are important foundationally, but do not follow from our usual canonical set theory (**ZFC**). We might then ask the natural question: What reason (aside from their usefulness[16]) do we have for accepting them?

When analysing whether or not we should accept an axiom, it is important to bear in mind the background concept of set against which we measure it. The contemporary conception of set is, for most philosophically-minded mathematicians, the *iterative conception*. There are other conceptions of set[17], however the iterative conception is normally the paradigm within which set-theoretically inclined mathematicians operate (especially those interested in large cardinals) and so is the conception we consider here. The discovery of the set-theoretic paradoxes at the turn of the century necessitated (assuming a revision of logic is not on the table) a conception of set on which not every condition $\phi(x)$ determines a set. The iterative conception incorporates this though the idea that sets are formed in *stages*. At the initial stage of construction we do not have anything, and so form the set containing nothing (i.e. the empty set). At the next stage, we form all possible sets available at previous stages. We continue going in this way, and at a limit stages collect together everything we have formed at previous stages, and continuing this for as long as possible.

The picture is informal, but is often formally construed through the repeated application of the powerset operation and union through the ordinals:

Definition 6. The *Cumulative Hierarchy* (or V) is defined by transfinite recursion over the ordinals as follows:

(i) $V_0 = \emptyset$,

(ii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, for successor ordinal $\alpha + 1$,

(iii) $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$, where λ is a limit ordinal,

(iv) $V = \bigcup_{\gamma \in On} V_\gamma$.

The iterative conception is often seen as theoretically appealing. First, it appears to block the set-theoretic paradoxes: Since the relevant problematic conditions have objects satisfying them unboundedly in the cumulative hierarchy, there is no set of all of them.[18] Secondly, it does so in a way that, one might think, is natural and seemingly well-motivated. Whether or not we would have come up with the iterative conception of set independently of the discovery of the paradoxes (as Boolos comes close to suggesting[19]) is a difficult question, but there is nonetheless a natural 'picture' behind this resolution of the paradoxes, and one that meshes well with our canonical theory of sets.

[16]Some authors (e.g. [Maddy, 2011]) regard the usefulness of an axiom as *key* to its acceptance (say because of the relevant foundational goals of set theory). Since we are focussed on the very specific issue of large cardinals are linked to *maximality*, we set aside this issue here.

[17]For discussions of different conceptions of set, the seminal [Fraenkel et al., 1973] is an important early text. More specifically, [Holmes, 1998] (Ch.8) and [Forster, 1995] provide some remarks about a possible conception for **NFU**, and Incurvati and Murzi (in [Incurvati, 2014], [Incurvati, 2012], and [Incurvati and Murzi, 2017]) discuss various different conceptions of set.

[18]A slight complication here for the Burali-Forti paradox is how we interpret the notion of *ordinal* in set theory. Usually a canonical representative is chosen with the property that such representatives appear unboundedly in a cumulative hierarchy: Common choices here are von Neumann ordinals (very much the canonical option), or Scott-Potter ordinals (see [Potter, 2004]).

[19]See [Boolos, 1971], p. 219.

1.3 Relating large cardinals and the iterative conception

The above serves as an introduction for the uninitiated, but will be familiar to specialists. Given these seemingly natural axioms and the usual iterative conception of set, a natural question is the extent to which there is a relationship between the two.

Note first that what is satisfied by the cumulative hierarchy[20] depends on two main factors:

(1.) What sets get formed at each additional stage.

(2.) How far the stages extend upwards.

The former issue we shall refer to as issues of *width* and the latter as issues of *height*. The relationship between the two determines what sentences are true in the cumulative hierarchy; once we fix what sets are formed at each additional stage and how far the stages go we thereby settle on the reference of our set-theoretic concepts and definitions. Given a principle that mathematics should be as unconstrained as possible, and that mathematical existence is relatively undemanding, the thoughts that there should be *as many* sets formed as possible at each additional stage, and that the stages should go on *as far* as possible are appealing (i.e. the cumulative hierarchy should be 'maximal'). Hao Wang concisely sums up this thought:

> "In a general way, hypotheses which purport to enrich the content of power sets (say that of integers) or to introduce more ordinals conform to the intuitive model. We believe that the collection of all ordinals is very 'long' and each power set (of an infinite set) is very 'thick'. Hence, any axioms to such effects are in accordance with our intuitive concept." ([Wang, 1974], p. 202)

We would thus like principles that axiomatise the idea that the length of the stages is 'long' (i.e. the universe is very high). Discussing height, Incurvati writes (in a survey on maximality principles in set theory):

> "We are told that the cumulative process of construction is indexed by ordinals, but how far does this process go? An initial and frequently given answer is that the process should be iterated as far as possible:
>
> **Height Maximality.** There are as many levels of the hierarchy as possible." ([Incurvati, 2017], p4)

As Incurvati notes, however:

> "However, Height Maximality does not tell us much until the idea of iteration 'as far as possible' is developed to some extent." ([Incurvati, 2017], p.4)

Here is where large cardinals come in. In order to capture height maximality, one might think that we should appeal to large cardinals. After all, don't large cardinals simply assert that 'very large' order-types exist? Incurvati continues:

[20] If there is a single such thing—for simplicity we shall assume that there is despite the subtlety of the question for the philosophy of set theory.

"To answer this question, a number of principles have been invoked. The ones that are probably best known are principles telling us, effectively, that the hierarchy goes at least as far as a certain ordinal. These include the Axiom of Infinity and the standard large cardinal axioms, such as (in order of increasing consistency strength): inaccessible, Mahlo, weakly compact, ω-Erdős, measurable, strong, Woodin, and supercompact." ([Incurvati, 2017], p.4)

Similar remarks are also found elsewhere in the literature, for example in the work of Maddy:

"As with any large cardinal, positing a supercompact can be viewed as a way of assuring that the stages go on and on; for example, below any supercompact cardinal κ there are κ measurable cardinals, and below any measurable cardinal λ, there are λ inaccessible cardinals." ([Maddy, 2011] pp. 125–126)

Since it will form the main target of our paper, we isolate the following assumption:

The Height **M***aximality though the* **E***xistence of* ***L****arge* ***C****ardinals Principle.* (MELC-Principle) One way axiomatise the claim that the sequence of stages is long is to postulate the existence of ordinals with large cardinal properties.

The thought then might be the following: Since large cardinals assert that the stages go as far as a particular large ordinal (i.e. they are good characterisations of height maximality), and since the iterative conception incorporates the idea that the stages should go as far as possible, then we should adopt large cardinals as axioms in virtue of their height-maximising properties. This argument goes back at least as far as Gödel who suggested that:

"...the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers." ([Gödel, 1947], p. 181)

as well Hauser, who refers to large cardinals as "global existence postulates motivated in part by a priori considerations about the inexhaustibility of the universe of all sets" ([Hauser, 2001], p. 257). The suggestion can also be found in set theory textbooks, such as Frank Drake's pleasantly written volume on large cardinals:

"But probably the main reason to study them [measurable cardinals] is the more open-minded interest in the properties which follow from assuming that very large cardinals exist; we want to consider the universe of set theory as being the cumulative type structure, continued through all possible ordinals, so that if it is possible to go so far that we get to a cardinal that is measurable, then we should do so." ([Drake, 1974], p. 186)[21]

[21] [Drake, 1974] is in fact sensitive to the broad shape of some of the considerations we shall mention later. Note that in the above passage he is careful to say that "...if it is *possible* to go so far that we get to a cardinal that is measurable, then we should do so."; it is precisely this possibility we will challenge.

Closely linked to the hypothesis that large cardinals maximise height is the idea that the consistency of a large cardinal principle provides sufficient (or at least good) evidence for its truth. For example Koellner writes (in an endnote to his PhD thesis):

> "Dodd and Jensen showed that this [a certain embedding principle] is equivalent to the statement that there is an inner model with a measurable cardinal. So we have a justification of such a model. Note, however, that this is quite different from a justification of the existence of a measurable cardinal. A further argument would be required to move from the consistency to the existence of a measurable cardinal. I suspect that such an argument can be supplied—large cardinals (in contrast, say, to an ω_2-well-ordering of the reals) seem to be the type of things which require for their existence only their consistency. But I will not pursue this thought here." ([Koellner, 2003], p100)

Similar ideas might be extracted from the work of Cantor. For example, the following is a famous quotation:

> "If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into "one thing", I call it a consistent multiplicity or a "set"." ([Cantor, 1899]: p.114)

Hallett, develops this Cantorian idea concerning 'consistent' multiplicities:

> "Let us grant that the Absolute is not counted in the scale of transfinite numbers. But why should numerability mean just numerability in the transfinite scale? Why does the Absolute not give rise to a further domain of mathematical activity, to super-transfinite numbers, Absolute numbers, or whatever? Why is it as Cantor says an Absolute maximum? One answer that Cantor would give is that to try to mathematize the Absolute would be simply a category mistake: everything mathematizable (or numerable) is already in the realm of the finite and transfinite and the Absolute is simply that which embraces all these. There are no numbers beyond all transfinite numbers waiting to enumerate the Absolute. This is not to say that we may not discover new types of number, perhaps with surprising properties. For example, Hausdorff later discovered numbers ω_α such that $\alpha = \omega_\alpha$, and since then much larger ordinals have been defined or isolated. But if—to take one example—'the smallest uncountable measurable cardinal' is a genuine number (i.e. if this concept is self-consistent or coherent) then it is not a new Absolute number, but a normal increasable transfinite number. We have discovered it within the realm of the transfinite. The same would hold of all numbers we might define or hope to introduce." ([Hallett, 1984], p43)

The idea then, for Hallett's Cantor, is that in the case of cardinals and ordinals, if you can isolate a *coherent* or *consistent* concept, then there is such an ordinal with the relevant property. That is just what it is for the universe to be Absolute; it contains all numbers we could coherently talk about.

Our main target for this paper is the MELC-Principle; we will argue that there are conceptions of maximality on which large cardinals are not height maximising. As we'll see, on these frameworks large cardinals are in fact *restrictive*; when we assert a large cardinal axiom we actually *leave out* subsets. As a corollary (though not the direct focus of this paper), we obtain the result there are significant challenges to arguments from the consistency of large cardinals to their truth.

2 Width Maximality through Absoluteness

The focus of our arguments will be on what holds under the formation of stages under the iterative conception. When we look at the iterative conception, we note that part of the idea is to *form as many sets as possible* at each additional stage, and *then* continue this process for *as long as possible*. Our core point later will be the following: It might be that the formation of certain subsets at each additional stage *precludes* the formation of a certain stage with a large cardinal property attached.

Some principles that have anti-large cardinal properties were already well-known.[22] Aside from an axiom asserting the brute non-existence (or inconsistency) of a large cardinal, good examples here are axioms of constructibility (such as $V = L$) and consequences thereof (such as $\square$-principles). However, it is not clear how any of these conceptions of subset formation should be linked to maximality. The hypothesis that every set is constructible, for example, seems to represent a *minimality* condition on what sets are formed at additional stages (we just take those sets that are definable and hence needed to satisfy **ZFC**) and so does not seem to cohere well with our concept of forming *all possible* sets at each additional stage. As Frank Drake puts it:

> "Note that this is a case where the word axiom [i.e. $V = L$] is used simply to indicate that we shall look at models of this sentence; there seems to be no very good argument to say that it should hold of the cumulative type structure. Most set theorists regard it as a restriction which may prevent one from taking every subset at each stage, and so reject it (this includes Gödel[23], who named it)." ([Drake, 1974], p131)

However, and importantly for our current discussion, there are principles that seek to *maximise* the *width* of the hierarchy (i.e. the sets formed at each additional stage) that have anti-large cardinal properties. This raises serious challenges for the idea that large cardinal principles can be directly inferred from maximality under the iterative conception; it might simply be that forming *as many* sets as possible at successor stages precludes the existence of a stage indexed by an ordinal with the relevant property. As we'll see, we can also have large cardinals *consistent* on these width-maximal pictures.

The core template for width maximality that we shall examine is the following:

> ***W****idth Maximality through* ***A****bsoluteness Principles.* (WA-Principles) Let Γ be a class of sentences in some appropriate logic. If $\phi \in \Gamma$ is true in some appropriate extension of V with the same ordinals (i.e. a *width* extension) then ϕ is already realised in some appropriate structure contained in V.

Clearly the WA-Principles are schematically formulated, and the content a WA-Principle has will be relative to the logical resources, extensions, and internal structures allowed. Some precedents do exist for justification of axioms by this means. *Forcing axioms* are a good example here. To facilitate understanding of the ideas later in this section, we first provide a very coarse and intuitive sketch of the forcing technique.

Forcing, loosely speaking, is a way of adding subsets of sets to certain kinds of model. For some model $\mathfrak{M}$ and atomless partial order $\mathbb{P} \in \mathfrak{M}$, we (via an ingenious

[22] I am grateful to the community on MathOverflow here. Special thanks are due to Monroe Eskew, Mohammad Golshani, Joel Hamkins, and Stefan Miedzianowski. An archive of the discussion is available at [Barton, 2017].

[23] Presumably Drake has in mind Gödel's remarks in [Gödel, 1947] and its rewrite [Gödel, 1964].

definition of ways of naming possible sets and evaluating these names) add a set G that intersects every dense set of $\mathbb{P}$ in $\mathfrak{M}$. The resulting model (often denoted by '$\mathfrak{M}[G]$'), can be thought of as the smallest object one gets when one adds G to $\mathfrak{M}$ and closes under the operations definable in $\mathfrak{M}$.

A *forcing axiom* expresses the claim that the universe has been saturated under forcing of a certain kind. For example we have the following axiom:

Definition 7. Let κ be an infinite cardinal. $\mathsf{MA}(\kappa)$ is the statement that for any forcing poset $\mathbb{P}$ in which all antichains are countable (i.e. $\mathbb{P}$ has the countable chain condition), and any family of dense sets $\mathcal{D}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter G on $\mathbb{P}$ such that if $D \in \mathcal{D}$ is a dense subset of $\mathbb{P}$, then $G \cap D \neq \emptyset$.

Definition 8. *Martin's Axiom* (or just MA) is the statement that for every κ smaller than the cardinality of the continuum, $\mathsf{MA}(\kappa)$ holds.

One can think of Martin's axiom in the following way: The universe has been saturated under forcing for all posets with a certain chain condition and less-than-continuum-sized families of dense sets.

There are several kinds of forcing axiom, each corresponding to different permissions on the kind of forcing poset allowed (the countable chain condition is quite a restrictive requirement). Many of these have interesting consequences for the study of independence, notably many (e.g. the Proper Forcing Axiom) imply that CH is false and that in fact $2^{\aleph_0} = \aleph_2$. If we think of forcing as a way of generating subsets we might think that saturation under forcing represents a good approximation to having all possible subsets at successor stages.

Some set theorists are sympathetic to this idea. For example, Magidor writes:

> "Forcing axioms like Martin's Axiom (MA), the Proper Forcing Axiom (PFA), Martin's Maximum (MM) and other variations were very successful in settling many independent problems. The intuitive motivation for all of them is that the universe of sets is as rich as possible, or at the slogan level: A set [whose] existence is possible and there is no clear obstruction to its existence [exists]...
>
> ...What do we mean by "possible"? I think that a good approximation is "can be forced to [exist]"... I consider forcing axioms as an attempt to try and get a consistent approximation to the above intuitive principle by restricting the properties we talk about and the the forcing extensions we use. " ([Magidor, U], pp. 15–16)

We can add precision to Magidor's intuition by considering different kinds of WA-Principle obtained by considering different classes of sentences and forcing extensions. Indeed some forcing axioms can be characterised this way. For example Martin's Axiom can be characterised as follows:

Definition 9. [Bagaria, 1997] *Absolute*-MA. We say that V satisfies *Absolute*-MA iff whenever $V[G]$ is a generic extension of V by a partial order $\mathbb{P}$ with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $V[G] \models \exists x \phi(x)$ then there is a y in V such that $\phi(y)$.

and we can characterise the Bounded Proper Forcing Axiom (BPFA) as folows:

Definition 10. [Bagaria, 2000] *Absolute*-BPFA. We say that V satisfies *Absolute*-BPFA iff whenever ϕ is a Σ_1 sentence with parameters from $H(\omega_2)$, if ϕ holds a forcing extension $V[G]$ obtained by proper forcing, then ϕ holds in V.

These formulations make it particularly perspicuous the sense in which some forcing axioms can be thought of as maximising the universe under 'possible' sets; if we could force there to be a set of kind ϕ (for a particular kind of ϕ and $\mathbb{P}$), one already exists in V.[24] Some authors (e.g. [Bagaria, 2005]) see this fact as evidence for the claim that such axioms are natural in virtue of their making precise a notion of maximality.

Of course, one immediate objection, especially for those that think there is a definite powerset operation, is that there are no non-trivial extensions of V. This would then result in the vacuity of WA-Principles; since there are no non-trivial extensions, vacuously anything satisfied in an extension is satisfied in V. This worry is assuaged by the fact that we can code satisfaction in forcing extensions of V by various means, for example by using forcing relations (often denoted, for conditions in $p \in \mathbb{P}$, by $p \Vdash_{\mathbb{P}} \phi$) that are definable over V.[25] In this way, one can think of these WA-Principles as stating that there are *actual* existents in V for certain sentences that can be satisfied *ideally* by coded extensions. For example, we might express an equivalent version of Absolute-MA as follows:

Definition 11. (**ZFC**) We say that V satisfies *Absolute*-$\mathsf{MA}^{\Vdash}$ iff whenever $\mathbb{P} \in V$ is a partial order with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula, if there is a $p \in \mathbb{P}$ and $\Vdash_{\mathbb{P}}$, such that $p \Vdash_{\mathbb{P}} \exists x \phi(x)$, then there is a y in V such that $\phi(y)$.

Thus there is no obstruction to formulating WA-Principles as long as the relevant talk of 'extensions' can be coded, even if one thinks that there is just one universe of sets, unique up to isomorphism. These forcing axioms have no anti-large cardinal properties themselves, however as we shall see there are close relatives that do.

3 Three troublesome cases for the MELC-Principle

Thus far, we've provided an account of two kinds of maximality principles. One, the MELC-Principle, purported to maximise the length of the stages by postulating the existence of large cardinals. The others, WA-Principles, seek to maximise the subsets formed at successor stages by holding that such an operation supports a high-degree of absoluteness between V and its (coded) extensions. As we'll see, there are areas in which the two come in to conflict. If we then accept that a rich process of subset formation (as given by WA-Principles) can preclude the existence of large cardinals, but nonetheless ratify their consistency, we have a challenge to the claim that large cardinals maximise height.

3.1 The Axiom of Choice and Reinhardt Cardinals

The first example serves as something of a warm-up, and is more speculative than the second two. It does, however, show the style of reasoning that we shall adopt in §3.2 and §3.3. Our strategy will be to argue that the Axiom of Choice can be recast as a WA-Principle, and that it then implies the falsity of a certain large cardinal principle (the existence of Reinhardt cardinals), whilst leaving the (epistemic) possibility that such axioms are realised in structures that leave out subsets.

We can formulate the Axiom of Choice as follows:

[24] For some discussion of the coding of Absolute-MA (and similar principles) for the philosopher inclined towards a "universist" picture of set-theoretic ontology see [Barton and Friedman, 2017].

[25] See here [Kunen, 2013], §IV.5 for a summary of these techniques.

Axiom 12. *Axiom of Choice* (AC). Let $\mathcal{F}$ be a non-empty family of pairwise disjoint non-empty sets. Then there is a set C that contains exactly one element of every member of $\mathcal{F}$.

While Choice is often regarded as receiving justification from a wide range of sources (especially notable here is its equivalence with diverse natural statements across mathematics) we might think that it follows naturally from the iterative conception. Suppose we have some family $\mathcal{F}$ of pairwise-disjoint non-empty sets first formed in some $V_{\alpha+1}$ (nothing new is formed at limit ordinals, the previous sets are simply collected together). Then we know that every element of $\mathcal{F}$ is first formed at latest at stage V_α, and hence all members of elements of sets are first formed at latest by stage V_α. But then, assuming that *all* subsets are formed at additional stages, a choice set must be formed at latest at stage $V_{\alpha+1}$ (what could possibly prevent it from existing?). Indeed, Kreisel went so far as to say:

> "For the fat (or "full") hierarchy, the axiom of choice is quite evident." ([Kreisel, 1980], p. 192)[26]

[Potter, 2004] (p. 257) explains how to recast the discussion in terms of second-order logic. The details need not detain us, but salient is that through using a *logical* choice function in second-order logic (rather than through coding a set of ordered-pairs), one can derive the second-order choice principle from the second-order separation principle (in conjunction with some other reasonably unobjectionable assumptions). We thus arrive at a position where, on the basis of the iterative conception, we hold that Choice receives a natural motivation from the idea that we form all possible sets at an additional stage, and this motivation can be recast in terms of an argument in second-order logic.[27]

Can we interpret this idea that Choice maximises subsets via a WA-Principle? Well, we do know that it is possible to characterise AC via forcing axioms.[28] We begin with the following definition:

Definition 13. Let κ be a cardinal and $\mathbb{P} = (P, \leq_\mathbb{P})$ be a partial order. $\mathsf{FA}_\kappa(\mathbb{P})$ is the statement that for all families $\mathcal{D} = \{D_\alpha | \alpha < \kappa\}$ of predense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ meeting all these predense sets.

Definition 14. Given a class Γ of partial orders $\mathsf{FA}_\kappa(\Gamma)$ holds iff $\mathsf{FA}_\kappa(\mathbb{P})$ holds for all $\mathbb{P} \in \Gamma$.

Definition 15. Let λ be a cardinal. A partial order $\mathbb{P}$ is $(< \lambda)$-closed iff every decreasing chain $\{p_\alpha | \alpha < \gamma\}$ indexed by some $\gamma < \lambda$ has a lower bound in $\mathbb{P}$.

[26] As [Potter, 2004] notes, similar remarks are to be found in [Ramsey, 1926]. However, by 1926 the iterative conception had not yet been fully isolated, and so it is questionable whether Ramsey's views flowed from a conception that was iterative *as well as* combinatorial, rather than a straight-up combinatorialism.

[27] Similar remarks can be found in [Paseau, 2007] concerning Boolos' views on the Axiom of Choice:

> "There is also an alternative to Boolos's suggestion that the Axiom of Choice should be derived from a stage version of Choice. One could instead see Choice as flowing from a combinatorial understanding of the set-formation process. If one thinks that any arbitrary combination of sets below some given stage constitutes a property, then a generalisation of Spec [i.e. Separation] to cover all possible properties whatsoever—as opposed to those expressible in some formal language, as in Boolos's initial presentation—expresses the intuitive thought that at any given stage all the possible sets available for formation are indeed formed. As it is usually conceived, and as Boolos himself conceives it, the iterative conception includes this combinatorial idea. And combinatorialism straightforwardly implies a Choice axiom." ([Paseau, 2007], pp. 35–36)

[28] This is very nicely explained in [Parente, 2012].

Definition 16. Γ_λ denotes the class of $(< \lambda)$-closed posets.

We can now point out the following:

Theorem 17. (Todorčević) $\mathsf{FA}_\kappa(\Gamma_\kappa)$ is equivalent (modulo **ZF**) to the Axiom of Choice.[29]

What should we take from this? If we accept the earlier argument that AC makes precise the claim that as many sets as possible are formed at successor stages, then there is a very clear sense in which this can be viewed as the existence of certain generics for forcing notions.[30] This supports the idea that forcing is a way of 'generating' new subsets, and perhaps we should view saturation under generics as part of taking 'all possible' sets at each additional stage.

In the context of a WA-Principle, we can make this idea precise using the following WA-Principle:

Definition 18. We say that V satisfies *Absolute*-AC iff for every ordinal κ and every $(< \kappa)$-closed pruned[31] κ-tree[32] T, if T has a cofinal branch in a forcing extension $V[G]$ of V, then T already has a cofinal branch in V.[33]

and note the following:

Proposition 19. Absolute-AC is equivalent (modulo **ZF**) to the usual Axiom of Choice.

Proof. By Theorem 12 of [Lévy, 1964], we know that AC is equivalent to $\forall\kappa\mathsf{DC}_\kappa$, where DC_κ is Dependent Choice for sequences of length κ. We also know that DC_κ is equivalent to the claim that every $(< \kappa)$-closed κ-tree has a cofinal branch.[34] We now claim that Absolute-AC is equivalent to $\forall\kappa\mathsf{DC}_\kappa$. One direction is immediate (since if every pruned $(< \kappa)$-closed κ-tree has a cofinal branch, then there is a forcing extension in which the tree has a branch). The other direction holds because one can just force using a tree T as a forcing partial order to obtain a cofinal branch in some $V[G]$, and hence a cofinal branch in V by Absolute-AC. Thus we have, Absolute-AC iff $\forall\kappa\mathsf{DC}(\kappa)$ iff AC. □

So, we can characterise AC as a WA-Principle. But how could there be a conflict with the MELC-Principle?

We can see this when we consider the definition of cardinals through elementary embeddings. We have already seen mention of measurable cardinals earlier, now the time has come to define them:

Definition 20. A cardinal κ is *measurable* iff it is the critical point of a non-trivial $j : V \longrightarrow M$, for some transitive inner model $\mathfrak{M} = (M, \in)$.[35]

[29] See [Viale, 2016], for discussion. A full proof is available in [Parente, 2012].

[30] In fact, it turns out that a wide variety of statements (including choice principles, Łos-style Theorems, and certain large cardinal axioms) can be unified in the guise of forcing axioms (again, see [Viale, 2016]). While the philosophical ramifications of these facts bear exploring, we lack the space to do so here.

[31] A κ-tree T is *pruned* iff κ is regular and above any node there are κ-many nodes.

[32] A κ-tree is a tree with height κ and such that every level of T has size smaller than κ.

[33] I am grateful to David Asperó for suggesting this idea, as well as subsequent discussion. More generally, I would like to thank David Fernández-Bretón and Asaf Karagila for some discussion of absoluteness and choice principles.

[34] The result that DC_κ is equivalent to the claim that every $(< \kappa)$-closed κ-tree has a cofinal branch is (to the best of my knowledge) folklore, but follows quickly from (a) a generalisation of the usual result that every pruned ω-tree has a cofinal branch is equivalent to DC_ω, and (b) the observation that DC_λ is continuous for singular λ. Thus, letting $cof(\lambda) = \kappa$, one can partition a singular λ-tree into pruned κ-trees and use DC_κ to obtain DC_λ.

[35] See [Drake, 1974] for a relatively friendly introduction to measurable cardinals, and [Kanamori, 2009] and [Jech, 2002] for detailed technical discussion. These cardinals admit of a wide variety of characterisations, many notably first-order in character.

One route to providing stronger definitions of large cardinals is to increase the similarity between V and $\mathfrak{M}$. For example, the following provides a definition of a cardinal far stronger than measurable:

Definition 21. Let λ be an ordinal. A cardinal κ is *λ-supercompact* iff κ is the critical point of a non-trivial elementary embedding $j : V \longrightarrow M$ for some transitive inner model $\mathfrak{M} = (M, \in)$, $j(\kappa) > \lambda$, and ${}^{\lambda}M \subseteq M$.

Here, for sufficiently large λ, specifying that M is closed under λ-sequences substantially strengthens the kind of cardinal defined. (An additional subtlety here is that part of what strengthens the cardinal is that j sends κ above λ, but we set this aside for now.) More generally, many such strengthenings of the notion of measurability make use of this strategy. Carrying this idea to its natural endpoint suggests the following principle:

Definition 22. A cardinal κ is *Reinhardt* iff κ is the critical point of a non-trivial elementary embedding $j : V \longrightarrow V$.

However, we can now state the following:

Theorem 23. [Kunen, 1971] There are no Reinhardt cardinals.

Importantly, Kunen's proof makes essential use of the Axiom of Choice[36], as do more recent proofs[37]. However, it is unknown whether or not the existence of a Reinhardt cardinal is inconsistent in **ZF**. In investigating this question, Koellner, Woodin, and Bagaria in unpublished work[38] have developed strengthenings of these axioms within **ZF**. For example:

Definition 24. A cardinal κ is *Super-Reinhardt* iff for every ordinal λ there is a $j : V \longrightarrow V$ with critical point κ and such that $j(\kappa) > \lambda$.

Interestingly, it turns out that these 'choiceless cardinals' in turn form an elegant hierarchy of consistency strengths[39]. What should our reaction to this situation be? In his PhD thesis, Koellner remarked:

> "There is an entire hierarchy of "choiceless cardinals" and it may be the case that the hierarchy of consistency strength outstrips that which assumes choice. In the end it may turn out to be reasonable to view AC as a limitative axiom on a par with $V = L$." ([Koellner, 2003], p. 100)

Assuming (highly non-trivially) that the existence of a Reinhardt cardinal is in fact consistent with **ZF**, and realised in an inner model of V, one might think that we should drop Choice. After all, then there is a natural theory of sets (**ZF**), one which can be given an iterative story, and under which it is consistent to have a Reinhardt cardinal.

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[36] This is because Kunen uses the theorem in [Erdős and Hajnal, 1966] that for any infinite ordinal λ, there is a function ${}^{\omega}\lambda \longrightarrow \lambda$ such that whenever $A \subseteq \lambda$ and $|A| = |\lambda|$ then $F"({}^{\omega}A) = \lambda$.

[37] For example those that use the Solovay Splitting Lemma that if κ is a regular uncountable cardinal then any stationary subset of κ can be partitioned into κ many disjoint stationary sets (such as the proof due to Woodin presented in [Schindler, 2014]).

[38] See [Koellner, 2014] for a summary of some of these ideas, and [Cutolo, 2018] for some recent work.

[39] There is a question of exactly what the consistency strength of a Reinhardt cardinal is within **ZF**, given that Choice has to be given up. [Woodin, 2011] (p. 456) mentions a result that the theory **ZF**+"There exists a weak Reinhardt cardinal" implies the consistency of **ZFC**+"There exists a proper class of ω-huge cardinals". Further discussion of this issue is available on MathOverflow at [Campion, 2016].

Insofar as we accept the earlier iterative story concerning the justification of the Axiom of Choice in terms of a WA-Principle, we should not be moved by the thought that AC might be limitative in a similar way to $V = L$. Simply put, we would already be confident that the formation of choice sets at each additional stage guarantees that there is no such cardinal. Continuing the hierarchy 'as far as possible' does not go so far as to include choiceless cardinals, since AC is true when we form 'all possible' sets at each additional stage.

Nonetheless, the consistency of a Reinhardt cardinal could still be witnessed. We could perfectly well have a Reinhardt cardinal in countable models of **ZF** or even an inner model of V satisfying **ZF**. It is just that such a model has to miss out some subsets, specifically those that guarantee the existence of the relevant choice functions.[40] Indeed, this has long been appreciated; for some time Jensen was a staunch adherent of $V = L$, yet held that we might have countable transitive models containing large cardinals. Drake is clear about the situation:

> "Perhaps it is worth indicating the sort of reason why the mere fact that a definition makes a cardinal look very large is not sufficient to convince us that such cardinals must exist in the cumulative type structure, if only we continue it far enough. Suppose there is, in some transitive model of **ZFC**, a strongly compact cardinal. Then there must be a countable, transitive model of **ZFC**, M say, in which there is a strongly compact cardinal (according to M); suppose α is an ordinal which is strongly compact in M. Then the various α-additive measures which must exist in M will only be measures in M because a great many possible subsets are missing in M, so that the purported measure does not have to consider them...This sort of consideration highlights the fact that even if we are convinced that strongly compact cardinals are consistent with **ZFC**, we have not answered the question whether they exist in the cumulative type structure." ([Drake, 1974], pp. 315–318)

Aside from countable models, we might have a model containing all ordinals, but leaving out some choice sets. Thus, it is at least epistemically possible that we have a κ that is Reinhardt in some inner model, but no Reinhardt in V. Given that we hold (for the purposes at hand) that Choice should be true, the only sense in which one 'could' continue the hierarchy to include a Reinhardt κ is to leave subsets out when iterating the powerset operation. Thus, in this possibly maximal context the large cardinal axiom "There exists a Reinhardt cardinal" is actually a *restricting* principle; necessitating the omission of subsets. It is in this sense, given a prior justified width maximality operation, that width is prior to height.

3.2 The Inner Model Hypothesis

We will see a similar feature with respect to a variety of principles known as *inner model hypotheses*. Again, we will see that this class of WA-principles provides a conception of maximality and forming as many sets as possible at each additional stage on which large cardinal axioms are width-restricting rather than height maximising.

We begin with the following:

[40] An additional subtlety here is that the consistency of a Reinhardt cardinal may be witnessed by an *outer* model of **ZF**, and not every outer model of **ZF** can be widened to a model of **ZFC**. In that case though, the witnessing model is (from the point of view of V) not a bona fide two-valued set-theoretic structure, but rather a Boolean-valued structure (which may be captured through the use of a forcing relation), and so we omit its consideration here.

Definition 25. [Friedman, 2006] Let ϕ be a parameter-free first-order sentence. The *Inner Model Hypothesis* (or IMH) states that if ϕ is true in an inner model of an outer model of V (all with the same ordinals), then ϕ is already true in an inner model of V.

Issues surrounding coding of the IMH are difficult if one thinks that there is a maximal universe of sets, since there are no restrictions on the kind of width extension that can be taken in finding a model satisfying ϕ (and hence no guarantee that we can definably capture satisfaction in all extensions). One can code the IMH in a strong impredicative class theory[41], however all we need for the results of the present paper is the following:

Definition 26. ($\mathbf{NBG}$) Let $(V, \in, \mathcal{C})$ be a $\mathbf{NBG}$ structure. The *Class-Generic Inner Model Hypothesis* (or CIMH) is the claim that if a (first-order, parameter free) sentence ϕ holds in an inner model of a tame class forcing extension $(V[G], \in, \mathcal{C}[G])$ (where where $V[G]$ consists of the interpretations of set-names in V using G, and $\mathcal{C}[G]$ consists of the interpretations of class-names in $\mathcal{C}$ using G), then ϕ holds in an inner model of V.[42]

Since forcing relations are definable for tame class forcings, we can formulate the above principle as follows:

Definition 27. ($\mathbf{NBG}$) $(V, \in, \mathcal{C})$ satisfies the *Absolute Class-Generic Inner Model Hypothesis* (or ACIMH) iff whenever $\mathbb{P} \subset V$ is a tame class forcing, and ϕ is a parameter-free first-order sentence, then if there is a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "ϕ is true in an inner model" then ϕ is true in an inner model of V.

Thus, the Class-Generic Inner Model Hypothesis can be formalised even the believer in just one universe of sets, using $\mathbf{NBG}$. How might we figure the CIMH into an iterative account? We have shown that it is a WA-Principle stating that any parameter-free first-order sentence consistent with the structure of V is already realised in V.

In fact, the CIMH can be formulated as other kinds of absoluteness principles:

Definition 28. ($\mathbf{NBG}$) A formula is *persistent-*Σ^1_1 iff it is of the following form:

$$(\exists M)M \models \psi$$

where ψ is first-order.

Definition 29. ($\mathbf{NBG}$) *Parameter-free persistent* Σ^1_1*-class-absoluteness* is then the claim that if ϕ is persistent-Σ^1_1 and true in a tame class-generic extension of V, then ϕ is true in V.

Theorem 30. [Friedman, 2006] ($\mathbf{NBG}$) The CIMH is equivalent to parameter-free persistent Σ^1_1-class-absoluteness.

This in turn can be viewed as a generalisation of the following theorem of $\mathbf{ZFC}$:

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[41] See here [Antos et al., S].

[42] In fact much of the strength of the IMH is captured by Lévy-absoluteness for Σ_1 formulas with parameter ω_1 for ω-preserving outer models which are tame, Δ_2-definable class forcing extensions. Thus, for many of the results stemming from the IMH, one does not need the full force of arbitrary outer models; the formula to which absoluteness is to be applied can just be first-order (Σ_1) with parameter ω_1. If satisfaction in *arbitrary* well-founded outer models is desired, $\mathbf{NBG} + \Sigma^1_1$-Comprehension suffices (with an assumption on the existence of isomorphisms) since satisfaction in arbitrary outer models can be coded as long as it is possible to produce a code for $Hyp(V)$—the least admissible set containing V. See here [Antos et al., S].

Theorem 31. (**ZFC**) *Lévy-Shoenfield Absoluteness*. Let ϕ be a Σ_1 sentence. If ϕ is true in an outer model of V, then ϕ is true in V.

We wish to take the following points from the above observations. First, the CIMH can be thought of as a principle that asserts that anything (of a particular kind) that 'could' have been realised by the formation of subsets already has a witness. In this way, it is clearly a WA-Principle. Second, it does so by generalising an idea already present in **ZFC**. In this respect, it resembles a reflection principle for height: A standard principle of absoluteness true in **ZFC** is generalised to a language of higher-order.[43]

Let us then, as before, suppose that we take this motivation for the CIMH to be sound. We immediately have the following result:

Theorem 32. [Friedman, 2006] (**NBG**) If the Class-Generic Inner Model Hypothesis holds, there are no inaccessible cardinals in V.

On our current understanding, this would mean that there could be no (significant) large cardinals in V; a conception of the formation of powersets on which there is a high degree of absoluteness between V and ideal class forcing extensions precludes their existence. Here though, an interesting contrast is highlighted with the example of choiceless cardinals. There we were only able to conjecture that it might be possible to leave out subsets to obtain large cardinals. In the current context, however, the existence of large cardinals in inner models is positively *implied*:

Theorem 33. [Friedman et al., 2008] (**NBG**) The Class-Generic Inner Model Hypothesis implies that there is an inner model with measurable cardinals of arbitrarily large Mitchell order.[44]

Thus, while the Inner Model Hypothesis does not permit the existence of large cardinals in V, it *does* vindicate their existence in inner models and thus their use in consistency proofs. Large cardinals, whilst not *true*, are acceptable for determining what combinations of sets are possible in satisfying particular formal theories, even if they are strictly incompatible with the *full* powerset operation.[45] However, on the current conception of maximality they act as *restrictive* principles; whilst they are witnessed as consistent we must omit subsets in order for them to hold.

[43]See [Barton et al., 2017] for an examination of other width reflection principles, and some explanations of the differences between height and width reflection.

[44]The Mitchell ordering is a way of ordering the normal measures on a measurable cardinal. Roughly, it corresponds to the strength of the measure, where a measure U_1 is below another U_2 in the Mitchell order if U_1 belongs to the ultrapower obtained through U_2. See [Jech, 2002] Ch. 19.

[45]Indeed, a worry we might have about the Inner Model Hypothesis is whether or not it is consistent. This is somewhat assuaged by the following:

Theorem 34. [Friedman et al., 2008] Assuming the consistency of the existence of a Woodin cardinal with an inaccessible above, the Inner Model Hypothesis is consistent.

Thus, we have a rough guide as to the consistency strength of the Inner Model Hypothesis (somewhere between many measurables and a Woodin with an inaccessible above). Should the believer in the Inner Model Hypothesis be (significantly) perturbed by the non-existence of Woodin cardinals or inaccessibles in V in getting this consistency proof? It is at least plausible that they should not; they hold that the subset operation supports the Inner Model Hypothesis, and thus supports many inner models with large cardinals. The hypothesis of the consistency of an inner model of a Woodin cardinal with an inaccessible above is thus substantially *less* worrisome than it would be otherwise—we already have some large cardinals in inner models, so why not have an inner model with a Woodin cardinal and an inaccessible above?

3.3 The ultimate set-forcing axiom

We have seen thus far that there are conceptions of maximality on which large cardinals are restricting rather than maximising principles. We will now consider a very extreme version of maximality which calls into question large-cardinal-like axioms of **ZFC**.

Earlier, we talked briefly about how Martin's Axiom could be viewed as a kind of WA-Principle expressing saturation under subsets. However, as will be well known to specialists, there are usually some limitations as to how far one can go. For instance, consider the following theorems:

Theorem 35. Letting $\mathfrak{c}$ denote the cardinality of the continuum, $\mathsf{MA}(\mathfrak{c})$ is inconsistent with **ZFC**.[46]

Theorem 36. In **ZFC** there is a non-countable-chain-condition $\mathbb{P}$ such that for a $(\leq \aleph_1)$-sized family of dense subsets $\mathcal{D}$ of $\mathbb{P}$, there is no filter G on $\mathbb{P}$ intersecting every member of $\mathcal{D}$ (i.e. $\mathsf{MA}_{\mathbb{P}}(\aleph_1)$ is false).[47]

It seems then that there are some limitations on what generics one can have. Given **ZFC**, we cannot just assert the existence of generic sets willy-nilly. However, the above two proofs depend on notions of uncountability; the first on the existence of $\mathfrak{c}$, and the second on the existence of $\aleph_1$.

Here then is a controversial suggestion: We might regard axioms asserting the existence of uncountable sets (e.g. the Powerset Axiom, or the claim that ω_1 exists) as certain kinds of large cardinal axiom, whilst using *forcing* (along with some *definable* powerset operation) as our way of generating all possible subsets.

These claims are certainly plausible when we take the theory of **ZFC**-Powerset. Since there are some subtleties in formulating this theory (see here [Gitman et al., 2011]), we make the following:

Definition 37. $\mathbf{ZFC}^-$ is the theory comprising of **ZFC** with the Replacement Scheme and usual Axiom of Choice removed, plus:

(i) Well-Ordering Principle: Every set can be well-ordered.

(ii) Collection Scheme: $\forall a \forall x \in a \exists y(\phi(x, y) \rightarrow (\exists z \forall x \in a \exists y \in z \phi(x, y)))$

(iii) Axiom Scheme of Separation: $\forall x \exists y \forall z((\phi(z) \wedge z \in x) \leftrightarrow z \in y)$

Definition 38. $\mathbf{NBG}^-$ is **NBG** with the Powerset axiom removed, and second-order versions of Collection and Separation substituted instead of Replacement.

We can then observe the following. For powerset, we have (trivially) that both $Con(\mathbf{ZFC}^-)$ is strictly weaker than $Con(\mathbf{ZFC}^- +$ Powerset), since the latter is just **ZFC**. In fact, the assertions that "ω_1 exists", "ω_2 exists" etc. have ever increasing consistency strength over the theory $\mathbf{ZFC}^-$. Thus, the existence of uncountable sets over the theory $\mathbf{ZFC}^-$ is something like the behaviour of small large cardinals over **ZFC**. Moreover, iterations of Powerset and uncountable sets behave something like a large cardinal axioms with respect to determinacy axioms; Borel determinacy requires ω_1-many iterations of the Powerset Axiom,[48] in a similar way to other determinacy axioms reversing to inner models with large cardinals (we shall see discussion of this fact regarding determinacy axioms later).[49]

[46] See [Kunen, 2013], p. 175, Lemma III.3.13.

[47] See [Kunen, 2013], pp. 175–176, Lemma III.3.15.

[48] See [Friedman, 1971].

[49] See [Koellner, 2014] for discussion of the links between large cardinals and determinacy.

With this in mind, we might view the limitative theorems concerning forcing axioms as indicative of a fundamental tension between forming every possible set given a *particular point* in the set-theoretic construction, and the formation of *all* subsets of an infinite set at a successor stage. Rather, we might think, in order to form *all possible* subsets in the hierarchy, they have to be formed in a piecemeal process; we get new subsets of certain sets appearing unboundedly in V. We can motivate such a position by generalising an idea of Cohen's:

> "A point of view which the author feels may eventually come to be accepted is that CH [the continuum hypothesis] is obviously false...$\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C. Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$ etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently." ([Cohen, 1966], p. 151)

Cohen's thought is based on the following idea: Given the immense richness of the powerset operation, and the flexibility afforded by the forcing technique, we can make the continuum have almost any value (of course though it can't have certain large cardinal properties or contradict König's Theorem). So perhaps we should say that it resists having a *specifiable* $\aleph$-number, instead being outside those we can define.[50]

But if the continuum is really generated by such a principle, why insist (aside from a prior adherence to the Powerset Axiom) that $\mathfrak{c}$ has an aleph value *at all*? Scott (in a forward to Bell's textbook on Boolean-valued models[51]) expresses the following thought:

> "I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible to have strong axioms, which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed. But really pleasant axioms have not been produced by me or anyone else, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models." ([Scott, 1977], p. xv)

The idea then is that perhaps that since we can force the continuum to be larger than any particular ordinal, maybe we should accept that it is, in fact, a proper class of V. Hallett, after appreciatively quoting the above two passages, sums the point up nicely:

[50] One interesting axiom that might capture this thought is the *Strong Inner Model Hypothesis*. Since our focus lies elsewhere for the moment, we do not consider it here, but see [Friedman, 2006] for discussion.

[51] See [Bell, 2011] for the third edition.

"Thus, the continuum evades all our attempts to characterize it by size (Cohen), so maybe we should start with this transcendence as a datum (Scott)." ([Hallett, 1984],

Building on Scott, our "new point of view" will be to regard the universe as generated not through the powerset axiom, but through saturation under *forcing* (combined with a definable power set operation).[52] We consider the following axiomatisation:

Definition 39. The theory of *Forcing Saturated Set Theory* or $\mathbf{FSST}$ comprises the following axioms:

1. All axioms of $\mathbf{ZFC}$ $-$ Powerset.
2. *Definable Powerset Axiom.* $(\forall x)(\exists y)y = Def(x)$ (where $Def(x)$ is the definable powerset of x).[53]
3. *Forcing Saturation Axiom.* (FSA) If $\mathbb{P}$ is a forcing poset, and $\mathcal{D}$ is a family of dense sets, then there is a filter $G \subseteq \mathbb{P}$ intersecting every member of $\mathcal{D}$.

Thus, under $\mathbf{FSST}$, we view the 'richness' of available subsets as given by saturation under forcing for any set-sized family of dense sets.

Below, we explain how one might think of $\mathbf{FSST}$ as arising from an iterative process. For now, we pause briefly to note some of the theory's properties.

Theorem 40. $\mathbf{FSST}$ is equivalent to the theory $\mathbf{ZFC}-$Powerset$+$"Definable powersets exist"+"Every set is countable".[54]

Proof. (1.) $\mathbf{FSST} \Rightarrow \mathbf{ZFC}-$Powerset$+$"Definable powersets exist"+"Every set is countable".

The only thing to show for this direction is to show that $\mathbf{FSST}$ implies that every set is countable. To see this, let α be the order-type of a well-ordering of an arbitrary set x (α is our putative 'uncountable' cardinal). Then, the poset $Col(\alpha, \omega)$ is obtainable by taking definable powersets. Letting $\mathcal{D}$ be an α-sized family of dense sets on $Col(\alpha, \omega)$ (again, obtained by the Definable Powerset Axiom) and using the Forcing Saturation Axiom, there is a generic G for this family, and so there is a collapsing function from α to ω.

(2.) $\mathbf{ZFC}-$Powerset$+$"Definable powersets exist"+"Every set is countable" $\Rightarrow \mathbf{FSST}$.

Again, we just have to show that we can obtain the Forcing Saturation Axiom from the axiom that every set is countable. So, let $\mathbb{P}$ be a forcing poset and $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$. Since every set is countable, we can enumerate $\mathcal{D}$ in order-type ω. So, without loss of generality, $\mathcal{D} = \langle D_n | n \in \omega \rangle$. Since every set is countable, $\mathbb{P}$ can also be enumerated in order-type ω, let 'f' denote the relevant enumerating function. We can then define via recursion (and using the parameter f) the following function π from $\mathcal{D}$ to $\mathbb{P}$:

$$\pi(D_0) = \text{"The } f\text{-least } p \in D_0\text{"}$$

[52] A salient alternative approach to ours, one which *expands* the notion of *continuum* to an 'absolute' continuum, uses Conway's notion of 'surreal number'. An explanation of this idea is available in [Ehrlich, 2012].

[53] This is in fact redundant, since for any set x, one can construct $L(x)$ in the theory $\mathbf{ZFC}-$Powerset. We include it simply to emphasise the iterative picture.

[54] I thank Sy-David Friedman for pointing out this fact to me and for discussion of the proof.

$$\pi(D_{n+1}) = \text{“The } f\text{-least } p \in D_{n+1} \text{ such that } p \leq_{\mathbb{P}} \pi(D_n)\text{”}$$

Effectively π successively picks elements of each member of $\mathcal{D}$, ensuring that we always go below our previous pick in the forcing order (this is allowed because each $D \in \mathcal{D}$ is dense in $\mathbb{P}$, and so such a p always exists). By Replacement, $ran(\pi)$ exists, and the object obtained is a generic for $\mathbb{P}$ and $\mathcal{D}$, and so the Forcing Saturation Axiom holds. □

By the above theorem, we have the immediate:

Corollary 41. $\mathbf{FSST}$ is consistent relative to the theory $\mathbf{ZFC}^-$+“ω_1 exists”.

Proof. This is a quick consequence of the fact that $\mathbf{FSST}$ is equivalent to $\mathbf{ZFC}^-$+“Every set is countable”, and the latter has a model in $H(\omega_1)$ (i.e. the heriditarily countable sets). □

Of course, as things stand, we haven't yet showed how to get $\mathbf{FSST}$ from a WA-Principle. However, we can with the following:

Definition 42. ($\mathbf{ZFC}^-$) We say that V, a model of $\mathbf{ZFC}^-$, satisfies the *Axiom of Set-Generic Absoluteness* (ASGA) iff whenever $\phi(\vec{a})$ is a sentence in the language of set theory *in the parameters* $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects *every* dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models \mathbf{ZFC}^-$, then $\phi(\vec{a})$ holds in V.

Clearly ASGA is a WA-Principle, stating that sentences holding in set forcing extensions are true in V. It also implies that every set is countable, since the cardinality of any set is collapsed to the countable in some extension $V[G]$. However it also goes substantially further than $\mathbf{FSST}$, since we can force the existence of a non-constructible set and so $\mathbf{ZFC}^- +$ ASGA implies that $V \neq L$, whereas $\mathbf{FSST}$ can be satisfied in models satisfying $V = L$ (e.g. the $H(\omega_1)$ of any model of $\mathbf{ZFC} + V = L$). The question then arises as to whether or not ASGA is consistent, which we can answer in the affirmative:

Proposition 43. $\mathbf{ZFC}^- +$ ASGA is consistent relative to $\mathbf{ZFC}$.

Proof. We begin with a model $\mathfrak{M}$ of $\mathbf{ZFC}$. Next, force using an $\aleph_1$-product of Cohen forcings with finite support (call this forcing $\mathbb{P}$), to form an extension $\mathfrak{M}[G]$.

We claim that $H(\omega_1)^{\mathfrak{M}[G]}$ satisfies $\mathbf{ZFC}^- +$ ASGA. The fact that $\mathbf{ZFC}^-$ holds is immediate, since the $H(\omega_1)$ of any model of $\mathbf{ZFC}$ satisfies $\mathbf{FSST}$. It just remains to argue that $H(\omega_1)^{\mathfrak{M}[G]}$ satisfies ASGA. To see this, we begin by noting that any finite sequence of parameters $\vec{a}$ from $H(\omega_1)^{\mathfrak{M}[G]}$ appears at some stage of the iteration. In other words, if we let G_α be the first α-many Cohen reals added by G, then $\vec{a}$ appears in $V[G_\alpha]$.

Since $\vec{a}$ is hereditarily countable, it can be coded by some real r. Moreover, r must belong to $V[G_\alpha]$ for some countable α. This is because $\mathbb{P}$ has the countable chain condition, which in turn implies that any real added by G has a countable $\mathbb{P}$-name, and hence, letting $\mathbb{P}_\alpha$ be the finite support α-length product of Cohen forcing, r has a $\mathbb{P}_\alpha$-name. In other words, any real r added by G is already added for some G_α, for countable α. Letting $G_{\alpha\rightsquigarrow}$ be the Cohen reals added after G_α by $\mathbb{P}$, we can then view $H(\omega_1)^{\mathfrak{M}[G]}$ as $H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, where $G_{\alpha\rightsquigarrow}$ is $H(\omega_1)^{\mathfrak{M}}[G_\alpha]$-generic for the ω_1-many Cohen forcings after the α^{th} stage of the iteration given by $\mathbb{P}$.

Now suppose that there is a countable forcing $\mathbb{Q}$ in $H(\omega_1)^{\mathfrak{M}[G]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, and generic $G_{\mathbb{Q}}$ such that $H(\omega_1)^{\mathfrak{M}[G]}[G_{\mathbb{Q}}] \models \phi(\vec{a})$ where $\vec{a} \in H(\omega_1)^{\mathfrak{M}[G]}$. To show that the ASGA is satisfied by $H(\omega_1)^{\mathfrak{M}[G]}$, we just have to show that $H(\omega_1)^{\mathfrak{M}[G]} \models \phi(\vec{a})$. Since $G_{\mathbb{Q}}$ is generic over $H(\omega_1)^{\mathfrak{M}[G]}$ for a countable forcing (i.e. $\mathbb{Q}$), we can assume without loss of generality that $G_{\mathbb{Q}}$ is generic for Cohen forcing, since Cohen forcing is the only countable forcing up to forcing-equivalence. Thus, since $H(\omega_1)^{\mathfrak{M}[G]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, we know that $H(\omega_1)^{\mathfrak{M}[G]}[G_{\mathbb{Q}}] = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}][G_{\mathbb{Q}}]$, and hence that $\phi(\vec{a})$ becomes true after forcing with the finite support product over $H(\omega_1)^{\mathfrak{M}[G_\alpha]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha]$, adding $G_{\alpha\rightsquigarrow}$ and $G_{\mathbb{Q}}$, i.e. adding $(\omega_1 + 1)$-many Cohen reals (which is just ω_1-many Cohen reals). It follows (using the homogeneity of Cohen forcing) that $H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}] = H(\omega_1)^{\mathfrak{M}}[G] \models \phi(\vec{a})$, as required.[55] □

We thus have a consistent WA-Principle that proves the FSA and hence that every set is countable. However there is a substantial challenge here; since we no longer have the Powerset Axiom, we no longer have the V-hierarchy. How might we understand theories containing the FSA or the ASGA as *iterative* theories?

What we want here is some recursive process of forming subsets along the ordinals, such that the resulting structure models **FSST**. One suggestion that is instructive, but ultimately flawed, is to take definable power sets and saturate under all generics at each successor stage:

Definition 44. *The Naive Forcing Saturated Hierarchy* is defined as follows (within **FSST**):

(i) $N_0 = \emptyset$

(ii) $N_{\alpha+1} = Def(N_\alpha) \cup \{G | \exists \mathbb{P} \in N_\alpha \exists \mathcal{D} \in N_\alpha$ "$\mathbb{P}$ is a forcing poset" $\wedge$ "$\mathcal{D}$ is a family of dense sets of $\mathbb{P}$" $\wedge$ "G intersects every member of $\mathcal{D}$"$\}$

(iii) $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$

(iv) $N = \bigcup_{\alpha \in On} F_\alpha$.

Such a hierarchy looks like it would satisfy **FSST** by design (since we have thrown in all generics at successor stages). Unfortunately the idea does not work. This is because once the Cohen poset has been formed, one immediately puts in all reals which are Cohen-generic for arithmetically-definable families of dense sets. But then we would immediately get uncountably-many reals at the following stage, and so the hierarchy breaks down.[56]

We thus need a more subtle perspective, and the N-hierarchy helps show what needs to be changed. Its failure highlighted that we couldn't just throw in *all* generics at successor stages, rather they need to be fed in slowly and unboundedly. We will therefore consider a well-order R on the universe, with countable initial segments. Let $\bar{R}$ be a predicate for this well-order, and define the following theory:

Definition 45. $\mathbf{FSST}^{\bar{R}}$ + ASGA comprises the following axioms:

(i) All axioms of $\mathbf{NBG}^-$ (in the expanded language with $\bar{R}$).

(ii) The ASGA.

(iii) The axiom that $\bar{R}$ is a class well-order of V with countable (i.e. set-sized) initial segments.

[55]The proof can actually be conducted over $\mathbf{ZFC}^-$+"ω_1 exists", but we conduct it in **ZFC** for simplicity. Many thanks to Sy-David Friedman for suggesting and discussing this proof.

[56]One could have a hierarchy in which stages need not be set-sized. We leave consideration of such class-like hierarchies as an open problem. I thank Joel-David Hamkins for pointing out this idea, and for subsequent discussion of possibilities in this direction.

We can then modify the definition of the hierarchy as follows:

Definition 46. *The Forcing Saturated Hierarchy* is defined as follows (within $\mathbf{FSST}^{\bar{R}}$):

(i) $F_0 = \emptyset$

(ii) $F_{\alpha+1} = Def(F_\alpha) \cup \{G | \exists \mathbb{P} \in F_\alpha \exists \mathcal{D} \in F_\alpha$"$\mathbb{P}$ is a forcing poset" $\wedge$ "$\mathcal{D}$ is a family of dense sets for $\mathbb{P}$" $\wedge$ "G intersects every member of $\mathcal{D}$" $\wedge$ "G is the R-least generic for $\mathbb{P}$ and $\mathcal{D}$"$\}$

(iii) $F_\lambda = \bigcup_{\beta<\lambda} F_\beta$

(iv) $F = \bigcup_{\alpha \in On} F_\alpha$.

On this perspective, we think of the hierarchy as formed by taking definable power sets, and adding at each stage a single generic for each pair of forcing poset and family of dense sets. Thus under this perspective, a set is 'possible' if it is obtainable by definable power set or the 'next' (codified in the sense of R) generic for some $\mathbb{P}$ and $\mathcal{D}$.

The situation is not quite as clean as with **ZFC** and the V_α, since there is no obvious theorem that *every* set of a model of $\mathbf{FSST}^{\bar{R}}$ must belong to the F-hierarchy. For example, if $0^\sharp$ exists, it can very well be a member of $H(\omega_1)$, but it is unclear whether or not it would be part of the F-hierarchy (it might depend, for example, on whether or not it is coded in by R). Despite this limitation, the F-hierarchy shows how we can have a perspective on which every set is countable and we have a meaningful hierarchy and notion of iterative set formation.

The existence of an iterative story not only gives us confidence that **FSST** is cogent, but also shows how we might have models of **ZFC** by missing out subsets. One way is somewhat obvious; letting Φ be large cardinal axiom, we could just bluntly assert the existence of a countable transitive model of $\mathbf{ZFC}+\Phi$. Clearly, such a model will have to miss out subsets in order to satisfy the large cardinal axiom in question. However, a less brutal approach is also possible. Since $\mathbf{ZFC}^-$ provides us with the resources to construct ultrapowers, we can construct inner models using mice. For example, one can define $0^\sharp$ equivalently as a *countable* structure $(L_\alpha, \in, U)$, where U is an ultrafilter with certain properties that allow us to construct (by iterating U) a non-trivial $j : L \to L$.[57] Perhaps then, we might adopt the theory **FSST**+"$0^\sharp$ exists". If we did, we obtain the following result:

Proposition 47. **FSST**+ "$0^\sharp$ exists" implies that there is an inner model (i.e. a transitive model containing all the ordinals) for **ZFC** (and indeed large cardinals).

Proof. Since we have $0^\sharp$ we can construct the usual ultrapower embedding by iterating U. We also know that ω_1 (i.e. On in this context) is inaccessible in L in the presence of $0^\sharp$. Thus $L \models$ **ZFC** in a model of **FSST**+"$0^\sharp$ exists". Moreover, since $\omega_1 = On$ is in fact Mahlo, in L, L will also satisfy the existence of many large cardinals, as well as **ZFC**.[58] □

[57] See [Schimmerling, 2001] for a pleasant explanation of mice, and [Schindler, 2014], §10.2 for details of this characterisation of $0^\sharp$.

[58] This observation shows that there is a fundamental difference between the kinds of large cardinal axioms that postulate that there are ordinals with genuine largeness properties (e.g. "there exists a measurable cardinal") versus those that merely have large cardinal strength (e.g. "$0^\sharp$ exists"); the latter can be true even in the total absence of large cardinals.

Thus, by viewing axioms asserting the existence of uncountable sets as species of *large cardinal* axioms, we have another theory where a WA-Principle, capturing maximality in the formation of subsets in the iteative process, is incompatible with the MELC-Principle. Nonetheless, **ZFC** with large cardinals added can be consistent on these pictures, but the action of asserting these axioms is to restrict the subsets we are allowed to consider. Regarding a similar state of affairs, Meadows writes:

> "Observing this situation and given our claim there are not any really uncountable infinities, we might imagine ourselves as, so to speak, navigating an endless collection of these countable models in something like the generic multiverse we have described. While the illusion of multiple infinite cardinalities is witnessed inside each of the universes, we are free to move between them...I would like to make the provocative suggestion that forcing is a kind of natural revenge or dual to Cantor's theorem: where Cantor gives us the transfinite, forcing tears it down." ([Meadows, 2015], p205–206)

In this way, $\mathbf{FSST}^{\bar{R}}$ codifies Meadow's intuition[59], and the picture we have described represents a peculiar fusion between so called 'actualist' and 'potentialist' frameworks. The universe of $\mathbf{FSST}^{\bar{R}}$ exists absolutely and tells us what sets exist. The **ZFC**-worlds however, are all ultimately countable transitive models or inner models, and can be extended in many and varied ways. Again, importantly, we have a picture on which the existence of certain cardinals is incompatible with a notion of taking 'all sets possible' at each additional stage, and the 'large cardinal' axioms "ω_1 exists", "ω_2 exists", Powerset, and the usual large cardinals only serve to restrict the subsets we consider rather than maximise, despite the fact that they can perfectly well be consistent. A theorist who holds that **FSST** is the right theory for capturing the iterative process of subset formation should not be moved by the consistency of the Powerset Axiom (or any other large cardinal axiom) to its truth; ironically, for the friend of **FSST**, you can only have the Powerset Axiom by *missing out* subsets.[60]

4 Restrictiveness and maximality

Thus far we've argued that there are natural interpretations of maximality (WA-Principles) based on iterative ideas that have anti-large cardinal features whilst allowing for the consistency of large cardinal axioms. As it stands, one might try and raise the charge that we are just intuition trading here; the friend of the MELC-Principle has their conception of maximality, and the friend of WA-Principles has their conception. The MELC-Principle theorist looks at the WA-Principle theorist and thinks that they are living in some small structure containing only countable sets (at least for the CIMH and **FSST**), the WA-Principle theorist looks at the MELC-Principle theorist and thinks that they also leave out subsets. We have already argued that iterative considerations might lead us to prioritise WA-Principles over the MELC-Principle, however aside from these general considerations, is there anything that points in favour of either the MELC-Principle or WA-Principles?

[59] At least insofar as sets are concerned. We actually have two infinite 'sizes'; countably infinite, and proper-class-sized (of which the continuum is one). Only the former, however, corresponds to sizes of *sets*.

[60] One interesting philosophical and technical question is how to handle the continuum which becomes a proper class in **FSST** and its relatives. Clearly some sort of second-order class theory is required, but we leave it open what form this might take.

There is at least one technically precise sense in which we can say that the MELC-Principle perspective is restrictive, at least for some of the axioms we've considered. For this we will examine [Maddy, 1998]'s notion of theories *maximising* over one another and (some thus being *restrictive*). Her account roughly takes the idea that one set theory $\mathbf{T}_1$ maximises over another $\mathbf{T}_2$ (and hence shows it to be restrictive) when one can use $\mathbf{T}_1$ to provably find an interpretation of $\mathbf{T}_2$ in an appropriately 'nice' context, and not vice versa, and the two theories are jointly inconsistent. More precisely, Maddy begins with the following definition.

Definition 48. [Maddy, 1998] A theory $\mathbf{T}$ *shows* ϕ *is an inner model* iff there is a fomula ϕ in one free variable such that:

(i) For all σ in $\mathbf{ZFC}$, $\mathbf{T} \vdash \sigma^\phi$ (i.e. σ holds relative to the ϕ-satisfiers).

(ii) $\mathbf{T} \vdash \forall\alpha\phi(\alpha)$ or $\mathbf{T} \vdash \exists\kappa((\text{“}\kappa \text{ is inaccessible”}) \wedge \forall\alpha(\alpha < \kappa \rightarrow \phi(\alpha)))$ (i.e. the ϕ-satisfiers either contain all ordinals, or all ordinals up to some inaccessible), and

(iii) $\mathbf{T} \vdash \forall x \forall y((x \in y \wedge \phi(y)) \rightarrow \phi(x))$ (i.e. ϕ defines a transitive interpretation).

This definition serves to specify the interpretations we are interested in; proper class inner models and truncations thereof at inaccessibles. She then defines:

Definition 49. [Maddy, 1998] ϕ is a *fair interpretation of* $\mathbf{T}_1$ *in* $\mathbf{T}_2$ iff:

(i) $\mathbf{T}_2$ shows ϕ is an inner model, and

(ii) For all σ in $\mathbf{T}_1$, $\mathbf{T}_2 \vdash \sigma^\phi$.

i.e. a fair interpretation of one theory $\mathbf{T}_1$ in another $\mathbf{T}_2$ is provided by finding some ϕ defining an inner model (or truncation thereof) in $\mathbf{T}_2$ that satisfies $\mathbf{T}_1$. Maddy then goes on to define what it means for a theory to maximise over another. First, she thinks that there should be new isomorphism types outside the interpretation, which, in the presence of Foundation, amounts to the existence of sets not satisfying ϕ:

Definition 50. [Maddy, 1998] $\mathbf{T}_2$ *maximizes* over $\mathbf{T}_1$ iff there is a ϕ such that:

(i) ϕ is a fair interpretation of $\mathbf{T}_1$ in $\mathbf{T}_2$, and

(ii) $\mathbf{T}_2 \vdash \exists x \neg\phi(x)$.

With this idea of maximisation in play, we next need to set up some additional definitions to make sure that weak but unrestrictive theories, whilst not maximising, do not count as restrictive. This is dealt with by the following definitions.

Definition 51. [Maddy, 1998] $\mathbf{T}_2$ *properly maximizes* over $\mathbf{T}_1$ iff $\mathbf{T}_2$ maximizes over $\mathbf{T}_1$ but not vice versa.

Definition 52. [Maddy, 1998] $\mathbf{T}_2$ *inconsistently maximizes* over $\mathbf{T}_1$ iff $\mathbf{T}_2$ properly maximises over $\mathbf{T}_1$ and $\mathbf{T}_2$ is inconsistent with $\mathbf{T}_1$.

Definition 53. [Maddy, 1998] $\mathbf{T}_2$ *strongly maximizes* over $\mathbf{T}_1$ iff $\mathbf{T}_2$ inconsistently maximizes over $\mathbf{T}_1$, and there is no consistent $\mathbf{T}_3$ extending $\mathbf{T}_1$ that properly maximizes over $\mathbf{T}_2$.

Thus we have a picture on which one theory $\mathbf{T}_2$ (strongly) maximises over $\mathbf{T}_1$ when we can prove in $\mathbf{T}_2$ that a certain formula ϕ defines a proper inner model (or truncation thereof), satisfying $\mathbf{T}_1$, and such that we cannot extend $\mathbf{T}_1$ to a theory capable of finding such an interpretation for $\mathbf{T}_2$. If there is a theory $\mathbf{T}_2$ strongly maximising over $\mathbf{T}_1$, then we say that $\mathbf{T}_1$ is *restrictive*. A natural example here is contrasting the theories $\mathbf{ZFC} + V = L$ and $\mathbf{ZFC}$+"There exists a measurable cardinal". The latter strongly maximises over the former, since we can always build L to find a model of $\mathbf{ZFC} + V = L$, but there are no fair interpretations with measurable cardinals under $V = L$ (though they can exist in other kinds of model, e.g. countable).

Maddy's definitions are not without their problems (notably some false negatives and positives), a fact which Maddy herself is admirably transparent about.[61] Subsequent developments of the notion have been considered by Löwe and Incurvati.[62] Our point here is not that Maddy's definitions provide *the* definitive word on restrictiveness, but rather that they provide an interesting perspective on which the rough ideas sketched earlier (about iterativity and leaving out subsets) could be made precise, if one so desired.

Indeed we get some interesting conclusions if we consider the WA-Principles advanced earlier in light of Maddy's definitions. Since the technical situation with Reinhardt Cardinals and AC is still relatively murky, we will set it to one side. However, both the CIMH and FSST can have maximising properties.

First, the CIMH. The CIMH is formulated in $\mathbf{NBG}$, and since Maddy's formulation is intended to apply only to first-order[63] set theories, we require some modification. It is, nonetheless, possible to prove the following:

Proposition 54. Let $\mathsf{ZFC}^{\mathsf{CIMH}}$ be the first-order part of $\mathbf{NBG} + \mathsf{CIMH}$. Then $\mathbf{ZFC}^{\mathsf{CIMH}}$ strongly maximises over $\mathbf{ZFC}$+"There exist α-many measurables" for every α.

Proof. We first need to show that $\mathbf{ZFC}^{\mathsf{CIMH}}$ shows that some ϕ is an inner model with α-many measurables, for any desired α (let α be fixed from now on). Theorem 2 of [Friedman et al., 2008] establishes that $\mathbf{NBG} + \mathsf{CIMH}$ proves that there is a definable inner model with measurable cardinals of arbitrarily large Mitchell order.[64] Thus, by going high enough in the Mitchell order, $\mathbf{ZFC}^{\mathsf{CIMH}}$ provides a fair interpretation of $\mathbf{ZFC}$+"There exist α-many measurables".

Moreover $\mathbf{ZFC}^{\mathsf{CIMH}}$ also *maximises* over $\mathbf{ZFC}$+"There exist α-many measurables", since there are always sets outside this interpretation. In particular, since $\mathbf{ZFC}^{\mathsf{CIMH}}$ implies that there are no inaccessible cardinals, for any particular β that is measurable in our interpretation, the interpretation misses out the sets witnessing the accessibility of β. Clearly, the two theories are also inconsistent with one another.

It just remains to show that $\mathbf{ZFC}$+"There exist α-many measurables" does not maximise over $\mathbf{ZFC}^{\mathsf{CIMH}}$ (for inconsistent maximisation), nor can any consistent extension (for strong maximisation). These are established by the following claim:

[61] In the original [Maddy, 1998].

[62] See here [Löwe, 2001], [Löwe, 2003], and [Incurvati and Löwe, 2016] (which responds to some criticisms of [Hamkins, 2014]).

[63] A brief note on nomenclature: In set theory is usual to refer to theories that do not have class variables as first-order, and those that do as second-order. This is despite the fact that, strictly speaking, $\mathbf{NBG}$ and its cousins are two-sorted first-order theories, even if they could be given a second-order formulation in which we quantify into predicate position.

[64] Note: Friedman, Welch, and Woodin are explicit about the fact that none of their theorems depend on arbitrary outer models, but rather could be formulated in terms of the CIMH. See [Friedman et al., 2008] pp. 391–392.

Claim 55. No consistent extension of $\mathbf{ZFC}$+"There exist α-many measurables" can provide a fair interpretation of $\mathbf{ZFC}^{\mathsf{CIMH}}$.

To show this, we need to show that under any extension of $\mathbf{ZFC}$+"There exist α-many measurables", none of (i) there is an inner model of $\mathbf{ZFC}^{\mathsf{CIMH}}$, (ii) there is a truncation at an inaccessible with $\mathbf{ZFC}^{\mathsf{CIMH}}$, or (iii) there is a truncation at an inaccessible of an inner model with $\mathbf{ZFC}^{\mathsf{CIMH}}$, are possible. For (i) it suffices to note that being accessible is upwards absolute. Since all cardinals are accessible under $\mathbf{ZFC}^{\mathsf{CIMH}}$, if $\mathbf{ZFC}^{\mathsf{CIMH}}$ holds in an inner model, then all cardinals are accessible, ruling out (i). For (ii) and (iii) note that if κ is inaccessible, then any inner model C of V_κ contains a proper class of wordly cardinals[65], since $(C, \in, \mathcal{P}^L(C)) \models \mathbf{MK}$, and $\mathbf{MK}$ implies that there is a proper class of worldly cardinals. However, the CIMH implies that there is a definable inner model of the form $L[R]$, where R is a real, with no worldly cardinals (see Theorem 15 of [Friedman, 2006]). Thus neither a truncation at an inaccessible nor a truncation of an inner model at an inaccessible can satisfy $\mathbf{ZFC}^{\mathsf{CIMH}}$, proving Claim 55 and hence Proposition 54.[66] □

We can thus see that the CIMH has maximising properties with respect to large cardinals, and shows them to be restrictive in a precise sense.[67]

Turning to the $\mathbf{FSST}$ case, we can again get restrictiveness properties, given a modification of Maddy's definition. Maddy considers extensions of $\mathbf{ZFC}$, and since $\mathbf{FSST}$ contradicts $\mathbf{ZFC}$, it is unclear how we can use Maddy's notion of restrictiveness without a change of definition. In what follows, then, we will consider Maddy's definition as applying to theories extending $\mathbf{ZFC}^-$.

Of course, neither $\mathbf{FSST}$ nor $\mathbf{FSST}$+ASGA can maximise over $\mathbf{ZFC}$ or its cousins, since the latter are strictly stronger in consistency strength. However, one might nonetheless adopt the existence of various mice as new axioms:

Proposition 56. $\mathbf{FSST}$+"$0^\sharp$ exists" strongly maximises over $\mathbf{ZFC}$+"There exists a proper class of inaccessible cardinals".

Proof. We can (exactly as in Proposition 47) iterate the ultrafilter U provided by $0^\sharp$ (conceived of as a mouse) through the (countable) ordinals, since $0^\sharp$ exists and $\mathbf{FSST}$ has the resources to construct the ultrapower. Under the existence of $0^\sharp$, ω_1 (which in this case is Ord) is Mahlo in L, and so there is a proper class of L-inaccessible cardinals (in fact a stationary class thereof) in L. Thus $\mathbf{FSST}$+"$0^\sharp$ exists" provides a fair interpretation of $\mathbf{ZFC}$+"There exists a proper class of inaccessible cardinals". $\mathbf{FSST}$+"$0^\sharp$ exists" also clearly maximises over this theory, since $\mathbf{ZFC}$-based theories will always miss out collapsing functions, and is trivially inconsistent with $\mathbf{ZFC}$-based set theories. All that is needed to show strong maximisation is then that no extension of $\mathbf{ZFC}$ can provide a fair interpretation of $\mathbf{FSST}$+"$0^\sharp$ exists", but this is immediate, since any transitive interpretation of $\mathbf{FSST}$-based theories contains only hereditarily countable sets.[68] □

[65] β is *worldly* iff $V_\beta \models \mathbf{ZFC}$.

[66] I am grateful to Kameryn Williams for some useful discussions concerning this proof.

[67] Of course, for stronger large cardinals that are capable of proving the CIMH consistent (e.g. anything stronger than the existence of a Woodin cardinal with an inaccessible above), it is not possible to provide a fair interpretation of those large cardinals within $\mathbf{ZFC}^{\mathsf{CIMH}}$ alone. However, if we were to augment our theory (somewhat artificially) with the claim that there is a definable inner model for the large cardinals (say by asserting the existence of the required mouse), then parallel reasoning would yield the same restrictiveness result. This can only be done up to a point since the CIMH contradicts PD, however there is the possibility of modifying the CIMH to make it consistent with PD §5.

[68] For ease we have stated this proof just for inaccessibles in L. However, exactly the same argument can be generalised to any large cardinal properties holding in an L-like model under the existence of a suitable countable mouse.

Thus, both the CIMH and FSST-based set theories seem to have maximisation properties over set theories with large cardinals, lending credence to the claim that there is a genuine sense in which some WA-Principles maximise subsets as compared to MELC-Principle-inspired set theory.

5 The proper place for large cardinals

We have thus far argued that (i) there is a distinction between Width Absoluteness Principles and asserting Maximality through the Existence of Large Cardinals, (ii) that the iterative conception seems to speak in favour of WA-Principles for maximising subsets at successor stages, and (iii) that MELC-inspired set theory seems restrictive as compared to some set theories inspired by WA-Principles. However, we do not wish to deny that large cardinals are rightly viewed as some of the most central principles in set-theoretic mathematics. There is thus a remaining substantive question: What happens to the study of large cardinals if we *do* adopt one of these anti-large-cardinal perspectives? In this section, we'll examine some uses of large cardinals in foundational discussions and argue that these uses are not necessarily threatened by the arguments we've put forward.

The uses of large cardinals in foundations.[69] We have already seen two uses for large cardinals in §1: (1.) To index the consistency strength of theories in a linearly ordered fashion, and (2.) To provide a framework theory that maximises interpretative power.

Point (1.) can be dealt with very quickly. In order to study the consistency strengths of mathematical theories, we only require that the theories be true in *some* model or other, not necessarily in V. More generally, note that there are the following 'levels' to where an axiom A can be true:

(i) A could be true in V.

(ii) A could be true in an inner model.

(iii) A could be true in a transitive model.

(iv) A could be true in a countable transitive model.

(v) A could be true in some model (whatever it may be).

For consistency statements, any model will do, and so any of (i)–(v) are acceptable places for considering A. As we have seen, there is no obstacle to having any of (ii)–(v) for the friend of anti-large-cardinal principles.

Point (2.) can also be dealt with reasonably easily. In order to maximise interpretive power we just need some appropriately 'nice' or 'standard' (e.g. well-founded, containing all ordinals) place where the relevant mathematics can be developed. But the observations of §4 show that we can perfectly well have large cardinals in inner models in many anti-large cardinal frameworks, and so there is not necessarily any loss of interpretive power; we can always assert (and often prove) that large cardinals exist in inner models, even if not in V. Thus any interpretability work that could be done using a large cardinal could be done by a large cardinal in an inner model on the frameworks we have considered.

[69] See also [Arrigoni and Friedman, 2013] for discussion of some of these uses.

We will consider a further two uses for large cardinals in foundations; (3.) large cardinals are used in the case for so called "axioms of definable determinacy", and (4.) large cardinals are used to build certain kinds of models (e.g. in the inner model programme that tries to construct L-like models for large cardinals).

For (3.), the full details will be familiar to specialists and obscure to non-specialists, so we omit them here.[70] Nonetheless, a coarse description will be helpful in stating our arguments. Roughly put, axioms of definable determinacy assert (schematic) claims about second-order arithmetic, postulating the existence of winning strategies for games played with natural numbers[71]. Importantly, some authors have argued that these axioms have various pleasant consequences we would like to capture.[72] One salient fact is that Projective Determinacy yields high degree of completeness for the hereditarily countable sets [i.e. there are no known statements apart from Gödel style diagonal sentences independent from the theory $\mathbf{ZFC} + \mathsf{PD} + V = H(\omega_1)$].[73] Moreover, whilst it is a theorem of **ZFC** that not all games are determined, certain restricted forms can be proved from large cardinals. For example:

Theorem 57. Borel Determinacy is provable in **ZFC**, but any proof requires ω_1-many applications of the Powerset Axiom.

Theorem 58. Analytic Determinacy is provable in **ZFC**+"There exists a measurable cardinal", but is independent from **ZFC**.

Theorem 59. Projective Determinacy is provable in **ZFC**+ "For every $n \in \mathbb{N}$, there are n-many Woodin cardinals", but is independent from **ZFC**+ "There exists a measurable cardinal".

Theorem 60. The Axiom of Determinacy for $L(\mathbb{R})$ is provable in **ZFC**+ "There are ω-many Woodin cardinals with a measurable above them all", but is not provable in **ZFC**+ "For every $n \in \mathbb{N}$, there are n-many Woodin cardinals"

Again, we will not go through the definitions of Borel, Analytic, Projective, or $L(\mathbb{R})$ here. Suffice to say, each admits progressively more sets of reals with a more permissive notion of definability, and each is resolved by strictly stronger large cardinal axioms. So, assuming that out 'best' theory of sets should contain axioms of definable determinacy, it remains to explain how we might obtain them in the absence of large cardinals.

Our core point is that it is *not* the case that a principle having anti-large cardinal features *immediately* disqualifies the justificatory case for PD found in the literature. This is because axioms of definable determinacy do not require the *literal truth* of large cardinal axioms, but rather only the truth of the large cardinals axioms in inner models. Generally speaking this is where there are equivalences (rather than strict implications from the large cardinals to axioms of definable determinacy). For example[74]:

Theorem 61. (Woodin) The following are equivalent:

[70] The interested reader is directed to [Schindler, 2014] for a recent presentation of the technical details, and [Koellner, 2006], [Maddy, 2011], and [Koellner, 2014] for a philosophical discussion.

[71] There are also versions of determinacy for real-valued games, or games of longer length. We put aside these issues here.

[72] See, for example, [Maddy, 2011] and [Welch, 2017].

[73] [Koellner, 2014] provides a detailed survey of the literature here, and is quick to point out that axioms of definable determinacy seem to be the consequence of any strong 'natural' theory extending **ZFC** (e.g. **ZFC** + PFA). Given the focus of this paper, we shall concern ourselves only with the argument from large cardinals.

[74] For a list see [Koellner, 2011].

(a) Projective Determinacy (schematically rendered).

(b) For every $n < \omega$, there is a fine-structural, countably iterable inner model $\mathfrak{M}$ such that $\mathfrak{M} \models$ "There are n Woodin cardinals".

Thus it may very well be the case that PD holds, there are plenty of Woodin cardinals in inner models, but no actual Woodin cardinals in V. More must be done to argue why the existence of such models must be *explained* by truth of the large cardinals, rather than the apparent consistency of the practice.[75] The friend of anti-large cardinal principles may acknowledge that the existence of an inner model theory is good evidence that the axiom is consistent (perhaps even in an inner model), agreeing that the diverse theoretical relationships between models of large cardinals and axioms of definable determinacy are evidence for the consistency of the practice. For them, however, this consistency is to be explained by the existence of an inner model rather than the strict truth of the axiom. Perhaps a supplementary argument can be provided. However, for the moment, any such claim stands in need of support and clarification.

For the specific theories $\mathbf{FSST}$ and $\mathbf{NBG} + \mathsf{CIMH}$ that we have considered, two additional points are salient. For $\mathbf{FSST}$, if we add PD to $\mathbf{FSST}$ we would obtain a *highly* complete set theory, since (as noted above) PD implies a high degree of completeness concerning the hereditarily countable sets. One might think that in the case of $\mathbf{FSST}$ that this provides additional strength to the 'extrinsic' justification of PD from completeness; we would obtain completeness about our whole theory of sets, not just a substructure thereof.

For the CIMH, the issue is somewhat subtle. Whilst there is no prima facie reason why such principles would interfere with a case for axioms of definable determinacy, as a matter of fact the CIMH implies that PD is false outright.[76] However, it is open whether there could be an CIMH-like principle with anti-large cardinal features that is nonetheless consistent with axioms of definable determinacy.[77] So it is not obviously the case that it is the anti-large cardinal features or even the status of the CIMH

[75]This is perhaps what lies behind the following idea of Woodin:

> "**A Set Theorist's Cosmological Principle:** The large cardinal axioms for which there is an inner model theory are consistent; the corresponding predictions of unsolvability are true because the axioms are true." ([Woodin, 2011], p. 458)

Woodin's idea is that on the basis of consistency statements, we can make predictions. For example, "There will be no discovery of an inconsistency in the theory $\mathbf{ZFC}$+"There is a Woodin cardinal" in the next 10'000 years" is a prediction ratified by the truth of the theory $\mathbf{ZFC}$+"There is a proper class of Woodin cardinals".

[76]This is because the CIMH implies that it is not the case that for every real x, $x^\sharp$ exists.

[77]For a somewhat speculative example, suppose that one is moved by justifications for Woodin cardinals and adopts $\mathbf{ZFC}$+"There is a proper class of Woodin cardinals" as one's canonical theory of sets. Suppose further that one holds that some IMH-like principle should hold on the basis of absoluteness considerations. We might then formulate the following principle:

Definition 62. *The Class-Generic Inner Model Hypothesis for Woodins* (CIMHW) states that if (first-order, parameter-free) ϕ is true in an inner model of a tame class-generic extension of V containing a proper class of Woodin cardinals, then ϕ is true in a inner model of V.

Assuming that the existence of a proper class of Woodin cardinals can be given an inner model theory (i.e. there is a model of the form $L[E]$ such that $L[E] \models$ "There is a proper class of Woodin cardinals"), the results of [Friedman, 2006] (in particular Theorem 15) might well then be generalised to show that over the base theory $\mathbf{ZFC}$+"There is a proper class of Woodin cardinals", the Inner Model Hypothesis for Woodins implies that there is no inaccessible limit of Woodin cardinals in V in the presence of PD. Since the required inner-model-theoretic questions are still to be answered here, we will leave this question open.

as a WA-Principle that results in the falsity of PD, since there may well be similar principles with these features consistent with PD.

Another use of large cardinals (4.) is in the building and studying of different models. In particular, we want to construct various 'L-like' inner models from large cardinals: For many large cardinal axioms we can (using large cardinals) build a model containing the cardinal, but also with a good deal of information (in particular, these L-like models satisfy various so called 'fine-structural' properties). Again, the details are rather technical, so we omit them.[78] The point is the following: Often in set theory we have very little information about the properties of certain sets, as exhibited by the independence phenomenon. This is not so for large cardinals with L-like inner models, where (whilst there are open questions) there is a large amount of highly tractable information concerning the objects. The construction of inner models from large cardinals thus represents an important and technically sophisticated area of study. Can we construct these models when we don't have the literal truth of the large cardinals?

A response to this is available in a similar fashion to (3.); there are no obvious obstacles to having various kinds of model within an anti-large cardinal framework. In fact, these fine-structural model building techniques are performed by iterating countable mice, and again one can have the relevant mice in play and construction of the ultrapower (pace, of course, the consistency of the existence of these mice with the theory in question). Thus it is not clear that building various kinds of models really requires the *truth* of a large cardinal axiom.

6 Open questions and concluding remarks

Thus far we have argued that:

(1.) There is a distinction to be made between maximality conceived of through large cardinal principles (the MELC-Principle) and maximality as conceived of through width absoluteness (WA-Principles).

(2.) The iterative conception seems to speak in favour of maximising subsets at successor stages, and *then* iterating this process as far as possible, and thus supports WA-Principles.

(3.) There are set theories based upon WA-Principles on which large cardinals are false but consistent, and serve to leave out subsets. This challenges the claim that large cardinals are genuine maximality principles.

(4.) This point can be further codified by showing that large cardinals are restrictive in senses derived from Maddy's notion of restrictiveness.

(5.) Large cardinals can still play their usual foundational roles in these anti-large cardinal frameworks, despite their falsity (putting aside some questions regarding possible modifications of the CIMH to incorporate PD).

Do we wish to repudiate large cardinals on this basis as definitively false? We wish to emphasise that this is *not* our intention. All we have identified is that the fact that large cardinals appear to postulate 'big' sets cannot be taken to straightforwardly imply that they are maximality principles, since there seem to be conceptions

[78] For the state of the art concerning inner model theory and the challenges faced, see [Sargsyan, 2013] and [Woodin, 2017].

of maximality (given by the WA-Principles we discussed) on which they serve to restrict the subsets we consider. It may well be that more precision concerning the notion of maximality in set theory is able to rehabilitate large cardinal axioms as genuine maximality principles. We therefore ask:

Question 63. Is there a criterion for 'height maximality' that can operate more independently of background theory, unlike the MELC-Principle?

One conjecture is to say that only those height principles that are downward absolute to inner models should definitely count as maximising height, other principles are too dependent on prior commitments regarding the (possibly presently unclear) notion of arbitrary subset. Thus, any large cardinal absolute between V and L should count as definitely height-maximising.[79] This idea, however, does not get us very high in the large cardinal hierarchy (it would fail to take us outside L).

It thus an interesting question whether there are other criteria that would allow us to draw a clean distinction between 'height' and 'width'. One suggestion is to regard large cardinals themselves as species of a kind of absoluteness as understood as reflection principles (as suggested in [Bagaria, 2012]). Given this, the results of the current paper would suggest there is a kind of *tension* in the iterative conception between different kinds of *absoluteness*. This is further suggested by the fact that second-order reflection is inconsistent with the CIMH in the $\mathbf{NBG}$-setting (since it implies the existence of inaccessibles). Moreover, in the $\mathbf{NBG}^-$-setting, the existence of ω_1 can be seen as an instance of reflection. More precisely over $\mathbf{NBG}^-$, we might formulate:

Definition 64. ($\mathbf{NBG}^-$) Let *weak second-order reflection* be the claim that if $\phi(A)$ is a sentence of $\mathbf{NBG}^-$ in the parameter A, then if $\phi(A)$ holds then there is a transitive set S such that $(S, \in, \mathcal{P}(S)) \models \phi(A)^S$.

Since the universe is uncountable, this principle would imply (over $\mathbf{NBG}^-$) that ω_1 exists (in fact, that there is a proper class of uncountable cardinals). This further suggests that there is a tension between height and width absoluteness, since this weak form of height absoluteness is inconsistent with the comparatively strong width absoluteness given by the ASGA.

Perhaps a greater understanding of these questions will in turn yield a better understanding of how maximality in set theory is linked to absoluteness, and whether there is an optimal middle-ground to be found. For now we conclude that it is at least unclear how large cardinals are related to notions of height maximality.

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[79] I thank Sy-David Friedman for pressing this suggestion.

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